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Ultraviolet Cancellations in Half-Maximal Supergravity as a Consequence of the Double-Copy Structure

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Abstract

We show that the double-copy structure of gravity forbids divergences in pure half-maximal (16 supercharge) supergravity at four and five points at one loop in $D < 8$ and at two loops in $D < 6$. We link the cancellations that render these supergravity amplitudes finite to corresponding ones that eliminate forbidden color factors from the divergences of pure nonsupersymmetric Yang-Mills theory. The vanishing of the two-loop four-point divergence in $D = 5$ half-maximal supergravity is an example where a valid counterterm satisfying the known symmetries exists, yet is not present. We also give explicit forms of divergences in half-maximal supergravity at one loop in $D = 8$ and at two loops in $D = 6$.

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I. INTRODUCTION

Recent years have made it clear that even at loop level perturbative scattering amplitudes in gravity theories are closely related to corresponding ones in gauge theories. In particular, a recent conjecture holds that whenever a duality between color and kinematics is made manifest, the integrands of (super)gravity loop amplitudes can be obtained immediately from corresponding gauge-theory ones [1, 2]. It has been clear since the original loop-level double-copy construction that it would have important implications for resolving long-standing questions on the ultraviolet properties of gravity theories. An obvious question is whether it can be used to show that $\mathcal{N} = 8$ [3] and other supergravity theories have a tamer than expected ultraviolet behavior. If each order of the perturbative expansion were finite, it would imply a deep new structure of the theory.

The double-copy structure has been used to simplify new nontrivial calculations of the ultraviolet properties of supergravity amplitudes, demonstrating behavior remarkably similar to corresponding gauge-theory amplitudes. In explicit calculations of amplitudes one can, of course, directly confirm that the conjectured duality and double-copy properties hold. For example, through at least four loops, the ultraviolet divergences of $\mathcal{N} = 8$ supergravity in the critical dimension where they first occur are proportional to divergences appearing in the subleading-color terms of corresponding $\mathcal{N} = 4$ super-Yang-Mills amplitudes [4–6]. The double-copy construction also played a key role in a recent computation showing that all three-loop four-point amplitudes in $\mathcal{N} = 4$ supergravity in $D = 4$ [7] are ultraviolet finite [8], contrary to expectations based on the availability of an apparently supersymmetric and duality invariant [9] R^4 counterterm [10].

In this paper, we use the double-copy construction to explain ultraviolet finiteness in a simpler example: the four- and five-point two-loop potential divergences in $D = 5$ half-maximal supergravity [11]. We directly link the finiteness of the half-maximal supergravity four- and five-point amplitudes at one loop in $D < 8$ and at two loops in $D < 6$ to ultraviolet cancellations of forbidden color factors in gauge-theory amplitudes. This can be understood in terms of generalized gauge invariance [1, 2, 12, 13], which links the symmetries and cancellations of gauge theory to those of gravity. We note that the absence of the potential $D = 5$ two-loop four-point divergence has been seen from string-theory calculations as well, so it offers a good way to expose cancellations in the theory [14].

At present there does not appear to be an argument restricting counterterms using the conventional symmetries of the theory to rule out this divergence. Indeed as shown in ref. [15] the counterterm appears to be expressible as a duality invariant full superspace integral of a density (which itself is not duality invariant). It would be very important to fully understand the extent to which duality symmetry and supersymmetry by themselves can shed light on counterterm restrictions in half-maximal supergravity at two loops in $D = 5$.

Some cases we study here are especially simple to analyze because the $\mathcal{N} = 4$ super-Yang-Mills amplitudes used on one side of the double-copy construction have diagrammatic numerators that are independent of loop momenta. Because of this property, even after performing the loop integration, the corresponding amplitudes in pure supergravity theories with sixteen or more supercharges are simple linear combinations of corresponding gauge-theory amplitudes [16, 17]. Indeed, using the double-copy construction, in ref. [18] one-loop four- and five-point and two-loop four-point gravity amplitudes were expressed directly in terms of certain subleading-color amplitudes of corresponding gauge theories. There the authors found cancellations leading to the relatively mild infrared singularities in gravity, similar to the way we find tamer ultraviolet behavior in gravity than in the gauge-theory amplitudes from which they are built.

Besides the double-copy relation between gravity and gauge theory, there are other reasons to believe that the ultraviolet behavior of gravity might be better than expected from applying standard symmetry arguments. Even pure Einstein gravity at one loop exhibits remarkable cancellations as the number of external legs increases, essentially scaling with the number of external legs in the same way as gauge theory [19, 20]. Through unitarity, such cancellations feed into nontrivial ultraviolet cancellations at all loop orders [21]. Very recently, resummations of $\mathcal{N} \geq 4$ supergravity amplitudes were shown to have surprisingly good behavior in the high-energy Regge limit [22], suggestive of a connection to the surprisingly good ultraviolet behavior of loop amplitudes in these theories.

Whether the observed cancellations are sufficient to render the theory ultraviolet finite remains an open question. (For a recent optimistic opinion in favor of ultraviolet finiteness of $\mathcal{N} = 8$ supergravity see ref. [23]. For a recent pessimistic opinion see ref. [24].) In $\mathcal{N} = 8$ supergravity in $D = 4$, in particular, no divergence can occur before seven loops, but a consensus holds that a valid D^8R^4 counterterm exists at seven loops [25]. This may seem to suggest that in $D = 4$ the theory diverges at seven loops [25]. Interestingly, the candidate

full-superspace integral for the counterterm turns out to vanish [9], leaving only a BPS candidate counterterm represented by an integral over 7/8 of the superspace. The potential three-loop counterterm of $\mathcal{N} = 4$ supergravity in $D = 4$ [7] is analogous in this regard, as it too is BPS. In a previous paper [8], we proved by direct computation that the coefficient of the expected three-loop counterterm in $\mathcal{N} = 4$ supergravity vanishes. (See ref. [14] for a string-theoretic argument of this vanishing and ref. [26] for a conjecture linking it to a hidden superconformal invariance.) While no nonrenormalization theorems are known for these cases, an important open question remains whether the BPS nature of the counterterm plays a role in explaining the finiteness. In any case, based on the vanishing of divergences in explicit calculations presented here and in ref. [8], we see that arguments based on applying the known symmetries of supergravity theories can be misleading. It is therefore important to carry out explicit computations to guide future studies. In particular, the arguments suggesting a seven-loop divergence in $D = 4$ also suggest that in higher dimensions, $\mathcal{N} = 8$ supergravity will be worse behaved than $\mathcal{N} = 4$ super-Yang-Mills theory starting at five loops due to the availability of a D^8R^4 counterterm. It should be possible to test this by direct computation [27].

Besides explaining the nontrivial cancellation of two-loop four-point divergences in half-maximal supergravity in $D = 5$, we also present the explicit forms of one-loop four- and five-point divergences in $D = 8$ and two-loop four-point divergences in $D = 6$. We obtain these using the same double-copy construction as used to demonstrate the vanishing of all three-loop divergences of $\mathcal{N} = 4$ supergravity in $D = 4$ [8]. In this construction, one copy is a maximally supersymmetric Yang-Mills amplitude in a form in which the duality between color and kinematics holds manifestly [2], while the second copy uses ordinary Feynman rules in Feynman gauge. The diagrams are then expanded for large loop momenta (or equivalently small external momenta) and integrated to extract the ultraviolet divergences [28]. The explicit expressions for divergences presented here should be useful in future studies of the symmetries and structure of half-maximal supergravity.

This paper is organized as follows. In Section II, we briefly review some basic features of the duality between color and kinematics and the double-copy construction of gravity. In Section III, we show that at four and five points the potential one-loop divergences in half-maximal supergravity cancel in $D < 8$ by linking them to forbidden divergences in corresponding gauge-theory amplitudes. We also present the explicit form of one-loop

divergences in $D = 8$. Then in Section IV we show that the two-loop four- and five-point amplitudes of half-maximal supergravity do not have divergences in $D < 6$. In addition, this section contains an explicit expression for four-point $D = 6$ divergences. We give our conclusions and outlook in Section V. An appendix computing the two-loop four-point divergence of pure Yang-Mills theory in $D = 5$ is also included. These results are used in Section IV to explicitly demonstrate ultraviolet cancellations in the corresponding half-maximal supergravity amplitude.

II. REVIEW OF BCJ DUALITY

In this section we review the duality between color and kinematics conjectured by Carrasco, Johansson and one of the authors (BCJ) and the related double-copy construction of gravity loop amplitudes [1, 2]. These properties underlie our ability to analyze the divergence structure of half-maximal supergravity amplitudes. Recent applications to the half-maximal theory of $\mathcal{N} = 4$ supergravity in $D = 4$ can be found in refs. [8, 16–18].

A. Duality between color and kinematics

We can write any m-point L-loop gauge-theory amplitude with all particles in the adjoint representation as

$$\mathcal{A}_m^{L-\text{loop}} = i^L g^{m-2+2L} \sum_{\mathcal{S}_m} \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2} \,. \tag{2.1}$$

The sum labeled by j runs over the set of distinct non-isomorphic m-point L-loop graphs with only cubic (i.e. trivalent) vertices. S_j is the symmetry factor of graph j, removing overcounts from the sum over $m!$ permutations of external legs indicated by $\mathcal{S}_m$ and from internal automorphism symmetry. The product in the denominator runs over all Feynman propagators of graph j. The integrals are over L independent D-dimensional loop momenta. The c_j are the color factors obtained by dressing every three-vertex with a group-theory structure constant,

$$\tilde{f}^{abc} = i\sqrt{2} f^{abc} = \text{Tr}([T^a, T^b]T^c) \,, \tag{2.2}$$

and n_j are kinematic numerators of graph j depending on momenta, polarizations and spinors. For supersymmetric amplitudes expressed in superspace, there will also be Grass-

mann parameters in the numerators. Contact terms in the amplitude are expressed in this form by multiplying and dividing by appropriate propagators. We note that there is enormous freedom in the choice of numerators, due to generalized gauge invariance [1, 2, 12, 13].

The conjectured duality of refs. [1, 2] states that to all loop orders there should exist a form of the amplitude where kinematic numerators satisfy the same algebraic relations as color factors. For Yang-Mills theory this amounts to imposing the same Jacobi identities on the kinematic numerators as satisfied by the color factors,

$$c_i = c_j - c_k \;\Rightarrow\; n_i = n_j - n_k \,, \tag{2.3}$$

where the indices i, j, k denote the diagram to which the color factors and numerators belong. Moreover, the numerator factors are required to have the same antisymmetry property as color factors under interchange of two legs attaching to a cubic vertex,

$$c_i \to -c_i \;\Rightarrow\; n_i \to -n_i \,. \tag{2.4}$$

As explained in some detail in refs. [6, 29, 30], the numerator relations are functional equations. For four-point tree amplitudes such relations were noticed long ago [31]. Beyond the four-point tree level, the relations are rather nontrivial and work only after appropriate rearrangements of the amplitudes.

At tree level, explicit forms of amplitudes satisfying the duality have been found for an arbitrary number of external legs [32]. An interesting consequence of the duality is that color-ordered partial tree amplitudes satisfy nontrivial relations [1]. These have been proven both in gauge theory and in string theory [33]. The duality is natural to understand using the heterotic string because of the parallel treatment of color and kinematics [13]. Although we do not yet have a satisfactory Lagrangian understanding, some progress in this direction can be found in refs. [12, 34]. The duality (2.3) has also been expressed in terms of an alternative trace-based representation [35], emphasizing the underlying group-theoretic structure of the duality. Indeed, progress has been made in understanding the underlying infinite-dimensional Lie algebra [34, 36]. Interestingly, the duality between color and kinematics also appears to hold in more exotic three-dimensional theories [37], as well as in certain cases with higher-dimension operators [38]. Relations similar to tree-level ones have also been shown to hold for the identical helicity one-loop amplitudes of pure Yang-Mills theory [39].

At loop level, although the duality remains a conjecture, a number of nontrivial checks have been carried out. The duality has been confirmed to hold up to four loops for the four-point amplitudes of $\mathcal{N} = 4$ super-Yang-Mills theory [2, 6], and for the five-point one- and two-loop amplitudes of this theory [5]. It is also known to hold for the identical-helicity one- and two-loop four-point amplitudes of pure Yang-Mills theory [2].

B. Gravity as a double copy of gauge theory

Once the gauge-theory amplitudes have been arranged into the form (2.1) where the numerators satisfy the duality (2.3), the corresponding gravity loop integrands become remarkably simple to obtain [1, 2] via the replacement,

$$c_i \to \tilde{n}_i \,. \tag{2.5}$$

The $\tilde{n}_i$ are diagram numerators from a second gauge theory. Making the substitution (2.5) in eq. (2.1) gives us the double-copy form of gravity amplitudes [1, 2],

$$\mathcal{M}_m^{L-\text{loop}} = i^{L+1} \Big(\frac{\kappa}{2}\Big)^{n-2+2L} \sum_{\mathcal{S}_m} \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2} \,, \tag{2.6}$$

where $\mathcal{M}_m^{L-\text{loop}}$ are m-point L-loop gravity amplitudes. In the double-copy formula (2.6), only one of the two sets of numerators n_j or $\tilde{n}_j$ needs to satisfy the duality relation (2.3).

Here we are interested in half-maximal supergravity in $D > 4$ dimensions. This theory is obtained via the double-copy formula by taking the direct product of pure nonsupersymmetric Yang-Mills theory with maximally supersymmetric Yang-Mills theory. This construction is the same one used to construct one- and two-loop amplitudes in $\mathcal{N} = 4$ supergravity in $D = 4$ [8, 16–18]. While the maximally supersymmetric Yang-Mills theory has exactly the same number of states as $\mathcal{N} = 4$ super-Yang-Mills theory does in four dimensions, the pure nonsupersymmetric theory used in this construction has additional gluon states compared to the $D = 4$ case.

At tree level, eq. (2.6) encodes the Kawai-Lewellen-Tye [40] relations between gravity and gauge theory [1]. The double-copy formula has been proven at tree level when the duality (2.3) holds in the corresponding gauge theories [12]. It has also been studied in some detail in a number of cases through four loops in $\mathcal{N} = 8$ supergravity [2, 5, 6], and through three loops in $\mathcal{N} = 4$ supergravity [8, 16, 17].

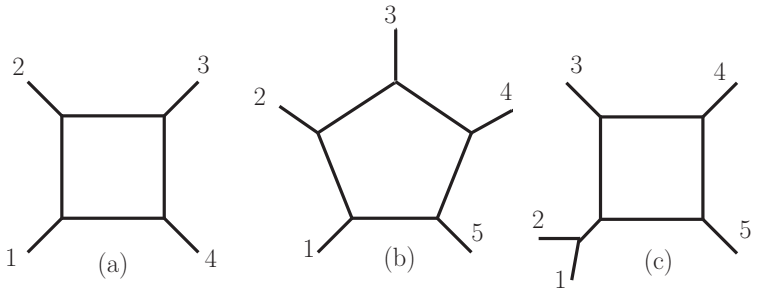

FIG. 1: Diagram (a) specifies the four-point color factor $c^{(1)}_{1234}$ used in eq. (2.7), and diagram (b) specifies the color factor $c^{(1)}_{12345}$ in eq. (2.14). Diagram (a) and its permutations appear in the four-point amplitude of maximal super-Yang-Mills theory. At five points both (b) and (c) and their permutations appear.

C. Supergravity with $Q + 16$ supercharges at one and two loops

A color-dressed four-point one-loop (super) Yang-Mills amplitude can be expressed as [41]

$$\mathcal{A}^{(1)}_Q(1,2,3,4) = g^4\Big[c^{(1)}_{1234}A^{(1)}_Q(1,2,3,4) + c^{(1)}_{1342}A^{(1)}_Q(1,3,4,2) + c^{(1)}_{1423}A^{(1)}_Q(1,4,2,3)\Big]\,. \quad (2.7)$$

The $c^{(1)}_{1234}$ are the color factors of a box diagram with consecutive external legs $(1,2,3,4)$, illustrated in Fig. 1(a), and dressed with structure constants $\tilde{f}^{abc}$. Here, $A^{(1)}_Q$ are one-loop color-ordered amplitudes [42]. The label Q specifies the number of supercharges. For maximally supersymmetric Yang-Mills ($Q = 16$), the amplitude is given by the one-loop scalar box integral, with the corresponding diagram numerators given by [43]

$$n_{1234} = n_{1342} = n_{1423} = stA^{\rm tree}_{Q=16}(1,2,3,4)\,, \quad (2.8)$$

where $A^{\rm tree}_{Q=16}(1,2,3,4)$ is the color-ordered tree amplitude of maximal super-Yang-Mills theory in any dimension and for any states of the theory. The Mandelstam invariants are defined as $s = (k_1 + k_2)^2$, $t = (k_2 + k_3)^2$ and $u = (k_1 + k_3)^2$. It is straightforward to check that this form satisfies the duality between color and kinematics.

To obtain pure supergravity amplitudes with $Q + 16$ supercharges, we simply replace the color factors with the corresponding numerators (2.8), yielding a rather simple formula,

$$\mathcal{M}^{(1)}_{Q+16} = i\Big(\frac{\kappa}{2}\Big)^4 stA^{\rm tree}_{Q=16}(1,2,3,4)\Big[A^{(1)}_Q(1,2,3,4) + A^{(1)}_Q(1,3,4,2) + A^{(1)}_Q(1,4,2,3)\Big]\,. \quad (2.9)$$

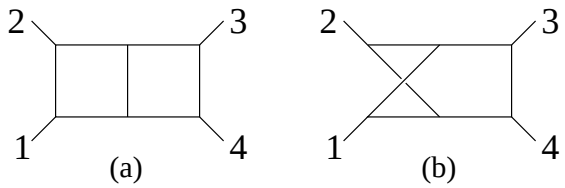

FIG. 2: The two-loop planar and nonplanar double-box diagrams.

In four dimensions, the prefactor in eq. (2.9) can be written in a supersymmetric form [44],

$$stA^{\rm tree}_{Q=16}(1,2,3,4) = -i\delta^{(8)}(\mathcal{Q})\,\frac{[1\,2][3\,4]}{\langle 1\,2\rangle\langle 3\,4\rangle}\,, \tag{2.10}$$

which makes half the supersymmetries manifest. Here $\langle 1\,2\rangle$ and $[1\,2]$ are the usual four-dimensional spinor-inner products for Weyl spinors (see e.g. ref. [45]). In this form all states of the $\mathcal{N}=4$ super-Yang-Mills multiplet are encoded in the Grassmann-valued delta function of the supercharges $\mathcal{Q}$. Simple superspace expressions also have been constructed in six dimensions [46]. Here we do not use any superspace properties, other than the fact that all states are encoded in one simple prefactor.

The two-loop four-point case is also relatively simple. The color-dressed two-loop four-point (super) Yang-Mills amplitude can be conveniently written as [16, 17]

$$\begin{aligned}\mathcal{A}^{(2)}_Q(1,2,3,4) \;=\; g^6\Big[&c^{\rm P}_{1234}A^{\rm P}_Q(1,2,3,4) + c^{\rm P}_{3421}A^{\rm P}_Q(3,4,2,1) \\ &+ c^{\rm NP}_{1234}A^{\rm NP}_Q(1,2,3,4) + c^{\rm NP}_{3421}A^{\rm NP}_Q(3,4,2,1) + {\rm cyclic}(2,3,4)\Big]\,,\end{aligned} \tag{2.11}$$

where 'cyclic$(2,3,4)$' indicates a sum over the remaining two cyclic permutations of legs 2, 3 and 4. Here $c^{\rm P}_{1234}$ and $c^{\rm NP}_{1234}$ are the color factors obtained by dressing the planar and nonplanar double-box diagrams in Fig. 2 with structure constants $\tilde{f}^{abc}$. The $A^{\rm P}_Q$ and $A^{\rm NP}_Q$ are the integrated planar and nonplanar kinematic parts of the amplitudes. The form (2.11) matches the one used in $\mathcal{N}=4$ super-Yang-Mills theory [47]. This form is valid for any theory with only adjoint representation particles, as can be shown using color Jacobi-identity rearrangements [41].

For maximal ($Q=16$) super-Yang-Mills theory in any dimension, the standard loop-integral representation of the two-loop four-point amplitude [2, 48] satisfies the duality between color and kinematics (2.3). An important simplifying feature is that the numerator factors do not have loop-momentum dependence, and are

$$n^{\rm P}_{1234} = s^2tA^{\rm tree}_{Q=16}(1,2,3,4)\,, \qquad\qquad n^{\rm NP}_{1234} = s^2tA^{\rm tree}_{Q=16}(1,2,3,4)\,, \tag{2.12}$$

corresponding to the two partial amplitudes $A^{\rm P}_{Q=16}(1,2,3,4)$ and $A^{\rm NP}_{Q=16}(1,2,3,4)$ in eq. (2.11). When constructing gravity amplitudes via the replacement (2.5), the numerator of the $\mathcal{N}=4$ super-Yang-Mills copy comes outside the integral, and thus one can express the integrated supergravity amplitude as a linear combination of integrated (super) Yang-Mills amplitudes [16]. Using this, the integrated four-point two-loop supergravity amplitude is [17]

$$\mathcal{M}^{(2)}_{Q+16}(1,2,3,4) \,=\, i\Big(\frac{\kappa}{2}\Big)^6 st A^{\rm tree}_{Q=16}(1,2,3,4)\Big[s\Big(A^{\rm P}_Q(1,2,3,4)+A^{\rm NP}_Q(1,2,3,4) \\ + A^{\rm P}_Q(3,4,2,1)+A^{\rm NP}_Q(3,4,2,1)\Big)+\text{cyclic}(2,3,4)\Big]\,, \quad (2.13)$$

which holds in any dimension $D \leq 10$ and for pure supergravity theories with $Q+16$ supercharges.

Now consider the five-point case. At one loop, a five-point gauge-theory amplitude with only adjoint representation particles can be written in the form [41],

$$\mathcal{A}^{(1)}_Q(1,2,3,4,5) = g^5 \sum_{\mathcal{S}_5/(\mathbb{Z}_5\times\mathbb{Z}_2)} c^{(1)}_{12345}\, A^{(1)}_Q(1,2,3,4,5)\,, \quad (2.14)$$

where the color factor $c^{(1)}_{12345}$ is that of the pentagon diagram, displayed in Fig. 1(b). The sum runs over all permutations with the five cyclic ones and reflections removed, signified by $\mathcal{S}_5/(\mathbb{Z}_5\times\mathbb{Z}_2)$. For the maximally supersymmetric ($Q=16$) case, only pentagon and box integrals contribute in the BCJ form [5], illustrated in Fig. 1(b) and (c).

Using the substitution rule (2.5), together with the observation that at four and five points, the duality-satisfying maximal super-Yang-Mills numerators with states restricted to a four-dimensional subspace are independent of loop momenta, we immediately obtain the simple expression [16],

$$\mathcal{M}^{(1)}_{Q+16}(1,2,3,4,5) = i\Big(\frac{\kappa}{2}\Big)^5 \sum_{\mathcal{S}_5/(\mathbb{Z}_5\times\mathbb{Z}_2)} \tilde{n}_{12345}\, A^{(1)}_Q(1,2,3,4,5)\,. \quad (2.15)$$

For $Q=0$ the obtained amplitudes are those of half-maximal pure supergravity theory.

In a four-dimensional external subspace, the maximal super-Yang-Mills kinematic numerators appearing in eq. (2.15) for external gluons in an MHV configuration are given by [5]

$$\tilde{n}_{12345} = \beta_{12345} \equiv \langle i\, j\rangle^4\, \frac{[1\,2][2\,3][3\,4][4\,5][5\,1]}{4\varepsilon(1,2,3,4)}, \quad (2.16)$$

where i and j label the two negative-helicity legs and $\varepsilon(1,2,3,4) \equiv \varepsilon_{\mu\nu\rho\sigma} k_1^\mu k_2^\nu k_3^\rho k_4^\sigma = \mathrm{Det}(k_i^\mu)$. The anti-MHV case is given by the parity conjugate. Five-point amplitudes with other states beside gluons have also been discussed in ref. [5], but we will not use them here. A conjectured D-dimensional generalization of these numerator functions may be found in ref. [30].

While only numerators corresponding to the pentagon diagram in Fig. 1(b) are required for eq. (2.15), in Section III B we will use the expressions for the numerators of the box diagrams, illustrated in Fig. 1(c), as well. Since the maximal super-Yang-Mills numerators satisfy the duality (2.3), the box numerators can be written in terms of the pentagon numerators by the kinematic Jacobi relations: $\tilde{n}_{[12]345} = \tilde{n}_{12345} - \tilde{n}_{21345}$, where $\tilde{n}_{[12]345}$ is the numerator for the box diagram in Fig. 1(c). This gives

$$\tilde{n}_{[12]345} = \gamma_{12} \equiv \gamma_{12345} \equiv \langle i\, j\rangle^4 \, \frac{[1\,2]^2 [3\,4][4\,5][3\,5]}{4\varepsilon(1,2,3,4)} \,. \tag{2.17}$$

The γ's are symmetric in their last three indices, so they can be specified by the first two indices only. They also satisfy the relations,

$$\sum_{i=1}^{5} \gamma_{ij} = 0, \qquad \gamma_{ij} = -\gamma_{ji} \,, \tag{2.18}$$

from which we see that there are six linearly independent γ's. They are completely interchangeable with the β's because,

$$\begin{aligned} \gamma_{12} \;&=\; \beta_{12345} - \beta_{21345}, \\ \beta_{12345} \;&=\; \frac{1}{2}\left(\gamma_{12} + \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} + \gamma_{34}\right) \,, \end{aligned} \tag{2.19}$$

so there are also six linearly independent β's.

III. ULTRAVIOLET STRUCTURE OF HALF-MAXIMAL SUPERGRAVITY AT ONE LOOP

In this section, we illustrate how the double copy links cancellations of supergravity divergences to those of forbidden color factors in gauge-theory divergences using simple one-loop examples. In particular, we discuss the divergence properties of the four- and five-point amplitudes in higher dimensions from this vantage point. The one-loop four- and five-point double-copy formulas (2.9) and (2.15) give integrated supergravity amplitudes

with 16 or more supercharges directly in terms of corresponding integrated (super) Yang-Mills amplitudes. This allows us to obtain the divergences of these supergravity amplitudes simply by plugging in known Yang-Mills counterterm amplitudes.

We note that in $D = 4$ the one-loop amplitudes of $\mathcal{N} < 8$ supergravity theories have been extensively studied recently in refs. [20, 49]. For the cases of four and five points, a double-copy construction has been given in ref. [16]. Very recently the one-loop four-graviton amplitude for $\mathcal{N} = 4$ supergravity coupled to $\mathcal{N} = 4$ vector multiplets has also been obtained by taking the field-theory limit of string-theory results [50]. Here we are mainly interested in higher dimensions.

A. Four-point divergences at one loop

We now demonstrate that pure half-maximal supergravity does not have four-point divergences at one loop for $D < 8$. In dimensional regularization at one loop, there can be no divergences in any dimension other than even integer dimensions. We will start with a warm up in $D = 4$ before turning to the more interesting cases of $D = 6$ and $D = 8$.

1. $D = 4$ warm up

We start by reproducing the well-known result that the four-point amplitude of pure $\mathcal{N} = 4$ supergravity has no divergence at one loop [51]. The renormalizability of Yang-Mills theory in $D = 4$ implies that the full one-loop divergence must be proportional to the color-dressed tree amplitude:

$$\mathcal{A}_Q^{(1)}\Big|_{D=4\,\text{div.}} = \frac{\beta_0^Q}{\epsilon}\mathcal{A}_Q^{\text{tree}}\,. \tag{3.1}$$

Here β_0^Q is a constant proportional to the one-loop beta function of the theory. The only part of the renormalizability of the theory that we need is that it implies that the color structure of the divergence must match exactly the color structure of the tree amplitude. This holds for any (super) Yang-Mills theory in four dimensions, though for $\mathcal{N} = 4$ super-Yang-Mills theory the beta-function coefficient vanishes, since the theory is ultraviolet finite [52]. The color-ordered tree amplitudes satisfy $U(1)$ decoupling relations,

$$A_Q^{\text{tree}}(1,2,3,4) + A_Q^{\text{tree}}(1,3,4,2) + A_Q^{\text{tree}}(1,4,2,3) = 0\,, \tag{3.2}$$

which are a simple consequence of the color structure. Finiteness of the four-point supergravity amplitude follows immediately by applying eqs. (3.1) and (3.2) to the supergravity amplitude (2.9),

$$\begin{aligned}\mathcal{M}^{(1)}_{Q+16}(1,2,3,4)\Big|_{D=4\,\text{div.}} &= i\Big(\frac{\kappa}{2}\Big)^4 stA^{\text{tree}}_{Q=16}(1,2,3,4) \\ &\quad\times \Big[A^{(1)}_Q(1,2,3,4) + A^{(1)}_Q(1,3,4,2) + A^{(1)}_Q(1,4,2,3)\Big]\Big|_{D=4\,\text{div.}} \\ &= 0\,. \end{aligned} \tag{3.3}$$

In six dimensions, Yang-Mills theory is not renormalizable. However, the counterterm has a color structure similar to the $D = 4$ one. For this reason, it is useful to slightly rephrase the $D = 4$ cancellation in terms of a basis of independent color tensors. As we shall see in the following section, this approach will also clarify the two-loop finiteness of four-point half-maximal supergravity in $D = 5$.

We start with tree level, where there are two independent color tensors corresponding to the color factors of s- and t-channel diagrams,

$$b^{(0)}_1 \equiv c^{(0)}_{1234} = \tilde{f}^{a_1a_2b}\tilde{f}^{ba_3a_4}\,, \qquad\qquad b^{(0)}_1 \equiv c^{(0)}_{1423} = \tilde{f}^{a_2a_3b}\tilde{f}^{ba_4a_1}\,. \tag{3.4}$$

The remaining u-channel color factor $c^{(0)}_{1324}$ is given in terms of the previous two by the color Jacobi equation, $c^{(0)}_{1324} = -b^{(0)}_1 - b^{(0)}_2$. At one loop there is one additional independent color tensor (see for example Appendix B of ref. [53]),

$$b^{(1)}_1 \equiv c^{(1)}_{1234} = \tilde{f}^{a_1b_2b_1}\tilde{f}^{a_2b_3b_2}\tilde{f}^{a_3b_4b_3}\tilde{f}^{a_4b_1b_4}\,. \tag{3.5}$$

The other color factors in the one-loop amplitude (2.7) are given in terms of these color tensors after using the color Jacobi identity and the ability to reduce the color factors with triangle or bubble subdiagrams to tree color tensors. For example, we have

$$c^{(1)}_{1342} = b^{(1)}_1 - \frac{1}{2}C_A b^{(0)}_1\,, \qquad c^{(1)}_{1423} = b^{(1)}_1 - \frac{1}{2}C_A b^{(0)}_2\,, \tag{3.6}$$

where C_A is the adjoint representation quadratic Casimir. For an $SU(N_c)$ group, $C_A = 2N_c$ with our nonstandard normalization.

Rewriting the gauge-theory amplitude (2.7) in terms of these independent color tensors gives

$$\begin{aligned}\mathcal{A}^{(1)}_Q(1,2,3,4) \;=\; g^4\Big[b^{(1)}_1\Big(&A^{(1)}_Q(1,2,3,4) + A^{(1)}_Q(1,3,4,2) + A^{(1)}_Q(1,4,2,3)\Big) \\ &- \frac{1}{2}C_A b^{(0)}_1 A^{(1)}_Q(1,3,4,2) - \frac{1}{2}C_A b^{(0)}_2 A^{(1)}_Q(1,4,2,3)\Big]\,. \end{aligned} \tag{3.7}$$

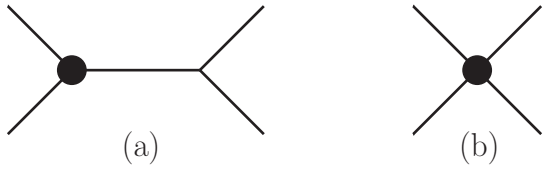

FIG. 3: The four-point diagrams generated by the F^3 counterterm in pure Yang-Mills at one loop $D = 6$ or at two loops in $D = 5$. The large dot indicate an insertion of a counterterm vertex, while a vertex without a dot represents an ordinary Yang-Mills vertex. In (a) a three-point counterterm vertex appears while in (b) a four-point counterterm vertex appears.

Since the Yang-Mills divergence in $D = 4$ contains only the tree color tensors, it cannot contain the one-loop color tensor $b_1^{(1)}$, implying that

$$A_Q^{(1)}(1,2,3,4) + A_Q^{(1)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3)\Big|_{\text{div.}} = 0\,. \tag{3.8}$$

This is equivalent to the tree-level decoupling relation (3.2), except in eq. (3.8) there is no explicit requirement that the divergence of the color-ordered amplitude be proportional to the tree amplitude, only that the one-loop color tensor $b_1^{(1)}$ not appear in the divergence. Thus, we obtain the vanishing of the supergravity divergence (3.3) purely from group-theoretic properties of the corresponding gauge theory.

2. $D = 6$ finiteness

In six dimensions, while the pure graviton R^3 counterterm is ruled out by supersymmetry, naively one might worry about counterterms of the form $\phi^k R^3$. As we now show, the same group-theoretic cancellations apply just as well in $D = 6$. Since the maximal super-Yang-Mills theory in six dimensions has $\mathcal{N} = (1,1)$ supersymmetry, the supergravity theory we are considering is the non-chiral $\mathcal{N} = (1,1)$ theory, in contrast to the chiral $\mathcal{N} = (2,0)$ theory.

From simple power-counting considerations, the one-loop $D = 6$ pure Yang-Mills counterterm operator is of the form [54],

$$F^3 \equiv \tilde{f}^{abc} F^{a\mu}{}_{\nu} F^{b\nu}{}_{\rho} F^{c\rho}{}_{\mu}\,, \tag{3.9}$$

because there are no other gauge-invariant operators of the proper dimensions that give nonvanishing matrix elements. (The gauge-invariant operator D^2F^2 is also allowed by dimensional analysis, as noted in ref. [54]. However, these can be removed via field redefinitions.) The symmetric color tensor d^{abc} does not appear because the combination of field strengths has an overall antisymmetry. The F^3 counterterm is forbidden in super-Yang-Mills theories because, in a four-dimensional external subspace, it generates a nonvanishing amplitude with helicities $(\pm,+,+,+)$ that is disallowed by supersymmetry Ward identities [55]. However, in nonsupersymmetric pure Yang-Mills theory in $D = 6$, it is a perfectly valid counterterm with a nonvanishing coefficient.

The key observation is that the counterterm diagrams displayed in Fig. 3 cannot generate color tensors other than the tree-level ones $b_1^{(0)}$ and $b_2^{(0)}$, defined in eq. (3.4). This follows because the counterterm three-vertex has a single $\tilde{f}^{abc}$ and the four-vertex has a pair of these, so the diagrams in Fig. 3 each have a pair of $\tilde{f}^{abc}$'s. Since the one-loop color tensor $b_1^{(1)}$ is built from four $\tilde{f}^{abc}$'s, it cannot appear in the Yang-Mills divergence. Thus, the situation is quite similar to the $D = 4$ case where only tree color tensors can appear in the divergence.

Following the $D = 4$ discussion, we demand that the one-loop color tensor $b_1^{(1)}$ not appear in the divergence. From eq. (3.7) we see that the $U(1)$ decoupling equation (3.8) holds for the $D = 6$ divergences. Plugging this into eq. (2.9) immediately shows that the one-loop divergence for $D = 6$ pure supergravity with 16 or more supercharges must vanish:

$$\mathcal{M}^{(1)}_{Q+16}(1,2,3,4)\Big|_{D=6\,\text{div.}} = 0\,. \tag{3.10}$$

For $Q > 0$, the divergence not only vanishes because the decoupling equation (3.8) holds, but also because F^3 is not a valid supersymmetric counterterm of the corresponding gauge theory.

It is straightforward to confirm that the decoupling identity (3.8) holds using the explicit forms of pure Yang-Mills counterterm amplitudes generated by the diagrams in Fig. 3. For example, the all-plus helicity counterterm amplitude in a four-dimensional subspace is [56]

$$A(1^+,2^+,3^+,4^+) = \frac{\alpha}{\epsilon}\frac{stu}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\,, \tag{3.11}$$

where α is a proportionality constant which can be fixed by explicit computation, but its value is unimportant for our discussion. This expression does indeed satisfy the required

$U(1)$ decoupling identity (3.8) because

$$\frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} + \frac{1}{\langle 13\rangle\langle 34\rangle\langle 42\rangle\langle 21\rangle} + \frac{1}{\langle 14\rangle\langle 42\rangle\langle 23\rangle\langle 31\rangle} = 0\,. \tag{3.12}$$

The vanishing of the $Q = 16$ counterterm can also be understood using a color-trace basis. Using the double-copy formula, ref. [18] showed that the one-loop $Q \geq 16$ supergravity amplitudes can be written as the double-trace Yang-Mills amplitude multiplied by a kinematic-dependent factor. This immediately leads us to conclude that six-dimensional $\mathcal{N} \geq 4$ supergravity must be one-loop ultraviolet finite since counterterm amplitudes generated with the F^3 operator do not contain a double-trace contribution.

We have also computed the coefficient of the $D = 6$ divergence of half-maximal supergravity using the procedure of ref. [8] and have confirmed that it vanishes. In this construction, one copy is the maximally supersymmetric Yang-Mills amplitude, while the second copy is based on ordinary Feynman rules. As mentioned earlier, the pure Yang-Mills numerators do not need to satisfy the duality eq. (2.3) since the maximal super-Yang-Mills side already does. A key simplifying feature of this method is that pure Yang-Mills numerators are required only for the box diagram since four-point maximal super-Yang-Mills numerators vanish for all other diagram topologies. In addition, there are no subdivergences since we are dealing here with one loop. We find the divergence cancels completely, in complete agreement with the above much simpler counterterm considerations.

3. $D = 8$ divergences

We now consider the $D = 8$ case. From ref. [57], the pure nonsupersymmetric Yang-Mills divergence is described by an F^4 operator of the form,

$$F^4 = c^{abcd}\Big[d_1 F^{a\,\mu\nu} F^{b}{}_{\nu\sigma} F^{c\,\sigma\rho} F^{d}{}_{\rho\mu} + d_2 F^{a\,\mu\nu} F^{b}{}_{\nu\mu}\, F^{c\,\rho\sigma} F^{d}{}_{\sigma\rho}\Big]\,, \tag{3.13}$$

where

$$c^{abcd} \equiv \tilde{f}^{a\,e_1 e_2}\tilde{f}^{b\,e_2 e_3}\tilde{f}^{c\,e_3 e_4}\tilde{f}^{d\,e_4 e_1}\,, \tag{3.14}$$

is the box-diagram color factor using the normalization in eq. (2.2). Only the linearized part of the field strength contributes to the divergent part of the four-point amplitude, $F_{\mu\nu} \equiv \partial_{[\mu} A_{\nu]}$. In $D = 8$ at one loop the constants appearing in the operator are

$$d_1 = \frac{g^4}{8}\frac{1}{(4\pi)^4}\frac{1}{\epsilon}\frac{238 + D_s}{360}\,, \qquad\qquad d_2 = \frac{g^4}{8}\frac{1}{(4\pi)^4}\frac{1}{\epsilon}\frac{D_s - 50}{288}\,, \tag{3.15}$$

where D_s is a state-counting parameter. It comes from contracting the metric $\eta_{\mu\nu}$ from gluon propagators around the loop. In pure Yang-Mills theory in eight dimensions, we take the state-counting parameter to be $D_s = 8$. (This is equivalent to the four-dimensional helicity regularization scheme [58] but with the state count adjusted to match the one of eight dimensions.) The divergence was first derived in the trace basis in ref. [54]. For the four-gluon amplitude at one loop, $n_s = D_s - 8$ counts additional minimally-coupled scalars circulating in the loop. The key difference between the $D = 8$ case and the previous $D = 4, 6$ cases is that the gauge-theory divergence contains the independent color tensor $b^{(1)}$. Thus the $U(1)$ decoupling equation (3.8) does *not* hold.

The amplitude is given by replacing the vector potential with the polarization vector ε_j, giving a polarization field strength for each leg j,

$$F_j^{\mu\nu} \equiv i(k_j^\mu \varepsilon_j^\nu - k_j^\nu \varepsilon_j^\mu)\,. \tag{3.16}$$

For notational convenience, we define the contractions of these polarization field strengths as

$$(F_i F_j) \equiv F_i^{\mu\nu} F_{j\mu\nu}, \qquad (F_i F_j F_k F_l) \equiv F_i{}^{\mu\nu} F_{j\nu\rho} F_k{}^{\rho\sigma} F_{l\sigma\mu}\,. \tag{3.17}$$

In terms of these, the nonvanishing divergence in the nonsupersymmetric pure Yang-Mills amplitude is

$$\begin{aligned}\mathcal{A}^{(1)}_{Q=0}(1,2,3,4)\Big|_{D=8\,\text{div.}} = \frac{i}{8\epsilon}\frac{1}{(4\pi)^4}\, g^4 c^{a_1 a_2 a_3 a_4} \Bigg[8\frac{238+D_s}{360}(F_1F_2F_3F_4) \\ + 4\,\frac{D_s-50}{288}\Big((F_1F_2)(F_3F_4) + (F_2F_3)(F_4F_1)\Big)\Bigg] \\ + \text{cyclic}(2,3,4)\,,\end{aligned} \tag{3.18}$$

where, as before, 'cyclic$(2,3,4)$' indicates that one should in addition include the two cyclic permutations of legs 2, 3 and 4 along with their color indices. Matching eq. (3.18) with eq. (2.7) and replacing color factors by the corresponding $Q = 16$ super-Yang-Mills numerators immediately gives the explicit form of the $Q = 16$ eight-dimensional supergravity divergence:

$$\begin{aligned}\mathcal{M}^{(1)}_{Q=16}(1,2,3,4)\Big|_{D=8\,\text{div.}} = -\frac{1}{\epsilon}\frac{1}{(4\pi)^4}\Big(\frac{\kappa}{2}\Big)^4 st A^{\text{tree}}_{Q=16}(1,2,3,4) \\ \times\Big[\frac{238+D_s}{360}(F_1F_2F_3F_4) + \frac{D_s-50}{288}(F_1F_2)(F_3F_4)\Big] + \text{cyclic}(2,3,4)\,,\end{aligned} \tag{3.19}$$

where $D_s = 8$ in the pure supergravity case. The factor $A^{\rm tree}_{Q=16}(1,2,3,4)$ is just the maximally supersymmetric four-point tree amplitude, for any of the states in the theory. The corresponding states in the $Q = 16$ supergravity theory are just the tensor product of these states with gluon states of the pure nonsupersymmetric Yang-Mills theory.

The explicit four-graviton R^4 counterterm for half-maximal supergravity in $D = 8$ is given in ref. [59]. It is built from the seven linearly independent R^4 forms in $D = 8$ [60] (in $D < 8$ these are no longer independent):

$$\begin{aligned}
T_1 &= (R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2, \\
T_2 &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\lambda}R_{\gamma\delta\kappa}{}^{\sigma}R^{\gamma\delta\kappa\lambda}, \\
T_3 &= R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\gamma}R^{\lambda\gamma}{}_{\delta\kappa}R^{\rho\sigma\delta\kappa}, \\
T_4 &= R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\gamma}R^{\rho\lambda}{}_{\delta\kappa}R^{\sigma\gamma\delta\kappa}, \\
T_5 &= R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\gamma}R^{\rho}{}_{\delta}{}^{\lambda}{}_{\kappa}R^{\sigma\delta\gamma\kappa}, \\
T_6 &= R_{\mu\nu\rho\sigma}R^{\mu}{}_{\lambda}{}^{\rho}{}_{\gamma}R^{\lambda}{}_{\delta}{}^{\gamma}{}_{\kappa}R^{\nu\delta\sigma\kappa}, \\
T_7 &= R_{\mu\nu\rho\sigma}R^{\mu}{}_{\lambda}{}^{\rho}{}_{\gamma}R^{\lambda}{}_{\delta}{}^{\nu}{}_{\kappa}R^{\gamma\delta\sigma\kappa}\,.
\end{aligned} \tag{3.20}$$

On shell the combination,

$$-\frac{T_1}{16} + T_2 - \frac{T_3}{8} - T_4 + 2T_5 - T_6 + 2T_7\,, \tag{3.21}$$

is a total derivative, so only 6 of the T_i are independent on shell. This gives us some freedom in how we write the explicit counterterm, which we give as [59]

$$\begin{aligned}
\frac{1}{\epsilon}\frac{1}{(4\pi)^4}\frac{1}{11520}[&(-126+3D_s)T_1 + (1968-24D_s)T_2 + (-252+6D_s)T_3 \\
&+ (8-4D_s)T_4 + 3840T_5 - 1920T_6 + (-3776-32D_s)T_7]\,,
\end{aligned} \tag{3.22}$$

where there is a relative i between the operators and amplitudes. The appropriate powers of the coupling are generated by expanding the metric around flat space, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$.

Since the kinematic numerators for one-loop four-point $\mathcal{N} = 4$ super-Yang-Mills theory are independent of loop momenta, we can also write the counterterm in a manner more suggestive of the double-copy structure. The four-point numerators in eq. (2.8) are given by an F^4 operator:

$$F^4 = -2\left[(F_1F_2F_3F_4) - \frac{1}{4}(F_1F_2)(F_3F_4) + \text{cyclic}(2,3,4)\right]\,. \tag{3.23}$$

Up to an overall constant including color factors, this is the same F^4 counterterm for four-point one-loop $\mathcal{N}=4$ sYM in $D=8$. Using this operator as a replacement for the kinematic numerator in eq. (3.19), we build an R^4 counterterm by making the association,

$$F_{i\,\mu\nu}F_{i\,\rho\sigma} \to -2R_{i\,\mu\nu\rho\sigma}\,. \tag{3.24}$$

At the linearized level, both terms in eq. (3.24) give the same contribution to the amplitude. On the gravity side, we replace the graviton field h by the polarization tensor $\epsilon_{\mu\nu}$, which can itself be replaced by the symmetrization of two polarization vectors $\epsilon_{\mu\nu} \to \epsilon_{(\mu}\epsilon_{\nu)}$. For the case of gravitons, we treat the two polarization vectors as being identical since the two possible replacements are $\epsilon^{++}_{\mu\nu} \to \epsilon^{+}_{\mu}\epsilon^{+}_{\nu}$ and $\epsilon^{--}_{\mu\nu} \to \epsilon^{-}_{\mu}\epsilon^{-}_{\nu}$. We then have

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda}\left(\partial_\rho\Gamma^\lambda_{\nu\sigma} - \partial_\sigma\Gamma^\lambda_{\nu\rho}\right) \\ &= \frac{1}{2}ik_\rho(ik_\sigma\epsilon_\mu\epsilon_\nu + ik_\nu\epsilon_\mu\epsilon_\sigma - ik_\mu\epsilon_\nu\epsilon_\sigma) - \frac{1}{2}ik_\sigma(ik_\rho\epsilon_\mu\epsilon_\nu + ik_\nu\epsilon_\mu\epsilon_\rho - ik_\mu\epsilon_\nu\epsilon_\rho) \\ &= \frac{1}{2}(k_\mu\epsilon_\nu - k_\nu\epsilon_\mu)(k_\rho\epsilon_\sigma - k_\sigma\epsilon_\rho)\,. \end{aligned} \tag{3.25}$$

Comparing to the polarization field strength tensor in eq. (3.16) gives us the replacement rule (3.24). After taking into account permutations, this replacement rule gives us the following contributing R^4 forms:

$$\begin{aligned} U_1 &= R_{\mu\nu\lambda\gamma}R^{\nu\ \ \gamma}_{\ \rho\ \ \delta}R^{\rho\ \ \delta}_{\ \sigma\ \ \kappa}R^{\sigma\mu\kappa\lambda}, \\ U_2 &= R_{\mu\nu\lambda\gamma}R^{\nu}_{\ \rho\delta\kappa}R^{\rho}_{\ \sigma}{}^{\gamma\delta}R^{\sigma\mu\kappa\lambda}, \\ U_3 &= (R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2, \\ U_4 &= R_{\mu\nu\lambda\gamma}R^{\mu\nu}_{\ \ \ \delta\kappa}R_{\rho\sigma}{}^{\lambda\gamma}R^{\rho\sigma\delta\kappa}, \\ U_5 &= R_{\mu\nu\lambda\gamma}R^{\nu\ \ \lambda\gamma}_{\ \rho}R^{\rho}_{\ \sigma\delta\kappa}R^{\sigma\mu\delta\kappa}, \\ U_6 &= R_{\mu\nu\lambda\gamma}R^{\nu}_{\ \rho\delta\kappa}R^{\rho}_{\ \sigma}{}^{\lambda\gamma}R^{\sigma\mu\delta\kappa}\,, \end{aligned} \tag{3.26}$$

and the counterterm is given by

$$\begin{aligned} \frac{1}{\epsilon}\frac{1}{(4\pi)^4}\frac{1}{23040}\,[&(-3808-16D_s)U_1 + (-7616-32D_s)U_2 + (-250+5D_s)U_3 \\ &+(-500+10D_s)U_4 + (3904-32D_s)U_5 + (1952-16D_s)U_6]\,. \end{aligned} \tag{3.27}$$

At the linearized level this is equivalent to eq. (3.22), but instead the index structure has been reorganized to expose the double-copy structure of gravity.

In terms of spinor helicity in a four-dimensional external subspace for the four-graviton case with external helicities $(1^+, 2^+, 3^-, 4^-)$, the divergence in $D = 8$ is

$$\mathcal{M}^{(1)}_{Q=16}(1^+, 2^+, 3^-, 4^-)\Big|_{D=8 \text{ div.}} = \frac{i}{\epsilon}\frac{1}{(4\pi)^4}\Big(\frac{\kappa}{2}\Big)^4 \frac{58 + D_s}{180}\langle 34\rangle^4[12]^4\,, \tag{3.28}$$

where we have plugged in spinor helicity for the $(1^+, 2^+, 3^-, 4^-)$ configuration on the right side of eq. (3.19). Similarly, any of the other helicity amplitudes can be extracted from eq. (3.19).

As in $D = 4, 6$ dimensions, we have also used the procedure described in ref. [8] for explicitly computing the divergences in half-maximal supergravity, finding agreement with the divergence in eq. (3.19).

B. Five points at one loop

To make the vanishing of ultraviolet divergences of half-maximal supergravity in $D = 4$ and $D = 6$ manifest at five-point one-loop, we write the one-loop supergravity amplitude (2.15) in terms of a basis of six independent β's (defined in eq. (2.16)):

$$\begin{aligned}
\mathcal{M}^{\text{1-loop}}_{Q+16}(1,2,3,4,5) &= i\left(\frac{\kappa}{2}\right)^5 \\
&\times \Big(\beta_{12345}(A^{(1)}_Q(1,2,3,4,5) + A^{(1)}_Q(2,1,3,4,5) + A^{(1)}_Q(2,3,1,4,5) + A^{(1)}_Q(2,3,4,1,5)) \\
&+ \beta_{12354}(A^{(1)}_Q(3,1,2,5,4) + A^{(1)}_Q(1,3,2,5,4) + A^{(1)}_Q(1,2,3,5,4) + A^{(1)}_Q(1,2,5,3,4)) \\
&+ \beta_{12435}(A^{(1)}_Q(2,1,4,3,5) + A^{(1)}_Q(1,2,4,3,5) + A^{(1)}_Q(1,4,2,3,5) + A^{(1)}_Q(1,4,3,2,5)) \\
&+ \beta_{12453}(A^{(1)}_Q(4,1,2,5,3) + A^{(1)}_Q(1,4,2,5,3) + A^{(1)}_Q(1,2,4,5,3) + A^{(1)}_Q(1,2,5,4,3)) \\
&+ \beta_{13245}(A^{(1)}_Q(5,1,3,2,4) + A^{(1)}_Q(1,5,3,2,4) + A^{(1)}_Q(1,3,5,2,4) + A^{(1)}_Q(1,3,2,5,4)) \\
&+ \beta_{13425}(A^{(1)}_Q(3,1,4,2,5) + A^{(1)}_Q(1,3,4,2,5) + A^{(1)}_Q(1,4,3,2,5) + A^{(1)}_Q(1,4,2,3,5))\Big)\,.
\end{aligned} \tag{3.29}$$

This expression is valid for all amplitudes where the external gluons on the super-Yang-Mills side of the double copy are in an MHV configuration in a four-dimensional subspace. The $\overline{\text{MHV}}$ result is just the parity conjugate. From the form (3.29), it is clear that when the gauge-theory divergences satisfy five-point $U(1)$ decoupling relations,

$$A^{(1)}_Q(1,2,3,4,5) + A^{(1)}_Q(2,1,3,4,5) + A^{(1)}_Q(2,3,1,4,5) + A^{(1)}_Q(2,3,4,1,5)\Big|_{\text{div.}} = 0\,, \tag{3.30}$$

and relabelings thereof, the supergravity amplitude (3.29) is finite, in much the same way as at four points. At tree level, these decoupling identities and their related Kleiss-Kuijf relations [61] are purely a consequence of color considerations [41]. Alternatively, as noted in ref. [1], they follow from the requirement that the color-ordered amplitudes can be described by diagrams with antisymmetric cubic vertices. As discussed above for the four-point case, in both $D = 4$ and $D = 6$ the Yang-Mills counterterms generate exactly the same color structures as at tree level, so the decoupling equation (3.30) indeed holds. Therefore, we immediately conclude that

$$\mathcal{M}^{(1)}_{Q+16}(1,2,3,4,5)\Big|_{D=4\,\text{div.}} = 0\,, \qquad\qquad \mathcal{M}^{(1)}_{Q+16}(1,2,3,4,5)\Big|_{D=6\,\text{div.}} = 0\,. \tag{3.31}$$

Had we used a different basis of β's, there could have been more terms multiplying a given β, but at the end the divergences still cancel due to the $U(1)$ decoupling identity.

We have also directly confirmed the vanishing of the divergences in $D = 4, 6$, and computed the nonvanishing divergence of half-maximal supergravity in $D = 8$, using the procedure in ref. [8]. In this procedure we take one copy to be maximal $Q = 16$ super-Yang-Mills theory and the other copy pure nonsupersymmetric Yang-Mills theory. From the double-copy formula (2.15), we have

$$\mathcal{M}^{(1)}_{Q=16}(1,2,3,4,5) = -\left(\frac{\kappa}{2}\right)^5 \sum_{\mathcal{S}_5} \left(\frac{1}{10}\beta_{12345} \int \frac{d^D p}{(2\pi)^D} \frac{n_{12345}}{\prod_{\alpha_j} p^2_{\alpha_j}} + \frac{1}{4}\gamma_{12} \int \frac{d^D p}{(2\pi)^D} \frac{n_{[12]345}}{\prod_{\alpha_j} p^2_{\alpha_j}} \right). \tag{3.32}$$

Here n_{12345} and $n_{[12]345}$ are numerators of pure Yang-Mills pentagon (shown in Fig. 1(b)) and box diagrams (shown in Fig. 1(c)) respectively, derived from Feynman diagrams in Feynman gauge. As described in ref. [8], the derived numerators include ghost contributions and contributions from four-point contact terms assigned according to their color factors. The β_{12345} given in eq. (2.16) and γ_{12} given in eq. (2.17) are the corresponding pentagon and box numerators of maximal super-Yang-Mills theory. The propagators are those of each graph. The sum $\mathcal{S}_5$ runs over all 5! permutations of the external legs, with symmetry factors included to adjust for the overcount. The symmetry factors for Fig. 1(b) and Fig. 1(c) are 10 and 4 respectively. The expression (3.32) is valid when the external gluons on the super-Yang-Mills side of the double copy are in an MHV configuration in the four-dimensional external subspace. The $\overline{\text{MHV}}$ configuration is obtained using parity.

Restricting the integrals to the divergent part, we find the divergences in $D = 4, 6$ to vanish, as was the case at four points. In $D = 8$ we find a nonvanishing divergence, the

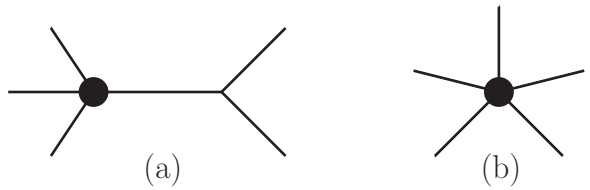

FIG. 4: The counterterm diagrams describing the one-loop divergences of either pure Yang-Mills theory or half-maximal supergravity in $D = 8$. The large dots indicate an insertion of a counterterm vertex generated by either an F^4 operator in Yang-Mills theory or an R^4 operator in supergravity.

explicit form of which we have included in an accompanying Mathematica attachment [62]. The first two terms of this expression are

$$\mathcal{M}^{(1)}_{Q=16}(1,2,3,4,5)\Big|_{D=8\,\text{div.}} = \frac{1}{(4\pi)^4}\Big(\frac{\kappa}{2}\Big)^5\Big[\frac{238+D_s}{180\sqrt{2}\epsilon}\gamma_{34}\,\varepsilon_1\cdot\varepsilon_4 k_1\cdot\varepsilon_2 k_1\cdot\varepsilon_5 k_2\cdot\varepsilon_3 \\ - \frac{D_s-122}{180\sqrt{2}\epsilon}\gamma_{14}\frac{\varepsilon_1\cdot\varepsilon_2\varepsilon_4\cdot\varepsilon_5 k_1\cdot\varepsilon_3 s_{23}s_{24}}{s_{45}}+\cdots\Big], \tag{3.33}$$

where the γ_{ij} are the box numerators defined in eq. (2.17) and the ε_i are gluon polarization vectors. As always the supergravity states are simply tensor products of the maximal super-Yang-Mills states with those of pure Yang-Mills theory.

As a nontrivial check, we have reproduced the $D = 8$ result in an additional way, which we briefly summarize. We used the Yang-Mills F^4 operator in eq. (3.13) to obtain the five-point pure Yang-Mills divergence using the Feynman diagrams illustrated in Fig. 4. Plugging the color-ordered Yang-Mills divergences into eq. (2.15) yields the gravity divergence:

$$\mathcal{M}^{(1)}_{Q=16}(1,2,3,4,5)\Big|_{D=8\,\text{div.}} = i\Big(\frac{\kappa}{2}\Big)^5 \sum_{\mathcal{S}_5/(\mathcal{Z}_5\times\mathcal{Z}_2)} \beta_{12345}\, A^{(1)}_{Q=0}(1,2,3,4,5)\Big|_{D=8\,\text{div.}}. \tag{3.34}$$

Remarkably, this suggests that the entire five-loop divergence in $D = 8$ for half-maximal supergravity is contained in the operators that describe four-point divergences and that no further independent operators should appear at five points.

As a first test of this, we used the R^4 counterterm as determined at four points to compute the five-graviton divergence, again using diagrams of the form shown in Fig. 4. We used both forms of the counterterm (eq. (3.22) and eq. (3.27)); in both cases we find agreement with eq. (3.34) for four-dimensional external graviton states. For cases with other external states, we suspect again all two-loop supergravity divergences are locked to the four-point

divergences given that no new independent five- or higher-point counterterms arise in pure Yang-Mills theory (by simple gauge invariance and dimensional considerations). It would be interesting to investigate this further.

C. Comments on the four-point one-loop $\mathcal{N} = 4$ gravity-matter system

The above group theoretic analysis can also be applied to understand the divergence structure of $Q \geq 16$ supergravity with matter. A particularly interesting case is $\mathcal{N} = 4$ supergravity in $D = 4$ coupled to n_v $\mathcal{N} = 4$ vector multiplets. These theories naturally arise from dimensional reduction of half-maximal pure supergravity models in higher dimensions. Over 30 years ago, Fischler showed that this theory is ultraviolet divergent at one loop [63]. This result can be simply understood from the double-copy vantage point.

In the double-copy picture, $\mathcal{N} = 4$ supergravity amplitudes with vector multiplets are constructed using $\mathcal{N} = 4$ super-Yang-Mills amplitudes for one copy and a Yang-Mills theory with adjoint scalars that interact with gluons. In the latter theory, the only allowed interactions of the scalar are the standard minimal interactions with gluons or self interactions via a ϕ^4 operator for the second copy. With either interaction, simple renormalizability constraints in $D = 4$ show that the only gauge-theory operators that can act as counterterms are the form F^2, $(D_\mu \phi)^2$ or ϕ^4. The first two operators generate amplitudes containing only tree-level color tensors, so the divergences satisfy $U(1)$ (3.8) decoupling relations. Hence from eq. (2.9), we immediately have that amplitudes with only supergravity multiplet states on the external lines or with two-graviton and two-vector multiplet states are finite irrespective of the number of vector multiplets. However, one-loop four-point amplitudes where all external legs belong to the matter multiplet are different. In the scalar-Yang-Mills system, the four-scalar counterterm operator with a one-loop color tensor of the form,

$$c^{abcd} \phi^a \phi^b \phi^c \phi^d \,, \tag{3.35}$$

is allowed, where c^{abcd} is defined in eq. (3.14). Here the generated divergence does not satisfy $U(1)$ decoupling, and when fed through eq. (2.9), the corresponding supergravity amplitude diverges. Indeed this is consistent with the divergence in the four-matter-multiplet amplitude found long ago by Fischler [63]. The same conclusion was also reached in ref. [64] with a corrected overall constant.

The case of $D = 6$ is a bit different. Here a divergence for the two-matter two-gravity matrix element appears. The presence of this supergravity divergence can be understood from the double-copy viewpoint as originating from a counterterm of the nonsupersymmetric scalar-Yang-Mills system:

$$c^{abcd} F^a{}_{\mu\nu} F^{b\mu\nu} \phi^c \phi^d \,. \tag{3.36}$$

In the double-copy formula, when this is combined with maximally supersymmetric Yang-Mills theory, we obtain a nonvanishing two-graviphoton and two-matter-photon counterterm of the form D^2F^4. This is related by supersymmetry to the two-graviton two-matter-photon counterterm R^2F^2. While we have not explicitly computed this divergence, it would be an interesting exercise to do so.

IV. HALF-MAXIMAL SUPERGRAVITY AT TWO LOOPS

We now turn to the main topic of this paper, which is the divergence structure of half-maximal supergravity at two loops. We follow similar reasoning as for the cases of $D = 4, 6$ at one loop. In particular, we demonstrate that the same cancellations that prevent forbidden color structures from appearing in pure Yang-Mills divergences are responsible for making the half-maximal pure supergravity two-loop four-point amplitude finite in $D = 5$. On dimensional grounds, we expect the $D = 5$ two-loop four-point counterterm of supergravity to be a supersymmetric completion of an R^4 operator [11, 15]. Nevertheless the corresponding divergence vanishes. We also explicitly demonstrate the ultraviolet finiteness of a subset of five-point amplitudes with external states in a four-dimensional subspace; specifically we look at those amplitudes where the external supergravity states are those obtained as a tensor product of gluon states in the four-dimensional subspace. Besides explaining the lack of a two-loop divergence in these amplitudes in $D = 5$, we also obtain the explicit value of the four-point divergence in $D = 6$.

A. Four-point divergence cancellations at two loops

1. *Group theory considerations*

Ordinary nonsupersymmetric Yang-Mills theory in $D = 5$ is, of course, divergent. At two loops in $D = 5$, the available counterterm in this theory is of the same F^3 form (3.9) as at one loop in $D = 6$. In $D = 5$ there are no one-loop divergences in dimensional regularization, so we do not need to concern ourselves with subdivergences.

Following the same logic as at one loop, we impose the constraint that the F^3 operator generates only the tree-level color structures. Using the color basis described in Appendix B of ref. [53] (see also ref. [65]), we express the color factors in eq. (2.11) in terms of the independent tree and one-loop color tensors given in eqs. (3.4) and (3.5), as well as two additional two-loop color tensors, $b_1^{(2)}$ and $b_2^{(2)}$. For the planar color factors we have

$$\begin{aligned}
c^{\rm P}_{1234} &= b_1^{(2)}\,, & c^{\rm P}_{3214} &= b_2^{(2)}\,,\\
c^{\rm P}_{3421} &= b_1^{(2)} - \frac{1}{4}C_A^2 b_1^{(0)}\,, & c^{\rm P}_{1423} &= b_2^{(2)} - \frac{1}{4}C_A^2 b_2^{(0)}\,,\\
c^{\rm P}_{1342} &= -b_1^{(2)} - b_2^{(2)} + \frac{3}{2}C_A b_1^{(1)} - \frac{1}{4}C_A^2 b_1^{(0)}\,,\\
c^{\rm P}_{3142} &= -b_1^{(2)} - b_2^{(2)} + \frac{3}{2}C_A b_1^{(1)} - \frac{1}{4}C_A^2 b_2^{(0)}\,.
\end{aligned} \tag{4.1}$$

Similarly for the nonplanar color factors we have

$$\begin{aligned}
c^{\rm NP}_{1234} &= c^{\rm P}_{1234} - \frac{1}{2}C_A b_1^{(1)}\,,\\
c^{\rm NP}_{3214} &= c^{\rm P}_{3214} - \frac{1}{2}C_A b_1^{(1)}\,,\\
c^{\rm NP}_{3421} &= c^{\rm P}_{3421} - \frac{1}{2}C_A b_1^{(1)} - \frac{1}{4}C_A^2 b_1^{(0)}\,,\\
c^{\rm NP}_{1423} &= c^{\rm P}_{1423} - \frac{1}{2}C_A b_1^{(1)} - \frac{1}{4}C_A^2 b_2^{(0)}\,,\\
c^{\rm NP}_{1342} &= c^{\rm P}_{1342} - \frac{1}{2}C_A b_1^{(1)} - \frac{1}{4}C_A^2 b_1^{(0)}\,,\\
c^{\rm NP}_{3142} &= c^{\rm P}_{3142} - \frac{1}{2}C_A b_1^{(1)} - \frac{1}{4}C_A^2 b_2^{(0)}\,.
\end{aligned} \tag{4.2}$$

Inserting these into the gauge-theory amplitude (2.11) and demanding that the divergent parts cannot have the two-loop tensor structures $b_1^{(2)}$ and $b_2^{(2)}$, we find constraints that must be satisfied by the divergent parts:

$$0 \;=\; t(A_Q^{\rm P}(1,3,4,2) + A_Q^{\rm P}(1,4,2,3) + A_Q^{\rm P}(3,1,4,2) + A_Q^{\rm P}(3,2,1,4)$$

$$
\begin{aligned}
&\quad + A_Q^{\rm NP}(1,3,4,2) + A_Q^{\rm NP}(1,4,2,3) + A_Q^{\rm NP}(3,1,4,2) + A_Q^{\rm NP}(3,2,1,4)) \\
&\quad + s(A_Q^{\rm P}(1,3,4,2) + A_Q^{\rm P}(3,1,4,2) + A_Q^{\rm NP}(1,3,4,2) + A_Q^{\rm NP}(3,1,4,2))\Big|_{D=5\,\text{div.}}\,, \\
0 \;=\; & s(A_Q^{\rm P}(1,2,3,4) + A_Q^{\rm P}(1,3,4,2) + A_Q^{\rm P}(3,1,4,2) + A_Q^{\rm P}(3,4,2,1) \\
&\quad + A_Q^{\rm NP}(1,2,3,4) + A_Q^{\rm NP}(1,3,4,2) + A_Q^{\rm NP}(3,1,4,2) + A_Q^{\rm NP}(3,4,2,1)) \\
&\quad + t(A_Q^{\rm P}(1,3,4,2) + A_Q^{\rm P}(3,1,4,2) + A_Q^{\rm NP}(1,3,4,2) + A_Q^{\rm NP}(3,1,4,2))\Big|_{D=5\,\text{div.}}\,.
\end{aligned}
\tag{4.3}
$$

Solving this system for the divergent parts of two of the partial amplitudes and plugging the solution into the supergravity expression (2.13), we immediately find that the corresponding two-loop supergravity divergence in $D = 5$ vanishes:

$$
\mathcal{M}^{(2)}_{16+Q}(1,2,3,4)\Big|_{D=5\,\text{div.}} = 0\,. \tag{4.4}
$$

It is interesting that there is no need to impose the vanishing of the contribution proportional to the one-loop color tensor $b_1^{(1)}$ to deduce this. This demonstrates that the cancellations that eliminate the $D = 5$ divergence in the two-loop four-point amplitude of half-maximal pure supergravity are identical to the ones that eliminate forbidden color tensors from the corresponding nonsupersymmetric pure Yang-Mills divergences. For supergravity theories with more than 16 supercharges, not only does the divergence vanish for this reason, but it also vanishes because the F^3 operator (3.9) in the corresponding super-Yang-Mills theory is no longer a valid counterterm.

In $D = 6$, pure Yang-Mills has a two-loop divergence described by an F^4 operator containing color factors not appearing at tree level. (See eq. (3.13), but also containing a two-loop color tensor.) Feeding the F^4 counterterm of pure Yang-Mills theory into the double-copy formula (2.13) immediately shows that half-maximal supergravity diverges in $D = 6$. Below we compute the explicit value of this divergence.

2. Explicit cancellations in $D = 5$

We can see the supergravity divergence cancellation more directly in a four-dimensional external subspace starting with the explicit values of the $D = 5$ pure Yang-Mills divergences computed in Appendix A for identical external helicity states. For Yang-Mills this external helicity configuration is sufficient because it detects the divergence generated by the F^3

operator. We note that the $(-+++)$ external helicity configuration is also divergent, but not the $(--++)$ case. This is because the allowed F^3 counterterm cannot generate the latter helicity configuration. The fact that the $D = 5$ pure Yang-Mills amplitude with helicities $(--++)$ in the four-dimensional subspace does not diverge at two loops immediately tells us that four-graviton amplitudes in the four-dimensional subspace must also be finite: the $(\pm+++)$ graviton amplitude vanishes due to supersymmetric Ward identities [55], while the $(--++)$ graviton amplitude is finite due to the lack of the corresponding Yang-Mills divergence. On the other hand, the presence of $(-+++)$ or $(++++)$ pure Yang-Mills divergences implies possible divergences in the supergravity theory with one or two external scalars unless there are additional cancellations beyond these helicity arguments, which, in fact, are present, as described above.

To explicitly see these additional cancellations in the four-dimensional external subspace, we use the results for the planar and nonplanar contributions to the divergence given in the appendix,

$$\begin{aligned} A^{\rm P}(1^+,2^+,3^+,4^+)\Big|_{D=5\,\rm div.} &= -i\,\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle} s\,(D_s-2)\frac{\pi}{70\epsilon}\frac{1}{(4\pi)^5}\,, \\ A^{\rm NP}(1^+,2^+,3^+,4^+)\Big|_{D=5\,\rm div.} &= i\,\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle} s\,(D_s-2)\frac{\pi}{70\epsilon}\frac{1}{(4\pi)^5}\,, \end{aligned} \tag{4.5}$$

where we take the state-counting parameter to be $D_s = 5$ for the pure Yang-Mills theory.

Plugging the above result back into the two-loop gravity amplitude (2.13), we immediately see that the divergences in the nonplanar contributions cancel with those in the planar contributions,

$$\begin{aligned} \mathcal{M}^{(2)}(1^+,2^+,3^+,4^+)\Big|_{D=5\,\rm div.} &= (D_s-2)\frac{\pi}{70\epsilon}\frac{1}{(4\pi)^5}\Big(\frac{\kappa}{2}\Big)^6 s t A^{\rm tree}_{Q=16}(1,2,3,4) \\ &\qquad \times\Big[s^2\,\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle}\,(1-1+1-1)+\text{cyclic}(2,3,4)\Big] \\ &= 0\,, \end{aligned} \tag{4.6}$$

valid for any external states in the graviton multiplet that are a tensor product of the states of the $\mathcal{N} = 4$ super-Yang-Mills multiplet and identical-helicity gluons. This explicit cancellation highlights the fact that supergravity can be less divergent than the component gauge-theory amplitudes because of cancellations between planar and nonplanar contributions.

B. Two loops and four points in $D = 6$

In $D = 6$, on dimensional grounds one expects an F^4 counterterm in pure Yang-Mills theory of the form in eq. (3.13), but with two-loop color tensors. As for the one-loop $D = 8$ case, the appearance of multiloop color tensors in the gauge-theory divergence implies that the corresponding supergravity divergences will not cancel.

In order to obtain the explicit value of the divergences, we follow the same procedure as carried out in ref. [8] for three-loop $\mathcal{N} = 4$ supergravity in $D = 4$. The ultraviolet divergences are then extracted by expanding in external momenta and integrating, while all subdivergences are subtracted integral by integral.

This construction yields the explicit form of the two-loop four-point divergence for any external states in the graviton multiplet,

$$\begin{aligned}\mathcal{M}^{(2)}(1,2,3,4)\Big|_{D=6\,\text{div.}} = \frac{1}{(4\pi)^6}\left(\frac{\kappa}{2}\right)^6 stA^{\text{tree}}_{Q=16}(1,2,3,4)\Big\{\Big(\frac{(D_s-6)(26-D_s)}{576\epsilon^2}+\frac{(19D_s-734)}{864\epsilon}\Big)\\ \times\Big[s\,(F_1F_2)(F_3F_4)+t\,(F_1F_4)(F_2F_3)+u\,(F_1F_3)(F_2F_4)\Big]\\ +\frac{(48D_s-1248)}{864\epsilon}\Big[u\,(F_1F_2F_3F_4)+t\,(F_1F_3F_4F_2)+s\,(F_1F_4F_2F_3)\Big]\Big\}\,,\end{aligned} \tag{4.7}$$

including the subtraction of one-loop subdivergences that appear for $D_s \neq 6$. These subdivergences come from extra states that circulate in the loop when $D_s \neq 6$. For pure half-maximal supergravity (where the state-counting parameter is $D_s = 6$), the $1/\epsilon^2$ divergence vanishes as expected since, as discussed in Section III, there are no one-loop subdivergences in pure half-maximal supergravity.

We can simplify the expression for the divergences in a four-dimensional external subspace using spinor helicity. For example, for four external gravitons with helicity configuration $(--++)$ we have

$$\mathcal{M}^{(2)}(1^-,2^-,3^+,4^+) = -\frac{i}{(4\pi)^6}\left(\frac{\kappa}{2}\right)^6\Big(\frac{(D_s-6)(26-D_s)}{576\epsilon^2}+\frac{19D_s-734}{864\epsilon}\Big)\,s\langle 12\rangle^4[34]^4\,, \tag{4.8}$$

for the one-loop-subtracted result. Among the $(F_iF_j)(F_kF_l)$ terms on the pure Yang-Mills side, only $(F_1F_2)(F_3F_4)$ gives a nonvanishing contribution, while the contributions of the $(F_iF_jF_kF_l)$ terms cancel among themselves. We note that the expression (4.8) has the helicity structure and dimensions of a D^2R^4 counterterm.

C. Two loops and five points in $D = 5$

We now turn our attention to five points. While the previous discussion rules out an R^4 divergence in $D = 5$, one may worry about a counterterm of the form ϕR^4 and its supersymmetric completion, which would lead to a divergence at five points. However, from the SO(1,1) duality symmetry obeyed by half-maximal supergravity in $D = 5$ [66], we know that ϕR^4 is not a valid counterterm because it is not invariant under the $\phi \to \phi + v$ shift symmetry. Nevertheless, it is interesting to see how the potential divergence cancels from the double-copy vantage point.

For the two-loop five-point amplitudes, the numerators of maximal super-Yang-Mills theory depend on loop momenta [5]. This complicates the analysis of the corresponding half-maximal supergravity theory, though it is straightforward to work out the divergences in $D = 5$ following the procedure of ref. [8].

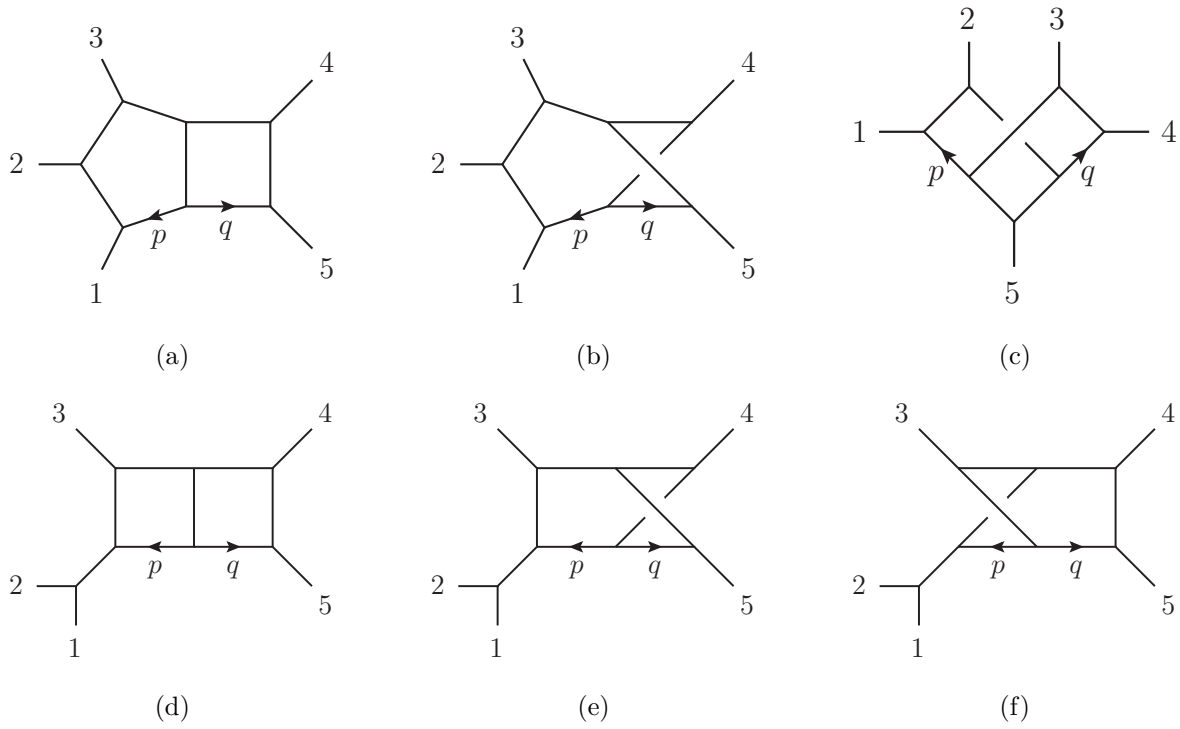

FIG. 5: Diagrams contributing to the five-point two-loop amplitude of maximal super-Yang-Mills theory. From ref. [5].

Once again we employ the double-copy construction (2.6) to obtain the results for pure half-maximal supergravity. In ref. [5], a form of the maximal super-Yang-Mills amplitude that satisfies BCJ duality was found for any internal dimension with the external states

TABLE I: The numerator factors of the graphs in Fig. 5. The first column indicates the integral, the second column the numerator factor for maximal $\mathcal{N} = 4$ super-Yang-Mills five-gluon MHV amplitudes, where the external momenta and states live in a four-dimensional subspace. From ref. [5].

$\mathcal{I}^{(x)}$	maximal super-Yang-Mills numerator
(a),(b)	$\frac{1}{4}\Big(\gamma_{12}(2s_{45} - s_{12} + \tau_{2p} - \tau_{1p}) + \gamma_{23}(s_{45} + 2s_{12} - \tau_{2p} + \tau_{3p}) + 2\gamma_{45}(\tau_{5p} - \tau_{4p}) + \gamma_{13}(s_{12} + s_{45} - \tau_{1p} + \tau_{3p})\Big)$
(c)	$\frac{1}{4}\Big(\gamma_{15}(\tau_{5p} - \tau_{1p}) + \gamma_{25}(s_{12} - \tau_{2p} + \tau_{5p}) + \gamma_{12}(s_{34} + \tau_{2p} - \tau_{1p} + 2s_{15} + 2\tau_{1q} - 2\tau_{2q}) + \gamma_{45}(\tau_{4q} - \tau_{5q}) - \gamma_{35}(s_{34} - \tau_{3q} + \tau_{5q}) + \gamma_{34}(s_{12} + \tau_{3q} - \tau_{4q} + 2s_{45} + 2\tau_{4p} - 2\tau_{3p})\Big)$
(d)-(f)	$\gamma_{12}s_{45} - \frac{1}{4}\Big(2\gamma_{12} + \gamma_{13} - \gamma_{23}\Big)s_{12}$

restricted to a four-dimensional subspace. We employ this here for pure gluon amplitudes. In the double copy this gives us access to all states obtained by tensoring two gluon states in the subspace. The graphs with nonvanishing numerators for maximal super-Yang-Mills are shown in Fig. 5, and the corresponding numerators are in Table I [5]. Since there is no need to have a BCJ form in the second copy, we follow ref. [8] and use ordinary Feynman-gauge Feynman diagrams on the pure Yang-Mills side to generate a set of suitable numerators. (See ref. [8] for a description of this procedure.) While using Feynman diagrams as a starting point is not efficient, enormous simplifications arise from the fact that we do not need contributions corresponding to those with vanishing numerators on the maximal super-Yang-Mills side. Unlike the cases covered earlier, the maximal ($Q = 16$) super-Yang-Mills two-loop five-point numerators contain loop momenta and therefore cannot be pulled out of the integral.

A generic integral for a graph in Fig. 5 is of the form,

$$I^{(x)} = \int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{n^{(x)}_{Q=16}(1,2,3,4,5;p,q) n^{(x)}_{Q=0}(1,2,3,4,5;p,q)}{\prod_{\alpha_{(x)}} l^2_{\alpha_{(x)}}}, \tag{4.9}$$

where $n_{Q=16}$ denotes the maximal super-Yang-Mills numerators specified in Table I and $n_{Q=0}$ the pure Yang-Mills numerator found via Feynman rules. Including the symmetry factors,

Graph	(divergence)/$(i\gamma_{12}\,\varepsilon_1\cdot\varepsilon_3\,\varepsilon_4\cdot\varepsilon_5\,k_1\cdot\varepsilon_2\,s_{12}/(4\pi)^5)$
(a)	$\frac{-64497+925D_s}{362880\sqrt{2}}\frac{1}{\epsilon}$
(b)	$\frac{820641-149788D_s}{1451520\sqrt{2}}\frac{1}{\epsilon}$
(c)	$\frac{-27555+8116D_s}{80640\sqrt{2}}\frac{1}{\epsilon}$
(d)	$\left(\frac{20605+912D_s}{53760\sqrt{2}}+\frac{-38+D_s}{240\sqrt{2}}\frac{s_{14}}{s_{13}}+\frac{655-161D_s}{1680\sqrt{2}}\frac{s_{23}}{s_{13}}+\frac{-5171-148D_s}{6720\sqrt{2}}\frac{s_{24}}{s_{13}}\right)\frac{1}{\epsilon}$
(e)	$\left(\frac{-71986+4511D_s}{241920\sqrt{2}}+\frac{935+6D_s}{6720\sqrt{2}}\frac{s_{14}}{s_{13}}+\frac{-907+342D_s}{6720\sqrt{2}}\frac{s_{23}}{s_{13}}+\frac{27859+844D_s}{60480\sqrt{2}}\frac{s_{24}}{s_{13}}\right)\frac{1}{\epsilon}$
(f)	$\left(\frac{-31847-8615D_s}{241920\sqrt{2}}+\frac{129-34D_s}{6720\sqrt{2}}\frac{s_{14}}{s_{13}}+\frac{-1713+302D_s}{6720\sqrt{2}}\frac{s_{23}}{s_{13}}+\frac{2335+61D_s}{7560\sqrt{2}}\frac{s_{24}}{s_{13}}\right)\frac{1}{\epsilon}$

TABLE II: The graph-by-graph divergent coefficients of the term containing the factor $i\gamma_{12}\,\epsilon_1\cdot\epsilon_3\,\epsilon_4\cdot\epsilon_5\,k_1\cdot\epsilon_2\,s_{12}/(4\pi)^5$ for the two-loop five-point half-maximal supergravity amplitude in $D=5$. As discussed in the text we have reduced each expression to a set of terms independent under momentum conservation and spinor identities. Each expression in the table includes a permutation sum over external legs, with the symmetry factor appropriate to the indicated graph. The sum of contributions over all graphs vanishes for any value of the state-counting parameter D_s; all other divergent terms amplitude similarly cancel.

the gravity amplitude is then given by

$$\mathcal{M}^{(2)}_{Q+16}(1,2,3,4,5) = -i\left(\frac{\kappa}{2}\right)^7\sum_{\mathcal{S}_5}\left(\frac{1}{2}I^{(\rm a)}+\frac{1}{4}I^{(\rm b)}+\frac{1}{4}I^{(\rm c)}+\frac{1}{2}I^{(\rm d)}+\frac{1}{4}I^{(\rm e)}+\frac{1}{4}I^{(\rm f)}\right), \tag{4.10}$$

where the sum $\mathcal{S}_5$ is over all permutations of external legs.

We carry out the extraction of the potential ultraviolet divergences exactly as in ref. [8], to which we refer the reader. In brief, we extract the ultraviolet divergences by expanding the external momenta [28], as has been recently carried out in various determinations of ultraviolet divergences in super-Yang-Mills theory and supergravity [4, 6, 8, 27, 53, 67, 68]. The resulting vacuum integrals are reduced to a basis using FIRE [69], giving integrals that are straightforward to evaluate. In $D=5$, there are no subdivergences to subtract, simplifying the construction compared to ref. [8].

As was the case at four points, we find the divergence to vanish:

$$\mathcal{M}^{(2)}_{Q=16}(1,2,3,4,5)\Big|_{D=5\,\rm div.} = 0\,. \tag{4.11}$$

This result is valid for any states obtained by tensoring a pair of gluon states restricted

to a four-dimensional subspace. The cancellation of the divergence between graphs for one independent term is shown in Table II. Each row gives the divergent coefficient of the term $i\gamma_{12}\,\varepsilon_1\cdot\varepsilon_3\,\varepsilon_4\cdot\varepsilon_5\,k_1\cdot\varepsilon_2\,s_{12}/(4\pi)^5$ from the indicated graph in Fig. 5. This includes the sum over permutations of external legs. We have applied momentum conservation as well as taken a basis of six γ_{ij}. Our choice is to eliminate k_5 via

$$k_5\cdot\varepsilon_i = -(k_1+k_2+k_3+k_4)\cdot\varepsilon_i\,, \qquad k_4\cdot\epsilon_5 = -(k_1+k_2+k_3)\cdot\epsilon_5\,, \qquad k_i\cdot\varepsilon_i = 0\,. \tag{4.12}$$

We use the five independent Mandelstam invariants $s_{12}, s_{13}, s_{14}, s_{23}$ and s_{24}. The six independent numerator factors are $\gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{23}, \gamma_{24}$ and γ_{34}. This gives a total of thirty monomials $\gamma_{ij}s_{kl}$; however, as explained in ref. [5], there are actually only twenty-five independent ones due to nontrivial additional relations amongst them. We have used this fact to eliminate the following monomials from our graph-by-graph results:

$$\gamma_{12}s_{14}\,, \qquad \gamma_{12}s_{23}\,, \qquad \gamma_{13}s_{12}\,, \qquad \gamma_{13}s_{13}\,, \qquad \gamma_{34}s_{24}\,. \tag{4.13}$$

After reducing to this basis (or any similar one), all divergences completely cancel in a manner similar to the cancellation obtained by summing the contributions in Table II. It is interesting that this cancellation is independent of the state-counting parameter D_s.

V. CONCLUSIONS AND OUTLOOK

In a previous paper [8], we proved that at three loops in $\mathcal{N}=4$ supergravity an R^4 counterterm—valid under all currently known supersymmetry and duality constraints [9]—has vanishing coefficient. In the present paper, we analyzed the simpler two-loop case of pure half-maximal supergravity in $D=5$, which has a valid counterterm under all known supersymmetry and duality constraints. However, using the double-copy structure, we showed that the corresponding divergences completely cancel. Indeed we found that there are no four-point divergences in $D<8$ at one loop and in $D<6$ at two loops, and we linked these cancellations to similar ones occurring in corresponding pure Yang-Mills amplitudes that prevent forbidden color structures from appearing in divergences. We also reached the same conclusions for the five-point amplitudes that we analyzed at one and two loops. This is consistent with previous explicit calculations showing that ultraviolet divergences of supergravity theories can bear a strong resemblance to those of corresponding gauge theories, not only in their general structure but in their details [6, 53].

For the half-maximal supergravity one- and two-loop four- and five-point cases studied here, when divergences of the corresponding pure-Yang-Mills amplitudes contain color structures other than the tree ones, then the supergravity amplitudes also diverge. In $D = 8$ and at two loops in $D = 6$ the pure Yang-Mills divergences have such color factors so the half-maximal supergravity amplitudes also diverge. In lower dimensions, only tree color tensors appear, so the corresponding supergravity amplitudes are finite. Using the double-copy formula we also presented explicit expressions for the valid supergravity divergences in terms of Yang-Mills ones.

The above results are suggestive of a strong link between the divergences of the two theories when the number of loops or legs increases. With large numbers of loops or legs, loop momenta can appear in both gauge-theory numerator factors of certain diagrams in the double-copy formula. This makes it is more difficult to directly tie the integrated divergence properties of supergravity theories to gauge theories. Nevertheless, it is rather striking that the finiteness of the three-loop four-point $\mathcal{N} = 4$ supergravity amplitude [8] is correlated with the lack of multiloop color tensors in the corresponding pure Yang-Mills divergences, suggesting a general pattern. Similarly, we found nontrivial cancellations in $D = 5$ five-point two-loop amplitudes of half-maximal supergravity, even though both gauge-theory copies have loop momenta in their numerators. An obvious conjecture is that the pattern continues to higher loops, with divergences possible in $(Q + 16)$-supercharge supergravity only when the divergences of corresponding Q-supercharge gauge theory contain independent color tensors other than tree ones. In $D = 4$ this would suggest ultraviolet finiteness of pure $\mathcal{N} \geq 4$ supergravity.

In order to test this and to guide future studies, it is, of course, crucially important to carry out further explicit studies of divergences with larger numbers of loops or legs. In particular, a computation of the five-loop four-point divergence in $\mathcal{N} = 8$ supergravity should be within reach [27], now that the corresponding $\mathcal{N} = 4$ super-Yang-Mills integrand has been obtained [27] (although not in a BCJ format). The calculation of the four-loop four-point divergence of $\mathcal{N} = 4$ supergravity in $D = 4$ is also doable with the procedure of ref. [8] since the BCJ form of the corresponding $\mathcal{N} = 4$ super-Yang-Mills amplitude required by the double-copy formula is known [6].

There are a number of other obvious directions for future research. A key issue is to find the extent to which supersymmetry and duality symmetries by themselves can be used

to place restrictions on counterterms corresponding to the results described here. Very interestingly, the potential two-loop four-point $D = 5$ counterterm does appear to be a duality satisfying full superspace integral of a density (which itself is not duality invariant) so such an explanation would be nontrivial [15]. It would be interesting to see if any of the recent developments in tree-level gravity amplitudes [70] can shed any light on the nontrivial ultraviolet cancellations we see at loop level.

In summary, in this paper we linked the divergences of half-maximal supergravity to those of pure Yang-Mills theory. In particular, for the $D = 5$ two-loop four-point amplitudes of half-maximal supergravity, the divergences vanish via the same cancellations that remove forbidden color factors from the divergences of corresponding pure Yang-Mills amplitudes. This case was particularly simple to analyze because the maximal super-Yang-Mills numerators used in the double-copy construction are independent of loop momenta. The next challenge is to fully unravel the ultraviolet cancellations implied by the double-copy structure at higher-loop orders.

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Appendix A: Two-loop pure Yang-Mills divergence in $D = 5$

In this appendix, we explicitly compute the $D = 5$ divergence of the two-loop pure Yang-Mills four-point amplitude. The counterterm in this case is the F^3 operator (3.9).

To simplify the analysis we restrict ourselves to a four-dimensional external subspace. In this subspace, the operator generates nonvanishing contributions to the $(++++)$ helicity states. The all-plus helicity two-loop integrand in Yang-Mills was given in ref. [71] in a form valid for arbitrary internal dimensions. Here we integrate this expression in $D = 5 - 2\epsilon$

to obtain the explicit form of the ultraviolet divergence. We then use this expression to explicitly confirm our more general discussion of the cancellations of the divergences in $D = 5$ half-maximal supergravity.

The unintegrated form of the pure Yang-Mills amplitude with identical external helicities is [71]

$$\begin{aligned} A^{\rm P}(1^+,2^+,3^+,4^+) \;&=\; i\,\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle}\Big\{s\,\mathcal{I}_4^{\rm P}(s,t) + 4(D_s-2)\,\mathcal{I}_4^{\text{bow-tie}}[(\lambda_p^2+\lambda_q^2)\,(\lambda_p\cdot\lambda_q)](s) \\ &\qquad\qquad + \frac{(D_s-2)^2}{s}\,\mathcal{I}_4^{\text{bow-tie}}\Big[\lambda_p^2\,\lambda_q^2\,((p+q)^2+s)\Big](s,t)\Big\}\,, \\ A^{\rm NP}(1^+,2^+,3^+,4^+) \;&=\; i\,\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle}\,s\,\mathcal{I}_4^{\rm NP}(s,t)\,, \end{aligned} \tag{A1}$$

where D_s is the state-counting parameter [58]. In pure half-maximal supergravity we take $D_s = 5$. Here, the external kinematics are four-dimensional, while the loop momenta are in $D = 5 - 2\epsilon$, and (λ_p, λ_q) are the $(D-4)$-dimensional components of the two-loop momenta. The planar and nonplanar double-box integrals are defined as

$$\mathcal{I}_4^{\rm P}(s,t) \equiv \int \frac{d^Dp}{(2\pi)^D}\,\frac{d^Dq}{(2\pi)^D}\,\frac{(D_s-2)(\lambda_p^2\,\lambda_q^2+\lambda_p^2\,\lambda_{p+q}^2+\lambda_q^2\,\lambda_{p+q}^2)+16\Big[(\lambda_p\cdot\lambda_q)^2-\lambda_p^2\,\lambda_q^2\Big]}{p^2\,q^2\,(p+q)^2(p-k_1)^2\,(p-k_1-k_2)^2\,(q-k_4)^2\,(q-k_3-k_4)^2}\,,$$

$$\begin{aligned} &\mathcal{I}_4^{\rm NP}(s,t) \\ &\qquad \equiv \int \frac{d^Dp}{(2\pi)^D}\,\frac{d^Dq}{(2\pi)^D}\,\frac{(D_s-2)(\lambda_p^2\,\lambda_q^2+\lambda_p^2\,\lambda_{p+q}^2+\lambda_q^2\,\lambda_{p+q}^2)+16\Big[(\lambda_p\cdot\lambda_q)^2-\lambda_p^2\,\lambda_q^2\Big]}{p^2\,q^2\,(p+q)^2\,(p-k_1)^2\,(q-k_2)^2\,(p+q+k_3)^2\,(p+q+k_3+k_4)^2}\,, \end{aligned} \tag{A2}$$

with corresponding diagrams shown in Fig. 2. The 'bow-tie' double-triangle integrals, displayed in Fig. 6, are defined as

$$\begin{aligned} &\mathcal{I}_4^{\text{bow-tie}}[\mathcal{P}(\lambda_i,p,q,k_i)](s) \\ &\qquad \equiv \int \frac{d^Dp}{(2\pi)^D}\,\frac{d^Dq}{(2\pi)^D}\,\frac{\mathcal{P}(\lambda_i,p,q,k_i)}{p^2\,q^2\,(p-k_1)^2\,(p-k_1-k_2)^2\,(q-k_4)^2\,(q-k_3-k_4)^2}\,. \end{aligned} \tag{A3}$$

We now compute the divergent parts of the integrals. In five dimensions, there are no infrared divergences so all divergences are ultraviolet in nature.

The bow-tie integrals are finite in five dimensions:

$$\mathcal{I}_4^{\text{bow-tie}}[\lambda_p^2\lambda_q^2](s) \;=\; \frac{\pi^3 s}{576}\,\frac{1}{(4\pi)^5}\,,$$

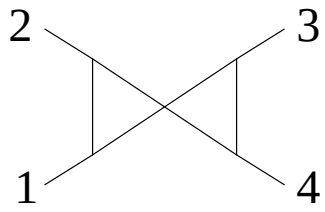

FIG. 6: The bow-tie integral.

$$\begin{aligned}\mathcal{I}_4^{\text{bow-tie}}[\lambda_p^2\lambda_q^2(p+q)^2](s,t) &= \frac{\pi^3 s(2t-15s)}{18432}\frac{1}{(4\pi)^5}\,,\\ \mathcal{I}_4^{\text{bow-tie}}[\lambda_p^2\,(\lambda_p\cdot\lambda_q)](s) &= 0\,,\end{aligned} \tag{A4}$$

thus the ultraviolet divergence comes solely from the double-box integrals.

Using Schwinger parameters, we write the planar double-box integral in eq. (A2) with constant numerator as

$$\mathcal{I}_4^{\text{P}}[1](s,t) = \prod_{i=1}^{7}\int_0^\infty dt_i\,[\Delta_{\text{P}}(T)]^{-\frac{D}{2}}\exp\left[-\frac{Q_{\text{P}}(s,t,t_i)}{\Delta_{\text{P}}(T)}\right]\,, \tag{A5}$$

where

$$\Delta_{\text{P}}(T) = (T_pT_q + T_pT_{pq} + T_qT_{pq})\,, \tag{A6}$$

with

$$T_p = t_3+t_4+t_5,\quad T_q = t_1+t_2+t_7,\quad T_{pq} = t_6\,. \tag{A7}$$

As the subscripts indicate, T_p, T_q and T_{pq} are the sum of Schwinger parameters whose corresponding propagators contain loop momenta p, q and $p+q$ respectively. Finally, we also have

$$Q_{\text{P}}(s,t,t_i) = -s\left(t_1t_2T_p + t_3t_4T_q + t_6(t_1+t_3)(t_2+t_4)\right) - t\,t_5t_6t_7\,. \tag{A8}$$

The effects of λ_p^2, λ_q^2 and λ_{p+q}^2 in the numerators are derived by taking derivatives on

$$\int d\lambda_p^{1-2\epsilon}d\lambda_q^{1-2\epsilon}\exp\left[-T_p\lambda_p^2 - T_q\lambda_q^2 - T_{pq}\lambda_{p+q}^2\right] \propto [\Delta_{\text{P}}(T)]^{-\frac{1}{2}+\epsilon}\,, \tag{A9}$$

with respect to T_p, T_q and T_{pq}. This leads to the following extra factors, for example, to be inserted in the integrand of eq. (A5),

$$\begin{aligned}\lambda_p^4 &\to (\epsilon-\frac{1}{2})(\epsilon-\frac{3}{2})\frac{(T_{pq}+T_q)^2}{\Delta_{\text{P}}^2(T)}\,,\\ \lambda_p^2\lambda_{p+q}^2 &\to \frac{(\epsilon-\frac{1}{2})^2}{\Delta_{\text{P}}(T)} + \frac{(\epsilon-\frac{1}{2})(\epsilon-\frac{3}{2})T_q^2}{\Delta_{\text{P}}^2(T)}\,.\end{aligned} \tag{A10}$$

We account for the extra factors of $\Delta_{\rm P}^a(T)$ by shifting the dimension $D \to D - 2a$. We now change six of the Schwinger parameters to Feynman parameters such that the delta-function constraint on the Feynman parameters is $\sum_{i\neq 6} \alpha_i = 1$. We then have

$$\mathcal{I}_4^{\rm P}[\mathcal{P}(\lambda_p,\lambda_q)](s,t) = \Gamma[7-D+\gamma] \int_0^\infty d\alpha_6 \prod_{i\neq 6} \int_0^1 d\alpha_i \delta\left(1 - \sum_{i\neq 6} \alpha_i\right) \frac{[\Delta_{\rm P}(T)]^{7-\frac{3D}{2}+\gamma}}{[Q_{\rm P}(s,t,\alpha_i)]^{7-D+\gamma}} D(\alpha_i)\,, \tag{A11}$$

where $D(\alpha_i)$ are the extra factors in eq. (A10), with $t_i \to \alpha_i$. If the extra factors in eq. (A10) depend on T_p, T_q and T_{pq}, then $\gamma = 2$; otherwise, we have $\gamma = 0$. Following Smirnov [72], we perform a change of variables that imposes the delta-function constraint:

$$\begin{aligned} &\alpha_1 = \beta_1\xi_3\,, \quad \alpha_2 = (1-\xi_5)(1-\xi_4)\,, \quad \alpha_3 = \beta_2\xi_1\,, \quad \alpha_4 = \xi_5(1-\xi_2)\,, \\ &\alpha_5 = \beta_2(1-\xi_1)\,, \quad \alpha_7 = \beta_1(1-\xi_3)\,, \quad \beta_1 = (1-\xi_5)\xi_4\,, \quad \beta_2 = \xi_5\xi_2\,. \end{aligned} \tag{A12}$$

The parameters can then be straightforwardly integrated to obtain a Mellin-Barnes representation, and explicit integration gives

$$\begin{aligned} \mathcal{I}_4^{\rm P}[\lambda_p^2\lambda_q^2] &= \frac{\pi}{70\epsilon}\frac{1}{(4\pi)^5} + \mathcal{O}(\epsilon^0)\,, \\ \mathcal{I}_4^{\rm P}[\lambda_p^2\lambda_{p+q}^2] &= -\frac{\pi}{70\epsilon}\frac{1}{(4\pi)^5} + \mathcal{O}(\epsilon^0)\,, \\ \mathcal{I}_4^{\rm P}[\lambda_p^4] &= -\frac{\pi}{70\epsilon}\frac{1}{(4\pi)^5} + \mathcal{O}(\epsilon^0)\,, \\ \mathcal{I}_4^{\rm P}[\lambda_{p+q}^4] &= \mathcal{O}(\epsilon^0)\,. \end{aligned} \tag{A13}$$

Inserting these results into eq. (A1), the all-plus helicity planar amplitude is

$$A^{\rm P}(1^+,2^+,3^+,4^+) = i\,\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle}\Big\{-s\,(D_s-2)\frac{\pi}{70\epsilon}\frac{1}{(4\pi)^5} + \mathcal{O}(\epsilon^0)\Big\}\,. \tag{A14}$$

The evaluation of the nonplanar double-box integrals follows the same steps as the planar ones, with $\Delta_{\rm NP}(T)$ taking the same form as $\Delta_{\rm P}(T)$, but now identifying:

$$T_p = t_1 + t_2, \quad T_q = t_3 + t_4, \quad T_{pq} = t_5 + t_6 + t_7\,. \tag{A15}$$

Similarly, we also have

$$Q_{\rm NP}(s,t,u,t_i) = -s\,(t_1t_3t_5 + t_2t_4t_7 + t_5t_7(T_p+T_q)) - t\,t_2t_3t_6 - u\,t_1t_4t_6\,. \tag{A16}$$

However, here we find it advantageous to change only four Schwinger parameters to Feynman parameters. Performing this change gives

$$\mathcal{I}_4^{\rm NP}[\mathcal{P}(\lambda_p,\lambda_q)] \;=\; \Gamma[7-D+\gamma] \tag{A17}$$

$$\times \prod_{i=5}^{7} \int_0^\infty d\alpha_i \prod_{j=1}^{4} \int_0^1 d\alpha_j \delta\left(1 - \sum_{i=1}^{4} \alpha_i\right) \frac{\left[\Delta_{\rm NP}(T)\right]^{7-\frac{3D}{2}+\gamma}}{\left[Q_{\rm NP}(s,t,u,\alpha_i)\right]^{7-D+\gamma}} D(\alpha_i)\,.$$

The delta-function constraint can be imposed via further redefinition:

$$\alpha_1 = \xi_3(1-\xi_1)\,, \qquad \alpha_2 = \xi_3\xi_1\,, \qquad \alpha_3 = (1-\xi_3)(1-\xi_2)\,, \qquad \alpha_4 = (1-\xi_3)\xi_2\,. \tag{A18}$$

The parameters can once again be straightforwardly integrated, and we arrive at

$$\begin{aligned}
\mathcal{I}_4^{\rm NP}[\lambda_p^2\lambda_q^2] &= -\frac{\pi}{42\epsilon}\frac{1}{(4\pi)^5} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm NP}[\lambda_p^2\lambda_{p+q}^2] &= \frac{2\pi}{105\epsilon}\frac{1}{(4\pi)^5} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm NP}[\lambda_p^4] &= \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm NP}[\lambda_{p+q}^4] &= \frac{\pi}{35\epsilon}\frac{1}{(4\pi)^5} + \mathcal{O}(\epsilon^0)\,.
\end{aligned} \tag{A19}$$

Inserting these results into eq. (A1), the all-plus helicity nonplanar amplitude is given by

$$A^{\rm NP}(1^+,2^+,3^+,4^+) = i\,\frac{[1\,2]\,[3\,4]}{\langle 1\,2\rangle\,\langle 3\,4\rangle}\Big\{s\,(D_s-2)\frac{\pi}{70\epsilon}\frac{1}{(4\pi)^5} + \mathcal{O}(\epsilon^0)\Big\}\,. \tag{A20}$$

We use the results for the two-loop divergences in eqs. (A14) and (A20) in Section IV to explicitly demonstrate the cancellation of the corresponding divergence of $D = 5$ half-maximal supergravity.

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