Compactification of QCD_{4} to QCD_{2} in a flux tube

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The Compactification of QCD4 to QCD2 in a Flux Tube

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We show from the action integral that in the special environment of a flux tube, QCD4 in (3+1) dimensional space-time can be approximately compactified into QCD2 in (1+1) dimensional space-time. In such a process, we find out how the coupling constant g_{2D} in QCD2 is related to the coupling constant g_{4D} in QCD4. We show how the quark and the gluon in QCD2 acquire contributions to their masses arising from their confinement within the tube, and how all these quantities depend on the excitation of the partons in the transverse degree of freedom. The compactification facilitates the investigation of some dynamical problems in QCD4 in the simpler dynamics of QCD2 where the variation of the gluon fields leads to a bound state.

I. INTRODUCTION

Previously, t'Hooft showed that in the limit of large N_c with fixed $g^2 N_c$ in single-flavor QCD4, planar diagrams with quarks at the edges dominate, whereas diagrams with the topology of a fermion loop or a wormhole are associated with suppressing factors of $1/N_c$ and $1/N_c^2$, respectively [1]. In this case a simple-minded perturbation expansion with respect to the coupling constant g cannot describe the spectrum, while the $1/N_c$ expansion may be a reasonable concept, in spite of the fact that N_c is equal to 3 and is not very big. The dominance of the planar diagram allows one to consider QCD in one space and one time dimensions (QCD2) and the physics resembles those of the dual string or a flux tube, with the physical spectrum of a straight Regge trajectory [2]. Since the pioneering work of t'Hooft, the properties of QCD in two-dimensional space-time have been investigated by many workers [1–16].

The flux tube picture of longitudinal dynamics is phenomenologically supported in hadron spectroscopy [17], in hadron collisions, and in e^+e^- annihilations at high energies [18–24]. In these high-energy processes, the average transverse momenta of produced hadrons are observed to be limited, of the order of a few hundred MeV. In contrast, the longitudinal momenta of the produced hadrons can be very large, as described by a rapidity plateau with a large average longitudinal momentum. This average longitudinal momentum increases with the collision energy. The limitation of the average transverse momenta of the produced hadrons means that the average momenta of partons in produced hadrons are also limited,[1] consistent with the picture that the produced partons as constituents of the produced hadrons are transversely confined in a flux tube. Further idealization of the three-dimensional flux tube as a one-dimensional string leads to the picture of the particle production process as a string fragmentation in (1+1) space-time dimensions. The particle production description of Casher, Kogut, and Susskind [18] in (1+1) dimensional Abelian gauge theory led to results that mimics the dynamics of particle production in hadron collisions and in the annihilation of e^+e^- pairs at high energies. Furthermore, the Lund model of classical string fragmentation has been quite successful in describing quantitatively the process of particle production in these high energy processes [19, 22].

With the successes of the theoretical description of Casher *et al.* and the Lund model of string fragmentation, it should be possible to compactify quantum chromodynamics in (3+1) dimensional space-time (QCD4) approximately to quantum chromodynamics in (1+1) dimensional space-time (QCD2), in the special environment appropriate for particle production at high-energies. It is useful to examine the circumstances under which such a compactification is possible. Such a link was given earlier in [23, 24] and reported briefly in [25]. Here, we would like to examine the problem from the more general viewpoint of the action integral.

We note that the process of string fragmentation occurs when a valence quark-antiquark pair pull part from each other at high energies, as described in [18, 19]. It is therefore reasonable to examine the QCD4 compactification under the dominance of longitudinal dynamics in the center-of-mass frame of the receding valence $q\bar{q}$ pair. Under such a longitudinal dominance in this frame, not only are the magnitudes of the longitudinal momenta of the leading valence quark and antiquark dominant over their transverse momenta, so too are the magnitudes of longitudinal

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[1] Even though the average transverse momenta of the partons are limited, the tails of the parton transverse momentum distribution of partons in the produced hadrons can still extend to the high p_T region, but with small probabilities.

momenta of the produced $q\bar{q}$ parton pairs. The spatially one-dimensional string is an idealization of a more realistic three-dimensional flux-tube. The description of produced $q\bar{q}$ parton pairs residing within the string or flux tube presumes the confinement of these produced partons in the string. Hence, it is reasonable to examine further the QCD4 compactification under transverse confinement. As transverse confinement is a nonperturbative process and is beyond the realm of perturbative QCD, we can describe the transverse confinement property in terms of a confining scalar interaction $S(\boldsymbol{r}_\perp)$ in transverse coordinates $\boldsymbol{r}_\perp$, with the quark mass function described by $m(\boldsymbol{r}_\perp){=}m_0{+}S(\boldsymbol{r}_\perp)$ where m_0 is the quark rest mass.

Having spelled out explicitly the circumstances under which the QCD4 compactification may occur, we proceed to start with the QCD4 action integral and begin our process of compactification. We need to find out how we can relate the field variables in four-dimensional space-time to those in 2-dimensional space-time in such a way that the four-dimensional action integral can be simplified to contain only field quantities in two-dimensional space-time. What is the form of the two-dimensional action integral after compactification? How are the coupling constant g_{2D} in the two-dimensional action integral related to the coupling constant $g{=}g_{4D}$ in QCD4 in four-dimensional space-time? Are there additional terms in the two-dimensional action integral that arise from the compactification? How do all these quantities depend on the excitation of the partons in the transverse degrees of freedom?

We shall show that the compactification for QCD4 in a flux tube leads to an action integral of a QCD gauge field coupled to the quark field in two-dimensional space time, which can be appropriately called QCD2. The QCD gauge field coupling constant is found to depend on the quark transverse wave function the flux tube. There are additional quark- and gluon-mass terms that arise from the confinement of the quark and the gluon within the tube.

The success of the compactification program facilitates the examination of some problems in QCD4 in the simpler dynamics of QCD2. The QCD2 action integral allows one to obtain the equations of motion for the quark field and the gauge field. We find self-consistent solution of a boson state with a mass in the flux tube environment, similar to Schwinger's solution of a massive boson in two-dimensional Abelian gauge field theory.

It should be noted that the occurrence of a massive composite bound state in gauge field theories has been known in many previous investigations [26]. While the basic principles of the massive bound state as arising from interactions of the gauge fields in these theories are the same as in the present investigation in a flux tube, the physical environments and the constraints are quite different. How the massive boson in a flux tube environment examined here can be related to the massive boson formed by purely gluons as a pole in the three gluon vertex in 4-dimensional space-time [27, 28] is a subject worthy of further investigation.

This paper is organized as follows. In Sec. II, we show how the action integral in QCD4 can be compactified into QCD2, under the assumption of longitudinal dominance and transverse confinement. The relationship between the 4-dimensional quantities and those two-dimensional counterpart are expressed explicitly. The fermions and gauge bosons acquire contributions to their masses that arises from the confinement. In Sec. III, we solve the Dirac field equation in $(1+1)$ space-time, and obtain the relation between the current and the gauge field. In Sec. IV, we examine the gauge field degrees of freedom in two-dimensional space-time. In Sec. V, we determine the equation of transverse motion for fermions in a tube. In Sec. VI, we present our conclusions and discussions.

II. $4D \to 2D$ COMPACTIFICATION IN THE ACTION INTEGRAL

We employ the convention that before compactification is achieved, all field quantities and gamma matrices are in four-dimensional space-time unless specified otherwise. With fermions interacting with an SU(N) gauge field and a scalar field $m(x)$ in the $(3+1)$ Minkowski space-time, the SU(N) gauge invariant action integral $\mathcal{A}$ is given by [32]

$$\mathcal{A} = \int d^4x \left\{ Tr\left[\frac{1}{2}\left[\bar{\Psi}\gamma^\mu \Pi_\mu\ \Psi - \bar{\Psi}\ m(x)\ \Psi \right] - \frac{1}{2}\left[\bar{\Psi}\gamma^\mu \overleftarrow{\Pi}_\mu\ \Psi + \bar{\Psi}\ m(x)\ \Psi \right] \right] - \frac{1}{4}F^a_{\mu\nu}\ F^{\mu\nu}_a \right\}, \tag{1}$$

where A^a_μ and Ψ are the gauge and fermion fields respectively in the Minkowski $(3+1)$-dimensional space-time with coordinates $x \equiv x^\mu = (x^0, \boldsymbol{x}) = (x^0, x^1, x^2, x^3)$ and transverse coordinates $\boldsymbol{r}_\perp = (x^1, x^2)$. Here in Eqs. (1) ,

$$\Pi_\mu \ = \ i\partial_\mu + g_{4D}\ T_a A^a_\mu,$$

$$F^a_{\mu\nu} \ = \ \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_{4D}\ f^a_{bc}\ A^b_\mu\ A^c_\nu, \tag{2}$$

$$\equiv \ \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ig_{4D}[A^b_\mu,\ A^c_\nu]^a, \tag{3}$$

γ^ν are the standard Dirac matrices, $\partial_\mu = (\partial/\partial t, \nabla)$, and $a, b, c = 1 \ldots N^2 - 1$ are SU(N) group indices. We use the signature of (1, -1, -1, -1) for the diagonal elements of metric tensor $g_{\mu\nu}$. We should note that the action integral is gauge invariant since $m(x)$ is independent of the SU(N) group generators T_a.

A. Fermion part of the action integral

The $4D$-action integral $\mathcal{A}$ resides in four-dimensional $(3+1)$ space-time. There are environments in which the full four-dimensional space-time is necessary, as for example in the discussion of the phase transition in a hot quark-gluon plasma [29–31]. There are however environments which are susceptible for compactification to two-dimensional $(1+1)$ space-time, in which the dynamics can be greatly simplified.

A proper environment for the compactification of QCD4 can be found in the special case in which a valence quark and antiquark pull part from each other at high energies, as in the case examined by Casher, Kugut, and Susskind [18]. It is convenient to work in the center-of-mass frame of the receding quark-antiquark pair in which the magnitudes of the longitudinal momenta of the valence quark pairs are very large, much larger than the magnitudes of their transverse momenta. Under such a dominance of longitudinal dynamics, not only are the magnitudes of the longitudinal momenta of the leading valence quark and antiquark pair large, so are also those of the produced q and $\bar{q}$ partons. It is then convenient to choose the Lorentz gauge

$$\partial^\nu A_\nu^a = 0. \tag{4}$$

In this Lorentz gauge, A_ν^a is given by an integral of the current J_ν^a. For a system with longitudinal dominance, the magnitudes of the transverse currents are much smaller than the magnitudes of the longitudinal currents. As a consequence, the magnitudes of the gauge field transverse components, A_1^a and A_2^a, along the transverse directions are small in comparison with those of A_0^a and A_3^a. The gauge field components A_1^a and A_2^a can be neglected. The absence of the transverse components of the gauge fields in the Lorentz gauge provides a needed simplification for compactification. However, both A_0^a and A_3^a still depend on the $4D$ space-time variables, $A_0^a(x^0, \boldsymbol{x})$, $A_3^a(x^0, \boldsymbol{x})$.

The dominance of the longitudinal motion implies that the valence leading quark and anti-quark lie inside a longitudinal tube. The limiting average transverse momentum suggests further that the produced quarks reside within the longitudinal tube with a radius inversely proportional to this limiting average transverse momentum. As the confinement of the produced quarks within the tube is a nonperturbative process that is beyond the realm of perturbative QCD, we can represent the confinement property in terms of a confining scalar interaction $S(\boldsymbol{r}_\perp)$ in transverse coordinates $\boldsymbol{r}_\perp$, with the quark mass function $m(\boldsymbol{r}_\perp) = m_0 + S(\boldsymbol{r}_\perp)$. The origin of $\boldsymbol{r}_\perp$ coordinates lies along the longitudinal axis of the receding valence quark pair. Because of the presence of a scalar interaction $m(\boldsymbol{r}_\perp)$, our dynamical problem does not maintain general Lorentz in all directions. There remains however approximate Lorentz invariance with respect to a finite boost along the longitudinal axis and the range of this finite boost increases as the energy of the receding quark pair increase.

Under such circumstances, we can carry out the compactification of QCD4 in $(3+1)$ dimensions as follows. The fermion part of the $4D$-action $\mathcal{A}_F$ in (1) is given by

$$\mathcal{A}_F = Tr \int d^4x \left\{ \frac{1}{2} \bar{\Psi} \gamma^\mu \Pi_\mu \Psi - \frac{1}{2} \bar{\Psi} \gamma^\mu \overleftarrow{\Pi}_\mu \Psi - \bar{\Psi} m(\boldsymbol{r}_\perp) \Psi \right\}, \tag{5}$$

where $\mu = 0, 1, 2, 3$ and γ^μ is the $4D$-Dirac matrices,

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{6}$$

To relate the field variables in four-dimensional space-time to those in 2-dimensional space-time in such a way that the four-dimensional action integral can be simplified, we write the Dirac fermion field $\Psi(x)$ in terms of the following bispinor with transverse functions $G_\pm(\boldsymbol{r}_\perp)$ and x^0-x^3 functions $f_\pm(x^0, x^3)$ [21],

$$\Psi(x) \equiv \begin{pmatrix} \varphi(x^0; \boldsymbol{x}) \\ \chi(x^0; \boldsymbol{x}) \end{pmatrix} \equiv \begin{pmatrix} \varphi_1(x^0; \boldsymbol{x}) \\ \varphi_2(x^0; \boldsymbol{x}) \\ \chi_1(x^0; \boldsymbol{x}) \\ \chi_2(x^0; \boldsymbol{x}) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} G_1(\boldsymbol{r}_\perp) \left(f_+(x^0; x^3) + f_-(x^0; x^3) \right) \\ -G_2(\boldsymbol{r}_\perp) \left(f_+(x^0; x^3) - f_-(x^0; x^3) \right) \\ G_1(\boldsymbol{r}_\perp) \left(f_+(x^0; x^3) - f_-(x^0; x^3) \right) \\ G_2(\boldsymbol{r}_\perp) \left(f_+(x^0; x^3) + f_-(x^0; x^3) \right) \end{pmatrix}, \tag{7}$$

where $\boldsymbol{r}_\perp$ is a vector in the plane perpendicular to the x^3 axis. Using this explicit form of the Dirac bispinor Ψ, we

can carry out simplifications (with detailed derivation given in Appendix A) that lead from Eq. (5) eventually to

$$
\begin{aligned}
\mathcal{A}_F = \; & Tr \int d^2X \left\{ \frac{1}{2}\bar{\Psi}(2D,X) \left[i\gamma^{\mu}(2D)\partial_{\mu} + g_{2D}\gamma^{\mu}T_a A_{\mu}^a(2D,X) \right] \Psi(2D,X) \right. \\
& - \frac{1}{2}\bar{\Psi}(2D,X) \left[i\gamma^{\mu}(2D)\overleftarrow{\partial}_{\mu} - g_{2D}\gamma^{\mu}(2D)T_a A_{\mu}^a(2D,X) \right] \Psi(2D,X) \\
& \left. - \bar{\Psi}(2D,X)\, m_{qT}\, \Psi(2D,X) \right\} \equiv \mathcal{A}_F(2D), \qquad \mu = 0,3,
\end{aligned}
\tag{8}
$$

where we have introduced in the Dirac fermion field $\Psi(2D,X)$, γ-matrices, and metric tensor $g_{\mu\nu}$, according to the following specifications in the $(1+1)$-dimensional QCD2 space-time

$$
\Psi(2D,X) = \begin{pmatrix} f_+(X) \\ f_-(X) \end{pmatrix}, \qquad X = (x^0; x^3),
\tag{9}
$$

$$
\gamma^0(2D) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^3(2D) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad g_{\mu\nu}(2D) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\tag{10}
$$

The $2D$ coupling constant, g_{2D}, is defined by the following equation (see Appendix A)

$$
g_{2D} = \int dx^1 dx^2 g_{4D}[|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{3/2},
\tag{11}
$$

where the transverse wave functions $G_{1,2}(\boldsymbol{r}_\perp)$ are normalized according to

$$
\int dx^1 dx^2 \left(|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2 \right) = 1.
\tag{12}
$$

In the special case of the transverse ground state, we can approximate the transverse density by a uniform distribution with a sharp transverse radius $R_{T\mathrm{sharp}}$,

$$
\left(|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2 \right) \sim \frac{1}{\pi R_{T\mathrm{sharp}}^2}\Theta(R_{T\mathrm{sharp}} - |\boldsymbol{r}_T|),
\tag{13}
$$

we then obtain for a sharp distribution in the transverse ground state the approximate relation [23]

$$
g_{2D} \sim \frac{g_{4D}}{\sqrt{\pi}R_{T\mathrm{sharp}}}.
\tag{14}
$$

If we characterize the transverse ground state with a Gaussian profile and a root-mean-square transverse radius $R_T = \sqrt{2}\sigma_T$ as

$$
\left(|G_1(\vec{r}_\perp)|^2 + |G_2(\vec{r}_\perp)|^2 \right) = \frac{1}{2\pi\sigma^2}\exp\{-\frac{r^2}{2\sigma_T^2}\},
\tag{15}
$$

then the corresponding g_{2D} coupling constant becomes

$$
g_{2D} = \frac{g_{4D}}{R_T}\sqrt{\frac{2}{9\pi}}.
\tag{16}
$$

The transverse quark mass m_{qT} in Eq. (8) is given by (see Appendix A)

$$
m_{qT} = \int dx^1 dx^2 \left\{ m(\boldsymbol{r}_\perp)\left(|G_1(\boldsymbol{r}_\perp)|^2 - |G_2(\boldsymbol{r}_\perp)|^2 \right) + (G_1^*(\boldsymbol{r}_\perp)(p_1 - ip_2)G_2(\boldsymbol{r}_\perp)) - (G_1(\boldsymbol{r}_\perp)(p_1 + ip_2)G_2^*(\boldsymbol{r}_\perp)) \right\}.
\tag{17}
$$

The transverse quark mass m_{qT} contains a contribution from the quark rest mass (through $m(r)$), in addition to a contribution arising from the confinement of the quark in the flux tube (through the confining wave functions $G_{1,2}(\boldsymbol{r}_\perp)$). In obtaining these results, we have considered $2D$ gauge fields $A_{\mu}^a(2D, x^0, x^3)$ to be related to the $4D$-field gauge fields $A_{\mu}^a(x^0, x^3, \boldsymbol{r}_\perp)$ by

$$
A_{\mu}^a(x^0, x^3, \boldsymbol{r}_\perp) = \sqrt{|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2}\, A_{\mu}^a(2D, x^0, x^3), \quad \mu = 0,3.
\tag{18}
$$

The above equation means that along with the confinement of the fermions, for which the wave function $G_{1,2}(\boldsymbol{r}_\perp)$ is confined within a finite region of transverse coordinates $\boldsymbol{r}_\perp$, the gauge field $A_{\mu}^a(x)$, $\mu = 0,3$, is also considered to be confined within the same finite region of transverse coordinates, as in the case for a flux tube. Note that because of the longitudinal dominance, we have assumed that $A_{\mu}^a(x^0, x^3, \boldsymbol{r}_\perp) = 0$ for $\mu = 1,2$.

B.   Gauge field part of the action integral

Having reduced the fermion part of the action integral $\mathcal{A}_F$, we come to examine the gauge field part of the action integral $\mathcal{A}_A$,

$$\mathcal{A}_A = -\frac{1}{4}\int d^4x F_{\mu\nu}^a \ F_a^{\mu\nu}. \tag{19}$$

Our task is to find out what will be the form of $\mathcal{A}_A$ involving the gauge fields $A_\mu(2D)$ the two-dimensional space-time, when $A_\mu(2D)$ the A_μ in four-dimensional space-time are related by Eq. (18).

In Eq. (19) the summation over μ, ν includes terms with $\mu, \nu = 1, 2$. Previously, in going from $\mathcal{A}_F$ in Eq. (5) to $\mathcal{A}_F(2D)$ in the action integral of Eq. (8), we have assumed that the currents in the x^0 and x^3 directions are much greater in magnitude than the currents in the transverse directions so that A_1^a and A_2^a are small in comparison and can be neglected. As a consequence, $F_{12}(4D) = 0$ (we omit the superscript color index a for simplicity).

We consider now the contribution one of the terms, $F_{03}F^{03}$, in Eq. (19). Equation (18) gives $F_{03}(x^0, x^3, \boldsymbol{r}_\perp)$ in four-dimensional space-time as

$$
\begin{aligned}
F_{03}(x^0, x^3, \boldsymbol{r}_\perp) &= [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}[\partial_0 A_3(2D, x^0, x^3) - \partial_3 A_0(2D, x^0, x^3)] \\
&\quad -ig_{4D}[|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2][A_0(2D, x^0, x^3), A_3(2D, x^0, x^3)].
\end{aligned}
\tag{20}
$$

On the other hand, the gauge field $F_{03}(2D, x^0, x^3)$ in two dimensional space-time is given by definition as

$$F_{03}(2D, x^0, x^3) = \partial_0 A_3(2D) - \partial_3 A_0(2D) - ig_{2D}[A_0(2D), A_3(2D)], \tag{21}$$

where for brevity of notation, the coordinates (x^0, x^3) in $A_\mu(2D, x^0, x^3)$ will be understood. As a consequence, $F_{03}(2D, x^0, x^3)$ in two-dimensional space-time and $F_{03}(x^0, x^3, \boldsymbol{r}_\perp)$ in four-dimensional space-time are related by

$$
\begin{aligned}
F_{03}(x^0, x^3, \boldsymbol{r}_\perp) &= [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\{F_{03}(2D, x^0, x^3) + ig_{2D}[A_0(2D), A_3(2D)]\} \\
&\quad -ig_{4D}[|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2][A_0(2D), A_3(2D)].
\end{aligned}
\tag{22}
$$

The above equation can be re-written as

$$
\begin{aligned}
F_{03}(x^0, x^3, \boldsymbol{r}_\perp) &= [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\Bigg\{F_{03}(2D, x^0, x^3) \\
&\quad + \bigg[ig_{2D} - ig_{4D}[|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\bigg][A_0(2D), A_3(2D)]\Bigg\}.
\end{aligned}
\tag{23}
$$

The product $F_{03}(x)F^{03}(x)$ in eq. (19) becomes

$$
\begin{aligned}
F_{03}(x^0, x^3, \boldsymbol{r}_\perp)F^{03}(x^0, x^3, \boldsymbol{r}_\perp) &= [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]\Bigg\{F_{03}(2D, x^0, x^3)F^{03}(2D, x^0, x^3) \\
&\quad + \bigg[ig_{2D} - ig_{4D}\{(|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2)\}^{1/2}\bigg]\Big(F_{03}(2D)[A^0(2D), A^3(2D)] + [A_0(2D), A_3(2D)]F^{03}(2D)\Big) \\
&\quad + \bigg[ig_{2D} - ig_{4D}[(|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2)]^{1/2}\bigg]^2\Big([A_0(2D), A_3(2D)][A^0(2D), A^3(2D)]\Big)\Bigg\}.
\end{aligned}
\tag{24}
$$

The action integral $\mathcal{A}_A$ in Eq. (19) involves the integration of the above quantity over x_1 and x_2. Upon integration over x^1 and x^2, the second term inside the curly bracket of the above equation, is zero,

$$\int dx^1 dx^2 [(|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2)]\bigg[ig_{2D} - ig_{4D}[(|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2)]^{1/2}\bigg] = 0, \tag{25}$$

where we have used the relation between g_{2D} and g_{4D} as given by Eq.(11) and the normalization condition of (12). As a consequence, the integral of $F_{03}(x)F^{03}(x)$ in Eq. (19) becomes

$$
\begin{aligned}
\int dx F_{03}(x^0, x^3, \boldsymbol{r}_\perp)F^{03}(x^0, x^3, \boldsymbol{r}_\perp) &= \int dx [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]\Bigg\{F_{03}(2D, x^0, x^3)F^{03}(2D, x^0, x^3) \\
&\quad + \bigg[ig_{2D} - ig_{4D}[(|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2)]^{1/2}\bigg]^2\Big([A_0(2D), A_3(2D)][A^0(2D), A^3(2D)]\Big)\Bigg\}.
\end{aligned}
\tag{26}
$$

For the second term in the curly bracket, the integral over dx^1 and dx^2 is

$$\int dx^1 dx^2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2] \left[ig_{2D} - ig_{4D}[(|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2)]^{1/2} \right]^2 \tag{27}$$

which can be considered as an integral over g_{2D} in the form

$$2i \int dg_{2D} \int dx^1 dx^2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2] \left[ig_{2D} - ig_{4D}[(|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2)]^{1/2} \right]. \tag{28}$$

Because of Eq. (25), the above integral gives an irrelevant constant which we can set to zero. After these manipulations, we obtain

$$\int dx^1 dx^2 F_{03}(x^0, x^3, \boldsymbol{r}_\perp) F^{03}(x^0, x^3, \boldsymbol{r}_\perp) = \int dx^1 dx^2 [(|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2)] F_{03}(2D, x^0, x^3) F^{03}(2D, x^0, x^3)$$
$$= F_{03}(2D, x^0, x^3) F^{03}(2D, x^0, x^3). \tag{29}$$

Following the same way (see Appendix B), we calculate terms containing $F_{01}(4D)$, $F_{02}(4D)$, $F_{31}(4D)$, and $F_{32}(4D)$. For the gauge field part of the action integral, we obtain

$$\frac{1}{4} \int dx F_{\mu\nu}^a F_a^{\mu\nu} = \frac{1}{4} \int dx^0 dx^3 F_{03}^a(2D, x^0, x^3) F_a^{03}(2D x^0, x^3)$$
$$- \int \frac{dx^0 dx^3}{4} \int dx^1 dx^2 \left(\{\partial_1 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 \right.$$
$$\left. + \{\partial_2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 \right)$$
$$\times [A_0(2D, x^0, x^3) A^0(2D, x^0, x^3) + A_3(2D, x^0, x^3) A^3(2D, x^0, x^3)]. \tag{30}$$

It is useful to introduce the gluon mass m_{gT} that arises from the confinement of the gluons in the transverse direction

$$m_{gT}^2 = \frac{1}{2} \int dx^1 dx^2 \left[\{\partial_1 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 + \{\partial_2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 \right]. \tag{31}$$

Equation (30) becomes

$$\frac{1}{4} \int dx F_{\mu\nu}^a \ F_a^{\mu\nu} = \frac{1}{4} \int dx^0 dx^3 \left\{ F_{03}^a(2D) F_a^{03}(2D) - 2m_{gT}^2 [A_0^a(2D) A_a^0(2D) + A_3^a(2D) A_a^3(2D)] \right\}. \tag{32}$$

We collect all the fermion and gauge field parts of the action in $\mathcal{A}(4D)$ in Eq. (1). The action integral $\mathcal{A} = \mathcal{A}_F + \mathcal{A}_A$ that was an integral in four-dimensional space-time now turns into an integral only in two-dimensional space-time. All quantities in $\mathcal{A}(4D)$ are completely defined in $(1+1)$ dimensional space-time coordinates, we rename this action integral $\mathcal{A}(2D)$ that is given explicitly by

$$\mathcal{A}(2D) = \int d^2 X \ \left\{ \ Tr \left[\frac{1}{2} \left[\bar{\Psi}(2D, X) \gamma^k(2D) \Pi_k(2D) \Psi(2D, X) - \bar{\Psi}(2D, X) m_{qT} \Psi(2D, X) \right] \right. \right.$$
$$\left. - \frac{1}{2} \left[\bar{\Psi}(2D, X) \gamma^k(2D) \overleftarrow{\Pi}_k(2D) \ \Psi(2D, X) + \bar{\Psi}(2D, X) m_{qT} \Psi(2D, X) \right] \right]$$
$$\left. - \frac{1}{4} F_{\mu\nu}^a(2D) \ F_a^{\mu\nu}(2D) + \frac{1}{2} m_{gT}^2 [A_a^\mu(2D) A_\mu^a(2D)] \right\}, \tag{33}$$

where $\{\mu, \nu\} = 0, 3$, and

$$\Pi_\mu(2D) = i\partial_\mu + g_{2D} T_a A_\mu^a(2D, X) = p_\mu + g_{2D} T_a A_\mu^a(2D, X). \tag{34}$$

Here, all terms (including matrices and coefficients) in the action integral of Eq. (33) are in the $(1+1)$ Minkowski space-time. Thus, in the environment of longitudinal dominance and transverse confinement, we succeed in compactifying

the action integral in four-dimensional space-time to two-dimensional space-time by judiciously relating the field operators in four-dimensional space-time to the corresponding field operators in two-dimensional space-time.

The result in this subsection indicates that the compactified two-dimensional action integral has the same form as QCD in two-dimensional space-time, and the compactified field theory can be appropriately call QCD2. It has the feature that the coupling constant g_{2D} in QCD2 acquires the dimension of a mass, and is related to g_{4D} and the wave functions of the confined fermions in the flux tube. Fermions in different excited states inside the tube will have different coupling constants as indicated in Eq. (11). Furthermore, the action integral gains additional transverse mass terms with an effective quark mass m_{qT} and gluon mass m_{gT} that also depend on the transverse fermion wave functions, as given in Eqs. (17) and (31) respectively. The transverse quark mass includes a contribution from the quark rest mass, in addition to a contribution due to the confinement of the flux tube. In the lower two-dimensional space-time, fermions in excited transverse states have a quark transverse mass different from those in the ground transverse states. All the transverse flux tube information is subsumed under these quantities.

Provided that the fields $A_\mu^a(4D, x^1, x^2, x^0, x^3)$ are governed by the standard gauge tarnsformation [32], the 2D gauged fields $A_\mu^a(2D, x^0, x^3)$ introduced according to Eq. (18) are found to transform under a gauge tarnsformation as follows (see Appendix C) :

$$A_\mu^a(2D, x^0, x^3) \rightarrow \tilde{A}_\mu^a(2D, x^0, x^3) = A_\mu^a(2D, x^0, x^3) + f^a_{\ bc}\varepsilon^b(x^0, x^3)A_\mu^c(2D, x^0, x^3) \tag{35}$$

As a consequence, the term $A_\mu^a(2D)A_a^\mu(2D)$ transforms under a gauge transformation as

$$A_\mu^a(2D, x^0, x^3)A_a^\mu(2D, x^0, x^3) \rightarrow \tilde{A}_\mu^a(2D, x^0, x^3)\tilde{A}_a^\mu(2D, x^0, x^3), \tag{36}$$

which indicates that the mass term in Eq. (33) does not violate gauge invariance and does not violate the Slavnov-Taylor [34] identities (see Appendix C) due to the 2D gauge transformations given by Eqs.(35), (C1).

III. SOLUTION OF THE DIRAC FIELDS IN (1+1) SPACE-TIME

Having completed the program of compactification of QCD4 to QCD2, we shall employ the new notation henceforth that all field quantities and gamma matrices are in two-dimensional space-time with $\mu = 0, 3$, unless specified otherwise. We can use the QCD2 action integral to get the equation of motion for the field. Varying the action integral $\mathcal{A}(2D)$ given by Eq.(33) with respect to $\bar{\Psi}$, we derive the 2D Dirac equation,

$$\left\{ i\gamma^\mu \left(\partial_\mu - ig_{2D} \cdot A_\mu^a(X) T_a \right) - m_{qT} \right\} \Psi(X) = 0, \tag{37}$$

where $2D$ Dirac matrices are those given in Eq. (10). The gauge field A_μ^a written in component form is

$$A_\mu^a(X) = (A_0^a, -A_3^a), \qquad X = (x^0; x^3). \tag{38}$$

We express the fermion field Ψ in terms of $f_+(X)$ and $f_-(X)$ as in Eq. (9),

$$\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} f_+(X) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} f_-(X). \tag{39}$$

Then, the Dirac equations becomes

$$i\frac{\partial f_+(t,z)}{\partial t} + i\frac{\partial f_-(t,z)}{\partial z} + g_{2D} T_a A_0^a f_+(t,z) - g_{2D} T_a A_3^a f_-(t,z) = m_{qT}f_+(t,z)$$

$$-i\frac{\partial f_-(t,z)}{\partial t} - i\frac{\partial f_+(t,z)}{\partial z} - g_{2D} T^a A_a^0 f_-(z) + g_{2D} T_a A_3^a f_+(t,z) = m_{qT}f_-(t,z); \qquad t \equiv x^0; z \equiv x^3. \tag{40}$$

We introduce new functions as sum and difference of f_+ and f_-:

$$\eta(t,z) = f_+(t,z) - f_-(t,z),$$
$$\zeta(t,z) = f_+(t,z) + f_-(t,z). \tag{41}$$

As a result, we obtain

$$i\frac{\partial \eta(t,z)}{\partial t} - i\frac{\partial \eta(t,z)}{\partial z} + g_{2D}\hat{A}_1\eta(t,z) = m_{qT}\zeta(t,z),$$
$$i\frac{\partial \zeta(t,z)}{\partial t} + i\frac{\partial \zeta(t,z)}{\partial z} + g_{2D}\hat{A}_2\zeta(t,z) = m_{qT}\eta(t,z), \tag{42}$$

where

$$\hat{A}_1 = T_a(A_0^a + A_3^a); \quad \hat{A}_2 = T_a(A_0^a - A_3^a). \tag{43}$$

We look for a solution of Eq. (42) in the form

$$\eta(t, z) = F(t, z)\chi(t, z),$$
$$\zeta(t, z) = G(t, z)\chi(t, z), \tag{44}$$

where the functions $\chi(t, z)$, $F(t, z)$ and $G(t, z)$ satisfy the following equations:

$$i\frac{\partial\chi(t, z)}{\partial t} - i\frac{\partial\chi(t, z)}{\partial z} + g_{2D}\hat{A}_1\eta(t, z) = 0,$$
$$i\frac{\partial\chi(t, z)}{\partial t} + i\frac{\partial\chi(t, z)}{\partial z} + g_{2D}\hat{A}_2\zeta(t, z) = 0, \tag{45}$$

while

$$i\frac{\partial F(t, z)}{\partial t} - i\frac{\partial F(t, z)}{\partial z} = m_{qT}G(t, z),$$
$$i\frac{\partial G(t, z)}{\partial t} + i\frac{\partial G(t, z)}{\partial z} = m_{qT}F(t, z). \tag{46}$$

The solution of Eq. (45) can be formally written in the operator form as follows:

$$\chi(t, z) = \{T_{l(M_0;M)}\exp\}\left\{ig_{2D}T_a\int dx^\mu A_\mu^a\right\}, \tag{47}$$

where the symbol $\{T_{l(M_0;M)}\exp\}$ means that the integration is to be carried out along the line on the light cone from the point M_0 to the point M such that the factors in exponent expansion are chronologically ordered from M_0 to M. Eq. (46) are the free 2D Dirac equations. When m_{qT} is a constant, the solution can be found as the superposition of 2D plane waves:

$$F(t, z) = \int \frac{d^2P}{2\pi}F(P)e^{(-iPx)} = \int \frac{d^2P}{2\pi}F(P)e^{-i(\omega t - p_z z)},$$
$$G(t, z) = \int \frac{d^2P}{2\pi}G(P)e^{(-iPx)} = \int \frac{d^2P}{2\pi}G(P)e^{-i(\omega t - p_z z)}. \tag{48}$$

Substituting the last expansion into Eq. (46), we obtain

$$F(P)(\omega + p) - m_{qT}G(P) = 0,$$
$$(\omega - p)G(P) - m_{qT}F(P) = 0,$$
$$P \equiv P^\mu = (\omega; \boldsymbol{p}) = (\omega; p_z) \equiv (\omega; p). \tag{49}$$

As a result, we have

$$f_+ = \frac{\zeta + \eta}{2} \propto G(P) + F(p),$$
$$f_- = \frac{\zeta - \eta}{2} \propto G(P) - F(p). \tag{50}$$

Taking $F(P) = 1$ and $G(P) = (\epsilon + p)/m_{qT}$, we derive

$$f_+ \propto \frac{\omega + p}{m_{qT}} + 1,$$
$$f_- \propto \frac{\omega + p}{m_{qT}} - 1. \tag{51}$$

The solution of Eq. (37) becomes

$$\Psi(2D, X) = \Psi(x^0; x^3) = f_+(x^0; x^3)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + f_-(x^0; x^3)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= \int \frac{d^2P}{2\pi}e^{-i(\omega t - p_z z)}N(\omega, p)\begin{pmatrix} \frac{\omega + p}{m_{qT}} + 1 \\ \frac{\omega + p}{m_{qT}} - 1 \end{pmatrix} \cdot \chi(X), \tag{52}$$

where $N(\omega, p)$ is some normalization multiplier. We can take the normalization condition

$$
\left(N(\omega, p) \begin{pmatrix} \frac{\omega+p}{m_{qT}} + 1 \\ \frac{\omega+p}{m_{qT}} - 1 \end{pmatrix} \right)^{\dagger} N(\omega, p) \begin{pmatrix} \frac{\omega+p}{m_{qT}} + 1 \\ \frac{\omega+p}{m_{qT}} - 1 \end{pmatrix} = \frac{1}{L},
\tag{53}
$$

where L is the flux tube length. Then, we obtain

$$
N(\omega, p) = \frac{m_{qT}}{2\sqrt{L(\omega^2 + p\omega)}}.
\tag{54}
$$

As a result, the general solution is

$$
\begin{aligned}
\Psi &= \int\limits_{-\infty}^{+\infty} \frac{d\omega}{2\sqrt{L}} \sum_p \frac{m_{qT}}{\sqrt{\omega^2 + p\omega}} \ \exp(-iP_\mu x^\mu) \ a(p,\omega) \left[\delta(\omega - \varepsilon(p)) + \delta(\omega + \varepsilon(p))\right] \begin{pmatrix} \frac{\omega+p}{m_{qT}} + 1 \\ \frac{\omega+p}{m_{qT}} - 1 \end{pmatrix} \\
&\quad \times \ \{T_{l(M_0; M)} \exp\} \left\{ ig_{2D} T_a \int dx^\mu A_\mu^a \right\},
\end{aligned}
\tag{55}
$$

where $a(p, \omega)$ are coefficients related to either particles or anti-particles under the field quantization. We have not deliberately separated out positive and negative frequency terms in Eq. (55) because the structure of the fermion vacuum is strongly dependent on the explicit form of the external field $A_\mu^a(X)$. Furthermore, when the external field depends on time, there will be no stationary particles and antiparticles states.

A. Fermion current and gauge fields

We envisage that a perturbative gauge field is introduced inside the flux tube, such a field will generate a current, and the current in turn will produce a gauge field self-consistently. How do these quantities relate to each other? We therefore need to obtain a relationship between the fermion current and the gauge field.

The fermion field solution in Eq. (55) leads to a fermion current J_a^μ

$$
J_a^\mu(2D) = g_{2D} \ Tr\left\{ \bar{\Psi}(X)\gamma^\mu T_a \Psi(X') \right\}, \qquad X' \to X.
\tag{56}
$$

Owing to the operation of trace calculation in the last formula, the current (56) contains the factor

$$
(T\exp)\left\{ ig_{2D} \ T_a \int\limits_X^{X'} A_\mu^a dX^\mu \right\}.
\tag{57}
$$

We expand the operator exponent in the last equation as a series with respect to $(X' - X) \to 0$,

$$
\begin{aligned}
(T\exp)\left\{ ig_{2D} \ T_a \int\limits_X^{X'} A_\mu^a dX^\mu \right\} &= 1 + ig_{2D} \ T_a(X' - X)^\mu A_\mu^a(\xi) + \frac{i}{2} g_{2D} \ T_a(X' - X)^\mu (X' - X)^\nu \partial_\nu A_\mu^a(\xi) \\
&\quad - g_{2D}^2 \ (T_a T_b)(\tilde{X}' - \tilde{X})^\mu (X' - X)^\nu A_\mu^a(\tilde{\xi}) A_\nu^b(\xi) \theta(\tilde{\xi} - \xi),
\end{aligned}
\tag{58}
$$

where $\tilde{\xi} \in [\tilde{X}, \tilde{X}']$, and $\xi \in [X, X']$ $X' \to X$. We take the limits $(\tilde{X}' - \tilde{X}) \to 0$ and $(X' - X) \to 0$ such that

$$
\frac{(\tilde{X}' - \tilde{X})}{(X' - X)} \to 0.
\tag{59}
$$

Then, the last term in the expansion in Eq. (58) is equal to zero. Substituting $(T\exp)\{ig_{2D} \ T_a \int\limits_X^{X'} A_\mu^a dx^\mu\}$ into Eq.(56), we obtain for $(X' - X) \to 0$

$$
\begin{aligned}
J_a^\mu &= \frac{g_{2D}}{L} Tr \int d\omega \sum_{p,f} \left\{ <a_f^\dagger(p,\omega) a_f(p,\omega)> (P^\mu T_a) \left(-\frac{\partial}{\partial P_\nu} \exp\left(-iP(X'-X)\right) \right) \left[\delta(\omega + \varepsilon(p)) + \delta(\omega - \varepsilon(p))\right] \right. \\
&\quad \left. \times \left(g_{2D} \ T_b A_\nu^b(\xi) + \frac{1}{2} g_{2D} \ T_b(X' - X)^\lambda \partial_\nu A_\lambda^b(\xi) \right) \right\},
\end{aligned}
\tag{60}
$$

where f denotes flavor states. In reaching the last equation, we have successively calculated a trace, gone from summation to integration, introduced the additional integration with respect to the p variable, and integrated by parts. We note that upon taking the partial derivative $\partial^2 \equiv \partial^2_{(X)}$ on $(X' - X)^\lambda \partial_{\nu(X)} A^b_\lambda(\xi)$, we get

$$
\begin{aligned}
\partial^2_{(X)} \lim_{X' \to X} \left\{ (X' - X)^\lambda \partial_{\nu(X)} A^b_\lambda(\xi) \right\} &= \lim_{X' \to X} \partial^2_{(X)} \left\{ (X' - X)^\lambda \partial_{\nu(X)} A^b_\lambda(X) \right\} \\
&= \lim_{X' \to X} \left\{ -2\partial^\lambda_{(X)} \partial_{\nu(X)} A^b_\lambda(X) + (X' - X)^\lambda \partial_{\kappa(X)} \partial^\kappa_{(X)} \{ \partial_{\nu(X)} A^b_\lambda(X) \} \right\}
\end{aligned}
\tag{61}
$$

Upon taking the limit $X' \to X$, the second term vanishes . Therefore, we have in the limit of $X' \to X$,

$$
\lim_{X' \to X} \left((X' - X)^\lambda \partial_\nu A^b_\lambda(\xi) \right) = -2\frac{\partial^\lambda \partial_\nu}{\partial^2} A^b_\lambda(X).
\tag{62}
$$

It should be noted that as QCD4 in the $(3+1)$ dimensional space-time is gauge invariant, and we have chosen the Lorentz gauge (4) in QCD4 to simplify the compactified action integral in QCD2. We need to continue to use the Lorentz gauge in QCD2 for consistency. In the Lorentz gauge, the last term in the second circular brackets in Eq.(60) is equal to zero because of Eq.(62). Calculating a trace with respect to the color variables according to Eq.(6), we can represent the current J^μ_a in the following form

$$
\begin{aligned}
J^\mu_a(2D, X) &= \frac{g^2_{2D} \, \mathcal{S}}{4} \, A^\mu_a(2D, X), \\
\mathcal{S} &= \frac{1}{2\pi} \sum_f \int d^2P \frac{\partial}{\partial P^\mu} \left\{ [\delta(\omega + \varepsilon(p)) + \delta(\omega - \varepsilon(p))] \, P^\mu \, < a^\dagger(p,\omega)_f \, a_f(p,\omega) > \right\}; \quad d^2P = d\omega \, dp
\end{aligned}
\tag{63}
$$

where P^μ is the momentum introduced in Eq. (49). We can introduce a boson mass m_{gfT} by

$$
m^2_{gf\,T} = \frac{g^2_{2D}\mathcal{S}}{4}.
\tag{64}
$$

Then, the current in Eq.(63) can be written as

$$
J^\mu_a(2D, X) = m^2_{gf\,T} \, A^\mu_a(2D, X).
\tag{65}
$$

We can calculate the quantity $\mathcal{S}$. Changing p by $-p$ in the term corresponding to the negative ω, we have

$$
\mathcal{S} = \frac{1}{2\pi} \sum_f \int d^2P \frac{\partial}{\partial P^\mu} \left\{ P^\mu \left(\delta(\omega - \varepsilon(p)) < a^\dagger(p,\omega)_f \, a_f(p,\omega) > + \delta(\omega + \varepsilon(p)) < a^\dagger(-p,\omega)_f \, a_f(-p,\omega) > \right) \right\}.
\tag{66}
$$

Integrating out the δ-functions, we obtain

$$
\mathcal{S} = \frac{1}{\pi} \sum_f \int dp \, \left(< a^\dagger_f(p, \varepsilon(p)) \, a_f(p, \varepsilon(p)) > + < a^\dagger_f(-p, -\varepsilon(p)) \, a_f(-p, -\varepsilon(p)) > \right) \left(1 - \frac{m^2_{qT}}{2\varepsilon^2(p)} \right).
\tag{67}
$$

Since a fermion moves either along or opposite to the only spatial axis, we have

$$
\int dp \, < a^\dagger_f(p, \varepsilon(p)) \, a_f(p, \varepsilon(p)) >= \int dp \, < a^\dagger_f(-p, -\varepsilon(p)) \, a_f(-p, -\varepsilon(p)) >= 1.
\tag{68}
$$

In the case of a flux tube for which $m_{qT} \gg p$, we obtain

$$
\mathcal{S} = \frac{2}{\pi} \, N_f,
\tag{69}
$$

where N_f is the number of flavors. We note in passing that in the special case of the massless QED2 [33], we obtain after summing over all spin states of a fermion

$$
\mathcal{S}_{\mathrm{QED2}} = \frac{4}{\pi}, \quad (N_f = 1),
\tag{70}
$$

and

$$
m^2_{gf\,T}(\mathrm{QED2}) = \frac{g^2}{\pi},
\tag{71}
$$

which agrees with the Schwinger massless QED2 result [33].

Finally, we note that under the gauge transformation

$$\delta A_a^\mu = \varepsilon_b f_a{}^{bc} A_c^\mu \tag{72}$$

the current (63) satisfies the gauge relation

$$\delta J_a^\mu = \varepsilon_b f_a{}^{bc} J_c^\mu. \tag{73}$$

IV. EQUATION OF MOTION FOR THE 2D GAUGE FIELDS

The action integral $\mathcal{A}$ allows us to obtain the equation of motion for the 2D gauge field. We rewrite the action integral (33) by expressing explicitly the term corresponding to the interaction between the fermion and the gauge field,

$$\mathcal{A}(2D) = \int d^2 X \ \left\{ \ \frac{i}{2} \left[\bar{\Psi} \gamma^k \partial_k \Psi - \bar{\Psi} m_{qT} \Psi \right] - \frac{i}{2} \left[\bar{\Psi} \gamma^k \overleftarrow{\partial}_k \Psi + \bar{\Psi} m_{qT} \Psi \right] + J_a^\mu A_\mu^a \right.$$
$$\left. - \frac{1}{4} F_{\mu\nu}^a \ F_a^{\mu\nu} + \frac{1}{2} m_{gT}^2 A_\nu^a A_a^\nu \right\}, \tag{74}$$

where $J_a^\mu(x)$ is the fermion current governed by Eq.(65). Substituting $J_a^\mu(x)$ given by Eq.(65) into the 2D action integral (74), we obtain

$$\mathcal{A}(2D) = \int d^2 X \ \left\{ \ \frac{i}{2} \left[\bar{\Psi} \gamma^k \partial_k \Psi - \bar{\Psi} m_{qT} \Psi \right] - \frac{i}{2} \left[\bar{\Psi} \gamma^k \overleftarrow{\partial}_k \Psi + \bar{\Psi} m_{qT} \Psi \right] \right.$$
$$\left. - \frac{1}{4} F_{\mu\nu}^a \ F_a^{\mu\nu} + \frac{1}{2} M_{gT}^2 A_\nu^a A_a^\nu \right\}. \tag{75}$$

Here the constant M_{gT} is given by

$$M_{gT}^2 = \frac{1}{2} \int dx^1 dx^2 \left[\{ \partial_1 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2} \}^2 + \{ \partial_2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2} \}^2 \right] + \frac{g_{2D}^2 \mathcal{S}}{2}$$
$$\equiv m_{gT}^2 + m_{gfT}^2 \geq 0. \tag{76}$$

To find out the meaning of M_{gT}, we consider the variation of the action integral (75) with respect to a variation of the gauge field $A_a^\nu(x)$. We obtain equation of motion for the variation $A_a^\nu(x)$. As a result, we derive the Klein-Gordon-like equation:

$$\Box \ A_a^\nu \ = \ M_{gT}^2 \ A_a^\nu. \tag{77}$$

We look for a solution for the variation of the gauge field in Eq. (77) of the form

$$A_a^\nu = b_a(k,\nu) \ e_a^\nu(k) \exp(-ik_\mu X^\mu),$$

$$k^\mu = (k^0; \boldsymbol{k}), \quad e_a^0 = \frac{|\boldsymbol{k}|}{M_{gT}} \ (1,0), \quad e_a^3 = \frac{|k^0|}{M_{gT}} (0,1), \tag{78}$$

where e_a^ν denotes a pair orthogonal vectors; $b_a(k,\nu)$ are some coefficients being independent on X. Substituting A_a^ν given by Eq. (78) into Eq. (77), we obtain:

$$(k^0)^2 = \boldsymbol{k}^2 + M_{gT}^2. \tag{79}$$

Because of both the positivity of M_{qT}^2 and Eq. (79), M_{qT} can be interpreted as a mass of the particle whose energy is

$$E(k) \equiv k^0 = +\sqrt{\boldsymbol{k}^2 + M_{qT}^2}. \tag{80}$$

Eqs. (78) and (80) allow us to write down the general solution of Eq. (77). Following the standard way [32], and separating the negative and positive frequency terms, we obtain

$$A_a^\nu(2D, X) = \sum_k \frac{e_a^\nu \, M_{qT}}{\sqrt{\left(\boldsymbol{k}^2 + M_{qT}^2\right)^3}} \left\{ \exp(-ikX) \, b_a(k, \nu) \, + \exp(+ikX) \, \bar{b}_a^\dagger(k, \nu) \right\}, \qquad (81)$$

where the symbols $b_a(k, \nu)$ and $\bar{b}_a^\dagger(k, \nu)$ are the operators of annihilation and creation of a boson with the mass M_{qT}. In this way, M_{qT} corresponds to the mass of the boson responding to the space-time variation of the gauge field variation.

The boson mass M_{gT} in the action integral Eq.(75) arises from the compactification of $4D \to 2D$ and from the interaction of the compactified fermions and gauge field. We should also note that all masses (boson and fermion field) as well as $2D$ coupling constant are governed by the functions of the transverse motion of a fermion, $G_{1,2}(\vec{r}_\perp)$. It is independent of the color index a. Out of the gauge field variations of different color components A_a^μ, one can construct a colorless variation of the type

$$A_{color-singlet}^\nu = \frac{1}{\sqrt{8}} \sum_a A_a^\mu |8, a\rangle, \qquad (82)$$

where $|8, a\rangle$ is the color-octet state with component a. Eq. (77) gives

$$\Box \, A_{color-singlet}^\nu = M_{gT}^2 \, A_{color-singlet}^\nu. \qquad (83)$$

Thus, we find that M_{gT} is also the mass corresponding to a colorless variation of the gauge field of different color components in a flux tube. Such a colorless variation should lead to an observable quantity. If one considers pion as the colorless dynamical response of the variations of the gage fields in a string, then M_{gT} may be presumed to be the mass of the pion within the environment of a flux tube under consideration.

V. EQUATIONS OF TRANSVERSE MOTION IN A TUBE AND THE FERMION EFFECTIVE MASS

To obtain the equations of motion for the functions $G_1(\vec{r}_\perp)$ and $G_2(\vec{r}_\perp)$, we vary the action integral $\mathcal{A}(4D)$ in Eq. (1) with the fermion fields $\Psi(4D, x)$ given by Eq.(7), under the constraint of the normalization condition Eq. (12). To do this we construct a new functional $\mathcal{F}$,

$$\mathcal{F} = \mathcal{A}(4D) + \frac{\lambda}{2} \int dx^1 dx^2 \left(|G_1(\vec{r}_\perp)|^2 + |G_2(\vec{r}_\perp)|^2 \right) \int dx^0 dx^3 \left(\bar{\Psi}(x^0, x^3) \, \Psi(x^0, x^3) \right), \qquad (84)$$

where λ is the Lagrange multiplier. The last term in Eq. (84) takes into account the unitarity of a fermion field in the 4D dimensional space-time. Varying the last equation with respect to the functions $G_1(\vec{r}_\perp)$ and $G_2(\vec{r}_\perp)$, we obtain

$$(p_1 + ip_2)G_1(\vec{r}_\perp) = (m(\vec{r}_\perp) + \lambda)G_2(\vec{r}_\perp),$$
$$(p_1 - ip_2)G_2(\vec{r}_\perp) = (\lambda - m(\vec{r}_\perp))G_1(\vec{r}),$$
$$(p_1 + ip_2)G_2^*(\vec{r}_\perp) = (m(\vec{r}_\perp) - \lambda)G_1^*(\vec{r}_\perp),$$
$$(p_1 - ip_2)G_1^*(\vec{r}_\perp) = -(m(\vec{r}_\perp) + \lambda)G_2^*(\vec{r}). \qquad (85)$$

Carrying out complex conjugation in the last two equations, we obtain

$$\lambda = \lambda^*. \qquad (86)$$

Combining Eq. (85), we get

$$\left(p_1^2 + p_2^2 - \lambda^2 + m^2(\vec{r}_\perp)\right) G_1(\vec{r}_\perp) = G_2(\vec{r}_\perp)(p_1 - ip_2)m(\vec{r}_\perp)$$
$$\left(p_1^2 + p_2^2 - \lambda^2 + m^2(\vec{r}_\perp)\right) G_2(\vec{r}_\perp) = -G_1(\vec{r}_\perp)(p_1 + ip_2)m(\vec{r}_\perp). \qquad (87)$$

Substituting the equations (85) for $G_{1,2}(\vec{r}_\perp)$ functions into the formula (17) for m_{qT}, we find that

$$m_{qT} = \lambda. \qquad (88)$$

Thus, the effective mass of the compactified 2D fermion field is equal to the energy eigenvalue for the transverse motion of the 4D fermion as described in Eqs. (85). We should note here that the 2D fermion can generally gain a mass even when the initial 4D fermion appears to be massless. The compactification effectively leads to a constraint in moving a fermion from one point of a space-time to another point due to decreasing the number of trajectories in the 2D space-time as compared with the 4D situation. This contraint leads to the presence of an effective mass.

VI. CONCLUSIONS AND DISCUSSIONS

Encouraged by the successes of the particle production model of Casher, Kogut, and Susskind using the Abelian gauge field theory in two-dimension space-time [18] and the Lund model of string fragmentation [19], we seek a compactification of QCD4 to QCD2 in the environment of a flux tube. Under the assumption of longitudinal dominance and transverse confinement, the SU(N) gauge invariant field theory of QCD4 can be compactified in the (1+1) Minkowski space-time, from the consideration of the action integral. This is achieved by finding a way to relate the field variables in 2-dimensional space-time to those in four-dimensional space-time.

The compactified 2D action integral $\mathcal{A}(2D)$ depends only on fields that are defined in two-dimensional space-time. It has the same structure as those in QCD in four-dimensional space-time and can therefore be appropriately called QCD2. In the compactified QCD2 quantum field theory, the coupling constant is found to be dimensional, and there are additional terms in the action associated with an effective quark mass and effective gauge field mass as a result of the flux tube confinement. These quantities depends on the transverse profile and the transverse state of the quarks in the flux tube.

On a basis of the derived QCD2 action integral, the equations of motion for the fields can be obtained for both the fermion field and the gauge field. The solution of 2D Dirac equation can then be formally obtained. The structure of the solution allows one to consider the effects of the fermion-gluon coupling. As a result, the 2D action integral can be re-written in the form such that the gauge field acquires an additional effective mass due to interaction with fermions. The structure of the derived mass term appears to be identical to the one obtained by Schwinger [33] in the special case of massless QED2.

The occurrence of a massive composite bound state in gauge field theories has been known in many previous investigations [26]. How the massive bosons as a pole in the three gluon vertex in 4-dimensional space-time [27, 28] can be produced in the flux tube environment in high-energy collisions will be an interesting subject worthy of further investigations.

Acknowledgement

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Appendix A

Substituting $\Psi(4D, x) \equiv \Psi(x)$ given by Eq. (7) into the first term in Eq. (5), we obtain

$$\bar{\Psi}(x)\gamma^\mu \Pi_\mu \; \Psi(x) = \chi^\dagger \left(\Pi_0 - (p_1\sigma_1 + p_2\sigma_2) - \Pi_3\sigma_3\right)\chi + \varphi^\dagger \left(\Pi_0 + (p_1\sigma_1 + p_2\sigma_2) + \Pi_3\sigma_3\right)\varphi$$

$$= \chi^\dagger \left[\begin{pmatrix} \Pi_0 - \Pi_3 & 0 \\ 0 & \Pi_0 + \Pi_3 \end{pmatrix} - \begin{pmatrix} 0 & p_1 - ip_2 \\ p_1 + ip_2 & 0 \end{pmatrix} \right] \chi + \varphi^\dagger \left[\begin{pmatrix} \Pi_0 + \Pi_3 & 0 \\ 0 & \Pi_0 - \Pi_3 \end{pmatrix} + \begin{pmatrix} 0 & p_1 - ip_2 \\ p_1 + ip_2 & 0 \end{pmatrix} \right] \varphi$$

$$= \chi_1^*(\Pi_0 - \Pi_3)\chi_1 + \chi_2^*(\Pi_0 + \Pi_3)\chi_2 - \chi_1^*(p_1 - ip_2)\chi_2 - \chi_2^*(p_1 + ip_2)\chi_1 + \varphi_1^*(\Pi_0 + \Pi_3)\varphi_1 + \varphi_2^*(\Pi_0 - \Pi_3)\varphi_2$$
$$+ \varphi_1^*(p_1 - ip_2)\varphi_2 + \varphi_2^*(p_1 + ip_2)\varphi_1$$

$$= \chi_1^*(\Pi_0 - \Pi_3)\chi_1 + \chi_2^*(\Pi_0 + \Pi_3)\chi_2 + \varphi_1^*(\Pi_0 + \Pi_3)\varphi_1 + \varphi_2^*(\Pi_0 - \Pi_3)\varphi_2$$
$$- \chi_1^*(p_1 - ip_2)\chi_2 - \chi_2^*(p_1 + ip_2)\chi_1 + \varphi_1^*(p_1 - ip_2)\varphi_2 + \varphi_2^*(p_1 + ip_2)\varphi_1. \tag{A1}$$

Integration of the last equation gives

$$\int d^4x \; \bar{\Psi}(x)\gamma^k \Pi_k \; \Psi(x) \;=\; \int d^4x \left\{ \chi_1^*(\Pi_0 - \Pi_3)\chi_1 + \chi_2^*(\Pi_0 + \Pi_3)\chi_2 + \varphi_1^*(\Pi_0 + \Pi_3)\varphi_1 + \varphi_2^*(\Pi_0 - \Pi_3)\varphi_2 \right\}$$

$$+ \int d^4x \left\{ -\chi_1^*(p_1 - ip_2)\chi_2 - \chi_2^*(p_1 + ip_2)\chi_1 + \varphi_1^*(p_1 - ip_2)\varphi_2 + \varphi_2^*(p_1 + ip_2)\varphi_1 \right\}$$

$$= \int d^4x \left\{ \chi_1^*(\Pi_0 - \Pi_3)\chi_1 + \chi_2^*(\Pi_0 + \Pi_3)\chi_2 + \varphi_1^*(\Pi_0 + \Pi_3)\varphi_1 + \varphi_2^*(\Pi_0 - \Pi_3)\varphi_2 \right\}$$

$$- \int d^4x \left\{ G_1^*(\vec{r}_\perp)(p_1 - ip_2)G_2(\vec{r}_\perp) \right\} \left(|f_+|^2 - |f_-|^2 \right)$$

$$= \int d^4x \left(|G_1(\vec{r}_\perp)|^2 + |G_2(\vec{r}_\perp)|^2 \right) \left[\; f_+^*\Pi_0 f_+ + \; f_-^*\Pi_0 f_- + \; f_+^*\Pi_3 f_- + \; f_-^*\Pi_3 f_+ \right]$$

$$- \int d^4x \left\{ G_1^*(\vec{r}_\perp)(p_1 - ip_2)G_2(\vec{r}_\perp) \right\} \left(|f_+|^2 - |f_-|^2 \right). \tag{A2}$$

Following the same way, we derive for the term $\bar{\Psi}(x)\gamma^k \overleftarrow{\Pi}_k \; \Psi(x)$

$$\int d^4x \; \bar{\Psi}(x)\gamma^k \overleftarrow{\Pi}_k \; \Psi(x) \;=\; \int d^4x \left(|G_1(\vec{r}_\perp)|^2 + |G_2(\vec{r}_\perp)|^2 \right) \left[\; f_+^*\overleftarrow{\Pi}_0 f_+ + \; f_-^*\overleftarrow{\Pi}_0 f_- + \; f_+^*\overleftarrow{\Pi}_3 f_- + f_-^*\overleftarrow{\Pi}_3 f_+ \right]$$

$$- \int d^4x \left\{ G_1(\vec{r}_\perp)(p_1 + ip_2)G_2^*(\vec{r}_\perp) \right\} \left(|f_+|^2 - |f_-|^2 \right). \tag{A3}$$

We substitute $\Psi(x)$ of Eq. (7) into the last term in Eq. (5), and we obtain

$$\bar{\Psi}(x)m(\vec{r}_\perp)\Psi(x) = m(\vec{r}_\perp) \; \left(|G_1(\vec{r}_\perp)|^2 - |G_2(\vec{r}_\perp)|^2 \right) \left[|f_+|^2 - |f_-|^2 \right]. \tag{A4}$$

Collecting the above results and introducing the 2D-fermion wave function $\Psi(X)$, 2D-gamma matrices γ^μ, and the metric tensor $g_{\mu\nu}(2D)$ as given in Eqs. (9) and (10), we obtain the Fermion part of the action integral in Eq. (8).

Appendix B

To compactify the gauge field parts of the $(3+1)$ dimensional space-time to $(1+1)$ dimensional space-time, we need to evaluate F_{01}, F_{02}, F_{31}, and F_{32}. Direct calculations give (color indexes are omitted for simplicity)

$$
\begin{aligned}
F_{01}(x^0, x^3, \boldsymbol{r}_\perp) &= -\partial_1 A_0(x^0, x^3, \boldsymbol{r}_\perp) \\
&= -\partial_1 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2} A_0(2D, x^0, x^3),
\end{aligned} \tag{B1}
$$

$$
\begin{aligned}
F_{01}(x^0, x^3, \boldsymbol{r}_\perp) F^{01}(x^0, x^3, \boldsymbol{r}_\perp) &= \{-\partial_1 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}\{-\partial^1 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\} \\
&\quad \times A_0(2D, x^0, x^3) A^0(2D, x^0, x^3), \quad \text{because } \partial^1 = -\partial_1, \\
&= -\{\partial_1 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 A_0(2D, x^0, x^3) A^0(2D, x^0, x^3),
\end{aligned} \tag{B2}
$$

which contribute a gauge field mass in $A_0(2D, x^0, x^3)$ gauge field. Similarly, we can calculate

$$
F_{02}(x^0, x^3, \boldsymbol{r}_\perp) F^{02}(x^0, x^3, \boldsymbol{r}_\perp) = -\{\partial_2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 A_0(2D, x^0, x^3) A^0(2D, x^0, x^3),
$$

$$
F_{31}(x^0, x^3, \boldsymbol{r}_\perp) F^{31}(x^0, x^3, \boldsymbol{r}_\perp) = -\{\partial_1 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 A_3(2D, x^0, x^3) A^3(2D, x^0, x^3), \tag{B3}
$$

which contribute a gauge field mass in $A_3(2D, x^0, x^3)$ gauge field. Similarly, we have also

$$
F_{32}(x^0, x^3, \boldsymbol{r}_\perp) F^{32}(x^0, x^3, \boldsymbol{r}_\perp) = -\{\partial_2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 A_3(2D, x^0, x^3) A^3(2D, x^0, x^3). \tag{B4}
$$

Combining all similar terms, we get

$$
\begin{aligned}
&[F_{01}F^{01} + F_{02}F^{02} + F_{31}F^{31} + F_{32}F^{32}](x^0, x^3, \boldsymbol{r}_\perp) \\
&= -\left(\{\partial_1 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 + \{\partial_2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 \right) \\
&\quad \times [A_0(2D, x^0, x^3) A^0(2D, x^0, x^3) + A_3(2D, x^0, x^3) A^3(2D, x^0, x^3)] \\
&= -\left(\{\partial_1 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 + \{\partial_2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 \right) \\
&\quad \times [A_0(2D, x^0, x^3) A^0(2D, x^0, x^3) + A_3(2D, x^0, x^3) A^3(2D, x^0, x^3)].
\end{aligned} \tag{B5}
$$

Then, due to the normalization relation (12) we have the following (see Eq. (26) and (27)) for the gauge field part:

$$
\begin{aligned}
\frac{1}{4} \int d^4 x F^a_{\mu\nu}(4D) \, F_a^{\mu\nu}(4D) &= \frac{1}{4} \int dx^0 dx^3 \int dx^1 dx^2 (|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2) F^a_{03}(2D) F_a^{03}(2D) \\
&\quad -\frac{1}{4} \int dx^0 dx^3 \int dx^1 dx^2 \left(\{\partial_1 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 + \{\partial_2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 \right) \\
&\quad \times [A_0(2D, x^0, x^3) A^0(2D, x^0, x^3) + A_3(2D, x^0, x^3) A^3(2D, x^0, x^3)] \\
&= \frac{1}{4} \int dx^0 dx^3 F^a_{03}(2D) \, F_a^{03}(2D) - \frac{1}{2} \int dx^0 dx^3 m^2_{gT} [A_0(2D) A^0(2D) + A_3(2D) A^3(2D)],
\end{aligned} \tag{B6}
$$

where m^2_{gT} is the mass term that arises from the confinement of the gluons in the transverse direction

$$
m^2_{gT} = \frac{1}{2} \int dx^1 dx^2 \left[\{\partial_1 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 + \{\partial_2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 \right]. \tag{B7}
$$

Note that using integration by parts, we get

$$
\begin{aligned}
&- \int dx^1 dx^2 \{\partial_1 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 \\
&= \int dx^1 dx^2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2} \partial_1^2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}.
\end{aligned} \tag{B8}
$$

Therefore, adding the terms together, we obtain:

$$m_{gT}^2 = \frac{1}{2} \int dx^1 dx^2 \left[\{\partial_1 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 + \{\partial_2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}\}^2 \right]$$

$$= -\frac{1}{2} \int dx^1 dx^2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2} (\partial_1^2 + \partial_2^2) [2|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}$$

$$= \int dx^1 dx^2 [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2} \left(-\frac{1}{2} \nabla_T^2 \right) [|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2]^{1/2}. \tag{B9}$$

Appendix C

1.   Transformation of a gauge field in the 2D space-time

We would like to write down the gauge transformation properties for $A_\mu^a(2D, x^0, x^3)$. For the corresponding gauge field $A_\mu^a(x)$ in the 4D space-time $x = (x^0, x^3, \boldsymbol{r}_\perp)$, it transforms under a gauge transformation as [32]

$$A_\mu^a(x) \to \tilde{A}_\mu^a(x) = A_\mu^a(x) + \delta A_\mu^a(x). \tag{C1}$$

where

$$\delta A_\mu^a(x) = f^a_{bc} \varepsilon^b(x) A_\mu^c(x) - \frac{1}{g(4D)} \partial_\mu \varepsilon^a(x) \tag{C2}$$

According to Eq. (18) the gauge fields in the 2D and 4D space-time are related to each other as follows:

$$A_\mu^a(2D, x^0, x^3) = \frac{A_\mu^a(x^0, x^3, \boldsymbol{r}_\perp)}{\sqrt{|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2}}, \quad \mu = 0, 3.$$

$$A_\mu^a(x^0, x^3, \boldsymbol{r}_\perp) = 0, \quad \mu = 1, 2 \tag{C3}$$

From the last equation we have

$$\delta A_\mu^a(x^0, x^3, \boldsymbol{r}_\perp) = 0, \quad \mu = 1, 2. \quad \Rightarrow \quad \partial_\mu \varepsilon^a(x) = 0, \quad \mu = 1, 2. \tag{C4}$$

Then, we have

$$\varepsilon^a(x) = \varepsilon^a(x^0, x^3) \tag{C5}$$

Next, we would like to transform $A_\mu^a(2D, x^0, x^3)$ by using the first relation in Eq.(C3).

$$\delta A_\mu^a(2D, x^0, x^3) = \frac{\delta A_\mu^a(x^0, x^3, \boldsymbol{r}_\perp)}{\sqrt{|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2}} + A_\mu^a(x^0, x^3, \boldsymbol{r}_\perp) \delta \left(\frac{1}{\sqrt{|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2}} \right) \tag{C6}$$

The last term in Eq.(C6) is equal to zero since $\varepsilon^a = \varepsilon^a(x^0, x^3)$. Then, substituting Eq. (C2) into Eq. (C6) we obtain

$$\delta A_\mu^a(2D, x^0, x^3) = f^a_{bc} \varepsilon^b(x^0, x^3) A_\mu^c(2D, x^0, x^3) - \frac{1}{g(4D) \sqrt{|G_1(\boldsymbol{r}_\perp)|^2 + |G_2(\boldsymbol{r}_\perp)|^2}} \partial_\mu \varepsilon^a(x^0, x^3) \tag{C7}$$

Since the left-hand side of Eq. (C7) depends on (x^0, x^3) the same must be for the right-hand side of this equation. This means that $\varepsilon^a(x^0, x^3) =$ constant and the transformation relation for $A_\mu^a(2D, x^0, x^3)$ is

$$\delta A_\mu^a(2D, x^0, x^3) = f^a_{bc} \varepsilon^b(x^0, x^3) A_\mu^c(2D, x^0, x^3) \tag{C8}$$

Varying the mass term in the 2D Lagrangian with respect to the group variables, we obtain

$$\delta \mathcal{L}_{m_{gT}} = \frac{1}{2} m_{gT}^2 \delta [A_a^\mu(2D) A_\mu^a(2D)] = m_{gT}^2 \delta [A_a^\mu(2D)] [A_\mu^a(2D)]$$

$$= m_{gT}^2 f^a_{bc} \varepsilon^b(x^0, x^3) [A_\mu^c(2D)] [A_a^\mu(2D)] = 0, \tag{C9}$$

due to the anti-symmetry of the structure constants f^a_{bc}. Using Eq. (C2) for the infinitesimal transformation of the gauge of the field $A^a_\mu(2D, x^0, x^3)$ we calculate the n-th variation of $A^a_\mu(2D, x^0, x^3)$. After such calculations we derive that the gauge transformation of the 2D gauge field has the form

$$
\begin{aligned}
\delta^{(n)} A^a_\mu(2D, x^0, x^3) &= f^a_{bc}\varepsilon^b(x^0, x^3)f^c{}_{b_1 c_1}\varepsilon^{b_1}(x^0, x^3)A^{c_1}_\mu(2D, x^0, x^3)\ldots f^{c_{n-2}}{}_{b_{n-1}c_{n-1}}\varepsilon^{b_{n-1}}(x^0, x^3)A^{c_{n-1}}_\mu(2D, x^0, x^3), \\
\tilde{A}^a_\mu(2D, x^0, x^3) &= e^{f^a_{bc}\varepsilon^b(x^0, x^3)}\, A^c_\mu(2D, x^0, x^3), \\
\tilde{A}^\mu_a(2D, x^0, x^3) &= A^c_\mu(2D, x^0, x^3)\, e^{-f^a_{bc}\varepsilon^b(x^0, x^3)}.
\end{aligned}
\tag{C10}
$$

As a consequence,

$$
\tilde{A}^a_\mu(2D, x^0, x^3)\, \tilde{A}^\mu_a(2D, x^0, x^3) = A^a_\mu(2D, x^0, x^3)\, A^\mu_a(2D, x^0, x^3),
\tag{C11}
$$

which maintains the 2D gauge invariance of the derived 2D action integral Eq. (33).

2. The Slavnov-Taylor identities in the 2D space-time in the Lorentz gauge

The Slavnov-Taylor identities in the standard 4D space-time in the Lorentz gauge has the form [34]:

$$
\int \exp\left(i\mathcal{A}(A)\right) \cdot \int dz \left(J^\nu_b(z)\partial_\nu(M^{-1})^{ba}(z,y) + g_{4D}f^{bc}_d\, J^d_\nu(z)\, A^\nu_b(z)\, (M^{-1})^{ca}(z,y)\right) \Delta(A)dA = 0,
\tag{C12}
$$

where $J^a_\nu(z)$ is a fermion current, $(M^{-1})^{ca}(z,y)$ is the propagator of a scalar field, and $\Delta(A)$ is the Faddeev-Popov determinate. After the compactification with respect to Eqs. (7), (12), and (18), the action integral $\mathcal{A}(A)$ becomes $\mathcal{A}[2D, A(2D)]$. Integrating the first term in the circular brackets by parts with respect to the z variable and using Eqs.(7) and (18), we obtain

$$
\begin{aligned}
\int dz \left(J^\nu_b(z)\partial_\nu(M^{-1})^{ba}(z,y)\right) &= -\int dz(M^{-1})^{ba}(z,y) \\
&\times \Bigg([|G_1(\boldsymbol{z}_\perp)|^2 + |G_2(\boldsymbol{z}_\perp)|^2]^2(\partial_0 J^0_b(z^0, z^3) + \partial_3 J^3_b(z^0, z^3)) \\
&\qquad - iTr\left\{(p_1 - ip_2)G^*_1(\boldsymbol{z}_\perp)G_2(\boldsymbol{z}_\perp)\bar{\Psi}(z^0, z^3)T_b\Psi(z^0, z^3)\right\}\Bigg).
\end{aligned}
\tag{C13}
$$

The integral involving the first term inside the above curly bracket is equal to zero because of the Lorentz gauge for the A^μ_a field and Eq.(65) while the second one is found to be the same due to the trace calculation. Thus the first term in the circular brackets in Eq. (C12) is equal to zero.

As for the second term in the circular brackets in Eq. (C12), by using Eqs. (12), (18), and (65) it can be written as

$$
\begin{aligned}
\int dz\quad &\left(g_{4D}f^{bc}_d\, J^d_\nu(z)\, A^\nu_b(z)\, (M^{-1})^{ca}(z,y)\right) \\
&= g_{4D}f^{bc}_d \int dz[|G_1(\boldsymbol{z}_\perp)|^2 + |G_2(\boldsymbol{z}_\perp)|^2]^{3/2}J^d_\nu(2D, z^0, z^3)\, A^\nu_b(2D, z^0, z^3)\, (M^{-1})^{ca}(z,y) \\
&= m^2_{gfT}\, g_{4D}f^{bc}_d \int dz[|G_1(\boldsymbol{z}_\perp)|^2 + |G_2(\boldsymbol{z}_\perp)|^2]^{3/2}A^d_\nu(2D, z^0, z^3)\, A^\nu_b(2D, z^0, z^3)\, (M^{-1})^{ca}(z,y) = 0.
\end{aligned}
\tag{C14}
$$

The last expression is equal to zero because of the anti-symmetry of the structure constant. Thus, the pre-exponent in Eq. (C12) is found to be equal to zero after the $4D \to 2D$ compactification. This means that the Slavnov-Taylor identities are not violated in the 2D space-time we have considered.