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Abstract

Inhomogeneous quantum cosmology is modeled as a dynamical system of discrete patches, whose interacting many-body equations can be mapped to a non-linear minisuperspace equation by methods analogous to Bose–Einstein condensation. Complicated gravitational dynamics can therefore be described by more-manageable equations for finitely many degrees of freedom, for which powerful solution procedures are available, including effective equations. The specific form of non-linear and non-local equations suggests new questions for mathematical and computational investigations, and general properties of non-linear wave equations lead to several new options for physical effects and tests of the consistency of loop quantum gravity. In particular, our quantum cosmological methods show how sizeable quantum corrections in a low-curvature universe can arise from tiny local contributions adding up coherently in large regions.

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I. INTRODUCTION

One of the main problems in deriving a reliable Planck-regime scenario in canonical quantum cosmology is the question of how to include inhomogeneity. While homogeneous models can easily be quantized, inhomogeneous degrees of freedom severely complicate mathematical evaluations. Even the formulation of consistent evolution equations, subject to the anomaly problem, remains incomplete: no consistent and covariant version of inhomogeneous modes valid at high density and including all relevant quantum effects is available at present.

As possible solutions, two approaches have been developed so far, mainly with the methods of loop quantum cosmology [1, 2]. First, effective equations have been successful in addressing the anomaly problem [3, 4] and in including all relevant quantum effects in sufficiently general form. (For details, see [5].) Potentially observable phenomena have been uncovered, showing physical consequences of discrete quantum geometry [4, 6–13] and making the theory falsifiable [14, 15]. At high density, the implications of quantum space-time can be dramatic: general properties of effective constrained systems show the presence of signature change, turning Lorentzian space-time into a quantum version of 4-dimensional Euclidean space [16–19]. Bounces as they had often been envisaged as non-singular versions of cosmology [20], and formulated in quite some detail [21] in homogeneous models of loop quantum cosmology, are then replaced by acausal pieces of 4-dimensional space devoid of deterministic evolution. Regarding specific field equations and details of the transition, however, present calculations remain incomplete because not all quantum effects could yet be implemented consistently at high density. Moreover, although state properties can be derived by canonical effective equations, finding full quantum states is difficult in this setting.

One of the alternatives is so-called hybrid quantization [22], in which one combines a loop-quantized homogeneous background model with Fock quantized inhomogeneous modes. Wave functions can then be solved for and evolved, at least numerically and with certain truncations [23]. However, using a Fock quantization for inhomogeneous modes, one does not directly deal with the discreteness of space-time. Moreover, the hybrid method does not address the anomaly problem; like related apporaches [24–27], it rather avoids dealing with the problem by fixing the gauge or using deparameterization, choosing a time variable before quantization. With these additional steps, it is unlikely that the correct space-time picture is obtained in covariant form, and in fact the models evaluated so far have missed the signature change at high density.

In this article, we introduce a new way of incorporating effects of inhomogeneity in loop quantum cosmology, dealing directly with wave functions. Adapting ideas of condensed-matter physics used to describe Bose–Einstein condensates,[73] some effects of inhomogeneity will not be described by individual degrees of freedom but rather by non-linearity of wave equations for homogeneous models. The relationship between difference equations of loop quantum cosmology [30, 31] and certain integrable non-linear Schrödinger equations has been noted in [32], providing additional motivation. The aim of the present article is to lay down the main ideas and to point out several new consequences for quantum cosmology. We find that tiny quantum corrections from inhomogeneous contributions to a large universe can add up coherently to produce sizeable effects on average, to be included in minisuperspace models.[74]

II. PRODUCT STATES

We start with a common way of dealing with inhomogeneity, viewing quantum space at a given time (a spatial slice used in canonical quantum gravity) as a collection of small homogeneous parts. As one moves between spatial slices, the geometry evolves, resembling a many-body system of "interacting" elementary building blocks. Each building block (called a patch) has a quantum geometry described by a wave function of one of the well-known homogeneous models of quantum cosmology, and they all interact dynamically according to the quantized gravitational Hamiltonian.

A. Classical model

For simplicity, in this article we will assume isotropic patch geometries, determined classically by a volume degree of freedom $V_{i,j,k}$ per patch, labelled by integers i, j, k to count patches in each spatial direction. A given spatial slice $\Sigma = \bigcup_{i,j,k=1}^{\mathcal{N}^{1/3}} \mathcal{V}_{i,j,k}$ — a differentiable manifold with a local atlas of coordinates — is then the union of $\mathcal{N}$ patches $\mathcal{V}_{i,j,k}$, or $\mathcal{N}^{1/3}$ in each spatial direction. (Had we used anisotropic but still homogeneous patches, we would in general have three independent factors in $\mathcal{N} = \mathcal{N}_1\mathcal{N}_2\mathcal{N}_3$.) For now, we will assume $\mathcal{N}$ to be constant, which should be good for sufficiently brief evolution times. In more realistic models, the number $\mathcal{N}$ of patches should change in time, either by a fundamental process of discrete geometries being refined [33, 34], or by an approximation procedure akin to adaptive mesh refinement that maintains the decomposition into isotropic patches as a good model. (A time-dependent number of degrees of freedom is a general problem, studied for instance in [35–39].)

For simplicity, we choose coordinates in space such that each patch has the same coordinate volume $\int_{\mathcal{V}_{i,j,k}} \mathrm{d}^3x = \ell_0^3$, with $\ell_0^3 = V_0/\mathcal{N}$ in terms of the total coordinate volume V_0 of Σ (or of a large compact subset). The geometrical volume of each patch is then determined by the spatial metric which, if it is inhomogeneous, gives rise to different patch volumes $V_{i,j,k}$. We assume that the metric is close to the one of a spatially flat, isotropic model with a longitudinal scalar mode, $h_{ab} = a(t)^2\delta_{ab} + 2L(t,x,y,z)\delta_{ab}$. (We will use a lapse function corresponding to proper time, $N = 1 - 2L/a^2$.) The patch volumes then take the values

$$\begin{aligned} V_{i,j,k} &= \int_{\mathcal{V}_{i,j,k}} \mathrm{d}^3x\sqrt{\det h} = a^3 \int_{\mathcal{V}_{i,j,k}} \mathrm{d}^3x(1+2L/a^2)^{3/2} \\ &\approx a^3\ell_0^3 + 3a\int_{\mathcal{V}_{i,j,k}} \mathrm{d}^3x L \approx \frac{V}{\mathcal{N}} + 3aL(x_{i,j,k})\ell_0^3 \end{aligned} \tag{1}$$

with the total volume $V = a^3V_0 = a^3\ell_0^3\mathcal{N}$. In the two approximations in the second line of this equation, we have first expanded the root and then replaced the patch-integrated L by its value at a point $x_{i,j,k} \in \mathcal{V}_{i,j,k}$, such as the center. Since we assume the patches to be nearly isotropic and smaller than the variation scale of the perturbative inhomogeneity, L, both approximations are well justified. Solving (1) for L, we can therefore replace the continuum function L by deviations of the discrete variables $V_{i,j,k}$ from the total volume $V = \mathcal{N}a^3\ell_0^3$:

$$L(x_{i,j,k}) = \frac{V_{i,j,k} - V/\mathcal{N}}{3a\ell_0^3}\,. \tag{2}$$

The dynamics of the $V_{i,j,k}$ as functions of time is governed by a discretized version of the Hamiltonian constraint

$$H_{\mathrm{grav}} + H_{\mathrm{matter}} = 0 \tag{3}$$

of general relativity, with contributions from the gravitational field and from matter. At this point, one will eventually have to face the problem of time and the anomaly problem.[75] In this article, however, we focus on laying out the details of the new model, and therefore circumvent these difficult problems by formulating the dynamics in a specific gauge. With this choice, we may be blind to the complete quantum space-time structure, but new qualitative effects should still become visible. To proceed and to be specific, we assume matter to be dust, with Hamiltonian $H_{\rm matter} = p_t/a^3$, where p_t is a momentum variable conjugate to a matter degree of freedom t that will play the role of time. The role of time is made clear if we rewrite the Hamiltonian constraint equation as

$$p_t = -a^3 H_{\rm grav} = -\frac{V}{\ell_0^3} H_{\rm grav} \,. \tag{4}$$

The variable p_t then appears formally as an energy, or a canonical Hamiltonian that generates evolution with respect to t. (More generally, we could assume matter to contribute to the Hamiltonian constraint by $H_{\rm matter} = p_{t'}/a^{3(1+w)}$ if there is a perfect fluid with equation-of-state parameter w. A time variable t' different from t then parameterizes evolution.)

To derive the dynamics in detail, we start with the classical Hamiltonian constraint of general relativity and write it in discrete canonical variables $V_{i,j,k}$ together with their momenta $\Pi_{i,j,k}$, related to $\dot{V}_{i,j,k}$. In the ADM formulation of canonical gravity, the spatial metric h_{ab} is canonically conjugate to

$$\pi^{ab} = \frac{\sqrt{\det h}}{16\pi G}(K^{ab} - K^c_c h^{ab}) \,. \tag{5}$$

(See [40] for an introduction to canonical gravity.) We compute the canonical variables in our perturbed situation by writing $h_{ab} = \bar{h}\delta_{ab} + \delta h_{ab}$ and $\pi^{ab} = (\bar{\pi}/V_0)\delta^{ab} + \delta\pi^{ab}$, split into background variables $\bar{h} = a^2$ and $\bar{\pi}$ (spatial constants) and inhomogeneity δh_{ab} and $\delta\pi^{ab}$. We divide $\bar{\pi}$ by V_0 in π^{ab} to ensure that the symplectic term $\int_\Sigma {\rm d}^3x \dot{h}_{ab}\pi^{ab} = \dot{\bar{h}}\bar{\pi} + \cdots$ assumes the canonical form in its background term. For scalar modes in longitudinal gauge, $\delta h_{ab} = 2L\delta_{ab}$ and $\delta\pi^{ab} = \delta\pi\delta^{ab}$.

To avoid overcounting of degrees of freedom, we require the inhomogeneity δf of any field $f = \bar{f} + \delta f$ to satisfy $\int_\Sigma {\rm d}^3x\delta f = 0$ when integrated over all of space. (We turn inhomogeneities of tensor fields such as δh_{ab} into scalars using the background metric δ_{ab}.) As a consequence, $\bar{f} = \int_\Sigma {\rm d}^3x f$ is indeed the spatial average. At this stage we do not assume that δf is of first or any specific order in perturbation theory; we have simply rearranged our degrees of freedom by splitting them into background variables and inhomogeneity. The symplectic structure, our current interest, only refers to degrees of freedom but not to orders of perturbation theory: higher perturbative orders do not introduce new degrees of freedom. We will introduce the perturbative expansion when we prepare our Hamiltonian for a derivation and analysis of equations of motion.

Any terms linear in δh_{ab} or $\delta\pi^{ab}$ in the Hamiltonian or symplectic term $\int_\Sigma \mathrm{d}^3x\dot{h}_{ab}\pi^{ab}$ therefore vanish. Inserting the inhomogeneous metric $h_{ab} = (a^2+2L)\delta_{ab}$ in (5) (with vanishing shift and longitudinal lapse in K_{ab}) results in the inhomogeneous momentum

$$\pi^{ab} = -\frac{1}{8\pi G}\left(\dot{a} + a\left(\frac{L}{a^2}\right)^{\bullet}\right)\delta^{ab} \tag{6}$$

from which we read off the momentum of $\bar{h} = a^2$ as $\bar{\pi} = -\dot{a}V_0/8\pi G$ with the total coordinate volume V_0, and the momentum of $\delta h_{ab} = 2L\delta_{ab}$ as $\delta\pi^{ab} = -(a(L/a^2)^{\bullet}/8\pi G)\delta^{ab}$. By a canonical transformation we can switch to volume variables as defined in our patch model: we have momenta

$$\Pi_V = -\frac{1}{12\pi G}\frac{\dot{V}}{V} \quad \text{and} \quad \Pi_{i,j,k} = -\frac{1}{12\pi G}\left(\frac{\mathcal{N}V_{i,j,k}}{V}\right)^{\bullet} \tag{7}$$

of V and $V_{i,j,k}$.

For small inhomogeneity, it is sufficient to expand the Hamiltonian constraint to second order in δh_{ab} (or L) and its time and space derivatives. Starting from

$$\begin{aligned} H_{\mathrm{grav}} &= \frac{1}{16\pi G}\int_\Sigma \mathrm{d}^3x N\left(K^{ab}K_{ab} - K^2 - {}^3\!R\right)\sqrt{\det h} \\ &= \int_\Sigma \mathrm{d}^3x \mathcal{H}_{\mathrm{grav}}\,, \end{aligned}$$

(with the lapse function N which we set equal to one in our gauge) we obtain

$$\mathcal{H}_{\mathrm{grav}} \approx -\frac{3}{8\pi G}\left(a\dot{a}^2 + \frac{\dot{L}^2 - 4(\dot{a}/a)\dot{L}L + 4(\dot{a}/a)^2L^2}{a} + a^{-3}\sum_{b=1}^{3}\left(\left(\frac{\partial L}{\partial x^b}\right)^2 - \frac{4}{3}L\left(\frac{\partial^2 L}{\partial x^{b^2}}\right)\right)\right)\,.$$

In the Hamiltonian $H_{\mathrm{grav}} = \int_\Sigma \mathrm{d}^3x\mathcal{H}_{\mathrm{grav}}$ we can integrate by parts in spatial derivatives, replacing second-order derivatives by first-order ones. (Boundary terms will play no role in what follows.) Moreover, it turns out that the time derivatives of L can be written more compactly if we use L/a^2, a combination of variables that is also more convenient when expressed by patch volumes: $\dot{L}^2 - 4(\dot{a}/a)\dot{L}L + 4(\dot{a}/a)^2L^2 = ((L/a^2)^{\bullet})^2$. The Hamiltonian density we use will therefore be

$$\mathcal{H}_{\mathrm{grav}} = \frac{3a^3}{8\pi G}\left(\left(\frac{\dot{a}}{a}\right)^2 + \left(\left(\frac{L}{a^2}\right)^{\bullet}\right)^2 + \frac{7}{3}\frac{1}{a^2}\sum_{b=1}^{3}\left(\frac{\partial(L/a^2)}{\partial x^b}\right)^2\right)\,. \tag{8}$$

We then introduce our background momentum $\dot{a}/a = -4\pi G\Pi_V$ and the patch momenta $(L/a^2)^{\bullet} \to -4\pi G\Pi_{i,j,k}$ after replacing the integral by a sum over patches, $\int_\Sigma \mathrm{d}^3x\mathcal{H}_{\mathrm{grav}} \approx$

$\sum_{i,j,k} \mathcal{H}_{i,j,k}$. Our discretized Hamiltonian then is

$$H_{\text{grav}}^{\text{disc}} = -6\pi G V \left(\Pi_V^2 + \frac{1}{\mathcal{N}} \sum_{i,j,k} \Pi_{i,j,k}^2 + \cdots \right) \tag{9}$$

where the dots indicate the derivative terms after discretization.

We have quadratic single-patch Hamiltonians in the first two terms, analogous to harmonic 1-particle Hamiltonians of our many-body problem. Spatial derivatives of L must be discretized before they can be expressed in terms of the $V_{i,j,k}$. The discretization procedure is a matter of choice and, to some degree, convenience; we will make use of

$$\frac{\partial}{\partial x^b} \frac{L(x_{i,j,k})}{a^2} \longrightarrow \frac{V_{(i,j,k)+\tilde{b}} - V_{(i,j,k)-\tilde{b}}}{6\ell_0(V/\mathcal{N})}\,, \tag{10}$$

indicating by $\tilde{b}$ the unit vector in the b-direction.[76]

Quadratic expressions of spatial derivatives in (8) then provide interaction terms that can be written as depending on either the patch geometries in product form, such as $V_{(i,j,k)+\tilde{b}} V_{(i,j,k)-\tilde{b}}$, or more conveniently, the difference $(V_{(i,j,k)+\tilde{b}} - V_{(i,j,k)-\tilde{b}})$ in discrete minisuperspace. The latter version is closer to interactions of many-body systems depending on the distance between particles.

In addition to interactions between neighboring patches, each patch volume interacts with the average volume V because it appears in some factors in the Hamiltonian. These variables are not independent but satisfy $\sum_{i,j,k} V_{i,j,k} = V$. In order to focus on the self-interaction of inhomogeneity, we will treat V as an external parameter for the dynamics of the $V_{i,j,k}$, corresponding to the common approximation in cosmology that ignores back-reaction of inhomogeneity on the background.

B. Quantization

Each patch of volume $V_{i,j,k}$ and expansion rate related to $\Pi_{i,j,k}$ is isotropic and may be quantized as a single minisuperspace model, corresponding to the 1-particle Hilbert space of a many-body system. One may follow either Wheeler–DeWitt quantization or loop quantization, both with volume representations in which $V_{i,j,k}$ becomes a multiplication operator. In the former case, one deals with wave functions $\psi(V_{i,j,k})$ in $L^2(\mathbb{R}_+, \mathrm{d}V_{i,j,k})$ and the momenta act by $\hat{\Pi}_{i,j,k} = -i\hbar \mathrm{d}/\mathrm{d}V_{i,j,k}$; in the latter, $\psi_{V_{i,j,k}}$ is an element of the non-separable sequence space $\ell^2(\mathbb{R})$ and exponentials of momenta, rather than momenta themselves, are quantized:

$$\exp(i\widehat{\delta_{i,j,k}\Pi_{i,j,k}}/\hbar)\psi_{V_{i,j,k}} = \psi_{V_{i,j,k}+\delta_{i,j,k}} \tag{11}$$

for real numbers $\delta_{i,j,k}$ (whose values are to be fixed as part of quantization choices).[77] The action of $\exp(i\widehat{\delta_{i,j,k}\Pi_{i,j,k}}/\hbar)$ on the sequence space is not continuous in $\delta_{i,j,k}$, and a derivative by $\delta_{i,j,k}$, which would otherwise result in an operator for $\Pi_{i,j,k}$, does not exist. (A second difference between the quantizations is that $V_{i,j,k}$ in Wheeler–DeWitt models is usually taken as the (positive) volume, while loop quantum cosmology is based on triad variables in which $V_{i,j,k}$ is the oriented volume, which can turn negative if the orientation is reversed. We therefore use the full real line $\mathbb{R}$ in the sequence space, rather than $\mathbb{R}_+$. Note that this resolves self-adjointness issues of derivative operators on $L^2(\mathbb{R}_+, \mathrm{d}V_{i,j,k})$.)

Both representations are well-defined but not unitarily related to each other; they lead to different physics. Especially at high curvature, where $\Pi_{i,j,k}$ is large, effects of the loop quantization can differ significantly from those of the Wheeler–DeWitt quantization. The discreteness inherent in shift operators (11) relating derivatives is then important, in addition to the discreteness implemented by our treatment of inhomogeneity.

If inhomogeneity is small, the patches evolve nearly independently of one another without strong correlations, and the evolved state remains a product state $\Psi(V_1, V_2, \ldots) = \psi_1(V_1)\psi_2(V_2)\cdots$ of the individual patch wave functions ψ_i if the initial state is of such a form. Each single-patch wave function evolves according to a differential (Wheeler–DeWitt [41]) or difference (loop quantum cosmology [30, 31]) equation if inhomogeneity can be ignored. With inhomogeneity included, interaction terms between the individual wave functions occur on superspace, complicating the dynamics. If inhomogeneity is sufficiently small, however, the interactions can be treated by approximation, such as perturbation theory.

Small inhomogeneity at the level of quantum geometry also implies that the individual wave functions are very similar to one another, so that the full state can approximately be written as $\Psi(V_1, V_2, \ldots) = \psi(V_1)\psi(V_2)\cdots$ with a single wave function ψ to be solved for. This form of product states allows one to map many-body dynamics to 1-particle dynamics in a specific potential, described by a wave equation that turns out to be non-linear. At this stage, standard techniques to describe matter condensates, in which individual wave functions of different particles are exactly equal to one another, can be applied.

C. Condensate

By our preceding considerations in cosmology, we have realized a mathematical formulation with all the ingredients used in the description of Bose–Einstein condensation. We interrupt our discussion of cosmology to recall salient features of this important system in condensed-matter physics. In this example, Ψ is a many-body state, and ψ the 1-particle wave function common to all constituents of the condensate. Taking the same ψ is not an assumption because condensed particles have exactly the same wave function.

Assuming pointlike interactions between the particles, described by a delta-function potential of strength α, we have the many-body Hamiltonian

$$\hat{H} = \sum_{i=1}^{n} \left(\frac{1}{2m} \hat{p}_i^2 + V(\hat{x}_i) \right) + \frac{1}{2} \alpha \sum_{i \neq j} \delta(\hat{x}_i - \hat{x}_j) \tag{12}$$

for n particles of mass m in individual potentials $V(x_i)$. With a product state $\Psi(x_1, x_2, \ldots) = \psi(x_1)\psi(x_2)\cdots$ for the condensate, we compute the expectation value of the Hamiltonian as

$$\langle \hat{H} \rangle_\Psi = n \langle \hat{p}^2/2m + V(\hat{x}) \rangle_\psi + \frac{1}{2} n(n-1) \alpha \int \mathrm{d}^3x |\psi(x)|^4 \,. \tag{13}$$

The first term just adds up the 1-particle expectation values computed for the wave function ψ. The second term is not equal to a 1-particle expectation value. However, we can formally interpret it as the expectation value of a "potential" $|\psi(x)|^2$ depending on the wave function. Accordingly, the 1-particle dynamics and energy spectra are governed by a non-linear Schrödinger equation, the Gross–Pitaevski equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi + \frac{1}{2}(n-1)\alpha |\psi(x)|^2 \psi \,. \tag{14}$$

For a full and rigorous derivation, see [42, 43].

Interacting many-body dynamics of the condensate wave function can therefore be mapped to non-linear 1-particle dynamics.

III. NON-LINEAR DYNAMICS IN QUANTUM COSMOLOGY

With the preparations presented in the preceding section, we propose a new method to deal with small cosmological inhomogeneity, making use of the same ideas and initial constructions employed to describe matter condensates. Well-established methods then provide a tractable approximate description by non-linear dynamics of a homogeneous model.

A. Equation of motion

Except for the differences in the conceptual nature, regarding for instance the approximations and assumptions used, our model for inhomogeneous quantum cosmology so far resembles those of matter condensates rather closely. The main mathematical difference lies in the interaction potential. For particles in a condensate, a delta function of the distance between particles is a good approximation for nearly pointlike interactions, which can be smeared out to more-complicated functions for realistic systems. The interaction potential we obtain in cosmology, expanding and discretizing the gravitational Hamiltonian constraint, is a quadratic polynomial in the distances in minisuperspace. Although the single-patch wave equation we obtain is still non-linear, as in the presence of any kind of interactions, it is more complicated than in the Gross-Pitaevski equation.

Another difference between the models is the discreteness of the quantum representation used in a loop quantization, in addition to the discretization of space by patches $\mathcal{V}_{i,j,k}$. Not only space but also superspace is then discrete. As a consequence, wave equations in loop quantum cosmology are difference equations, and with our method to include inhomogeneity we will be dealing with some version of a discrete non-linear Schrödinger equation, one example given by

$$i\hbar\frac{\partial\psi_n}{\partial t} = \frac{1}{2}(\psi_{n+1} - 2|\psi_n|^2\psi_n + \psi_{n-1})\,. \tag{15}$$

However, since we are not dealing with pointlike interactions in superspace, modeled by delta functions, but rather with polynomials, the non-linearity will be different. In fact, our equation will not only be non-linear but also non-local but nevertheless, as it turns out, well-suited to canonical effective methods.

Using the same starting point as in Bose–Einstein condensation, the key step is to evaluate the expectation value of the interaction Hamiltonian in a product state. To illustrate the main consequence, we consider just two variables V_1 and V_2 interacting with each other via a potential $W_{\rm int}(V_1,V_2) = \alpha(V_1-V_2)^2/V^2$ as in a discretized (8). We divide by the total volume squared, treated as an external but time-dependent parameter, in order to have the correct scaling behavior of the Hamiltonian under a change of the spatial region. The expectation value of the quantized $W_{\rm int}$ then produces a term

$$\langle\hat{W}_{\rm int}\rangle_\Psi \;=\; \frac{\alpha}{V^2}\int {\rm d}V_1{\rm d}V_2|\psi(V_1)|^2|\psi(V_2)|^2(V_1-V_2)^2 \tag{16}$$

$$= \frac{\alpha}{V^2}\int \mathrm{d}V_1|\psi(V_1)|^2\int \mathrm{d}\delta V|\psi(V_1+\delta V)|^2(\delta V)^2$$

where we introduce $\delta V := V_2 - V_1$.

We can perform the second integration independently of the first over V_1. It depends on the wave function, but if we assume that ψ is sharply peaked around the expectation value $\langle V_1\rangle$, the dominant contribution to $\langle\hat{W}_{\mathrm{int}}\rangle_\Psi$ comes from values of V_1 for which the second integration

$$\int \mathrm{d}\delta V|\psi(\langle V\rangle+\delta V)|^2(\delta V)^2 = (\Delta V)^2 \tag{17}$$

equals the quantum fluctuation of V in the state $\psi(V)$. Instead of a non-linearity potential depending on $\psi(V)$ or ψ_n as in (15), we have a non-linearity potential that depends on the wave function via moments such as ΔV. For instance, following the preceding arguments and noting that the minisuperspace V is quantized to a discrete parameter n, we need to consider an equation of the form

$$i\hbar\frac{\partial\psi_n}{\partial t} = \psi_{n+1} - 2\left(1-\frac{1}{2}\alpha\frac{(\Delta n)^2_\psi}{n^2}\right)\psi_n + \psi_{n-1}\,. \tag{18}$$

We note that equation (18) is not only non-linear but also non-local: the coefficient $(\Delta n)^2_\psi = \sum_n (n-\langle n\rangle_\psi)^2|\psi_n|^2$ depends on all values of ψ_n. Moreover, the equation as written is meaningful only for $n\neq 0$. At $n=0$, the volume vanishes and we encounter a cosmological singularity. By inverse-triad corrections [44], loop quantum cosmology resolves this singularity in such a way that $1/n$ is replaced by a bounded function. For simplicity, we will not discuss these terms here and instead focus on evolution at large n.

We must ensure that our assumption of a sharply peaked state remains true for the approximation to be valid. If the state is not sharply peaked or if the approximation is to be driven to higher orders, we can use a derivative expansion of ψ. Writing

$$|\psi(V_1+\delta V)|^2 = |\psi(\langle V\rangle + \delta V + (V_1 - \langle V\rangle))|^2$$

and expanding by $V_1 - \langle V\rangle$, we obtain

$$\langle\hat{W}_{\mathrm{int}}\rangle_\Psi = \int \mathrm{d}V_1|\psi(V_1)|^2 W_{\mathrm{nonlin}}(V_1)$$

with the non-linearity potential

$$W_{\mathrm{nonlin}}(V) = \sum_{j=0}^{\infty}\frac{1}{j!}(\Delta V)^2_{\rho^{(j)}}(V-\langle V\rangle)^j \tag{19}$$

where the moment $(\Delta V)^2_{\rho^{(j)}}$ is the V-fluctuation computed with the "distribution" $\rho^{(j)}$, defined as the j-th derivative of $\rho(V) = |\psi(V)|^2$. Note that these derivatives need not be normalized or positive, so that we do not have probability distributions and fluctuations in the statistical sense. Nevertheless, the resulting numbers are well-defined as parameterizations of the non-linearity potential.

We continue with a discussion of the leading-order equation (18).

B. Solution procedures

An inverse scattering transform is the method of choice to solve non-linear discrete or differential Schrödinger equations [45]. However, the equation we obtain here, (18), is not only non-linear but also non-local. Standard techniques are therefore not readily available.

Non-local equations can sometimes be treated by replacing the non-local coefficient by new auxiliary degrees of freedom, as in [46] in the context of the non-linear Schrödinger equation. If the new degree of freedom is subject to a differential or difference equation with a source term given by the original wave function ψ, its general solution is a non-local expression in ψ (integrating its product with the Green's function of the auxiliary equation). If the right equation is chosen, the general solution for the auxiliary variable may provide the non-local coefficient, $(\Delta n)^2$ in our case. Here, however, such a treatment is not obvious.

Instead, canonical effective methods [47] based on the dynamics of moments of a state provide solution techniques well-suited for equations such as (18). The non-local coefficient is a second-order moment of the wave function; using equations for the moments instead of ψ_n itself then provides a reformulation of the problem in variables in which the non-locality disappears. Morally, this procedure is a version of introducing new degrees of freedom related to the wave function non-locally, for moments[78] such as $\Delta(n^a) := \sum_n (n - \langle n\rangle)^a |\psi_n|^2$ with the expectation value $\langle n\rangle = \sum_n n|\psi_n|^2$ are non-local in ψ_n. However, in quantum physics the moments are not auxiliary variables but rather variables of prime physical interest. For $a = 2$, we have quantum fluctuations, and higher moments with $a > 2$ provide additional statistical information about the state.

For linear discrete or differential Schrödinger equations, canonical effective techniques [47] amount to a systematic expansion of Ehrenfest's equations, used not just to derive the semiclassical limit in rigorous terms [48] but also to compute quantum corrections to any desired order in $\hbar$. For our purposes, we need to generalize these methods to non-linear equations as encountered here.

In quantum mechanics, a set of N basic operators $\hat{J}_i$ with closed linear commutators

$$[\hat{J}_i, \hat{J}_j] = \sum_k C_{ij}{}^k J_k \tag{20}$$

(perhaps including the identity operator if some commutators are constants) provides a closed algebra for expectation values under Poisson brackets

$$\{\langle\hat{J}_i\rangle, \langle\hat{J}_j\rangle\} = \frac{\langle[\hat{J}_i, \hat{J}_j]\rangle}{i\hbar}\,. \tag{21}$$

If the operators are complete, any observable can be expressed as a function of the expectation values $\langle\hat{J}_i\rangle$ and moments

$$\Delta\left(\prod_i J_i^{a_i}\right) := \left\langle \prod_i (\hat{J}_i - \langle\hat{J}_i\rangle)^{a_i} \right\rangle_{\rm symm} \tag{22}$$

with operator products in totally symmetric ordering. Using linearity and the Leibniz rule for Poisson brackets, these expectation values and moments form a Poisson manifold. Their dynamics is determined by the Hamiltonian flow generated by the expectation value $H_Q := \langle\hat{H}\rangle$ of the Hamiltonian constraint, another observable interpreted as a function of expectation values and moments. Hamiltonian equations of motion usually couple infinitely many moments to the expectation values, but a semiclassical expansion to some finite order in $\hbar$ results in finitely coupled equations which can be solved at least numerically. Computer-algebra codes exist to automate the generation of equations to rather high orders [49] (so far restricted to canonical commutators).

Writing $\hat{J}_i = \langle\hat{J}_i\rangle + (\hat{J}_i - \langle\hat{J}_i\rangle)$ in the quantum Hamiltonian $H_Q = \langle H(\hat{J}_i)\rangle$ and performing a formal expansion in $(\hat{J}_i - \langle\hat{J}_i\rangle)$, the Hamiltonian flow is generated by

$$H_Q = H(\langle\hat{J}_i\rangle) + \sum_{a_i} \frac{1}{a_1!}\cdots\frac{1}{a_N!} \frac{\partial^{a_1+\cdots a_N} H(\langle\hat{J}_j\rangle)}{\partial\langle\hat{J}_1\rangle^{a_1}\cdots\partial\langle\hat{J}_N\rangle^{a_N}} \Delta\left(\prod_i J_i^{a_i}\right)\,. \tag{23}$$

The first term is the classical Hamiltonian evaluated in expectation values, and the series includes quantum corrections of progressing order $\sum_i a_i$. Equations of motion follow from Poisson brackets.

These constructions rely on commutators of linear operators and cannot be used directly for non-linear Schrödinger-type equations. Nevertheless, a closely related procedure can be followed for equations such as (18) in which the non-linearity comes from non-local coefficients depending on the moments. As one can readily confirm by computing time derivatives of expectation values directly using (18) for wave-function factors, the evolution of moments is now governed by a quantum Hamiltonian (23) in which one initially treats the moments that appear in the non-local coefficients as external functions; because they do not come from a linear operator, they do not appear in commutators or in Poisson brackets of the moments when equations of motion are derived. In the equations of motion, once derived, these variables are to be equated to the moments they signify, providing additional coupling terms between moments compared with a linear Hamiltonian.

In our case, the $\hat{J}_i$ are given by three basic operators, a multiplication operator by n (the volume operator) and two shift operators $\hat{h}$ and $\hat{h}^\dagger$ that change n by ± 1, implementing (11) with $\delta = 1$. In terms of canonical variables (n, P), we can write shift operators as quantizations of $h = \exp(iP)$. The commutators

$$[\hat{n}, \hat{h}] = -\hbar\hat{h} \quad , \quad [\hat{n}, \hat{h}^\dagger] = \hbar\hat{h}^\dagger \quad , \quad [\hat{h}, \hat{h}^\dagger] = 0 \tag{24}$$

then define the basic algebra (20) of our loop-quantized theory, and correspondingly the Poisson brackets of expectation values and moments of n and h. Moreover, since we introduced complex variables, the reality condition $\hat{h}\hat{h}^\dagger = 1$ as well as analogs for the moments (such as $\Delta(hh^*) = 1 - hh^*$) must be satisfied.

We can realize the linear part of (18) as the Schrödinger equation with Hamiltonian operator $\hat{h} + \hat{h}^\dagger - 2$, resulting in the quantum Hamiltonian $H_Q^{\rm lin} = h + h^* - 2$ depending only on expectation values but not on moments. Adding the non-linearity, we have an extra term $-\frac{1}{2}\alpha A\langle\widehat{n^{-2}}\rangle$ with A treated as a constant to be set equal to $A = (\Delta n)^2$ in equations of motion, and $\langle\widehat{n^{-2}}\rangle$ to be expanded by moments as in (23). For the difference equation (18), we then have the quantum Hamiltonian

$$\begin{aligned} H_Q &= h + h^* - (2 - \alpha A\langle\widehat{n^{-2}}\rangle) \\ &= h + h^* - 2 + \alpha A(3n^{-4}(\Delta n)^2 - 20n^{-5}\Delta(n^3) + \cdots) \end{aligned} \tag{25}$$

with the non-local coefficient A treated for now as an external parameter. (Instead of the inverse of n, which is ill-defined at $n = 0$, modifications due to inverse-triad corrections in loop quantum cosmology should be used at small n [44].)

We obtain the equations of motion from Poisson brackets, in which we then set $A = (\Delta n)^2$:

$$\begin{aligned}
\dot{n} &= i(h - h^*) && (26)\\
\dot{h} &= 12i\alpha\frac{h}{n}\left(\frac{\Delta n}{n}\right)^4 - 6i\alpha\frac{(\Delta n)^2\Delta(nh)}{n^4} + \cdots \\
\frac{\mathrm{d}(\Delta n)^2}{\mathrm{d}t} &= 2i(\Delta(nh) - \Delta(nh^*)) && (27)
\end{aligned}$$

and so on for further moments. Instead of $\ddot{n} = 0$ as in the linear case, we can combine the first two equations to obtain

$$\begin{aligned}
\ddot{n} &= i(\dot{h} - \dot{h}^*) && (28)\\
&= -24\alpha\frac{(\Delta n)^2}{n^4}\left(\frac{\mathrm{Re}h}{n}(\Delta n)^2 - \frac{1}{2}\mathrm{Re}(\Delta(nh))\right) + \cdots .
\end{aligned}$$

Non-zero moments imply acceleration of the volume expansion (which is negative unless correlations $n\mathrm{Re}\Delta(nh)/((\Delta n)^2\mathrm{Re}h)$ are large).

C. Interpretation

Irrespective of the precise form of non-linearity, its presence has several general consequences of potential importance for quantum cosmology. An obvious and seemingly problematic implication is a loss of unitarity: wave functions evolved by the non-linear equation do not have preserved scalar products with other evolved states. There is no linear operator that could serve as a Hamiltonian whose adjointness properties one could analyze by standard techniques. Still, a straightforward direct calculation shows that the norms $\langle\psi|\psi\rangle$ of states (but not scalar products $\langle\phi|\psi\rangle$ of different states) are preserved. However, the original many-body system is clearly unitary, and therefore non-unitarity is a consequence of the reductions and approximations used. In order to interpret the non-linearity correctly, we should therefore look back on the constructions used to descend from many-body dynamics to a 1-particle equation.

For a matter condensate, we obtain the non-linear wave equation (14) in a rather indirect way: We do not reduce the many-body wave equation for Ψ directly, but rather compute the expectation value of the Hamiltonian (13), rewrite it in terms of the 1-particle wave function ψ, and recognize the extra term as a formal analog of a potential depending on the wave function. This potential, inserted in the standard Schrödinger equation, then provides (14), a step which is again only formal. Experience shows that the resulting non-linear equation nevertheless captures crucial properties of the many-body problem, and rigorous proofs have been provided [42, 43].

One can avoid the last formal step by forgoing wave equations and instead using the expectation value (13) to compute the spectrum of the many-body Hamiltonian, for instance by variational methods applied to the 1-particle wave function ψ on the right-hand side of (13). If the spectrum of the Hamiltonian is known, evolution properties then follow without directly using the non-linear equation (14). Similarly, effective canonical equations in quantum mechanics refer to expectation values of the Hamiltonian, such as (13) rather than wave equations, and are therefore less sensitive to the apparent loss of unitarity.

The physics of the system therefore does not suffer from a lack of unitarity. Moreover, since the norm is still preserved, the probability interpretation of a single state remains meaningful. Instead of using (14) as a fundamental wave equation for some function ψ in a Hilbert space, the equation models other dynamical effects, such as the evolution of particle distributions or the approach and possible interaction of superposed states. Properties such as the overlap of superposed states or the distance between different distributions can be determined from moments of a single wave function for the superposition and are independent of scalar products of the wave function with other states; they can be analyzed with a formal equation lacking unitarity. These are also the properties that effective equations are sensitive to. In quantum cosmology, such questions are usually of most interest because the exact state or wave function of quantum space is not accessible by observations available now or in the foreseeable future. Our model and with analog (18) of the Gross-Pitaevski equation (14) is therefore reasonable.

IV. DISCUSSION

We have introduced a new model for inhomogeneous quantum cosmology, aiming to capture essential features of the interacting dynamics of different parts of quantum space. The processes we describe therefore provide the dynamics of structure formation at a fundamental level. Using several approximations, justified when inhomogeneity is sufficiently small, and importing ideas of condensed-matter physics, we have been able to map the complicated many-body dynamics to a non-linear minisuperspace equation.[79]

In addition to the approximate nature, several differences with the condensate model occur:

- In the cosmological model, "interactions" between different patches are realized in superspace, not in actual space. Patches do not interact depending on their spatial distance, but depending on what their geometries are: The gravitational Hamiltonian depends on inhomogeneous modes, or on deviations of patch geometries from the spatial average.

- There is no delta-function potential (for pointlike interactions) but rather a polynomial potential, obtained by expanding the gravitational Hamiltonian as a function of patch geometries. As a consequence, the non-linearity is realized non-locally in the configuration space of wave functions.

- While the many-body Hamiltonian of a condensate is well known but difficult to deal with, a consistent version of an inhomogeneous gravitational Hamiltonian in quantum gravity is still lacking. In particular, covariance conditions and the related problem of anomalies have not been evaluated in sufficient detail [50–54]. (But see [55–59] for recent progress.)

In this situation, having an approximate description of incompletely known dynamics, we cannot expect to derive detailed quantitative cosmological scenarios. (This statement does not only apply to our new method, but to all derivations possible in quantum cosmology so far.) Effective techniques, as used in our solution procedure for non-linear non-local equations, provide means to parameterize ambiguities and ignorance, and to discuss anomalies, but no details are available yet. We therefore focus our discussion on new qualitative features suggested by the non-linearity of the homogeneous model.

Non-linear wave equations provide new forms of minisuperspace effects that capture crucial properties of averaged inhomogeneity. These terms need not require high, near-Planckian densities to be significant because they could potentially be large when many patch contributions are added up, even if each of them is tiny. All leading contributions have the same sign because they come from volume fluctuations, required to be positive. No cancellations happen when one sums over all patches, potentially giving large effects. For certain behaviors of quantum fluctuations as functions of time or the volume, our non-linearity can be interpreted as a cosmological-constant term, which turns out to be negative. (Again, the sign is determined because quantum fluctuations are always positive.) It remains to be seen whether more-refined models, including those with anisotropic patches, or higher orders in the moments in (19), not all of which are restricted by positivity, as well as perturbed Hamiltonians beyond second order can turn the sign to provide an overall positive cosmological constant.

An interesting feature of non-linear wave equations is the existence of a particular type of solutions: solitons. These are sharply peaked wave packets which evolve without changing shape. Moreover, if solitons occur in superposition, moving in different directions, they may occasionally overlap but do not influence each other. After they have moved through the same spot, they retain their old shapes. Such states are a promising candidate for new dynamical coherent states in quantum cosmology. In contrast to kinematical coherent states (or Gaussians) commonly used in such cases, solitons are adapted to the dynamics and, in the indirect way that employs non-linear wave equations, capture properties of inhomogeneity; in fact, their existence as solutions relies on deviations from exact homogeneity.

The existence of solitons and the integrability of equations, together with the associated possibility of chaos, depends sensitively on the form of discrete equation [60]. The discreteness, in turn, is related to quantization and regularization ambiguities in canonical quantum gravity. The strong sensitivity of some physical features may allow one to find tight restrictions on ambiguities.

We end by mentioning another, more speculative consequence. In quantum cosmology, solitons in superposition would correspond to different universes superposed in the same state. Solitons may overlap but do not affect each other's motion; they always form separate contributions to the total state. Solitons and the non-linear wave equations they solve could therefore play a role in the description and analysis of multiverse models.

Acknowledgments

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[73] Bose–Einstein condensates have been used in cosmological models before [61], but for matter rather than quantum geometry. Moreover, we are not suggesting that there is physical condensation, but rather use related mathematical techniques to describe approximately homogeneous geometries. A geometrical condensation picture not unlike the one developed in this article has been mentioned in [28], using the perspective of group field theory [29].

[74] At this level, we will not yet address the anomaly problem and derive a detailed model, but we will demonstrate the prospects for this to be done at a later stage. Accordingly, reliable cosmological applications of the present model do not include the calculation of early-universe scalar power spectra which are very sensitive to anomaly issues. New cosmological effects instead refer to additional corrections in the background evolution compared to classical equations, including those that may cause acceleration. A modified background evolution affects power spectra of inhomogeneities as well, but a reliable computation can be performed

only when the anomaly problem is solved. (See [5] for a review.)

[75] Regimes of interest here, with small inhomogeneity, are usually semiclassical regarding quantum geometry. In such a situation, effective constraints [62–64] can be used to solve the problems of time [65–67] and anomalies [3].

[76] Choices in the discretization procedure affect physical implications and can therefore be tested for their consistency. These choices are related to quantization and regularization ambiguities in canonical quantum gravity from which our expressions should follow in some complicated way. If discretization choices can be restricted, the same will be true for ambiguities of an underlying fundamental theory.

[77] We work in a special Abelian sector of homogeneous loop quantum cosmology. The general non-Abelian structure is more complicated due to refinement features [68], but qualitative aspects are shown well by the Abelian simplification.

[78] Our notation is a variation of the common $(\Delta n)^2 = \Delta(n^2)$ at second order, the parenthesis displaced to be unambiguous at higher orders.

[79] Other versions of non-linear quantum cosmology have been proposed [69–72], motivated by non-commutativity and information-theoretic arguments.