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Enhancement of Critical Temperature of a Striped Holographic Superconductor

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We study the interplay between the stripe order and the superconducting order in a strongly coupled striped superconductor using gauge/gravity duality. In particular, we study the effects of inhomogeneity introduced by the stripe order on the superconducting transition temperature beyond the mean field level by including the effects of backreaction onto the spacetime geometry in the dual gravitational picture. We find that inhomogeneity *enhances* the critical temperature relative to its value for the uniform system.

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I. INTRODUCTION

One of the differences between conventional superconductors and high temperature superconductors is that the normal states of the conventional superconductors are well described by Fermi liquid, whose only (weak coupling) instability is to superconductivity. By contrast, the normal states of high temperature superconductors, such as cuprates and iron pnictides, are highly correlated and thus, exhibit other low temperature orders which interact strongly with superconductivity. One of the prominent orders is the unidirectional charge density wave "stripe" order [1–4].

It is therefore important to understand the nature of the interplay between superconductivity and the stripe order in the presence of strong correlation. In this article, we study the effects of the stripe order on the superconducting transition temperature of the strongly correlated superconductor using holography or gauge/gravity correspondence. In gauge/gravity duality, the strongly coupled systems are mapped to a weakly coupled Einstein-Maxwell-scalar theory on black hole spacetimes with negative cosmological constant, or the so-called anti de-Sitter (AdS) black holes. Just like the normal states of high temperature superconductors, AdS black holes feature numerous types of instability that lead to the formation of scalar 'hair' [5, 6] (which corresponds to superconductivity), striped phases [7–9] and nematic phases [10]. Ultimately, we would like to study the system where both the superconducting order and the stripe order emerge dynamically, however, since our focus in this article is the effects of the stripe order on the critical temperature, we will follow Refs. [11, 12] where the inhomogeneity is introduced via a modulated chemical potential, with wavenumber Q. These articles studied the holographic striped superconductor by neglecting the backreaction of the electromagnetic field on the spacetime geometry, which in field theory language corresponds to the mean field treatment of the system. To improve upon their results, we would like to consider the effects of fluctuations on the strongly coupled striped superconductor by studying the backreacted spacetimes.

We find that the critical temperature for the formation of the scalar hair is maximum in the case of an AdS Schwarzschild black hole, which is the solution to the Einstein equations when backreaction is neglected [11, 12]. This means that the superconducting transition temperature obtained by including fluctuations is lower than that obtained in the case where fluctuations are neglected. On the field theory side, this can also be understood using a Ginzburg-Landau type argument: introducing fluctuations costs us free energy, and thus lowering of the critical temperature. Usually, the corrections due to fluctuations are small enough that the qualitative behavior of T_c as a function of other parameters remains the same. However, when the system is inhomogeneous due to the presence of stripe order, the effects of fluctuations in the regime where the inverse fluctuation length scale is smaller than the wavenumber Q are rather drastic. Starting at $Q = 0$, as we increase Q, we see that T_c exhibits a steep jump when we turn on the modulation, and after reaching a maximum, it decreases monotonically, as Q increases, asymptotically approaching a constant value as $Q \to \infty$. As the critical temperature at finite Q is larger than the values at both $Q = 0$ and $Q \to \infty$, which correspond to homogeneous limits, we find that the critical temperature is enhanced by the presence of stripe order. This result shares similar qualitative features with the result of Ref. [13], in which the effects of local inhomogeneity on the critical temperature within the framework of BCS theory were studied.

II. SET-UP

We are interested in studying a strongly coupled striped superconductor using the gauge/gravity duality. To this end, consider a $U(1)$ gauge potential A^a and a scalar field ψ charged under this potential living in a $3+1$-dimensional spacetime with negative cosmological constant $\Lambda = -3/L^2$. The scalar field is dual to the scalar

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order parameter of the superconductor, i.e., the condensate, while the $U(1)$ gauge field is dual to the four-current in the strongly coupled system. For simplicity, we shall adopt units in which $L = 1$, $16\pi G = 1$.

To study the strong coupling regime of the superconductor, we only need to study the gravity theory at the classical level. In particular, we are interested in finding solutions to the classical equations of motion whose boundary values are related to the parameters of the superconductor.

The action for this system is

$$S = \int d^4x\sqrt{-g}\left[R + 6 - \frac{1}{4}F^2 - |D_a\psi|^2 - m^2|\psi|^2\right] , \tag{1}$$

where, $D_a = \partial_a - iqA_a$, $F_{ab} = \partial_a A_b - \partial_b A_a$ and $a, b \in \{t, r, x, y\}$. Here, m and q are the mass and charge of the scalar field, respectively.

The field equations consist of the Einstein equations,

$$R_{ab} - \frac{1}{2}g_{ab}R - 3g_{ab} = \frac{1}{2}T_{ab} , \tag{2}$$

where the stress-energy tensor is

$$\begin{aligned} T_{ab} &= F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F^{cd}F_{cd} \\ &\quad + D_a\psi(D_b\psi)^* + (a \leftrightarrow b) - g_{ab}\left[|D_a\psi|^2 + m^2|\psi|^2\right] , \end{aligned} \tag{3}$$

the Maxwell equations,

$$\frac{1}{\sqrt{-g}}\partial_b(\sqrt{-g}F^{ab}) = J^a , \tag{4}$$

where the $U(1)$ current is

$$J^a = -i[\psi^* D^a\psi - \text{c.c.}] , \tag{5}$$

and the Klein-Gordon equation for the scalar field,

$$-\frac{1}{\sqrt{-g}}D_a(\sqrt{-g}g^{ab}D_b\psi) + m^2\psi = 0 . \tag{6}$$

III. ABOVE THE CRITICAL TEMPERATURE

Above the critical temperature T_c, the scalar field vanishes ($\psi = 0$) and thus, the Einstein-Maxwell equations simplify to

$$R^a{}_b + 3\delta^a{}_b = \frac{1}{2}T^a{}_b , \quad \partial_b(\sqrt{-g}F^{ab}) = 0 , \tag{7}$$

where

$$T^a{}_b = F^{ac}F_{bc} - \frac{1}{4}\delta^a{}_b F^{cd}F_{cd} . \tag{8}$$

We are interested in finding a static black hole solution of flat conformal boundary which is sourced by a modulated chemical potential $\mu(\vec{x})$, where μ is a given spatially dependent function. The presence of this modulated chemical potential gives rise to inhomogeneities in the system. However, it should be emphasized that this is a phenomenological description of inhomogeneities. The chemical potential is an effective potential resulting from interactions within the system. Ultimately, one would like to understand the dynamical emergence of the modulated potential and attendant inhomogeneities. However, here, we are only interested in the consequences of the presence of the modulated potential, which we treat as fixed.

Let (x, y) be the Cartesian spatial coordinates of the two-dimensional conformal boundary. We concentrate on the case in which μ only depends on one of the coordinates, which is chosen to be x. For definiteness, we only consider the case in which only a homogeneous term and a single oscillating mode are present,

$$\mu(x) = \mu(1 - \delta) + \mu\delta\cos Qx . \tag{9}$$

To find a solution to the Einstein-Maxwell equations (7), consider the metric *ansatz*

$$ds^2 = -r^2e^{-\alpha}dt^2 + e^{\alpha}\frac{dr^2}{r^2} + r^2e^{-\beta}\left[e^{-\gamma}dx^2 + e^{\gamma}dy^2\right] , \tag{10}$$

where α, β, and γ are functions of (r, x). The boundary is at $r \to \infty$ and the horizon is at $r = r_+$, where r_+ is an arbitrary parameter. For a flat conformal boundary, we require α, β, $\gamma \to 0$, as $r \to \infty$, and in fact, we find $\alpha \sim \mathcal{O}(r^{-3})$ while β and $\gamma \sim \mathcal{O}(r^{-4})$.

For the $U(1)$ potential, we fix the gauge so that $A_r = A_x = A_y = 0$ and $A_t = A_t(r, x)$ with $A_t = 0$ at the horizon (for a finite norm, $A_aA^a < \infty$), whereas at the boundary,

$$A_t(r, x)\Big|_{r\to\infty} = \mu(x) . \tag{11}$$

We shall solve the Einstein-Maxwell equations (7) perturbatively by expanding around the Schwarzschild solution, which is obtained as $\mu \to 0$. This corresponds to the probe limit in which the scalar charge $q \to \infty$ so that the product $q\mu$ remains finite. Near the critical temperature, we have a radius of the horizon of the same order as $q\mu$ ($r_+ \sim q\mu$), so an expansion in μ is equivalent to an expansion in $1/q$. More precisely, the expansion is in the dimensionless parameter

$$\left(\frac{\mu}{r_+}\right)^2 \sim \frac{1}{q^2} , \tag{12}$$

which is the only parameter in the Einstein-Maxwell system (since the vector potential enters quadratically). This expansion is valid for large black holes (or small chemical potential), or more precisely for

$$\mu \lesssim r_+ . \tag{13}$$

Expanding in the small dimensionless parameter (12), we have

$$A_t = A_t^{(0)} + \left(\frac{\mu}{r_+}\right)^2 A_t^{(1)} + \dots$$
$$\alpha = \alpha^{(0)} + \left(\frac{\mu}{r_+}\right)^2 \alpha^{(1)} + \dots$$
$$\beta = \beta^{(0)} + \left(\frac{\mu}{r_+}\right)^2 \beta^{(1)} + \dots$$
$$\gamma = \gamma^{(0)} + \left(\frac{\mu}{r_+}\right)^2 \gamma^{(1)} + \dots \quad (14)$$

and consequently, the expansions of the metric, Ricci tensor and $U(1)$ field strength and stress-energy tensor, respectively,

$$g_{ab} = g_{ab}^{(0)} + \left(\frac{\mu}{r_+}\right)^2 g_{ab}^{(1)} + \dots$$
$$R_{ab} = R_{ab}^{(0)} + \left(\frac{\mu}{r_+}\right)^2 R_{ab}^{(1)} + \dots$$
$$F_{ab} = \left(\frac{\mu}{r_+}\right)^2 F_{ab}^{(0)} + \dots$$
$$T_{ab} = \left(\frac{\mu}{r_+}\right)^2 \mathcal{T}_{ab}^{(0)} + \dots \quad (15)$$

At zeroth order, the Einstein-Maxwell equations read

$$R^{(0)\,a}{}_b + 3\delta^a{}_b = 0\,, \qquad \partial_b\left(\sqrt{-g^{(0)}}F^{(0)ab}\right) = 0\,. \quad (16)$$

The Einstein equations decouple and are solved by the AdS Schwarzschild black hole[1]

$$e^{-\alpha^{(0)}} \equiv h = 1 - \left(\frac{r_+}{r}\right)^3\,, \qquad \beta^{(0)} = \gamma^{(0)} = 0\,. \quad (17)$$

To solve the Maxwell equations, it is convenient to introduce the coordinate

$$z = \frac{r_+}{r}\,, \quad (18)$$

so that the boundary is at $z = 0$ and the horizon at $z = 1$. Writing the $U(1)$ potential in terms of Fourier modes

$$A_t^{(0)} = \mu(1-\delta)\mathcal{A}_0(z) + \mu\delta\mathcal{A}_1(z)\cos Qx\,, \quad (19)$$

we deduce the mode equations

$$\mathcal{A}_n''(z) - \frac{n^2Q^2}{r_+^2 h(z)}\mathcal{A}_n(z) = 0 \qquad (n = 0, 1)\,, \quad (20)$$

to be solved together with the boundary conditions $\mathcal{A}_n(0) = 1$, $\mathcal{A}_n(1) = 0$. Here, $h(z) = 1 - z^3$ (Eq. (17)) and $'$ denotes a derivative with respect to z. For $n = 0$, we obtain

$$\mathcal{A}_0(z) = 1 - z\,. \quad (21)$$

For $n = 1$, a good analytic approximation to the solution is given by [11, 12]

$$\mathcal{A}_1(z) \approx \frac{\sinh\left[\frac{Q}{r_+}(1-z)\right]}{\sinh\frac{Q}{r_+}}\,. \quad (22)$$

The error vanishes at both ends ($z = 0, 1$) and attains a maximum value at an intermediate z. As $Q \to 0$, this maximum value decreases like Q^2, whereas as $Q \to \infty$, it decays exponentially. Numerically, for $Q/r_+ \sim 0.1, 1, 10$, we obtain a maximum error of $10^{-4}, 0.01, 0.001$, respectively.

With the choice of boundary conditions (9), the lowest-order stress-energy tensor $\mathcal{T}_{ab}^{(0)}$ has modes with $n \le 2$ due to the fact that it is quadratic in the $U(1)$ potential. The same should be true for the first-order corrections to the metric.

Explicitly, the non-vanishing components of the zeroth-order electromagnetic stress-energy tensor are

$$\mathcal{T}^{(0)\,t}{}_t = -\mathcal{T}^{(0)\,y}{}_y = -\frac{z^4}{4}\left[\frac{\mathcal{E}_x^2}{h} + \mathcal{E}_z^2\right]\,,$$
$$\mathcal{T}^{(0)\,z}{}_z = -\mathcal{T}^{(0)\,x}{}_x = \frac{z^4}{4}\left[\frac{\mathcal{E}_x^2}{h} - \mathcal{E}_z^2\right]\,,$$
$$\mathcal{T}^{(0)\,x}{}_z = \frac{1}{h}\mathcal{T}^{(0)\,z}{}_x = -\frac{z^4}{2\,h}\mathcal{E}_x\mathcal{E}_z\,, \quad (23)$$

given in terms of the components of the electric field

$$\mathcal{E}_x = \frac{\delta Q}{r_+}\mathcal{A}_1 \sin Qx\,,$$
$$\mathcal{E}_z = (1-\delta)\mathcal{A}_0'(z) + \delta\mathcal{A}_1'(z)\cos Qx\,. \quad (24)$$

To solve the Einstein equations at first order,

$$R^{(1)\,a}{}_b = \mathcal{T}^{(0)\,a}{}_b\,, \quad (25)$$

we set

$$\alpha^{(1)} = \alpha_0^{(1)}(z) + \alpha_1^{(1)}(z)\cos Qx + \alpha_2^{(1)}(z)\cos 2Qx\,,$$
$$\beta^{(1)} = \beta_0^{(1)}(z) + \beta_1^{(1)}(z)\cos Qx + \beta_2^{(1)}(z)\cos 2Qx\,,$$
$$\gamma^{(1)} = \gamma_0^{(1)}(z) + \gamma_1^{(1)}(z)\cos Qx + \gamma_2^{(1)}(z)\cos 2Qx\,. \quad (26)$$

We obtain five non-vanishing components for each set of functions $\{\alpha_i^{(1)}, \beta_i^{(1)}, \gamma_i^{(1)}\}$, where $i = 0, 1, 2$. Of the five equations, only three are independent and can be solved analytically for the three corresponding metric functions. After some algebra, we obtain the following system of equations for the modes of the metric functions.

[1] There are also other inhomogeneous black hole spacetimes obtained by perturbing Reissner-Nordström black hole [14].

For the Fourier zero modes, we obtain

$$\alpha_0^{(1)\prime} - \left(\frac{3}{z} - \frac{h'}{h}\right)\alpha_0^{(1)} - \frac{z}{2}\left(\frac{4}{z} - \frac{h'}{h}\right)\gamma_0^{(1)\prime} - \frac{z^3\left(\frac{Q^2}{r_+^2}\delta^2\,\mathcal{A}_1^2 - h\left(2(1-\delta)^2\mathcal{A}_0'^2 + \delta^2\mathcal{A}_1'^2\right)\right)}{8h^2} = 0\,,$$

$$\beta_0^{(1)\prime\prime} - \left(\frac{2}{z} - \frac{h'}{h}\right)\beta_0^{(1)\prime} - \frac{Q^2z^2\delta^2\mathcal{A}_1^2}{4r_+^2h^2} = 0\,,$$

$$\gamma_0^{(1)\prime\prime} + \frac{Q^2z^2\delta^2\mathcal{A}_1^2}{4r_+^2h^2} = 0\,. \tag{27}$$

We solve these equations by requiring that the functions be regular at the horizon ($z = 1$) and vanish sufficiently fast at the boundary ($z = 0$). We obtain

$$\gamma_0^{(1)}(z) = -\frac{Q^2\delta^2}{4\,r_+^2}\int_0^z dz'\int_0^{z'} dz''\;\frac{(z'')^2\,\mathcal{A}_1^2}{h^2}\,, \tag{28}$$

$$\beta_0^{(1)}(z) = -\frac{Q^2\delta^2}{4\,r_+^2}\int_0^z dz'\,\frac{(z')^2}{h}\int_{z'}^1 dz''\,\frac{\mathcal{A}_1^2}{h}\,, \tag{29}$$

$$\alpha_0^{(1)}(z) = \frac{z^3}{8h}\int_z^1 \overline{\alpha}_0^{(1)}(z')\,dz'\,, \tag{30}$$

where

$$\overline{\alpha}_0^{(1)}(z) = 2(1-\delta)^2\mathcal{A}_0'^{\,2} + \delta^2\mathcal{A}_1'^{\,2} - \frac{Q^2}{r_+^2}\delta^2\frac{\mathcal{A}_1^2}{h} - \gamma_0^{(1)\prime}\frac{4h - zh'}{z^3}\,. \tag{31}$$

For the Fourier first modes, we obtain

$$\alpha_1^{(1)\prime} - \left(\frac{3}{z} - \frac{\frac{Q^2}{r_+^2}\,z + 2h'}{2h}\right)\alpha_1^{(1)} - \frac{z}{2}\left(\frac{4}{z} - \frac{h'}{h}\right)\gamma_1^{(1)\prime} + \frac{Q^2z(\beta_1^{(1)} - \gamma_1^{(1)})}{2r_+^2h} + \frac{z^3\delta(1-\delta)\mathcal{A}_0'\mathcal{A}_1'}{2h} = 0\,,$$

$$\beta_1^{(1)\prime\prime} - \left(\frac{2}{z} - \frac{h'}{h}\right)\beta_1^{(1)\prime} = 0\,,$$

$$\gamma_1^{(1)\prime\prime} - \frac{Q^2}{r_+^2\,h}\alpha_1^{(1)} = 0\,. \tag{32}$$

The second equation readily yields

$$\beta_1^{(1)}(z) = 0\,. \tag{33}$$

By eliminating $\alpha_1^{(1)}$ between the other two equations, we obtain a third order differential equation for $\gamma_1^{(1)}$. Then the possible behavior of $\gamma_1^{(1)}$ at the horizon is found to be a linear combination of $1 - z$, $(1 - z)\ln(1 - z)$, and $(1 - z)^{1+Q^2/6r_+^2}$. We fix the three integration constants by demanding $\gamma_1^{(1)}(0) = 0$, $\gamma_1^{(1)\prime}(0) = 0$, and $\gamma_1^{(1)\prime\prime} \lesssim \mathcal{O}(1/(1-z))$ at the horizon ($z = 1$). The second boundary condition, together with Eqs. (32), ensure $\alpha_1^{(1)} \sim z^3$ at the boundary. The third boundary condition is necessary for the existence of a well-defined temperature (surface gravity), resulting in $\alpha_1^{(1)}(1) = 0$, on account of the third equation in (32).

Finally, for the Fourier second modes, we obtain

$$\alpha_2^{(1)\prime} - \left(\frac{3}{z} - \frac{2\frac{Q^2}{r_+^2}z + h'}{h}\right)\alpha_2^{(1)} - \frac{z}{2}\left(\frac{4}{z} - \frac{h'}{h}\right)\gamma_2^{(1)\prime} + \frac{2Q^2z}{r_+^2h}(\beta_2^{(1)} - \gamma_2^{(1)}) + \frac{\delta^2z^3\left(\frac{Q^2}{r_+^2}\mathcal{A}_1^2 + h\mathcal{A}_1'^{\,2}\right)}{8h^2} = 0\,,$$

$$\beta_2^{(1)\prime\prime} - \left(\frac{2}{z} - \frac{h'}{h}\right)\beta_2^{(1)\prime} + \frac{\delta^2Q^2z^2\mathcal{A}_1^2}{4r_+^2h^2} = 0\,,$$

$$\gamma_2^{(1)\prime\prime} - \frac{Q^2\left(16\,h\,\alpha_2^{(1)} + \delta^2z^2\mathcal{A}_1^2\right)}{4r_+^2h^2} = 0\,. \tag{34}$$

The second equation yields

$$\beta_2^{(1)}(z) = \frac{Q^2\,\delta^2}{4r_+^2}\int_0^z dz'\,\frac{(z')^2}{h}\int_{z'}^1 dz''\,\frac{\mathcal{A}_1^2}{h}\,. \tag{35}$$

We note that $\beta_2^{(1)} = -\beta_0^{(1)}$.

Eliminating $\alpha_2^{(1)}$ between the other two equations, we obtain, as before, a third-order differential equation for $\gamma_2^{(1)}$, from which we deduce the possible near horizon behavior, $1 - z$, $(1 - z)\ln(1 - z)$, and $(1 - z)^{1+2Q^2/3r_+^2}$. As before, we fix the three integration constants by demanding $\gamma_2^{(1)}(0) = 0$, $\gamma_2^{(1)\prime}(0) = 0$, and $\gamma_2^{(1)\prime\prime} \lesssim \mathcal{O}(1/(1 - z))$ at the horizon ($z = 1$). The second boundary condition, together with Eqs. (34), ensure $\alpha_2^{(1)} \sim z^3$ at the boundary. The third boundary condition is necessary for the existence of a well-defined temperature (surface gravity), resulting in $\alpha_2^{(1)}(1) = 0$, on account of the third equation in (34).

The equations for the various modes can be solved numerically subject to the boundary conditions outlined above. We have plotted $\alpha_n^{(1)}$ ($n = 0, 1, 2$) in Fig. 1 for representative values of Q, whereas $\beta_n^{(1)}$ ($n = 0, 2$; it vanishes for $n = 1$) is plotted in Fig. 2, and $\gamma_n^{(1)}$ is plotted in Figs. 3, 4, and 5, for $n = 0, 1$ and 2, respectively.

Note that, the $\beta_n^{(1)}$ and $\gamma_n^{(1)}$ components of metric perturbations are sourced by x-component of electric field (24), which vanishes at both small and large Q. The functions $\beta_n^{(1)}$s and $\gamma_n^{(1)}$s depend on Q via two terms: a direct proportionality factor Q^2 and area under the functions $\mathcal{A}_1^2/h$ or $z^2\mathcal{A}_1^2/h^2$. The first factor vanishes at $Q \to 0$, while the integrals vanish at $Q \to \infty$, due to $\mathcal{A}_1 \sim \frac{Q}{\sinh Q}(1 - z)$ near the horizon. Consequently,

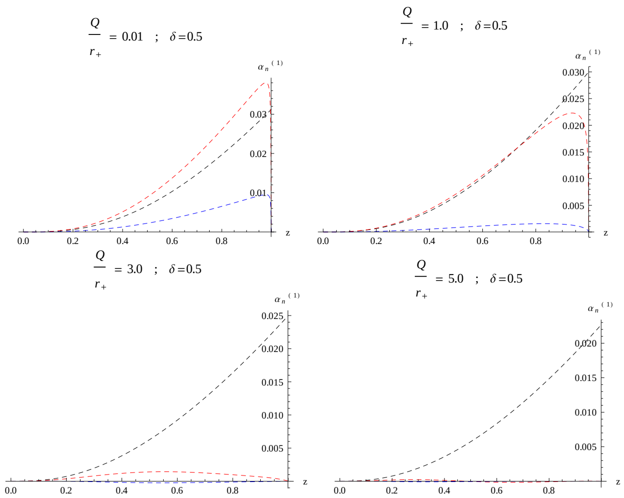

FIG. 1. $\alpha_0^{(1)}$ (black), $\alpha_1^{(1)}$ (red) and $\alpha_2^{(1)}$ (blue) for $\delta = 0.5$, and $Q/r_+ = 0.1$, 1, 3 and 5.

$\beta_n^{(1)}$ and $\gamma_n^{(1)}$ $(n = 0, 1, 2)$ are very small in both limits $Q \ll r_+$ and $Q \gg r_+$. These functions are more significant in the intermediate range $2 < Q/r_+ < 3$ and decay rapidly on both sides, but even when they reach their maximum, they remain well below unity (see Figs. 2–5). Thus, their contribution to physical quantities is negligible in the entire range of Q.

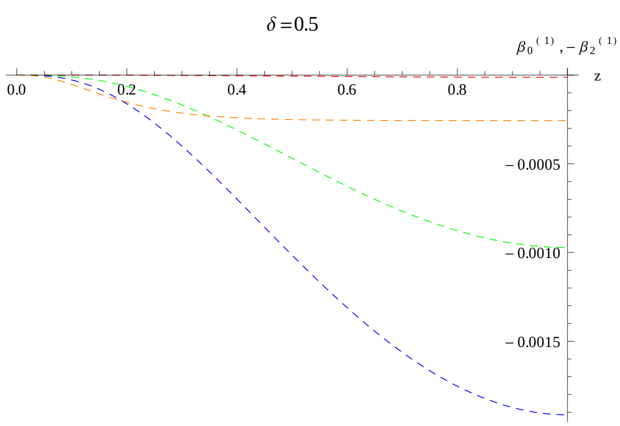

FIG. 2. $\beta_0^{(1)} = -\beta_2^{(1)}$ for $\delta = 0.5$, and $QL^2/r_+ = 0.1$ (red), 1.0 (green), 2.0 (blue) and 8.0 (orange).

Next, we discuss the behavior of $\alpha_n^{(1)}$ $(n = 0, 1, 2)$ which are physically important because they determine the temperature. Indeed, the Hawking temperature at first perturbative order is

$$T = \frac{3r_+}{4\pi}\left[1 - \frac{\mu^2}{r_+^2}\alpha^{(1)}(1)\right] . \tag{36}$$

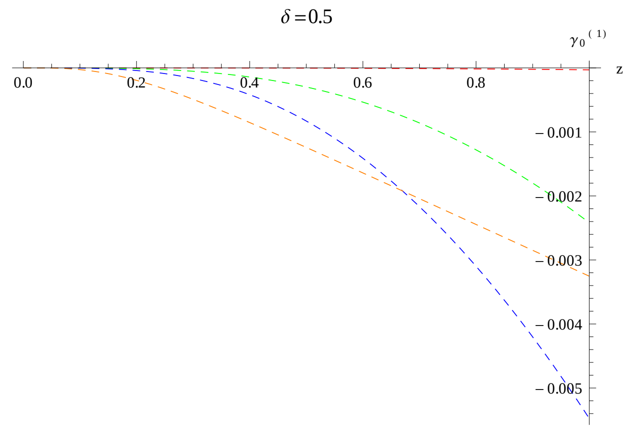

FIG. 3. $\gamma_0^{(1)}$ for $\delta = 0.5$, and $Q/r_+ = 0.1$ (red), 1.0 (green), 2.0 (blue) and 8.0 (orange).

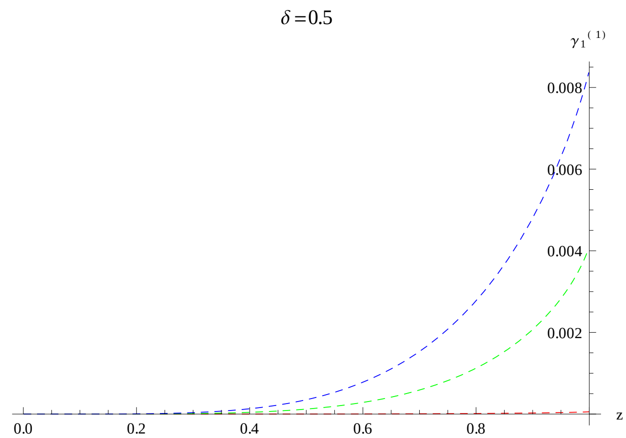

FIG. 4. $\gamma_1^{(1)}$ for $\delta = 0.5$, and $Q/r_+ = 0.1$ (red), 1.0 (green) and 2.0 (blue).

Since $\alpha_n^{(1)}(1) = 0$ for $n \geq 1$, we have

$$\alpha^{(1)}(1) = \alpha_0^{(1)}(1) = \frac{\overline{\alpha}_0(1)}{24} = \left.\frac{2(1-\delta)^2 + \delta^2 \mathcal{A}_1'^{\,2} - 3\gamma_0^{(1)\prime}}{24}\right|_{z=1} . \tag{37}$$

We can calculate these functions analytically in the two important limits: $Q \to 0$ and $Q \to \infty$.

In the limit $Q \to 0$, we obtain the analytic expressions

$$\begin{aligned}
\alpha_0^{(1)} &= \left((1-\delta)^2 + \frac{\delta^2}{2}\right)\frac{z^3}{4\,(1+z+z^2)} + \mathcal{O}\left(\frac{Q^2}{r_+^2}\right) , \\
\alpha_1^{(1)} &= \frac{(1-\delta)\,\delta\, z^3}{2\,(1+z+z^2)}(1-z)^{Q^2/6r_+^2} + \mathcal{O}\left(\frac{Q^2}{r_+^2}\right) , \\
\alpha_2^{(1)} &= \frac{\delta^2\, z^3}{8\,(1+z+z^2)}(1-z)^{2Q^2/3r_+^2} + \mathcal{O}\left(\frac{Q^2}{r_+^2}\right) .
\end{aligned} \tag{38}$$

At $Q = 0$ (or equivalently, $\delta = 0$), we recover the exact Reissner-Nordström solution representing the homo-

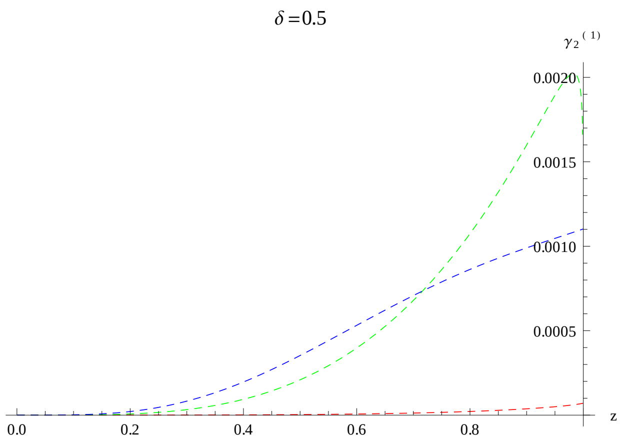

FIG. 5. $\gamma_2^{(1)}$ for $\delta = 0.5$, and $Q/r_+ = 0.1$ (red), 1.0 (green) and 2.0 (blue).

geneous system

$$e^{-\alpha} = e^{-\alpha^{(0)}} \left(1 + \frac{\mu^2}{r_+^2}\,\alpha^{(1)}\right) = 1 - \left(1 + \frac{\mu^2}{4r_+^2}\right) z^3 + \frac{\mu^2}{4r_+^2} z^4 \ , \tag{39}$$

where we used $\alpha^{(1)} = \alpha_0^{(1)} + \alpha_1^{(1)} + \alpha_2^{(1)}$ from Eq. (26) with $Q = 0$.

We recover the Schwarzschild solution, which is the solution in the probe limit, in the limit $\mu \to 0$. As we increase μ, we move further away from the probe limit and the effects of back reaction to the metric become more pronounced. We reach extremality at $\mu/r_+ = 2\sqrt{3}$, but this lies outside the regime of validity of our approximation (Eq. (13)).

It should be noted that the convergence to the homogeneous system is not uniform. At the horizon, $\alpha_n^{(1)}(1) \to 0$ for $n = 1, 2$, and therefore $\alpha^{(1)}$ does not converge to its homogeneous counterpart. In other words, the limits $Q \to 0$ and $z \to 1$ do not commute. It follows that there is a discontinuity in the temperature which depends on the behavior of $\alpha_n^{(1)}$ at the horizon. From Eq. (36) in the limit $Q \to 0$, we obtain

$$T \approx \frac{3r_+}{4\pi}\left[1 - \frac{\mu^2\left((1-\delta)^2 + \delta^2/2\right)}{12r_+^2}\right] , \tag{40}$$

which is valid for small Q. Comparing this result with the homogeneous case, which is recovered by setting $\delta = 0$, we obtain an *enhancement* in temperature upon turning on modulation

$$\frac{\Delta T}{T} = \frac{T}{T_{\delta=0}} - 1 \approx \frac{\mu^2}{12r_+^2}\delta\left(2 - \frac{3\delta}{2}\right) , \tag{41}$$

with a maximum enhancement for $\delta = \frac{2}{3}$. The change in temperature is discontinuous, but this is an artifact of keeping only the first order in perturbation theory. This change in the temperature is expected to become smooth (yet remain steep) as higher orders in the perturbative expansion are included.

On the other hand, in the $Q \gg r_+$ regime, the contribution of $\mathcal{A}_1$ becomes exponentially small, and all functions except $\alpha_0^{(1)}$ become negligible. In this regime, we have

$$\alpha_0^{(1)} \approx \frac{(1-\delta)^2\, z^3}{4\,(1 + z + z^2)} \,. \tag{42}$$

So in the $Q \to \infty$ limit, we recover another exact Reissner-Nordström solution, albeit with less charge density,

$$e^{-\alpha} \approx 1 - \left(1 + \frac{\mu^2(1-\delta)^2}{4r_+^2}\right) z^3 + \frac{\mu^2(1-\delta)^2}{4r_+^2} z^4 \,. \tag{43}$$

This coincides with the homogeneous solution (39) if $\delta = 0$, as expected.

We then deduce the temperature for large Q to be given by

$$T \approx \frac{3r_+}{4\pi}\left[1 - \frac{\mu^2(1-\delta)^2}{12r_+^2}\right] . \tag{44}$$

IV. THE CRITICAL TEMPERATURE

The Klein-Gordon equation for a static scalar field $\psi(z, x)$ of mass m and charge q reads

$$\sum_{i=z,x} \frac{1}{\sqrt{-g}}\,\partial_i\left(\sqrt{-g}g^{ii}\partial_i\psi\right) + \left(q^2\,A_t^2 - m^2\right)\psi = 0\,. \tag{45}$$

The mass is related to the conformal dimension Δ of the superconducting order parameter by

$$m^2 = \Delta\,(\Delta - 3)\ . \tag{46}$$

We have $\psi \sim z^\Delta$ as $z \to 0$.

For a given set of parameters $\{\Delta, \delta, q, Q, \mu\}$, the wave equation yields the critical value of the radius of the horizon,

$$r_+ = r_{+c}\ . \tag{47}$$

It is convenient to define the eigenvalue

$$\lambda = \frac{q\mu}{r_{+c}}\,. \tag{48}$$

The critical temperature is then found from (36) by setting $r_+ = r_{+c}$. We obtain

$$\frac{T_c}{q\mu} = \frac{3}{4\pi}\left[\frac{1}{\lambda} - \frac{\lambda}{q^2}\alpha^{(1)}(1) + \mathcal{O}\left(\frac{1}{q^4}\right)\right] . \tag{49}$$

To simplify the wave equation, we note that the electrostatic potential has Fourier modes with wavenumbers

nQ, where $n = 0, 1$, whereas the metric consists of modes with $n = 0, 1, 2$. It follows that ψ can be expanded in a Fourier series,

$$\psi(z,x) = z^{\Delta} F(z,x) \quad , \qquad F(z,x) = \sum F_n(z) \cos nQx \ , \tag{50}$$

where we factored out z^{Δ}, so that the modes $F_n(z) \sim$ const., as $z \to 0$. Using (50), the wave equation (45) can be written as an infinite system of coupled ordinary differential equations.

In the large Q regime, the higher modes become negligible, and the wave equation can be well approximated by the equation obeyed by the zero mode where all other modes have been set to zero,

$$F_0'' + \left[\frac{2(\Delta-1)}{z} + \frac{h_0'}{h_0} \right] F_0' + \frac{\Delta\left[(\Delta-3)(h_0-1) + z h_0'\right]}{z^2 h_0} F_0 + \lambda^2 (1-\delta)^2 \frac{(1-z)^2}{h_0^2} F_0 = 0 \, , \tag{51}$$

where

$$h_0 \equiv e^{-\alpha^{(0)} - \frac{\mu^2}{r_+^2}\alpha_0^{(1)}} \approx h \left[1 - \frac{\mu^2}{r_+^2} \alpha_0^{(1)} \right] . \tag{52}$$

Expanding the scalar field, as we did with the other fields,

$$F = F^{(0)} + \left(\frac{\mu}{r_+} \right)^2 F^{(1)} + \dots \ , \tag{53}$$

where $F^{(0)}$ is the scalar field in the probe limit, we obtain for each Fourier mode,

$$F_n = F_n^{(0)} + \left(\frac{\mu}{r_+} \right)^2 F_n^{(1)} + \dots \ . \tag{54}$$

We also need to expand the eigenvalue (48) similarly,

$$\lambda = \lambda_0 + \left(\frac{\mu}{r_+} \right)^2 \lambda_1 + \dots \ . \tag{55}$$

We deduce for the probe limit zero mode,

$$F_0^{(0)\prime\prime} + \left[\frac{2(\Delta-1)}{z} + \frac{h'}{h} \right] F_0^{(0)\prime} + \frac{\Delta\left[(\Delta-3)(h-1) + z h'\right]}{z^2 h} F_0^{(0)} + \lambda_0^2 (1-\delta)^2 \frac{(1-z)^2}{h^2} F_0^{(0)} = 0 \, . \tag{56}$$

which is the same as the equation for a homogeneous system in the probe limit, but with μ reduced to $\mu(1-\delta)$.

The correction to the zeroth-order eigenvalue can be found using standard first-order perturbation theory. After some algebra, we obtain

$$\lambda_1 = \frac{\int_0^1 dz\, z^{2(\Delta-1)} h F_0^{(0)} \mathcal{H} F_0^{(0)}}{2\lambda_0 (1-\delta)^2 \int_0^1 dz\, z^{2(\Delta-1)} \frac{(1-z)^2}{h} [F_0^{(0)}]^2} \ , \tag{57}$$

where

$$\mathcal{H}\mathcal{F} \equiv \alpha_0^{(1)\prime} \mathcal{F}' + \left[\frac{\Delta(\Delta-3)}{z^2 h} \alpha_0^{(1)} + \frac{\Delta}{z} \alpha_0^{(1)\prime} - 2\lambda_0^2 (1-\delta)^2 \frac{(1-z)^2}{h^2} \alpha_0^{(1)} \right] \mathcal{F} \ . \tag{58}$$

The above results are valid in the $Q \to \infty$ limit. From (49), we deduce the asymptotic value of the temperature in this limit.

As we decrease Q, an increasing number of Fourier modes become significant and one needs to solve a coupled system of ordinary differential equations of increasing complexity. This can be done numerically. The error in the numerical analysis can be reduced to the desired accuracy by including enough higher modes of the Fourier expansion.

As $Q \to 0$, all modes become significant. In this limit, numerical methods based on a Fourier expansion become cumbersome. Fortunately, we can obtain analytic results in the limit $Q \to 0$, because all functions are slowly varying functions of x, and therefore the x-dependence can be ignored. We deduce the wave equation in the limit $Q \to 0$,

$$F'' + \left[\frac{2(\Delta-1)}{z} + \frac{\overline{h}'}{\overline{h}} \right] F' + \frac{\Delta}{z^2 \overline{h}} \left[(\Delta-3)(\overline{h}-1) + z\, \overline{h}' \right] F + \lambda^2 \frac{(1-z)^2}{\overline{h}^2} F = 0 \, , \tag{59}$$

where

$$\overline{h} \equiv h \left[1 - \frac{\mu^2}{r_+^2} \overline{\alpha} \right] \quad , \quad \overline{\alpha} = \alpha_0^{(1)} + \alpha_1^{(1)} + \alpha_2^{(1)} \, . \tag{60}$$

At zeroth order, this reduces to the probe limit result of the homogeneous case

$$F^{(0)\prime\prime} + \left[\frac{2(\Delta-1)}{z} + \frac{h'}{h} \right] F^{(0)\prime} + \frac{\Delta}{z^2 h} \left[(\Delta-3)(h-1) + z h' \right] F^{(0)} + \lambda_0^2 \frac{(1-z)^2}{h^2} F^{(0)} = 0 \, , \tag{61}$$

to be compared with the $Q \to \infty$ result (56).

At first order, we obtain the correction to the eigenvalue in the limit $Q \to 0$,

$$\lambda_1 = \frac{\int_0^1 dz\, z^{2(\Delta-1)} h F_0^{(0)} \overline{\mathcal{H}} F_0^{(0)}}{2\lambda_0 (1-\delta)^2 \int_0^1 dz\, z^{2(\Delta-1)} \frac{(1-z)^2}{h} [F_0^{(0)}]^2} \ , \tag{62}$$

where

$$\overline{\mathcal{H}}\mathcal{F} \equiv \overline{\alpha}' \mathcal{F}' + \left[\frac{\Delta(\Delta-3)}{z^2 h} \overline{\alpha} + \frac{\Delta}{z} \overline{\alpha}' - 2\lambda_0^2 \frac{(1-z)^2}{h^2} \overline{\alpha} \right] \mathcal{F} \ . \tag{63}$$

From (49), we deduce the value of the temperature in the limit $Q \to 0$.

At small Q, we obtain from Eq. (41) the enhancement in the critical temperature,

$$\frac{\Delta T_c}{T_c} \approx \frac{\lambda^2}{12q^2}\delta\left(2 - \frac{3\delta}{2}\right) + \mathcal{O}\left(\frac{1}{q^2}\right) , \tag{64}$$

which vanishes at the probe limit ($q \to \infty$) and becomes significant away from it. However, we stress that the above results are not accurate in the small q limit, as they are only first-order $\mathcal{O}(1/q^2)$ results.

The wave equation is solved numerically subject to the boundary conditions $F_0 \sim z^\Delta$ at the boundary and the demand of regularity at the horizon ($F_0(1) < \infty$). The results are shown in Figs. 6 and 7, for $\Delta = 2$ and 3, respectively. In each case, we have chosen the other parameters so that the curves asymptote to the same temperature as $Q \to \infty$. We note that all curves exhibit a jump at $Q = 0^+$, showing the enhancement of the critical temperature once modulation is switched on, in agreement with our analytic result (64). As Q increases, the critical temperature decreases monotonically. The jump vanishes in the probe limit which is obtained for $\mu = 0$ (Schwarzschild black hole). For any given Q, the critical temperature attains its maximum value at this limit. Put differently, back reaction to the metric lowers the critical temperature. Correspondingly, in the dual boundary system, quantum fluctuations result in a reduction in the critical temperature for a given modulation vector Q.

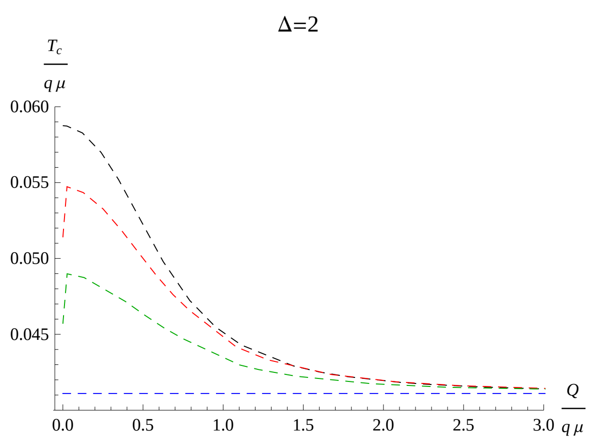

FIG. 6. From top to bottom: T_c *vs.* Q for $\Delta = 2$ and $(\delta, q^2) = (0.3, \infty)$, $(0.2, 7.92)$ and $(0.1, 4.22)$. Parameters are chosen so that curves asymptote to $T_c/(q\mu) = 0.041$ as $Q \to \infty$.

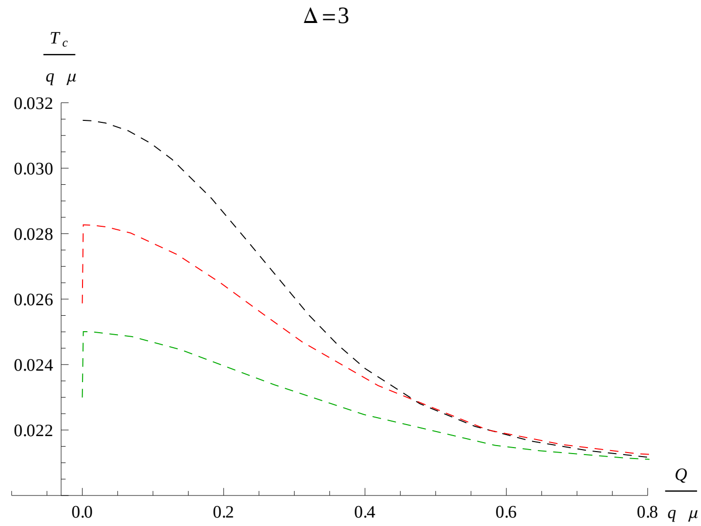

FIG. 7. From top to bottom: T_c *vs.* Q for $\Delta = 3$ and $(\delta, q^2) = (0.34, \infty)$, $(0.2, 20.65)$ and $(0.1, 13.13)$. Parameters are chosen so that curves asymptote to $T_c/(q\mu) = 0.020$ as $Q \to \infty$.

V. SUMMARY AND OUTLOOK

In this article, we have studied the effect of inhomogeneity on the superconducting transition temperature of the strongly coupled striped superconductor beyond the mean field level, by including backreaction of the electromagnetic field on the geometry of spacetime in the dual gravitational picture. We found that as we turn on the modulation, the critical temperature exhibits a steep jump. After that, as we increase Q, the critical temperature decreases until it reaches the asymptotic value. In other words, we found an enhancement of the critical temperature due to inhomogeneity that comes from the stripe order.

The discontinuous jump we see here is an artifact of only keeping the zero mode of the scalar field in the calculation and we expect that as we include the higher modes, this jump will become smooth but yet steep. It will be interesting to study whether the maximum of T_c corresponds to the value of Q being the inverse of superconducting correlation length scale as is seen in the BCS result.

Finally, we emphasize that the analysis was performed under the assumption that the inhomogeneities in the system were sourced by a fixed modulated chemical potential. This is, of course, a phenomenological description of the origin of inhomogeneities. It would be of great interest to understand the dynamical origin of the modulated chemical potential, which is due to interactions within the system, and how Q can be determined from the properties of the system (*cf.* with condensed matter systems in which the value of Q is tuned by changing the doping). Work in this direction is in progress.

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