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Vacuum Instability in Chern-Simons Gravity

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We explore perturbations about a Friedmann-Robertson-Walker background with a non-vanishing cosmological Chern-Simons scalar field in Chern-Simons gravity. At large momenta one of the two circularly polarized tensor modes becomes ghostlike. We argue that nevertheless the theory does not exhibit classical runaway solutions, except possibly in the relativistic nonlinear regime. However, the ghost modes cause the vacuum state to be quantum mechanically unstable, with a decay rate that is naively infinite. The decay rate can be made finite only if one interprets the theory as an effective quantum field theory valid up to some momentum cutoff Λ , which violates Lorentz invariance. By demanding that the energy density in photons created by vacuum decay over the lifetime of the Universe not violate observational bounds, we derive strong constraints on the two dimensional parameter space of the theory, consisting of the cutoff Λ and the Chern-Simons mass.

I. INTRODUCTION AND SUMMARY

General relativity has held up well to various tests over the years from experiments and astronomical observations [1], and is considered a pillar of standard cosmology. However, it is interesting to consider modifications to the theory, particularly in light of the observed acceleration of the Universe [2]. One useful approach is to think of Einstein gravity as an effective field theory, and consider higher order corrections to the Einstein-Hilbert action, either involving the metric alone or involving an additional posited scalar field. The goal then becomes to calculate the corrections to general relativity arising from these higher order terms, and using experiments to set bounds on the couplings parameterizing them.

One such extension to general relativity is Chern-Simons gravity [3, 4], where one assumes the existence of a scalar field ϑ coupled to gravity through a parity violating term. The theory is described by the action¹

$$S = S_{EH} + S_{CS} + S_{\vartheta} + S_{\text{mat}}, \quad (1.1)$$

where the various terms are respectively the Einstein-Hilbert term

$$S_{EH} = \frac{1}{2} m_{\text{p}}^2 \int d^4x \sqrt{-g} R, \quad (1.2a)$$

the Chern-Simons term

$$S_{CS} = \frac{1}{4} \alpha \int d^4x \sqrt{-g} \vartheta {}^*RR, \quad (1.2b)$$

the scalar term

$$S_{\vartheta} = -\frac{1}{2} \int d^4x \sqrt{-g} [\nabla_a \vartheta \nabla^a \vartheta + 2V(\vartheta)], \quad (1.2c)$$

and S_{mat} describes any other matter present. In these expressions $m_{\text{p}}^2 = (8\pi G)^{-1}$ is the square of the reduced Planck mass, g is the determinant of the metric, R is the Ricci scalar and α a coupling constant with dimensions of inverse mass. Here and throughout we use units with $\hbar = c = 1$. Also $V(\vartheta)$ is an arbitrary potential and the Pontryagin density is defined as

$${}^*RR = \frac{1}{2} \epsilon^{cdef} R^a{}_{bef} R^b{}_{acd}, \quad (1.3)$$

where ϵ^{cdef} is the four dimensional Levi-Civita tensor. For purposes of the present discussion, we will assume that matter is minimally coupled to the metric.

Consider now the dynamics of Chern-Simons gravity in perturbation theory about a Friedmann-Robertson-Walker (FRW) cosmological background. Now in the limit $\vartheta = \text{constant}$, the Chern-Simons term (1.2b) reduces to a surface term and we recover Einstein's equations for the space-time dynamics. Therefore, in the effective field theory that describes the perturbations, the operators that arise from the Chern-Simons term must be suppressed by a mass scale that is related to the derivative of the background scalar field. This mass scale is called the Chern-Simons mass m_{cs} , and is defined in the FRW context by [5]

$$m_{\text{cs}} \equiv \frac{m_{\text{p}}^2}{\alpha \dot{\vartheta}}, \quad (1.4)$$

where the dot denotes a derivative with respect to time. General relativity is recovered in the limit $m_{\text{cs}} \rightarrow \infty$ for linearized tensor perturbations. Because it is the Chern-Simons mass that enters into equations describing observables for linear perturbations, we choose to constrain it, rather than the more fundamental coupling α that appears in the action (1.2b). Also, since the background cosmological solution $\vartheta(t)$ need not be a linear function

¹ Throughout this paper we will restrict attention to the theory (1.1) in which the scalar field is dynamical; we will not consider the “non-dynamical” version of the theory in which the kinetic term for the scalar field is absent [3]. We note however that our derivation of the action (2.5) of the ghost graviton modes is valid for the non-dynamical theory, since the derivation does not involve any perturbations to the scalar field.

of time, the Chern-Simons mass will in general be a function of time or of redshift; we will focus in this paper on its value $m_{cs} = m_{cs}(t_0)$ today.

Past work constraining Chern-Simons gravity has utilized Solar System and binary pulsar tests of general relativity. Measurement of Lense-Thirring precession by LAGEOS give the bound [5]

$$m_{cs} \gtrsim 2 \times 10^{-13} \text{ eV}. \quad (1.5)$$

A bound 10^{11} times stronger has been claimed from binary pulsar studies [6], but the validity of this result has been questioned [7], and a corrected bound² from binary pulsars is [7]

$$m_{cs} \gtrsim 5 \times 10^{-10} \text{ eV}. \quad (1.6)$$

In this paper we study the vacuum stability of Friedmann-Robertson-Walker (FRW) solutions in Chern-Simons gravity as a function of the Chern-Simons mass parameter. In Sec. II we consider tensor perturbations to the FRW metric. We show that for spatial momenta above the Chern-Simons mass scale, one of the two polarization modes is ghostlike and can decay to radiation. Requiring that the radiation produced over the lifetime of the Universe not exceed observational bounds allows us to constraint the parameters of the theory, which we do in Sec. III. Finally in Section IV we argue that the theory does not exhibit classical runaway solutions, despite the existence of ghostlike modes, except possibly in the relativistic nonlinear regime.

II. THE EXISTENCE OF GHOST GRAVITON MODES

In Chern-Simons gravity, we assume that the background cosmological model is homogeneous and isotropic as in general relativity. The metric is of the form

$$ds^2 = a(\eta)^2 [-d\eta^2 + \delta_{ij} d\chi^i d\chi^j], \quad (2.1)$$

where $a(\eta)$ is the scale factor, η is conformal time, and χ^i are comoving coordinates, and the scalar field is $\vartheta = \vartheta(\eta)$. [For simplicity we assume spatial flatness.] The corresponding equations of motion do not contain any contribution from the Chern-Simons term, since the FRW symmetries lead to a vanishing Pontryagin density [8]. We therefore obtain the usual Friedmann equations with a scalar field

$$3m_p^2 \frac{a'^2}{a^4} = \rho_m + \frac{\vartheta'^2}{2a^2} + V(\vartheta), \quad (2.2)$$

² Although this bound was derived in the context of the non-dynamical Chern-Simons theory, without the kinetic term for the scalar field, it is also valid for the dynamical theory when the cosmological background solution $\vartheta(\eta)$ is nonzero, since the derivation is based on the modified gravitational Ampère equation which is valid in the dynamical theory [5].

and

$$\vartheta'' + 2\frac{a'}{a}\vartheta' + a^2 V'(\vartheta) = 0, \quad (2.3)$$

where primes denote derivatives with respect to η and ρ_m is the matter density.

We now fix a solution $a(\eta)$, $\vartheta(\eta)$ of the background equations (2.2) and (2.3). We assume that $\vartheta'(\eta)$ is not identically vanishing. While there may exist solutions with $\vartheta'(\eta) = 0$, if the potential has local minima, these will not be generic. Consider now linear perturbations about such a solution. As in general relativity, the symmetries of the background solution guarantee that perturbations can be decomposed into scalar, vector and tensor modes. In this paper we will focus on tensor perturbation modes, for which the perturbation to the scalar field vanishes and the metric takes the form

$$ds^2 = a^2(\eta) [-d\eta^2 + (\delta_{ij} + h_{ij}) d\chi^i d\chi^j]. \quad (2.4)$$

We adopt the transverse and traceless gauge conditions, $h^i_i = 0$, $\partial_i h^{ij} = 0$. Expanding the terms in the action (1.1) to quadratic order and simplifying using the background equations (2.2) and (2.3) yields [9]

$$\begin{aligned} S = & \frac{1}{8} \int d\eta d^3\chi \left[m_p^2 a^2(\eta) (h^i_{j,\eta} h^j_{i,\eta} - h^i_{j,k} h^j_{i,k}) \right. \\ & \left. - \alpha \vartheta_{,\eta} \epsilon^{ijk} (h^q_{i,\eta} h_{kq,j\eta} - h^q_{i,r} h_{kq,rj}) \right] \\ & + O(h^3), \end{aligned} \quad (2.5)$$

where ϵ^{ijk} is the Levi-Cevita symbol. We rewrite this action in terms of the Fourier transform of the metric perturbation, which is defined by

$$h_{ij}(\eta, \boldsymbol{\chi}) = \int d^3k \tilde{h}_{ij}(\eta, \mathbf{k}) e^{i\mathbf{k}\cdot\boldsymbol{\chi}}, \quad (2.6)$$

where \mathbf{k} is the comoving wavevector.

As noted by Refs. [9, 10], the dynamics is simplest when expressed in terms of a circular polarization basis. We define the left and right circular polarization modes $\tilde{h}_{A\mathbf{k}}(\eta)$ by

$$\tilde{h}_{ij}(\eta, \mathbf{k}) = \sum_{A=L,R} \tilde{h}_{A\mathbf{k}}(\eta) e^A_{ij}(\mathbf{n}). \quad (2.7)$$

where $\mathbf{n} = \mathbf{k}/k$ is a unit vector in the direction of propagation, $k \equiv |\mathbf{k}|$, and the polarization tensors $e^A_{ij}(\mathbf{n})$ satisfy the conditions

$$e^A_{ij} (e^B_{ij})^* = 2\delta^{AB}, \quad (2.8a)$$

$$n_i \epsilon^{ijk} e^A_{kl} = i\lambda_A (e^j_l)^A, \quad (2.8b)$$

with $\lambda_R = +1$ and $\lambda_L = -1$. The action can now be written as

$$\begin{aligned} S = & \frac{m_p^2}{4} \int d\eta d^3k \sum_{A=L,R} a^2(\eta) \\ & \left[1 + \frac{\lambda_A k}{a^2(\eta)} \frac{\alpha \vartheta_{,\eta}}{m_p^2} \right] \left[|\tilde{h}_{A\mathbf{k},\eta}|^2 - k^2 |\tilde{h}_{A\mathbf{k}}|^2 \right]. \end{aligned} \quad (2.9)$$

This can be recast using the Chern-Simons mass scale (1.4) as

$$S = \frac{m_{\text{p}}^2}{4} \int d\eta d^3k \sum_{A=L,R} a^2(\eta) \left[1 + \lambda_A \frac{k_{\text{phy}}}{m_{\text{cs}}} \right] \left[|\tilde{h}_{A\mathbf{k},\eta}|^2 - k^2 |\tilde{h}_{A\mathbf{k}}|^2 \right], \quad (2.10)$$

where $k_{\text{phy}} = k/a$ is the physical wavenumber. The action (2.10) is the usual action for tensor perturbations in FRW, except for the momentum dependent correction factor $(1 + \lambda_A k_{\text{phy}}/m_{\text{cs}})$. This factor becomes negative for the left handed polarization modes when $k_{\text{phy}} \geq m_{\text{cs}}$, giving rise to a kinetic term with the wrong sign, i.e. a ghost mode.

The action (2.10) can be simplified by changing the normalization of the graviton modes to attain canonical normalization. We define

$$\varepsilon_A(k) = \text{sgn} \left(1 + \frac{\lambda_A k_{\text{phys}}}{m_{\text{cs}}} \right), \quad (2.11)$$

which is +1 for normal modes and -1 for ghost modes. We define the mass scale

$$m_* = m_{\text{p}} \sqrt{\left| 1 + \frac{\lambda_A k_{\text{phys}}}{m_{\text{cs}}} \right|}, \quad (2.12)$$

and the canonically normalized graviton field modes

$$\tilde{h}_{A\mathbf{k}}^{\text{can}} = m_* \tilde{h}_{A\mathbf{k}}. \quad (2.13)$$

The action (2.10) can now be written as

$$S = \frac{1}{4} \int d\eta d^3k \sum_{A=L,R} a^2(\eta) \varepsilon_A(k) \left[|\tilde{h}_{A\mathbf{k},\eta}^{\text{can}} - (\ln m_*)_{,\eta} \tilde{h}_{A\mathbf{k}}^{\text{can}}|^2 - k^2 |\tilde{h}_{A\mathbf{k}}^{\text{can}}|^2 \right]. \quad (2.14)$$

Note that from a classical point of view the existence of ghost modes does not necessarily imply any inconsistency of the theory. The Hamiltonian of the theory may or may not be unbounded below; addressing this question would require a nonlinear analysis beyond the scope of this paper. Also it is not known at present whether the theory possesses a well posed initial value formulation: the sign flip at $k_{\text{phys}} = m_{\text{cs}}$ may be a hint that it does not. However, the ghost modes are a significant problem when quantum mechanical effects are taken into account, as we discuss in the next section.

III. CONSTRAINTS FROM VACUUM DECAY

We now specialize to perturbation modes today which are deep inside the horizon, that is, $k \gg H_0$, where $H_0 = a_{,\eta}/a^2$ is the Hubble parameter. We also assume that $m_{\text{cs}} \gg H_0$. In this limit, we can neglect in the action

(2.14) the time dependence of the prefactor $a(\eta)^2$, and also the term proportional to the time derivative of m_* . The result is just the standard action for graviton modes in Minkowski spacetime, except for the sign flip for the ghost modes³.

Because of this sign flip, decay of the vacuum of the theory is kinematically allowed. The vacuum decay rate per unit volume Γ is naively infinite, because of the infinite phase space available for the decay products that arises from Lorentz invariance⁴. Rather than ruling out the theory outright because of the presence of ghost modes, we adopt the viewpoint that the action (2.10) defines an effective quantum field theory as in Refs. [3, 11, 12], with some effective cutoff Λ on the physical wavenumber k_{phys} in cosmological rest frame. Thus, our cutoff explicitly violates Lorentz invariance; such a violation is inevitable if one wants to obtain a finite vacuum decay rate. Our viewpoint and treatment follow similar analyses of scalar ghost fields in cosmological models with equation of state parameter w that satisfies $w < -1$ [13, 14]. As in those analyses, we will find that stringent constraints on the parameters of the theory can be obtained by demanding that the total number of photons produced from vacuum decay over the lifetime of the Universe not be in conflict with observations.

Our theory is now parameterized by two parameters, the cutoff Λ and the Chern-Simons mass m_{cs} . Now since the ghost modes arise only for momenta k_{phys} satisfying $k_{\text{phys}} > m_{\text{cs}}$, there will be no ghost modes in the region $m_{\text{cs}} > \Lambda$ of parameter space. Therefore our arguments about vacuum decay do not constrain that region of parameter space; we focus from now on on the complementary region $m_{\text{cs}} < \Lambda$ (see Fig. 2). We note that many papers on Chern-Simons gravity implicitly work in this regime $m_{\text{cs}} < \Lambda$, since they involve effects arising on scales $k_{\text{phys}} \sim m_{\text{cs}}$, assumed to be within the domain of validity of the theory. We also note that the regime $m_{\text{cs}} \lesssim \Lambda$ is disfavored by naturalness arguments, i.e., it requires considerable fine tuning of the Lagrangian (see Appendix A).

If the vacuum perturbatively decays to stable particles, we can estimate the number of particles produced over the lifetime of the Universe from such a process. Given observational bounds on the energy density in this particle species, we can then constrain the decay rate. The strongest bounds will therefore come from production of particles whose energy density has been reliably measured, and which can be produced at a low order in

³ We have omitted the action describing the interaction of the graviton field modes with matter fields. This interaction takes the standard form when written in terms of $\tilde{h}_{A\mathbf{k}}$, but acquires correction factors of m_{p}/m_* when written in terms of $\tilde{h}_{A\mathbf{k}}^{\text{can}}$.

⁴ The full theory of the perturbations about FRW, including coupling to matter, does violate Lorentz invariance because of the k_{phys} dependence of the factor $1 + \lambda_A k_{\text{phys}}/m_{\text{cs}}$. However this violation does not affect the accessible volume of phase space.

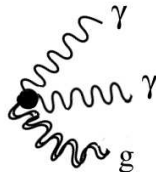


FIG. 1: Spontaneous production of photons and ghost gravitons from vacuum

perturbation theory. We argue that the photon is the best such candidate.

A. Derivation of photon energy spectrum

We next make an order of magnitude estimate of the decay rate Γ per unit volume to photons in the regime $m_{\text{cs}} < \Lambda$. The action for electromagnetism contains an interaction of the form

$$S_{\text{int}} \sim \int d^4x h (\partial A)^2 \sim \frac{1}{m_*} \int d^4x h^{\text{can}} (\partial A)^2, \quad (3.1)$$

where A^a is the 4-vector potential, h is the metric perturbation, and h^{can} is the canonically normalized version. Because the graviton is ghostlike, the process

$$0 \rightarrow g\gamma\gamma \quad (3.2)$$

is kinematically allowed, where g is a left polarized graviton and γ is a photon (Fig. 1). Hence graviton ghosts can decay to photons at first order in perturbation theory.

Next, the coefficient $1/m_*$ in the interaction (3.1) depends on the wavenumber k . However, for the purposes of our order of magnitude estimate, it will be sufficient to evaluate this coefficient at $k_{\text{phys}} \sim \Lambda$, since most of the decays will be at $k_{\text{phys}} \sim \Lambda$. Therefore we can treat m_* as a constant,

$$m_* \sim m_{\text{p}} \sqrt{\frac{\Lambda}{m_{\text{cs}}}}, \quad (3.3)$$

where we have used $m_{\text{cs}} < \Lambda$ and specialized to ghost modes. The decay rate Γ per unit time per unit volume must be proportional to the square of the coefficient of the operator, so $\Gamma \propto 1/m_*^2$. The constant of proportionality in this relation must be some function of Λ , and from dimensional analysis it now follows that⁵

$$\Gamma_{0 \rightarrow g\gamma\gamma} \sim \frac{\Lambda^6}{m_*^2} \sim \frac{m_{\text{cs}} \Lambda^5}{m_{\text{p}}^2}, \quad (3.4)$$

since Γ has dimension $(\text{mass})^4$ in units with $\hbar = c = 1$.

Consider now the production of photons by this mechanism over the lifetime of the Universe. We make the idealization that at any epoch, all the photons produced have an energy of exactly Λ . Then the evolution of the number $n(k, t)$ of photons per unit logarithmic wavenumber k per proper volume is given by

$$\frac{1}{a^3} \frac{d}{dt} (a^3 n) = \Gamma \Lambda \delta(k/a - \Lambda), \quad (3.5)$$

where a is the scale factor with $a = 1$ today. Solving this equation gives for the spectrum today

$$n(k) = \Theta(\Lambda - k) \frac{\Gamma a_*^3}{H_*}, \quad (3.6)$$

where Θ is the step function, $H = \dot{a}/a$ is the Hubble parameter, and $a_*(k)$ and $H_*(k)$ are the values of a and H at $k/a = \Lambda$. The result (3.6) is easy to understand: the factor of $1/H_*$ is the length of time during which photons of present-day energy $\sim k$ are produced, and the factor of a_*^3 is the volume expansion factor since then. For a Λ CDM cosmology with $H^2 = H_0^2(\Omega_M/a^3 + 1 - \Omega_M)$ the spectrum becomes

$$n(k) = \Theta(\Lambda - k) \frac{\Gamma}{H_0} \left[\Omega_M \left(\frac{\Lambda}{k} \right)^9 + (1 - \Omega_M) \left(\frac{\Lambda}{k} \right)^6 \right]^{-1/2}. \quad (3.7)$$

It follows that the energy density per unit logarithmic wavenumber $dE/d^3x d \ln k \sim kn(k)$ is peaked at $k \sim \Lambda$ with a peak value of

$$\frac{dE}{d^3x d \ln k} \sim \frac{\Gamma \Lambda}{H_0} \sim \frac{m_{\text{cs}} \Lambda^6}{m_{\text{p}}^2 H_0}. \quad (3.8)$$

Finally, we note that the above derivation implicitly assumed that the Chern-Simons mass m_{cs} is constant. In reality m_{cs} is a function of redshift or of time, since the background cosmological solution $\vartheta(t)$ is time dependent. However, our derivation shows most of the photons are produced at low redshift, $z \lesssim 1$. Since the background cosmological solution evolves on the Hubble timescale, the fractional change in the Chern-Simons mass out to redshifts of order unity is of order unity. Therefore the result (3.8) is still valid when taking the evolution of m_{cs} into account, up to a correction factor of order unity.

B. Comparison of photon energy spectrum with observations

We now compare the prediction (3.8) with observational data, in two different ways. First, as long as $\Lambda \gtrsim H_0$, the energy density (3.8) will contribute to the expansion of the Universe. Since observations of the expansion history tell us that the Universe is not radiation dominated today, we obtain as a very conservative upper bound that

$$\frac{dE}{d^3x d \ln k} \lesssim m_{\text{p}}^2 H_0^2. \quad (3.9)$$

⁵ This computation breaks down in the limit $m_{\text{cs}} \rightarrow \Lambda$, in which the volume of the region in phase space containing unstable modes shrinks to zero. The computation is valid to within a factor of order unity whenever $\Lambda/m_{\text{cs}} - 1 \gtrsim O(1)$.

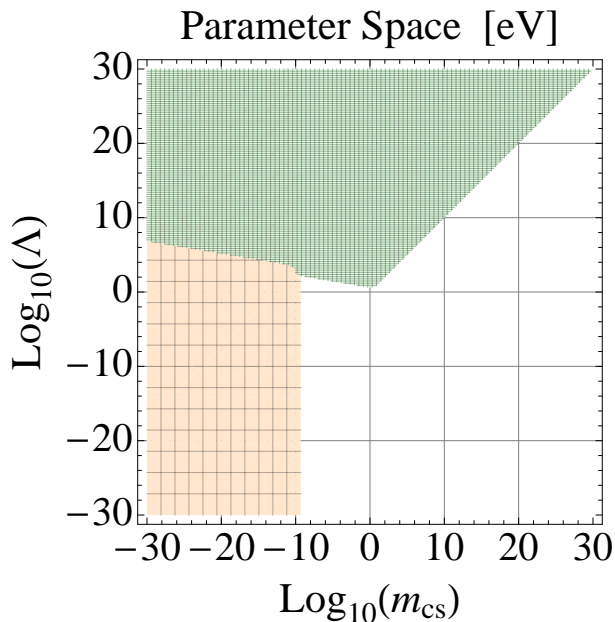


FIG. 2: Effective field theory parameter space consisting of the cutoff scale Λ and the Chern-Simons mass m_{cs} , in eV. The lightly shaded region is excluded from binary pulsar observations [7], and the darkly shaded region is excluded by consideration of vacuum decay.

It follows that

$$m_{cs}\Lambda^6 \lesssim m_p^4 H_0^3 \sim (38 \text{ eV})^7 \quad \text{for } 10^{-33} \text{ eV} \lesssim \Lambda, \quad (3.10)$$

where we have used $m_p = 2.4 \times 10^{27} \text{ eV}$ and $H_0 = 1.5 \times 10^{-33} \text{ eV}$.

Second, a more stringent constraint can be obtained by using direct observations of photons, albeit within a finite range of cutoffs Λ . Observations in various wavelength bands from radio to gamma rays gives the upper bound⁶ on the background radiation spectrum

$$\frac{dE}{d^3x d \ln k} \lesssim (2.5 \times 10^{-5} \text{ eV})^4 \quad (3.11)$$

for $10^{-10} \text{ eV} \lesssim k \lesssim 10 \text{ GeV}$ [15, 16]. Combining this with the prediction (3.8) yields the constraint

$$m_{cs}\Lambda^6 \lesssim (3.3 \text{ eV})^7 \quad \text{for } 10^{-10} \text{ eV} \lesssim \Lambda \lesssim 10 \text{ GeV}. \quad (3.12)$$

Figure 2 shows the constraints (1.6) and (3.10), (3.12) on the (m_{cs}, Λ) parameter space. It can be seen that the combined constraints rule out a large part of the region in parameter space with $\Lambda > m_{cs}$ in which interesting Chern-Simons phenomenology occurs, except for the window $10^{-10} \text{ eV} \lesssim m_{cs} \lesssim 3 \text{ eV}$.

⁶ This upper bound is exceeded by about two orders of magnitude by the CMB, but the CMB is known to be thermal to about 1 part in 10^5 , so the bound effectively holds.

We note that our constraint (3.10) on the parameter space extends up to arbitrarily large values of m_{cs} , even though the theory ostensibly reduces to general relativity in this limit to linear order in perturbation theory. This is because for any value of m_{cs} , there exist modes with $k_{\text{phys}} \sim m_{cs}$ for which the Chern-Simons correction factor in Eq. (2.10) is of order unity. Even if m_{cs} is arbitrarily large, the high energy modes above this scale are unstable and the vacuum decays. The high energy photons produced can then give an observable contribution to the expansion rate of the Universe.

C. Vacuum decay to gravitons

One can also compute the constraints obtained from vacuum decay to gravitons, in which one ghost graviton and two normal gravitons are produced. The interaction Lagrangian is

$$\mathcal{L}_{\text{int}} \sim m_p^2 h(\partial h)^2 \sim \frac{m_p^2}{m_*^3} h_{\text{can}}(\partial h_{\text{can}})^2. \quad (3.13)$$

The analysis now proceeds as before, with the resulting energy density in gravitons given by

$$\Omega_{\text{gw}} \sim \frac{1}{m_p^2 H_0^2} \frac{dE}{d^3x d \ln k} \sim \frac{\Lambda^4 m_{cs}^3}{m_p^4 H_0^3} \quad (3.14)$$

at $k \sim \Lambda$. Now observations of the cosmic microwave background give an integrated upper bound $\int d \ln k \Omega_{\text{gw}} \lesssim 10^{-5}$ for $k \gtrsim 10^{-30} \text{ eV}$ [17]. However this bound applies only to the gravitons produced before recombination, whose energy density will be smaller than the estimate (3.14) by a factor of the ratio of the age of the Universe at recombination to the age of the Universe today. This factor is approximately $z_{\text{rec}}^{-3/2}$, where $z_{\text{rec}} \sim 1000$ is the redshift of recombination. Also for gravitons produced before recombination, the Chern-Simons mass $m_{cs} = m_{cs}(t_0)$ in Eq. (3.14) should be replaced by $\tilde{m}_{cs} = m_{cs}(t_{\text{rec}})$, the value at recombination. We therefore obtain the constraint

$$\tilde{m}_{cs}^3 \Lambda^4 \lesssim z_{\text{rec}}^{3/2} m_p^4 H_0^3 (10^{-5}) \sim (32 \text{ eV})^7 \quad (3.15)$$

for $\Lambda \gtrsim 10^{-30} \text{ eV}$. The result (3.15) is slightly weaker than the constraint (3.12) from photons, and applies to a different version of the Chern-Simons mass.

Finally, one might also be able to place additional constraints on the theory by considering the decay of the vacuum to two gravitons and one quanta of the scalar field, rather than to three gravitons.

IV. CLASSICAL RUNAWAY SOLUTIONS

We next turn to a different question, the possible existence of classical runaway solutions that can occur in a theory whose Hamiltonian is unbounded below. We

will argue that such solutions do not arise in Chern-Simons gravity except possibly in the nonlinear relativistic regime $h \sim 1$ where our analysis is invalid anyway.

Generically, a Hamiltonian which is unbounded from below will only exhibit runaway solutions in the regime where the interaction terms in the Lagrangian are comparable to the negative kinetic energy terms [13]. We use this criterion to estimate the required occupation number of a ghost graviton mode for such a runaway behavior to develop in our system.

No runaway solutions will plague the effective field theory (2.10) because the ghost and non-ghost fields are decoupled to linear order. However, interaction terms will appear if we expand the action (1.1) to higher than quadratic order. Expanding up to quartic order we expect to find interaction terms of the form

$$S_3 \sim \int d^4x m_p^2 k^2 h_L^2 h_R + \dots, \quad (4.1a)$$

$$S_4 \sim \int d^4x m_p^2 k^2 h_L^2 h_R^2 + \dots. \quad (4.1b)$$

Consider a single localized wave packet mode with characteristic size λ . We would like to estimate the number of quanta N for an instability to form. The energy from the kinetic term is

$$E_K = \frac{1}{2} \int d^3x m_p^2 \dot{h}^2 \sim \lambda m_p^2 h^2 \sim \frac{N}{\lambda}. \quad (4.2)$$

Therefore the field fluctuation is approximately

$$h \sim \frac{N^{1/2}}{\lambda m_p}. \quad (4.3)$$

Each of the interaction terms will therefore contribute terms to the energy on the order of

$$E_3 \sim \int d^3x m_p^2 k^2 h^3 \sim \frac{N^{3/2}}{\lambda^2 m_p}, \quad (4.4a)$$

$$E_4 \sim \int d^3x m_p^2 k^2 h^4 \sim \frac{N^2}{\lambda^3 m_p^2}. \quad (4.4b)$$

Requiring that the kinetic energy be less than the interaction energy

$$E_3 \gtrsim E_K, \quad E_4 \gtrsim E_K, \quad (4.5)$$

and taking the wavelength $\lambda \sim m_{cs}^{-1}$ yields

$$N \geq \left(\frac{m_p}{m_{cs}} \right)^2 \gg 1. \quad (4.6)$$

We deduce from (4.3) that in this regime

$$h \gtrsim 1. \quad (4.7)$$

We conclude the ghost modes generated by perturbations to FRW do not exhibit runaway solutions within the domain of validity of our analysis.

V. OUTLOOK

We have shown that if the effective field theory cutoff Λ is above the Chern-Simons mass scale m_{cs} , and if the Chern-Simons mass is not infinite (the generic case), then the vacuum of Chern-Simons gravity is unstable and can decay to photons. Our estimate of the photon production rate was used to set constraints on the parameter space. One might imagine improving this bound by being less cavalier in the estimation of the decay rate and number density. However, because the production rate goes like a high power of the cutoff scale, orders of magnitude difference in the number density yield very modest improvements in final constraints on Λ and m_{cs} .

Whether the full nonlinear theory has a Hamiltonian which is unbounded below is an interesting open question.

We also note that it is possible that the decay of the vacuum could be a lot faster than indicated by our perturbative calculations, if it is mediated by a non-perturbative process. One might imagine a tunneling process similar to the decay of false vacua in scalar field theories via bubble nucleation. In particular, a homogeneous unstable region with small Chern-Simons might be expected to produce a stable endstate consisting of a homogeneous region with a Chern-Simons mass larger than the cutoff. However any such transition could not proceed via a spherically symmetric process, since the Chern-Simons term does not contribute to the dynamics in spherical symmetry.

Finally, we note that our analysis does not constrain the (non-generic) scenario where the background cosmological Chern-Simons field is sitting at a minimum of the potential, which corresponds to an infinite Chern-Simons mass. In this scenario, no ghostlike tensor modes appear. Deviations from general relativity for localized sources do arise, at nonlinear order in perturbation theory, via the Pontryagin density source term in the scalar field equation. It may be possible to use tests of general relativity to constrain the parameter α in the action (1.2b) in this scenario.

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Appendix A: Naturalness Argument

In this appendix we shall argue that the regime $\Lambda \gtrsim m_{\text{cs}}$ of parameter space is disfavored by naturalness arguments, that is, it requires considerable fine tuning of the Lagrangian. We specialize in this appendix to the version of the theory without a potential, $V(\vartheta) = 0$.

We start by assuming that Chern-Simons gravity arises from an effective field theory. In this framework we seek a description of the physics only at energies E small compared to a cutoff scale Λ , and we assume that at these energies the only relevant degrees of freedom are a metric g_{ab} and a scalar field ϑ , with the only symmetries being diffeomorphism invariance and the shift symmetry $\vartheta \rightarrow \vartheta + \text{constant}$. As is usual, the theory at energies $E \gtrsim \Lambda$ is unknown, but is assumed to be sufficiently random or generic (i.e. no additional symmetries are present) that the coefficients of all nonrenormalizable operators in the low energy theory can be estimated using dimensional analysis to within coefficients of order unity.

This framework leads to the following action, consisting of the Chern-Simons action discussed in the introduction together with a series of correction terms that are suppressed by powers of Λ :

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} m_p^2 R - \frac{1}{2} (\nabla\vartheta)^2 + \frac{1}{4} \alpha \vartheta * RR \right] + \int d^4x \sqrt{-g} \left[\frac{c_1}{\Lambda^4} (\nabla\vartheta)^4 + \dots \right]. \quad (\text{A1})$$

Here c_1 is dimensionless and of order unity. The coefficient α can be estimated by using

$$R \sim (\nabla h)^2 + \nabla\nabla h \sim \frac{1}{m_p^2} (\nabla h^{\text{can}})^2 + \frac{1}{m_p} \nabla\nabla h^{\text{can}}, \quad (\text{A2})$$

where h is the metric perturbation and $h^{\text{can}} = m_p h$ is the canonically normalized version. This yields

$$\alpha \vartheta * RR \sim \frac{\alpha}{m_p^2} \vartheta (\nabla\nabla h^{\text{can}})^2 \sim \frac{c_2}{\Lambda^3} \vartheta (\nabla\nabla h^{\text{can}})^2 \quad (\text{A3})$$

where the last equality follows from dimensional analysis and c_2 is dimensionless and of order unity. It follows that

$$\alpha = c_2 \frac{m_p^2}{\Lambda^3}. \quad (\text{A4})$$

This relationship between the coefficient α and the cutoff Λ was previously derived using a different method in Ref. [12].

By combining the relation (A4) with the definition (1.4) of the Chern-Simons mass we obtain

$$m_{\text{cs}} = \frac{\Lambda^3}{c_2 \vartheta}. \quad (\text{A5})$$

We now make use of the fact that the background cosmological solution must lie within the domain of validity of the effective field theory, which requires that

$$\frac{c_1}{\Lambda^4} (\nabla\vartheta)^4 \ll (\nabla\vartheta)^2 \quad (\text{A6})$$

or $\dot{\vartheta} \ll \Lambda^2 / \sqrt{c_1}$. Combining this with Eq. (A5) gives

$$m_{\text{cs}} \gg \frac{\sqrt{c_1}}{c_2} \Lambda, \quad (\text{A7})$$

and so $m_{\text{cs}} \lesssim \Lambda$ is disallowed unless $\sqrt{c_1}/c_2 \ll 1$, which would be a fine tuning.

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