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Deformed General Relativity and Effective Actions from Loop Quantum Gravity

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Canonical methods can be used to construct effective actions from deformed covariance algebras, as implied by quantum-geometry corrections of loop quantum gravity. To this end, classical constructions are extended systematically to effective constraints of canonical quantum gravity and applied to model systems as well as general metrics, with the following conclusions: (i) Dispersion relations of matter and gravitational waves are deformed in related ways, ensuring a consistent realization of causality. (ii) Inverse-triad corrections modify the classical action in a way clearly distinguishable from curvature effects. In particular, these corrections can be significantly larger than often expected for standard quantum-gravity phenomena. (iii) Finally, holonomy corrections in high-curvature regimes do not signal the evolution from collapse to expansion in a "bounce," but rather the emergence of the universe from Euclidean space at high density. This new version of signature-change cosmology suggests a natural way of posing initial conditions, and a solution to the entropy problem.

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I. INTRODUCTION

A major consequence expected for quantum gravity is the emergence of non-classical space-time structures such as discrete or non-commutative ones. Any such modification by quantum properties deforms the standard notion of covariance and thus gives rise to possible new actions and interaction terms. Developments in this direction are of interest for a fundamental understanding of space and time, and also for potential observations of quantum gravity: Unexpected structures may give rise to new effects, or magnify others. One example is early-universe cosmology. Assuming the classical space-time structure with the usual notion of covariance results in higher-curvature terms in an effective action, and only small quantum corrections are possible, suppressed by factors of $\ell_{\rm P}/\ell_{\mathcal{H}}$ of the Planck length by the Hubble distance. Non-classical space-time structures, on the other hand, can sometimes circumvent such limitations and magnify expected effects compared to what standard higher-curvature terms would deliver (as realized explicitly in [1, 2]).

However, relaxing conditions on covariance in a consistent way is not a straightforward task. Space-time properties such as discreteness or non-commutativity are often obtained at some kind of kinematical quantum level far removed from direct space-time analysis. One may, for instance, look at operators that quantize geometrical quantities such as distances or areas, or analyze the behavior of test particles or, mathematically, test functions on quantum space-time. These concepts are not directly related to the dynamics of space-time itself, and so it is initially not clear what form of deformed covariance principle could be used to formulate dynamics on such modified space-times and to find the possible correction terms analogous to higher-curvature effective actions.

Fortunately, an abstract but powerful substitute exists in canonical formulations: Any generally covariant theory in four space-time dimensions has a gauge algebra of four local generators per space-time point, which serve as constraints on suitable initial values and generate space-time transformations on phase-space functions by canonical transformations. If quantization leads one to modified expressions for these generators, covariance is realized — albeit perhaps deformed — if the generators still obey an algebra of the classical dimension. From the perspective of general gauge theory, the same number of spurious degrees of freedom is then removed by the constraints as classically, and all equations of motion derived for the system are guaranteed to be consistent. The theory is anomaly-free.

More specifically, in generally covariant theories there are three smeared constraints per point labeled by spatial vector fields, the diffeomorphism constraint $D[N^i]$ depending on an arbitrary shift vector N^i, and a fourth one labeled by a function, the Hamiltonian constraint $H[N]$ depending on the lapse N. Classically, these phase-space functions obey the hypersurface-deformation algebra [85]

$$\{D[N^i],D[M^j]\} = D[\mathcal{L}_{M^j}N^i] \tag{1}$$
$$\{H[N],D[N^j]\} = H[\mathcal{L}_{N^j}N] \tag{2}$$
$$\{H[N_1],H[N_2]\} = D[g^{ij}(N_1\partial_jN_2-N_2\partial_jN_1)] \tag{3}$$

with the spatial metric $g_{ij}(x)$. (In this article we denote the metric on a spatial 3-manifold in space-time by $g_{ij}(x)$, and by $\pi^{ij}(x)$ its conjugate momentum, using for the sake of easier comparison the notation of the articles [3, 4] which we will follow closely in some parts. For an overview of canonical methods, the reader is referred to [5].)

Gauge transformations $\delta F=\{F,H[N]+D[N^i]\}$ of a phase-space function F then agree with the changes implied by infinitesimal deformations of the spatial hypersurfaces in space-time. In a passive picture, this gauge transformation agrees with a coordinate change along a space-time vector field ξ^μ with components given in terms of the spatial fields N and N^i (see e.g. [6]). The whole hypersurface-deformation algebra presents a large extension of the local Poincaré algebra, which is recovered for linear N and N^i in a local coordinate patch [7]. A general property of the algebra is that it is largely insensitive to the dynamics of the underlying covariant theory: all higher-curvature theories have constraints obeying the same algebra; see e.g. [8] for an explicit calculation. This uniqueness statement can be reversed if the derivative order of one's theory is constrained to be at most two in the equations of motion, in which case the form of the action (up to the values of Newton's and the cosmological constant) can uniquely be recovered from the hypersurface-deformation algebra [3, 4]. Mimicking the usual tensorial arguments to fix the terms of the Einstein–Hilbert action, the dynamics, to the lowest order of derivatives, is thus uniquely determined by the algebra of constraints.

The algebra itself is rather rigid as well, making it difficult to implement new covariance principles and correction terms other than higher-derivative ones. A new result of recent years, however, is that loop quantum gravity, if it can be consistent at all, gives rise to modified hypersurface-deformation algebras. With different kinds of quantum corrections characteristic of the theory, this has been seen for perturbative inhomogeneity [9–11], in $2+1$ dimensions [12, 13] and in spherically symmetric models [14–16]. Different physical consequences for cosmology [1, 17, 18] and for properties of black holes [19–21] have resulted. As a common form of the modified constraint algebra, one can write

$$\{H_{(\beta)}[N_1], H_{(\beta)}[N_2]\} = D[\beta g^{ij}(N_1\partial_j N_2 - N_2\partial_j N_1)] \quad (4)$$

in terms of a phase-space function $\beta[g_{ij}, \pi^{ij}]$ determined by the quantum corrections considered. Poisson brackets in (1) involving the diffeomorphism constraint remain unmodified (except in the case of [10] which has been superseded by [11]).

That a closed algebra still arises is far from trivial, and shows that general relativity, at least in the models considered, can be deformed consistently. The systems obtained correspond to a more general form of space-time covariance than usually taken into account [86]. In this article, we will assume an algebra of the form (4) and analyze what the possible consequences for action principles are. With action principles at hand, the interpretation of deformed constraint algebras will become more intuitive. Moreover, they provide manifestly covariant (in the deformed sense) formulations of the underlying models of loop quantum gravity from which the quantum corrections have been extracted.

The conclusions we will be able to derive are surprisingly rich: (i) We will obtain a clear separation of some corrections from others. In particular, inverse-triad corrections in loop quantum gravity will play a much more characteristic role than holonomy corrections of the same theory, or higher-curvature corrections of general form. (ii) The dynamics of loop quantum gravity near a space-like classical singularity takes on a specific form in which spatial derivatives become subdominant. A scenario similar to but more generic than the BKL picture follows. (iii) Loop quantum gravity will be seen to give rise to signature change in strong-curvature regimes. This new feature of the theory, overlooked so far in minisuperspace models, gives rise to new and improved cosmological scenarios.

II. OVERVIEW OF DEFORMED CONSTRAINT ALGEBRAS IN LOOP QUANTUM GRAVITY

Canonically, the quantum effects of interacting gravitational theories, often expressed by higher-curvature effective actions, are derived from quantum back-reaction [22]: While expectation values of semiclassical states follow nearly the classical trajectories, additional state parameters such as fluctuations and other moments influence the quantum trajectory. Coupled equations of motion for expectation values and the moments can, in some regimes of adiabatic nature, be reformulated as the usual equations of low-energy effective actions [23].

Obviously, these effects should play a large role for quantum gravity and cosmology. But in addition to the ubiquitous quantum back-reaction (or corrections from loop diagrams in perturbative terms), there are characteristic quantum corrections expected for loop quantum gravity, providing two distinct quantum-geometry effects: (i) higher powers of spatial curvature components (intrinsic and extrinsic) stemming from the appearance of holonomies of the Ashtekar–Barbero connection instead of direct connection components in quantized constraints [24, 25], and (ii) natural cut-off functions of divergences of factors containing inverse components of the densitized triad, arising from spatial discreteness [25, 26]. The first type of quantum-geometry corrections is usually referred to as "holonomy corrections," the second as "inverse-triad corrections" (or, in the context of nearly isotropic cosmology, "inverse-volume corrections"). Both can be expanded as series of corrections by components of *spatial* tensors in the constraints, not by scalar invariants of space-time tensors as one is used to from covariant effective actions. Neither the reconstruction of an action principle from the constraints nor properties of covariance are obvious in such a situation, and the only systematic way to determine such features is an analysis of the constraint algebra. As shown in several model systems so far, the hypersurface-deformation algebra is generically deformed by quantum-geometry. In particular, corrections cannot be written purely as higher-curvature terms added to the Einstein–Hilbert action, as often expected for quantum gravity. One of the main questions to be addressed in this article is what actions and covariance properties could be realized instead [87].

In this section we summarize the models investigated so far for their properties of deformations of the constraint algebra, split into the two types of quantum-geometry corrections. (Quantum back-reaction has not yet been analyzed to completion in this context, but the procedure would follow [23, 27, 28].) The set of models in which consistent deformations have been achieved is quite diverse, but the general form of β appears to be insensitive to model specifications. The constraint algebra therefore displays universal implications for covariant space-time structure.

A. Inverse-triad corrections

In loop quantum gravity, space-time geometry is described by canonical fields A_i^I and E_I^i, a connection related to curvature and the densitized triad, instead of the spatial metric g_{ij} and its momentum π^{ij}. These fields have advantages for a background independent quantization because they can be smeared without reference to an auxiliary metric structure: The connection is integrated along curves e in space to obtain holonomies $h_e(A) = \mathcal{P}\exp(\int_e \tau_I A_i^I \dot{e}^i \mathrm{d}\lambda)$, and the densitized triad, dual to a 2-form, is integrated to fluxes $F_S(E) = \int_S \tau^I E_I^i n_i \mathrm{d}^2 y$ through surfaces S in space. Here, $\tau_I = \frac{1}{2} i\sigma_I$ are generators of su(2), related to the Pauli matrices. The canonical structure $\{A_i^I(x), E_J^j(y)\} = 8\pi\gamma G \delta_i^j \delta_J^I \delta(x,y)$ with the Barbero–Immirzi parameter γ [29, 30] provides a regular relation for $\{h_e(A), F_S(E)\}$ free of delta functions [88].

Holonomies and fluxes are promoted to basic operators of the resulting quantum theory, and they represent the canonical fields in all composite operators such as Hamiltonians. Both types of basic operators imply some form of non-locality because they are integrated rather than pointlike fields. Using holonomies for connection components, moreover, implies that there are higher-order corrections when the exponential is expanded, compared with classical expression which are usually polynomials of degree at most two in the connection. Fluxes also give rise to corrections in addition to their non-locality: They are quantized to operators with discrete spectra, containing zero as an eigenvalue. Such operators are not invertible, and yet an inverse of the densitized triad (or its determinant) is needed to quantize matter Hamiltonians (usually in the kinetic part) and the Hamiltonian constraint. Well-defined operators with inverse densitized triad components as their classical limit do exist [25], but they have strong quantum corrections for small values of the fluxes. Correction functions, obtained from expectation values of inverse-triad operators [31], then primarily depend on the fluxes, or on the densitized triad and the spatial metric. In non-Abelian situations, there can also be some dependence on the connection via higher-order terms [32].

Inverse-triad corrections cannot easily be formulated consistently in homogeneous models, where the rescaling freedom of the scale factor under changes of coordinates may be broken unless one properly refers to underlying discreteness scales. However, with some inhomogeneous input consistent formulations exist [2, 18, 33] and show the importance of these quantum-geometry corrections. Quantization of the dynamics can proceed only if a substitute for the non-existing inverse of an elementary flux $\hat{F}$ is found, which according to [25] is possible by using Poisson-bracket identities. If we write schematically [89] $|F|^{q-1}\mathrm{sgn}(F) = (8\pi G\gamma\delta q)^{-1} i \exp(i\delta A)\{\exp(-i\delta A), |F|^q\}$ with a connection component A, we have an inverse of F on the left-hand side for $q < 1$ while the right-hand side can be quantized without requiring an inverse of F if $q > 0$. The Poisson bracket can straightforwardly be quantized: There is an operator $\hat{F}$ whose positive power $|\hat{F}|^q$ can easily be taken via the spectral decomposition. While loop quantum gravity does not provide an operator for A, it does have well-defined quantizations of "holonomies" $h = \exp(i\delta A)$. Finally, we turn the Poisson bracket into a commutator divided by $i\hbar$, and achieve to quantize $|F|^{q-1}\mathrm{sgn}(F)$ in spite of the non-existence of an inverse of $\hat{F}$.

The resulting operator is well-defined and has an inverse power of F as its classical limit, approached on the part of the spectrum of $\hat{F}$ with large eigenvalues. There are quantization ambiguities which prevent one from finding a unique correction function [34, 35]. The typical form, however, follows from algebraic properties and results in $\widehat{|F|^{q-1}\mathrm{sgn}F} = \left(|\hat{F}+\Delta F|^q - |\hat{F}-\Delta F|^q\right)/2q\Delta F$ with a Planckian $\Delta F \approx \ell_{\mathrm{P}}^2 = \hbar G$. Such corrections with a tiny Planck area may seem small, but $\langle\hat{F}\rangle$ as a fundamental flux or area of a discrete state is typically Planckian as well. For small flux values, characteristic quantum corrections result [36], constituting inverse-triad corrections. We collect inverse-triad effects in a correction function

$$\begin{aligned}\bar{\alpha}(F) &= |F|^{1-q}\mathrm{sgn}F \cdot \langle \widehat{|F|^{q-1}\mathrm{sgn}}F\rangle_F \\ &= |F|^{1-q}\mathrm{sgn}F \frac{|F+\Delta F|^q - |F-\Delta F|^q}{2q\Delta F} + \text{moments}\end{aligned} \qquad (5)$$

up to $\hat{F}$-fluctuations and higher moments, using an expectation value in a state peaked at flux F.

1. *Cosmology*

The most general class of models shown so far to have a consistently deformed constraint algebra is the one of perturbative inhomogeneity around spatially flat Friedmann–Robertson–Walker models [9], including inverse-triad corrections. In this case, $\beta = \bar{\alpha}^2$ in (4) with the background function $\bar{\alpha}$ of inverse-triad corrections depends on the scale factor a. These corrections, as in the full theory, arise because loop quantum cosmology [37–39] quantizes the scale factor, or rather its square $|p| = a^2$ equipped with a sign for spatial orientation, to an operator $\hat{p}$ with discrete, equally spaced spectrum. The spectrum contains zero as an eigenvalue, and therefore $\hat{p}$ has no densely defined inverse.

Apart from their formal derivation, inverse-triad corrections in cosmology are characterized by cut-off effects of classically diverging quantities such as a^{-1}. For degenerate geometries, or near the big-bang singularity of isotropic models, discreteness effects lead to non-divergent quantities when shift-operators $\exp(i\delta A)$ instead of differential operators $-i\hbar\partial/\partial F$ are used in the commutators of inverse-triad operators. For fluxes in isotropic space-times, we write $F = \ell_0^2 a^2$ with the coordinate size ℓ_0 of elementary plaquettes in a regular-lattice discrete state (choosing F to be positive without loss of generality). The cut-off behavior is clearly visible from properties of the ratio

$$\frac{\bar{\alpha}(\ell_0^2 a^2)}{a^{2(1-q)}} = \frac{|\ell_0^2 a^2 + \Delta F|^q - |\ell_0^2 a^2 - \Delta F|^q}{2q\Delta F} \tag{6}$$

which approaches zero (instead of infinity) for $a \to 0$, and asymptotes to the classical expression $1/a^{2(1-q)}$ for $a \gg a_*$ well above a characteristic scale $a_* = \sqrt{\Delta F}/\ell_0$. The latter depends on the discreteness behavior of an underlying state, which is responsible for the quantum correction and the implicit cut-off of divergences associated with $1/a^{2(1-q)}$. Regarding the scaling behavior of a and a_*, the background behavior of inverse-triad corrections, as derived in [31, 36], has been made consistent in inhomogeneous settings in [2, 33]. (The precise form of $\bar{\alpha}$ as a function of phase-space variables will not be important in this article.)

For perturbative inhomogeneities in spatially flat isotropic models, a consistent deformation (4) results at least if $\bar{\alpha}$ is close to one [9]. Once it is ensured that the algebra of constraints closes, several consistency conditions for the correction functions arise. The background behavior of $\bar{\alpha}$ appearing in the gravitational part of the Hamiltonian constraint remains unrestricted, but analogous correction functions in possible matter contributions must be related to it and are no longer completely arbitrary. The case of a scalar field will be discussed in more detail below, Section V C. Moreover, in the perturbative terms by inhomogeneous perturbations of the phase-space fields, there are additional corrections called "counterterms" which are completely fixed by the consistency requirements. They can be understood as determining the dependence of $\bar{\alpha}$ on inhomogeneities going beyond the background behavior which is more straightforward to compute directly from expectation values of inverse-triad operators. Some counterterms also contain additional spatial derivatives compared to the classical terms, which can be interpreted as contributions from a derivative expansion of non-local inverse-triad effects, making use of surface integrations of the densitized triad, or flux operators, in inhomogeneous settings.

2. *Spherical Symmetry*

A second class of models in which inverse-triad corrections have been included consistently, this time non-perturbatively, is spherically symmetric models. Several different cases have been investigated: Poisson Sigma Models [14] (see [40–43] for the classical models) and different versions of Lemaître–Tolman–Bondi models [16, 44]. In these models, it is noteworthy that non-trivial quantum corrections are possible even without any deformation of the constraint algebra, a property which we will discuss in more detail later.

We quote the corrected constraints in terms of triad variables rather than the metric g_{ij} because one of the triad components is directly responsible for the corrections. (In the full theory, by comparison, it is primarily $\det g$ which gives rise to inverse-triad corrections. Because $\det g$ equals the squared determinant of the densitized triad, in the general case it will make no difference what variables we use.) As spherically symmetric phase space variables, with radial coordinate x (not necessarily the area radius) and azimuth angle φ, we then have the radial component E^x and angular component E^φ of the densitized triad together with the radial component K_x and angular component K_φ of extrinsic curvature [45, 46]. The metric is related to E^x and E^φ by $g_{xx} = (E^\varphi)^2/|E^x|$ and $g_{\varphi\varphi} = |E^x|\sin\vartheta$. Consistent deformations of the Hamiltonian constraint (with unmodified diffeomorphism constraint) have the form

$$H^Q_{\rm grav}[N] = -\frac{1}{2G}\int {\rm d}x\, N\left[\alpha|E^x|^{-\frac{1}{2}}K_\varphi^2 E^\varphi + 2\bar{\alpha}K_\varphi K_x|E^x|^{\frac{1}{2}} + \alpha|E^x|^{-\frac{1}{2}}E^\varphi - \alpha_\Gamma|E^x|^{-\frac{1}{2}}\Gamma_\varphi^2 E^\varphi + 2\bar{\alpha}_\Gamma\Gamma_\varphi'|E^x|^{\frac{1}{2}}\right] \tag{7}$$

with correction functions α, $\bar{\alpha}$, α_Γ and $\bar{\alpha}_\Gamma$. In the second line, $\Gamma_\varphi = -(E^x)'/2E^\varphi$ is the angular component of the spin connection.

The four correction functions are not independent but must satisfy [16]

$$(\bar{\alpha}\alpha_\Gamma - \alpha\bar{\alpha}_\Gamma)(E^x)' + 2(\bar{\alpha}'\bar{\alpha}_\Gamma - \bar{\alpha}\bar{\alpha}_\Gamma')E^x = 0 \tag{8}$$

for the Poisson bracket of two Hamiltonian constraints to be anomaly-free. If the two terms in this equation vanish separately, a case studied in [16], they imply $\alpha_\Gamma \propto \alpha$ and $\bar{\alpha}_\Gamma \propto \bar{\alpha}$ for a closed constraint algebra. For correction functions defined such that they approach the classical value one for large arguments, $\alpha_\Gamma = \alpha$ and $\bar{\alpha}_\Gamma = \bar{\alpha}$.

From the Poisson bracket $\{H[N], D[N^x]\}$, the only restriction is that both correction functions depend only on the radial triad component E^x, not on E^φ. (This fact is easily understandable from transformation properties of the components: E^x is a scalar in the radial manifold while E^φ is a scalar density [45].) Only the functions $\bar{\alpha}$ and $\bar{\alpha}_\Gamma$ appear in the deformed constraint algebra (4) via $\beta = \bar{\alpha}\bar{\alpha}_\Gamma$, not α or α_Γ. By changing only α and α_Γ, one can modify the spherically symmetric constraints while keeping their derivative order and the constraint algebra unchanged: In spherical symmetry, the classical dynamics does not follow uniquely from the hypersurface-deformation algebra.

3. $2+1$ gravity

In spherically symmetric models and for perturbative inhomogeneities around isotropic models, consistent deformations of the hypersurface-deformation algebra have been found by computing Poisson brackets of effective constraints, obtained by amending the classical constraints by correction functions. In $2+1$-dimensional models, there are detailed calculations [13] of partially off-shell constraint algebras even at an operator level. The results confirm the general form of consistent deformations seen with effective constraints.

B. Holonomy corrections

In spherically symmetric models, also a limited version of holonomy corrections has been implemented consistently, those that involve only the scalar component K_φ of extrinsic curvature but not the component K_x of density weight one [15]. One can therefore consistently substitute $-i\delta^{-1}\exp(i\delta K_\varphi)$ for K_φ, but there is no known consistent way to use $\exp(i\int_e K_x \mathrm{d}x)$ for K_x. Accordingly, no spatial integration or discretization is required to ensure the existing forms of consistent correction functions to be scalar and to keep the $\{H, D\}$-part of the constraint algebra unmodified. This type of correction thus does not lead to non-locality, as holonomy corrections usually do owing to the spatial integrations involved in their definition. In this case, the form of the deformation is similar to the one obtained for inverse-triad corrections, with a Poisson bracket (4) for a correction function now depending on extrinsic curvature instead of the densitized triad.

If we parameterize the Hamiltonian constraint as

$$H^Q_{\rm grav}[N] = -\frac{1}{2G}\int \mathrm{d}x\, N\left(\alpha|E^x|^{-\frac{1}{2}}E^\varphi f_1(K_\varphi,K_x)+2\bar{\alpha}|E^x|^{\frac{1}{2}}f_2(K_\varphi,K_x)+\alpha|E^x|^{-\frac{1}{2}}E^\varphi-\alpha_\Gamma|E^x|^{-\frac{1}{2}}\Gamma_\varphi^2E^\varphi+2\bar{\alpha}_\Gamma\Gamma'_\varphi|E^x|^{\frac{1}{2}}\right)$$

including inverse-triad corrections as well as holonomy corrections via two new functions f_1 and f_2, anomaly freedom can be realized if $f_1 = F_1^2$ and $f_2 = K_xF_2$ provided that $F_2 = F_1(\partial F_1/\partial K_\varphi)\alpha/\alpha_\Gamma$ [15]. If F_1 is independent of E^x, or at least depends on this triad variable in a way different from inverse-triad corrections, we obtain that $\alpha_\Gamma = \alpha$ and also $\bar{\alpha}_\Gamma = \bar{\alpha}$ must be realized, restricting the set of solutions of (8). Combinations of different corrections therefore can reduce the freedom of choices seen for just a single type. If we take $F_1 = (\delta\gamma)^{-1}\sin(\gamma\delta K_\varphi)$, as often done for holonomy corrections, we see that $F_2 = (2\gamma\delta)^{-1}\sin(2\gamma\delta K_\varphi)$. The algebraic deformation is then given by $\beta(E^x, K_\varphi) = \bar{\alpha}\bar{\alpha}_\Gamma\partial F_2/\partial K_\varphi$. For the example provided, this means $\beta(K_\varphi) = \cos(2\delta K_\varphi)$ if we include only holonomy corrections. Note that β can be negative for holonomy corrections, unlike for inverse-triad corrections. In particular, disregarding inverse-triad corrections, we can write $\beta = \partial F_2/\partial K_\varphi = \frac{1}{2}\partial^2 f_1/\partial K_\varphi^2$. At curvatures for which f_1 is at a maximum, $\beta < 0$.

A more general case of holonomy corrections, including even discretization and non-locality, has been implemented consistently in $2+1$-dimensional gravity with a non-vanishing cosmological constant [12]. (A vanishing cosmological constant in $2+1$ dimensions does not require deformations of the constraint algebra, which is much simpler in this case.) As with inverse-triad corrections in $2+1$ dimensions, also these calculations have been performed at an operator level. Similarly to the spherically symmetric case, the correction function is given by the trace of a holonomy used to write the Hamiltonian constraint in loop variables.

For some perturbative models around Friedmann–Robertson–Walker backgrounds, holonomy corrections have been included consistently, too. This is the case for tensor [47], vector [10], and scalar modes [11]. A new feature in [10], which did not show up in any other case of consistent deformations of (1) so far, is that the Poisson bracket $\{H, D\}$ could be consistently modified (even if D itself remains classical). However, this possibility has been ruled out by more restricted consistent deformations of scalar modes [11]. Also here, the correction function is of the form $\cos(2\delta c)$ with the isotropic connection component c. It is similar to the correction function for holonomy corrections in spherical symmetry, and also becomes negative for large curvatures, with $\delta c \sim \pi/2$. Implications will be discussed later. As in spherical symmetry, no non-locality effects have yet been implemented for holonomy corrections in nearly isotropic cosmology.

C. Discretization

Effective constraints of loop quantum gravity in inhomogeneous situations naturally include discretization (or a derivative expansion of spatially discretized terms) because the basic variables, holonomies and fluxes, are defined as spatial integrations of non-scalar quantities. Also spatial derivatives in Hamiltonians must be replaced suitably by finite differences. Modelling classical constraints with these variables to ensure the correct classical limit of the resulting theory then requires one to refer to the field values at different points even for classically local expressions. For this type of corrections, independently of holonomy corrections, no consistently deformed algebra has been formulated explicitly, but work on consistent discretizations exists [48–50] and indicates that deformations should occur also here.

III. HYPERSURFACE DEFORMATIONS

The meaning of the hypersurface-deformation algebra has been discussed in detail in the classic references [3] and [4]. Nevertheless, it is useful to go through some of the arguments once again with a fresh perspective suggested by the deformed algebras found recently and summarized in the preceding section.

A. Spatial diffeomorphisms

Most (but not all) deformations found so far in loop quantum gravity leave the spatial part of the hypersurface-deformation algebra intact, which will also be one of our assumptions in this article. There are several reasons for this assumption: First, spatial diffeomorphisms can be implemented directly in loop quantum gravity by moving graphs in the spatial manifold used to set up the canonical formulation. This action is the same as the one on classical fields, and so one would not expect corrections to the diffeomorphism constraint at an effective level. If one just assumes that the part of the constraint algebra associated with a vector field δN^i generates relabellings $x^i \mapsto x^i + \delta N^i$ of points in the spatial manifold, any field on space must automatically change by the Lie derivative along δN^i. Since spaces in a very general sense are described mathematically by labelling their elements in some way, while physics should be insensitive to how the labels are chosen, it is natural to expect a relabelling symmetry to be present at an effective level, even if the fundamental spatial structure may become discrete or non-commutative. From the relation $\{F, D[\delta N^i]\} = \mathcal{L}_{\delta N^i} F$ and the usual expressions for Lie derivatives of the fundamental fields, one can then uniquely derive the phase-space expression that the diffeomorphism constraint must take [3]. In particular, it is always linear in the momenta of the fields, a consequence which we will make use of later on. For the fields considered here, this implies $D_{\rm scalar}[N^i] = \int {\rm d}^3x N^i p_\phi \phi_{|i}$ for a scalar field and $D_{\rm grav}[N^i] = -2\int {\rm d}^3x N^i \pi_i{}^j{}_{|j}$ for gravity in ADM variables.

Once the diffeomorphism constraint is determined, it must obviously satisfy the spatial part (1) of the classical constraint algebra as well as (2), as long as the corrected $H_{(\beta)}[N]$ remains a scalar. The latter assumption (that $H_{(\beta)}[N]$ be a spatial scalar) appears safe, too, because of the nature of effective constraints as integrated functionals on a spatial manifold. In what follows, we will make use not only of the assumption that the spatial part of the hypersurface-deformation algebra remains unmodified, but also of several further consequences regarding the form of the diffeomorphism constraint. Most importantly, the diffeomorphism constraint appears on the right-hand side of (4); thus, the expression it takes will influence the Hamiltonian constraint determined from the constraint algebra.

B. Transversal deformations

The modification by β in (4) occurs for the commutator of two transversal deformations of spatial hypersurfaces along their normal vectors, by two different and position-dependent amounts N_1 and N_2. This part of the deformation algebra is distinguished from the spatial part not only in that it is of dynamical content, owing to the presence of the Hamiltonian constraint and matter Hamiltonians. Also, the use of the normal vector to point the deformation normally implies a dependence on the space-time metric $\mathbf{g}_{\mu\nu}$, containing phase-space degrees of freedom. The algebra, as a consequence, acquires structure functions rather than just structure constants as suffice for the part of spatial deformations. Implications of structure functions for canonical quantization, mainly negative ones owing to additional difficulties in commutator relationships, are well known; in the present context they are, perhaps more positively so thanks to interesting implications, realized as a general source of possible deformations by quantum corrections.

Unlike the spatial part of the deformation algebra, which directly shows its relation to infinitesimal deformations by the presence of the Lie derivative, relating the $\{H, H\}$ part of the algebra to transversal deformations is not so obvious. As indicated by the algebra, we consider two transversal deformations by lapse functions N_1 and N_2, done in a row but in the two different possible orders. Starting with an initial hypersurface $S_{\rm in}$, we obtain two intermediate ones, S_1 by deforming $S_{\rm in}$ by N_1 along the normal and S_2 by deforming $S_{\rm in}$ by N_2 along the normal. From those two intermediate hypersurfaces, we obtain two final hypersurfaces, $S_{\rm fin}^{(1)}$ by deforming S_1 by N_2 along the new normal of S_1 and $S_{\rm fin}^{(2)}$ by deforming S_2 by N_1 along the new normal of S_2. Comparing the two final hypersurfaces should then yield a commutator of deformations according to (3). In the process of computing the normals of $S_{\rm in}$, S_1 and S_2, the metric tensor must be used. We will not fix the signature $\sigma = \pm 1$ of the metric for our calculations in order to be able to incorporate possible sign changes due to quantum corrections, as suggested by holonomy corrections where β can turn negative. (For Lorentzian signature with $\sigma = -1$, we choose the time part of the metric to carry the minus sign.)

For simplicity, and without loss of generality, we choose space-time coordinates such that $S_{\rm in}$ is given by a constant-time slice, $S_{\rm in}\colon y^i \mapsto (t_{\rm in}, y^i)$ with some spatial embedding coordinates y^i. The general expression for the future-pointing unit normal to a hypersurface $y^i \mapsto x^\mu(y^i)$,

$$n^\mu = \sigma \frac{\mathbf{g}^{\mu\mu'}\epsilon_{\mu'\nu\lambda\kappa}\partial_{y^1}x^\nu \partial_{y^2}x^\lambda \partial_{y^3}x^\kappa}{||\cdot||} \qquad (9)$$

(with $||\cdot||$ denoting the norm of the numerator) then simplifies to $n^\mu_{\rm in} = \sigma \mathbf{g}^{\mu 0}/\sqrt{|\mathbf{g}^{00}|}$.

The intermediate hypersurfaces, with infinitesimal transversal deformations, are obtained as

$$\begin{aligned} S_1\colon y^i \mapsto x^\mu(y^i) + N_1(y^i) n^\mu_{\rm in} &= (t_{\rm in}, y^i) + \frac{\sigma N_1(y^i)\mathbf{g}^{\mu 0}(y^i)}{\sqrt{|\mathbf{g}^{00}|}} \\ S_2\colon y^i \mapsto x^\mu(y^i) + N_2(y^i) n^\mu_{\rm in} &= (t_{\rm in}, y^i) + \frac{\sigma N_2(y^i)\mathbf{g}^{\mu 0}(y^i)}{\sqrt{|\mathbf{g}^{00}|}}\,. \end{aligned}$$

From these expressions, we obtain the new normals by the general formula (9), expanded to first order in the lapse functions for infinitesimal deformations:

$$\begin{aligned} n_1^\mu &= \sigma \frac{\mathbf{g}^{\mu 0}}{\sqrt{|\mathbf{g}^{00}|}} + \left(-\sigma \mathbf{g}^{\mu i} + \frac{\mathbf{g}^{\mu 0}\mathbf{g}^{i0}}{|\mathbf{g}^{00}|}\right)\partial_i N_1 + N_1 X + O(N_1^2) = \sigma \frac{\mathbf{g}^{\mu 0}}{\sqrt{|\mathbf{g}^{00}|}} - \sigma g^{\mu i}\partial_i N_1 + N_1 X + O(N_1^2) \\ n_2^\mu &= \sigma \frac{\mathbf{g}^{\mu 0}}{\sqrt{|\mathbf{g}^{00}|}} + \left(-\sigma \mathbf{g}^{\mu i} + \frac{\mathbf{g}^{\mu 0}\mathbf{g}^{i0}}{|\mathbf{g}^{00}|}\right)\partial_i N_2 + N_2 X + O(N_2^2) = \sigma \frac{\mathbf{g}^{\mu 0}}{\sqrt{|\mathbf{g}^{00}|}} - \sigma g^{\mu i}\partial_i N_2 + N_2 X + O(N_2^2) \end{aligned}$$

with the spatial metric $g^{\mu\nu} = \mathbf{g}^{\mu\nu} - \sigma n^\mu_{\rm in} n^\nu_{\rm in}$ on the initial slice. The coefficient X denotes a combination of metric components and their derivatives; the precise form will not be important because these terms, depending on N_1 and N_2 but not on their derivatives, will drop out of the final commutator results. The two final hypersurfaces are then parameterized as

$$\begin{aligned} S^{(1)}_{\rm fin}\colon y^i &\mapsto x^\mu(y^i) + N_1(y^i) n^\mu_{\rm in} + N_2(y^i) n^\mu_1 \\ &= (t_{\rm in}, y^i) + \sigma N_1(y^i)\frac{\mathbf{g}^{\mu 0}}{\sqrt{|\mathbf{g}^{00}|}} + \sigma N_2(y^i)\left(\frac{\mathbf{g}^{\mu 0}}{\sqrt{|\mathbf{g}^{00}|}} - g^{\mu i}\partial_i N_1(y^i)\right) + N_1 N_2 X + O(N_1^2) \\ S^{(2)}_{\rm fin}\colon y^i &\mapsto x^\mu(y^i) + N_2(y^i) n^\mu_{\rm in} + N_1(y^i) n^\mu_2 \\ &= (t_{\rm in}, y^i) + \sigma N_2(y^i)\frac{\mathbf{g}^{\mu 0}}{\sqrt{|\mathbf{g}^{00}|}} + \sigma N_1(y^i)\left(\frac{\mathbf{g}^{\mu 0}}{\sqrt{|\mathbf{g}^{00}|}} - g^{\mu i}\partial_i N_2(y^i)\right) + N_2 N_1 X + O(N_2^2)\,. \end{aligned}$$

With these expressions it is easy to notice that, writing $S^{(1)}_{\rm fin}\colon y^i \mapsto x^\mu_{{\rm fin},1}(y^i)$, we have

$$S^{(2)}_{\rm fin}\colon y^i \mapsto x^\mu_{{\rm fin},2}(y^i) = x^\mu_{{\rm fin},1}(y^i) + \delta S^\mu(y^i)$$

with

$$\delta S^\mu(y^i) = -\sigma g^{\mu i}(N_1\partial_i N_2 - N_2 \partial_i N_1)\,. \qquad (10)$$

To leading order in the lapse functions, $\delta S^\mu(y^i)$ (depending only on spatial metric components $g^{\mu i}$) is orthogonal to the normal vector and thus amounts to an infinitesimal spatial diffeomorphism along the hypersurface. The spatial deformation δS^μ in (10) is obtained from the commutator of two normal deformations, and it reproduces the normal part of the algebra (3) for $\sigma = -1$. A change of sign in the structure function is equivalent to signature change. (Formally, this implication of signature change can also be seen by replacing N with iN.)

So far we have assumed the classical manifold structure and geometry in order to compute the normal vectors. The deformed algebra (4) can be achieved formally by using $\beta g^{\mu\nu}$ instead of the inverse metric $g^{\mu\nu}$. For inverse-triad corrections, such a modification would be expected because it affects all inverse components of the metric in Hamiltonians. Nevertheless, the appearance of the correction function in the constraint algebra must have a more general origin than just modifying any appearance of the inverse metric because a deformation of the same form is obtained for some versions of holonomy corrections. The latter do not affect inverse-metric components but rather appearances of extrinsic curvature or the Ashtekar–Barbero connection. However, only the spatial metric appears in the structure functions of the constraint algebra; deformations, therefore, cannot be reduced to simply applying the usual corrections of loop quantum gravity to the structure functions. Such a procedure would be questionable, anyway, because the structure functions are not quantized but rather arise from the algebra satisfied by effective quantum constraints, with corrections following in a more indirect way.

IV. CONSTRAINTS AND SPACE-TIME STRUCTURE

Quantum-geometry corrections change the hypersurface-deformation algebra and accordingly the space-time structure: Normal deformations of spatial slices then behave differently from the classical case. Corresponding actions cannot be covariant in the usual sense, but they are still covariant in a deformed sense, under an algebra of the type (4). In the absence of a corresponding space-time tensor calculus, it is difficult to imagine the form of actions covariant with respect to the new space-time structures. But fortunately, such actions can be systematically derived from the constraint algebra, or regained in the language of [3, 4].

In this and the following section we will go in some detail through the steps outlined in [3], focusing our discussion on those that use assumptions no longer valid if the classical space-time structure cannot be taken for granted. According to the form of the deformed constraint algebra used here, and as a rather general consequence of canonical quantum gravity, the spatial structure on the one hand and the structure of hypersurface deformations within space-time, on the other, will play rather different roles. The algebraic effects considered here are thus truly dynamical and do not arise at the kinematical level of spatial manifolds.

A. Locality

Once the spatial structure is fixed, the next object to consider is the change of the spatial metric under a normal deformation of a spatial slice. Classically, this deformation is given by the extrinsic-curvature tensor, $\{g_{ij}(x), H[\delta N]\} = 2K_{ij}(x)\delta N$, and it plays an important role in [3] in helping to show that the Hamiltonian constraint must be a local expression in the momentum: Identifying

$$\frac{\delta H[\delta N]}{\delta \pi^{ij}(x)} = \{g_{ij}(x), H[\delta N]\} = 2K_{ij}(x)\delta N(x) \qquad (11)$$

implies that $H[\delta N]$ must be local in the momentum $\pi^{ij}(x)$ without any dependence on π^{ij}-derivatives. The specific form of K_{ij} as extrinsic curvature does not matter for this conclusion, but it is important that it is a local function, and that no derivatives of δN appear on the right-hand side.

In the presence of deformed space-time structures, we cannot safely assume that transversal metric deformations are given in terms of extrinsic curvature. For the explicit examples of deformed constraint algebras, it is known that the relationship between momentum variables and extrinsic curvature deviates from the classical one; see e.g. the discussion in [16]. It should then be possible for the change of the metric under a transversal deformation, while still being related to the momentum of the metric, to have a modified relationship with extrinsic curvature. In the absence of a geometrical interpretation of the change of the metric, one can compute it only by using the canonical formula $\{g_{ij}(x), H[\delta N]\}$; but then, one piece of independent information is lost and we cannot derive locality properties of the Hamiltonian constraint. If $H[\delta N]$ is local in the momentum, $\{g_{ij}, H[\delta N]\}$ is local and vice versa, but there is no independent general statement that could determine whether locality is realized.

Instead, we will make use of the following line of arguments: We know that the classical constraint must be local without spatial derivatives of π^{ij}, and in most cases the form of corrections expected from loop quantum gravity tells us what locality properties new terms have. Most of them are indeed non-local, for instance those arising from the use of holonomies as exponentiated line integrals of a connection related to extrinsic curvature, or inverse-triad corrections depending on fluxes through extended surfaces. In derivative expansions, whole series of spatial derivatives of π^{ij} or g_{ij} will result. The form of the corrections and their impact on effective constraints can thus be used to decide whether local or non-local constraints should be expected. The arguments put forward to regain the form of the constraint will then have to be adapted, depending on the locality properties realized. In most cases, effective equations include a derivative expansion, approximating non-local features locally. We can then assume a local Hamiltonian constraint, but, in contrast to the classical case, must take into account additional derivatives, for instance of K_{ij}.

Similar considerations can be applied to the question of whether the matter Hamiltonian must be local in the momentum. Here, the assumptions made in [3] appear safer in the context of deformed algebras than those for the corresponding gravitational terms. Instead of looking at transversal deformations of the spatial metric, we look at transversal deformations of the matter field, assumed to be a scalar to be specific. The relation $\{\phi(x), H[\delta N]\} = V(x)\delta N$ then replaces the gravitational relation involving extrinsic curvature, with $V(x)$ interpreted as the velocity of the scalar field. In contrast to the gravitational part, there are some quantum corrections in matter Hamiltonians that, while changing the specific expression for $V(x)$, leave its local nature intact [90]. Thus, in some cases we can assume the matter Hamiltonian to be local in the momentum even in the presence of corrections making the gravitational part non-local without a derivative expansion. This difference between gravitational and matter Hamiltonians may play an important role for the interplay of different contributions to the constraints ensuring that the algebra closes.

B. Gravity and matter

There is a useful argument showing that the gravity and matter parts of the constraints $D[N^i] = D_{\rm grav}[N^i] + D_{\rm matter}[N^i]$ and $H[N] = H_{\rm grav}[N] + H_{\rm matter}[N]$ must satisfy the hypersurface-deformation algebra separately, provided that matter Hamiltonians do not depend on the gravitational momentum $\pi^{ij}(x)$ and the gravitational constraint does not contain spatial derivatives of $\pi^{ij}(x)$. In this case,

$$\{H[N_1], H[N_2]\} = \{H_{\rm grav}[N_1], H_{\rm grav}[N_2]\} + \{H_{\rm matter}[N_1], H_{\rm matter}[N_2]\} \,. \tag{12}$$

The assumption is realized classically for a scalar field, for instance, and so one can consider its simpler algebraic regaining procedure independently of the gravitational part. With quantum corrections, however, the assumption can be violated easily, depending on the type of the correction. Matter fields are usually introduced in loop quantum gravity via the values they take at the vertices of a spin network. Spatial derivatives as they occur in the Hamiltonians must be discretized and replaced by finite differences of the values at neighboring vertices before they can be quantized. Depending on how the differencing is done, one may have to refer to the gravitational connection, making the matter constraint dependent on the gravitational momentum. Another source of such a dependence may be counterterms as introduced in [9], required to close the constraint algebra. An extra momentum dependence can be avoided for a scalar field, but there may be reasons to prefer more complicated quantizations.

Coming back to the results found in the preceding subsection on locality, we can see a potential obstruction to the existence of consistent deformations of the classical constraint algebra. There are corrections expected from loop quantum gravity, most notably holonomy corrections, which are non-local in the connection and thus make the gravitational part of the Hamiltonian constraint non-local in the gravitational momentum. A scalar Hamiltonian in the presence of the same corrections, on the other hand, remains local in its momentum. If the gravitational part and the matter part are to satisfy the same deformed algebra for consistency, the mismatch of locality properties could be seen as an obstacle to the existence of a consistent deformation: The function β of (4) would be non-local in one contribution and local in another one, preventing one from adding up the constraint contributions to a consistent whole. However, the situation is not obviously inconsistent because the same property giving rise to the mismatch, non-locality and the presence of derivatives of π^{ij}, also violates the assumptions that led one to conclude that gravity and matter satisfy the hypersurface-deformation algebra independently. Non-locality, in a derivative expansion of holonomy corrections in effective constraints, makes the gravitational constraint depend on spatial derivatives of the momentum $\pi^{ij}(x)$, such that cross-terms between gravity and matter in (12) no longer cancel. It is reassuring that properties of non-locality thus restore the a-priori possibility of consistency, but the necessary appearance of gravity-matter cross-terms makes the explicit construction of consistent deformations for non-local momentum-dependent corrections more difficult than for local ones. As recalled in Sec. II B, results in spherical symmetry are indeed much easier to find in local versions of the corrections. Also for perturbative inhomogeneities as in [11] one so far assumes a local, pointwise form of holonomy corrections. The manipulations required for non-local modifications to be consistent appear to be rather complicated, a fact which may explain the difficulties found in constructing consistent deformations corresponding to the non-local holonomy or discreteness corrections. (On the other hand, tying matter terms more closely to gravitational ones rather than having them algebraically separated as in (12) may be of interest in the context of unification.)

V. ALGEBRAICALLY REGAINING HAMILTONIANS

With these preparatory discussions, we can now begin to enter details of regaining Hamiltonians from deformed constraint algebras. There are several interesting applications and generalizations of the methods of [3], which we develop in different cases.

A. Spherical symmetry

Before looking at the general theory, it is instructive to specialize the calculations to spherical symmetry. Some steps will simplify, and it will be interesting to compare the differences in uniqueness for different degrees of symmetry. As already noted in Sec. II A 2, in spherical symmetry the classical dynamics does not follow uniquely from the algebra.

For the sake of easier comparison with calculations of modified constraints motivated by loop quantum gravity, we will present equations in this subsection for triad variables. A spherically symmetric spatial densitized triad has two components E^x and E^φ, for the radial coordinate x and one angular coordinate φ, which determine the spatial metric by $g_{xx} = (E^\varphi)^2/|E^x|$ and $g_{\varphi\varphi} = \sin^2\vartheta|E^x|$. We will assume $E^x > 0$ to avoid some sign factors.

Instead of working with spatial curvature tensors, in this context it turns out to be useful to refer to the angular spin-connection component and its spatial and functional derivatives,

$$\Gamma_\varphi = -\frac{(E^x)'}{2E^\varphi} \tag{13}$$

$$\Gamma'_\varphi = -\frac{(E^x)''}{2E^\varphi} + \frac{(E^x)'(E^\varphi)'}{2(E^\varphi)^2} \tag{14}$$

$$\frac{\delta\Gamma_\varphi(y)}{\delta E^x(x)} = -\frac{1}{2E^\varphi(y)}\delta'(y,x) \tag{15}$$

$$\frac{\delta\Gamma_\varphi(y)}{\delta E^\varphi(x)} = \frac{(E^x)'(y)}{2(E^\varphi(y))^2}\delta(y,x) \tag{16}$$

$$\frac{\delta\Gamma'_\varphi(y)}{\delta E^x(x)} = -\frac{1}{2E^\varphi(y)}\delta''(y,x) + \frac{(E^\varphi)'(y)}{2(E^\varphi(y))^2}\delta'(y,x) \tag{17}$$

$$\frac{\delta\Gamma'_\varphi(y)}{\delta E^\varphi(x)} = \frac{(E^x)''(y)}{2(E^\varphi(y))^2}\delta(y,x) + \frac{(E^x)'(y)}{2(E^\varphi(y))^2}\delta'(y,x) \tag{18}$$

$$-\frac{(E^x)'(y)(E^\varphi)'(y)}{(E^\varphi(y))^3}\delta(y,x)\,. \tag{19}$$

(The radial component of the spin connection does not have any gauge-invariant contribution [45].)

Momenta of the densitized triad are classically given by extrinsic-curvature components K_x and K_φ with $\{K_x(x), E^x(y)\} = 2G\delta(x,y)$ and $\{K_\varphi(x), E^\varphi(y)\} = G\delta(x,y)$. With these properties, the commutator relationship (4) to exploit here reads

$$\begin{aligned}\{H(x),H(y)\} &= G\int \mathrm{d}^3z \left(2\frac{\delta H(x)}{\delta K_x(z)}\frac{\delta H(y)}{\delta E^x(z)} - 2\frac{\delta H(y)}{\delta K_x(z)}\frac{\delta H(x)}{\delta E^x(z)} + \frac{\delta H(x)}{\delta K_\varphi(z)}\frac{\delta H(y)}{\delta E^\varphi(z)} - \frac{\delta H(y)}{\delta K_\varphi(z)}\frac{\delta H(x)}{\delta E^\varphi(z)}\right)\\ &= \beta\frac{E^x(x)}{(E^\varphi(x))^2}D(x)\delta'(x,y) - (x\leftrightarrow y)\end{aligned} \tag{20}$$

with the local diffeomorphism constraint

$$D(x) = \frac{1}{2G}(2E^\varphi K'_\varphi - K_x(E^x)')\,. \tag{21}$$

With a modified Hamiltonian, K_x and K_φ may no longer be components of extrinsic curvature. However, they are still canonically conjugate to E^x and E^φ, and we continue to use the same letters for momentum variables.

For now, we will be looking only for constraints with quadratic "kinetic" term in momenta and no non-locality or spatial derivatives of K,

$$H = {}^{00}H + {}^{11}HK_xK_\varphi + {}^{20}HK_xK_x + {}^{02}HK_\varphi K_\varphi \tag{22}$$

(without linear terms, assuming time reversal symmetry), and have linear functional derivatives

$$G\frac{\delta H(x)}{\delta K_x(z)} = (A_1(x)K_x(x) + B_1(x)K_\varphi(x))\,\delta(x,z) \tag{23}$$

$$G\frac{\delta H(x)}{\delta K_\varphi(z)} = (A_2(x)K_x(x) + B_2(x)K_\varphi(x))\,\delta(x,z) \tag{24}$$

We then identify ${}^{11}HG = A_2 = B_1$, ${}^{02}HG = B_2/2$, ${}^{20}HG = A_1/2$, which may all depend on the triad components. The Poisson bracket of two Hamiltonian constraints becomes

$$\begin{aligned}\{H(x),H(y)\} &= \frac{\delta H(y)}{\delta E^x(x)}(2A_1K_x(x) + 2B_1K_\varphi(x)) + \frac{\delta H(y)}{\delta E^\varphi(x)}(A_2K_x(x) + B_2K_\varphi(x)) - (x\leftrightarrow y)\\ &= \beta\frac{E^x(x)}{G(E^\varphi(x))^2}\left(E^\varphi(x)K'_\varphi(x) - \frac{1}{2}K_x(x)(E^x)'(x)\right)\delta'(x,y) - (x\leftrightarrow y)\,.\end{aligned} \tag{25}$$

We evaluate its implications by comparing coefficients of K_x and K_φ. In this section, we will assume that β does not depend on K_x or K_φ, thus considering the case of inverse-triad corrections.

For $K_x = 0$, $K_\varphi = 0$, the equation is automatically satisfied. For the first-order coefficients in K_x, we operate with $\delta/\delta K_x$ and then set $K_x = 0$, $K_\varphi = 0$:

$$\left(2\frac{\delta\,{}^{00}H(y)}{\delta E^x(x)}A_1(x) + \frac{\delta\,{}^{00}H(y)}{\delta E^\varphi(x)}A_2(x)\right)\delta(x,z) - (x\leftrightarrow y) = -\frac{\beta E^x(x)(E^x)'(x)}{2G(E^\varphi(x))^2}\delta'(x,y)\delta(x,z) - (x\leftrightarrow y)\,. \tag{26}$$

For functional derivatives of ${}^{00}H$ by E^x and E^φ, we must know the general triad-dependent terms possible. In addition to a direct dependence on the fields, ${}^{00}H$ can depend on the triad via spatial curvature which, in turn, depends on the spin connection and its derivatives. We thus have to expect a dependence on E^x, E^φ, Γ_φ and Γ_φ'. Higher derivatives are not included because here, as in (22), we expand only to second order in momenta and derivatives.

We then have the chain rule

$$\frac{\delta\,{}^{00}H(y)}{\delta E^x(x)} = \frac{\partial\,{}^{00}H(y)}{\partial\Gamma_\varphi(y)}\frac{\delta\Gamma_\varphi(y)}{\delta E^x(x)} + \frac{\partial\,{}^{00}H(y)}{\partial\Gamma_\varphi'(y)}\frac{\delta\Gamma_\varphi'(y)}{\delta E^x(x)} + \frac{\partial\,{}^{00}H(y)}{\partial E^x(y)}\frac{\delta E^x(y)}{\delta E^x(x)} \tag{27}$$

and a similar relation for $\delta\,{}^{00}H(y)/\delta E^\varphi(x)$ to rewrite (26). We substitute our expressions (16)–(19) for $\delta\Gamma_\varphi(y)/\delta E^x(x)$ and so on, multiply with test functions $a(x)$, $b(y)$, and $c(z)$, and integrate over x, y, and z. We state the result obtained after several integrations by parts:

$$\int \mathrm{d}y\Bigg[-(a'cA_1)\frac{b}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi} + (a'cA_1)\frac{b(E^\varphi)'}{(E^\varphi)^2}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'} + (a'cA_2)\frac{b(E^x)'}{2(E^\varphi)^2}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'} + 2(a'cA_1)b\left(\frac{1}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'}\right)' \tag{28}$$

$$+(a''cA_1)\frac{b}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'} - \frac{\beta E^x(E^x)'}{2G(E^\varphi)^2}a'cb\Bigg] - (a\leftrightarrow b) = 0\,.$$

(Several terms that cancel in the antisymmetrization with respect to a and b have not been written explicitly.) Collecting the coefficients of $c(a''b-b''a)$ and $c(a'b-b'a)$, respectively, we get

$$\frac{A_1}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'} \;=\; 0\,, \tag{29}$$

$$-\frac{A_1}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi} + \frac{A_1(E^\varphi)'}{2(E^\varphi)^2}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'} + \frac{A_2(E^x)'}{2(E^\varphi)^2}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'} + \left(\frac{1}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'}\right)'2A_1 - \frac{\beta E^x(E^x)'}{2G(E^\varphi)^2} \;=\; 0\,. \tag{30}$$

Going back to (25) to look at the first order in K_φ (and zeroth in K_x), and performing similar operations, we get

$$\int \mathrm{d}y\Bigg[-(a'cB_1)\frac{b}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi} + (a'cB_1)\frac{b(E^\varphi)'}{(E^\varphi)^2}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'} + (a'cB_2)\frac{b(E^x)'}{2(E^\varphi)^2}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'} + 2(a'cB_1)b\left(\frac{1}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'}\right)'$$
$$+(a''cB_1)\frac{b}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'} - a''bc\frac{\bar\alpha^2E^x}{E^\varphi} - a'bc\left(\frac{\beta E^x}{GE^\varphi}\right)'\Bigg] - (a\leftrightarrow b) = 0\,. \tag{31}$$

Collecting the coefficients of $c(a''b-b''a)$ and $c(a'b-b'a)$, respectively, results in

$$\frac{B_1}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'} - \frac{\beta E^x}{GE^\varphi} \;=\; 0 \tag{32}$$

$$-\frac{B_1}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi} + \frac{B_1(E^\varphi)'}{2(E^\varphi)^2}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'} + \frac{B_2(E^x)'}{2(E^\varphi)^2}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'} + \left(\frac{1}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'}\right)'2B_1 - \left(\frac{\beta E^x}{GE^\varphi}\right)' \;=\; 0\,. \tag{33}$$

Equation (32) implies that $\delta\,{}^{00}H/\delta\Gamma_\varphi'$ cannot be zero. With this condition, we find $A_1=0$ from (29),

$$A_2 = \frac{\beta E^x}{G}\left(\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'}\right)^{-1} = B_1 \tag{34}$$

from (30) and (32), and

$$-\frac{B_1}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi} + \frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'}\left(\frac{B_1(E^\varphi)'}{(E^\varphi)^2} + \frac{B_2(E^x)'}{2(E^\varphi)^2}\right)$$
$$+2B_1\left(\frac{1}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi'}\right)' = \frac{1}{G}\left(\frac{\beta E^x}{E^\varphi}\right)'. \tag{35}$$

This tells us that

$$G\frac{B_1}{E^\varphi}\frac{\partial\,{}^{00}H}{\partial\Gamma_\varphi} = \left(\frac{B_2}{B_1}\frac{E^x}{2E^\varphi}+1\right)\frac{\beta(E^x)'}{E^\varphi} + \frac{E^x}{E^\varphi}\left(\beta'-2\frac{B_1'}{B_1}\beta\right). \tag{36}$$

To solve these equations, we introduce a function b_1 such that $B_1 = -\sqrt{|\beta|}b_1\sqrt{E^x} = A_2$. The factors are chosen so as to cancel several terms in (36):

$$\frac{\beta(E^x)'}{E^\varphi} + \frac{E^x}{E^\varphi}\left(\beta' - 2\beta B_1'/B_1\right) = -2\beta\frac{E^x}{E^\varphi}\frac{b_1'}{b_1}\,.$$

For the correct density weights in the first term in

(36), B_2 must be proportional to E^φ. (The other factors B_1 and E^x are scalar and cannot change the density weight.) With another free function b_2, we write $B_2 = -b_1 b_2 \sqrt{|\beta|} E^\varphi/\sqrt{E^x}$, with factors other than E^φ chosen for later convenience. The coefficients A_1, A_2, B_1 and B_2 determine the form of momentum contributions to the Hamiltonian constraint:

$$ {}^{11}H = \frac{B_1}{G} = -\frac{\sqrt{|\beta| E^x} b_1}{G} \tag{37} $$

$$ {}^{20}H = \frac{A_1}{2G} = 0 \tag{38} $$

$$ {}^{02}H = \frac{B_2}{2G} = -b_1 b_2 \sqrt{|\beta|} \frac{E^\varphi}{2G\sqrt{E^x}} \,. \tag{39} $$

With these solutions, we obtain $\partial\, {}^{00}H/\partial\Gamma'_\varphi = -\mathrm{sgn}(\beta)\sqrt{|\beta| E^x}/Gb_1$ from (36) and $\partial\, {}^{00}H/\partial\Gamma_\varphi = \mathrm{sgn}(\beta)\sqrt{|\beta|}(b_2 - 4(\mathrm{d}b_1/\mathrm{d}E^x)(E^x/b_1))E^\varphi\Gamma_\varphi/(Gb_1\sqrt{E^x})$ from (34), or integrated,

$$ {}^{00}H = -\frac{\mathrm{sgn}(\beta)\sqrt{|\beta|}}{G}\left(\frac{\sqrt{E^x}}{b_1}\Gamma'_\varphi - \frac{1}{b_1}\left(\frac{b_2}{2} - \frac{2E^x}{b_1}\frac{\mathrm{d}b_1}{\mathrm{d}E^x}\right)\frac{E^\varphi}{\sqrt{E^x}}\Gamma_\varphi^2\right) + f(E^x)E^\varphi \,. \tag{40} $$

Comparing with the general form (7), we read off

$$ \bar{\alpha} = \sqrt{|\beta|} b_1 \quad , \quad \alpha = \sqrt{|\beta|} b_1 b_2 \,, $$
$$ \bar{\alpha}_\Gamma = \mathrm{sgn}(\beta)\frac{\sqrt{|\beta|}}{b_1} \quad , \quad \alpha_\Gamma = \mathrm{sgn}(\beta)\frac{\sqrt{|\beta|}}{b_1}\left(b_2 - 4\frac{\mathrm{d}\log b_1}{\mathrm{d}\log E^x}\right) \,. $$

With these relationships, the correction functions can easily be seen to satisfy the condition (8) as well as $\beta = \bar{\alpha}\bar{\alpha}_\Gamma$.

Modifications to the spherically symmetric dynamics are not entirely determined by the constraint algebra, consistent with the results of [16, 44]. The function b_1 is related to the ratio of $\bar{\alpha}$ to $\bar{\alpha}_\Gamma$, and b_2 determines how α differs from $\bar{\alpha}$. The E^x-dependence of ${}^{00}H$ in (40) (which may include a cosmological-constant term) is not fully determined because E^x is a scalar with no density weight and can, for the purpose of the constraint algebra, be inserted rather freely in the constraints. In this feature we can see why the full dynamics is more unique than the spherically symmetric one: Without symmetry, there is less freedom in the choice of spatial tensors with the correct transformation properties. Indeed, as we will see later, spatial transformation properties play an important role for the regaining procedure. Without spherical symmetry Γ'_φ and Γ_φ^2 would be part of the same contribution ${}^{(3)}R$, which cannot be split apart by different correction functions if the spatial structure of geometry remains unmodified. The case of $\alpha = \bar{\alpha}$ ($b_2 = 1$) and $\alpha_\Gamma = \bar{\alpha}_\Gamma$ (b_1 constant and therefore $b_1 = 1$ for it to approach one at large fluxes) is then preferred, with all corrections determined by the algebraic deformation β.

B. Legendre transform

Instead of having to assume $\delta H/\delta\pi^{ij}$ (or $\delta H/\delta K_x$ and $\delta H/\delta K_\varphi$ in spherical symmetry with triad variables) to be linear in the momenta, it is more general to treat $\delta H/\delta\pi^{ij}(x) =: v_{ij}(x)$ as a new independent variable in place of π^{ij}, and then expand by this newly defined v_{ij}. This change amounts to a Legendre transformation from (g_{ij}, π^{ij}) with Hamiltonian H to (g_{ij}, v_{ij}) with Lagrangian $L = \pi^{ij} v_{ij} - H$, as proposed in [3]. We then have the equations

$$ H = \pi^{ij} v_{ij} - L \tag{41} $$

$$ \left.\frac{\delta H}{\delta g_{ij}(x')}\right|_{\pi^{ij}(x)} = -\left.\frac{\delta L}{\delta g_{ij}(x')}\right|_{v_{ij}(x)} \,. \tag{42} $$

There are now two differences to [3]. First, our v_{ij} here need not be geometrical extrinsic curvature because of modifications to space-time geometry. We simply define a new independent variable $v_{ij} = (\delta N)^{-1}\{g_{ij}, H[\delta N]\}$, which we interpret as the rate of change of the metric, eventually providing time derivatives in an effective action. Secondly, we cannot always assume that the Hamiltonian is local and free of derivatives of π^{ij}, which would imply that partial derivatives could be used to compute v_{ij}.

Using v_{ij}, we write the Poisson bracket of two smeared Hamiltonian constraints as

$$ \begin{aligned} \{H[N], H[M]\} &= \int \mathrm{d}^3x \frac{\delta H[N]}{\delta g_{ij}(x)} v_{ij}(x) M(x) - (N \leftrightarrow M) \\ &= -\int \mathrm{d}^3x \int \mathrm{d}^3y \frac{\delta L(y)}{\delta g_{ij}(x)} v_{ij}(x) N(y) M(x) \\ &\quad -(N \leftrightarrow M) \\ &= \int \mathrm{d}^3x \beta D^i(x)(NM_{|i} - MN_{|i}) \end{aligned} \tag{43} $$

with the local diffeomorphism constraint D^i. Taking functional derivatives by N and M, we arrive at the functional equation

$$ \frac{\delta L(x)}{\delta g_{ij}(x')} v_{ij}(x') + \beta D^i(x)\delta_{|i}(x, x') - (x \leftrightarrow x') = 0 \tag{44} $$

for $L(x)$, which can be solved once an expression for the diffeomorphism constraint D^i is inserted. With D^i linear in the momenta, a fact which remains true in the cases of deformed constraint algebras considered here, and momenta related to functional derivatives of L by v_{ij}, a linear equation for L is obtained. The importance of this consequence of the Legendre transform has been stressed in [3].

If gravity and matter split into independent constrained systems, as realized for matter constraints independent of the gravitational momentum and in the absence of derivatives of $\pi^{ij}(x)$ in $H_{\rm grav}$, equation (44) can be derived in an analogous form for the matter part, just using canonical matter variables and the matter diffeomorphism constraint. Because the following calculations, integrating the functional differential equation, are easier for scalar matter, we will first consider this case as an illustration of the general procedure. As we will see, the Lagrangian viewpoint provides a new interpretation of conditions of anomaly freedom found earlier for inverse-triad corrections of a scalar field.

C. Scalar matter

With the classical spatial structure, the Lagrangian density of a scalar field ϕ must be of the form $\mathcal{L} = \sqrt{\det g}L(\phi, V, \psi)$ where $V = (\delta N)^{-1}\{\phi, H[\delta N]\}$ is the normal scalar velocity introduced before and $\psi = g^{ij}\phi_{|i}\phi_{|j}$ is the only remaining scalar that can be formed from ϕ and its derivatives up to a total derivative order of at most two. Higher derivatives do not appear classically for equations of motion of second order, but they can easily be introduced by quantum effects. Higher spatial derivatives, in particular, are a natural consequence of discretization in loop quantum gravity, which in effective form combined with a derivative expansion will give rise to derivative terms of arbitrary orders. Higher time derivatives, on the other hand, follow from quantum back-reaction. The following considerations for matter assume the absence of higher-order derivatives, as realized for instance for inverse-triad corrections and some forms of holonomy corrections.

With the canonical variables of a scalar field and its diffeomorphism constraint $D^i = p_\varphi \phi^{|i}$, equation (44), adapted to a scalar field, assumes the form

$$\frac{\delta L(x)}{\delta\phi(x')}V(x') + \beta\frac{\partial L(x)}{\partial V(x)}\phi^{|i}(x)\delta_{|i}(x,x') - (x \leftrightarrow x') = 0\,. \tag{45}$$

As in [3], we write

$$\frac{\delta L(x)}{\delta\phi(x')} = \frac{\partial L(x)}{\partial\phi(x)}\frac{\delta\phi(x)}{\delta\phi(x')} + 2\frac{\partial L(x)}{\partial\psi(x)}\phi^{|i}(x)\delta)_{|i}(x,x'$$

and conclude, taking into account the additional factor of β, that

$$A^i := \phi^{|i}\left(\beta\frac{\partial L}{\partial V} + 2V\frac{\partial L}{\partial\psi}\right)$$

satisfies the equation $A^i(x)\delta_{|i}(x,x') - (x \leftrightarrow x') = 0$, shown in [3] to imply $A^i = 0$. Thus,

$$\beta\frac{\partial L}{\partial V} + 2V\frac{\partial L}{\partial\psi} = 0$$

and L must be of the form $L(\phi, \psi - V^2/\beta)$.

This is a concrete indication that the deformed hypersurface-deformation algebra implies a modification of the usual covariance and of the dispersion relation of fields: The kinetic term of scalar Lagrangians does not depend on $\psi - V^2 = \mathbf{g}^{\mu\nu}\phi_{|\mu}\phi_{|\nu}$ in space-time terms, but has its time derivatives in $\psi - V^2/\beta$ rescaled by the correction function β. Nevertheless, the system is covariant and consistent, albeit with a deformed notion of covariance as per the constraint algebra (4). The dependence of the Lagrangian on the potential remains unrestricted, leaving the form of some counterterms as introduced in [9] more open.

It is illustrative to compare this form of the kinetic term with the one obtained for the matter Hamiltonian in a consistent deformation [9]. One begins with a matter Hamiltonian density of the form

$$H = \nu\frac{p_\phi^2}{2\sqrt{\det g}} + \frac{1}{2}\sigma\sqrt{\det g}\psi + \sqrt{\det g}W(\phi)$$

with metric factors corrected by inverse-triad corrections ν and σ, and some potential $W(\phi)$. The corresponding Lagrangian density, with $V = \nu p_\phi/\sqrt{\det g}$, takes the form

$$\begin{aligned} L &= \sqrt{\det g}\left(\frac{V^2}{2\nu} - \frac{\sigma\psi}{2} - W(\phi)\right) \\ &= -\sqrt{\det g}\frac{\sigma}{2}\left(\psi - \frac{V^2}{\beta}\right) - \sqrt{\det g}W(\phi) \end{aligned} \tag{46}$$

with the kinetic dependence as derived above, provided that $\beta = \nu\sigma$. This condition, as derived in [9] for linear inhomogeneities around isotropic models, is exactly one of the requirements for anomaly freedom to ensure a closed constraint algebra of the form (4) for inverse-triad corrections with $\beta = \bar{\alpha}^2$ from the gravitational constraint. The Lagrangian view clearly shows how this condition of anomaly cancellation is necessary to ensure a (deformed) covariant kinetic term in the action. With the same corrections in d'Alembertians, propagation speeds of massless matter and gravitational waves naturally agree, as explicitly shown for electromagentic waves in [47].

From the new derivation of corrected scalar Lagrangians in this paper, we must expect corrections in matter terms also if β results from holonomy corrections, provided they can be consistently implemented. Explicit examples for holonomy modifications required in matter terms have already been found in [11, 51]. However, in a scalar Hamiltonian quantized by the methods of loop quantum gravity [26] we do not expect holonomy corrections. Consistent formulations of holonomy corrections in the presence of matter therefore seem to encounter stronger difficulties than inverse-triad corrections. Another peculiar feature can be seen by recalling that β for holonomy corrections can turn negative. The modified d'Alembertian $\psi - V^2/\beta$ then becomes one of Euclidean signature, or a 4-dimensional Laplacian, and fields no longer propagate. Also this property can explicitly be seen in the wave equations of [11] (but not in [51] where a gauge-fixing has veiled this effect). We will discuss further consequences of this new form of signature change in Sec. VI C.

D. Gravitational part

As in the case of scalar matter, we begin our discussion of the gravitational part by inserting the explicit expression of the diffeomorphism constraint in the general equation (44): In particular,

$$\beta D^i(x)\delta_{|i}(x,x') = -2\beta\pi^{ij}_{|j}(x)\delta_{|i}(x,x')\,. \tag{47}$$

We then proceed as in the example of spherical symmetry: We multiply this expression by two test functions $a(x)$ and $b(x')$ and integrate over x and x', observing that some terms symmetric in a and b cancel. After several steps, integrating by parts, discarding total derivatives and using the symmetry of π^{ij}, we arrive at

$$\int \mathrm{d}x \left[2\pi^{ij}(x)\beta_{|j}(x)\left(a(x)b_{|i}(x) - a_{|i}(x)b(x)\right) + 2\pi^{ij}(x)\beta\left(a(x)b_{|ij}(x) - a_{|ij}(x)b(x)\right)\right]$$

from the right-hand side of (47). Functional derivatives with respect to $a(y)$ and $b(z)$ give

$$\begin{aligned}&\int \mathrm{d}x \left[2\pi^{ij}(x)\beta_{|j}(x)\left(\delta(x,y)\delta_{|i}(x,z) - \delta_{|i}(x,y)\delta(x,z)\right) + 2\pi^{ij}(x)\beta\left(\delta(x,y)\delta_{|ij}(x,z) - \delta_{|ij}(x,y)\delta(x,z)\right)\right]\\ =\;& 2\pi^{ij}(y)\beta_{|j}(y)\delta_{|i}(y,z) + 2\pi^{ij}(y)\beta(y)\delta_{|ij}(y,z) - (y\leftrightarrow z) = 2\frac{\partial L(y)}{\partial v_{ij}(y)}\beta_{|j}(y)\delta_{|i}(y,z) + 2\frac{\partial L(y)}{\partial v_{ij}(y)}\beta(y)\delta_{|ij}(y,z) - (y\leftrightarrow z)\end{aligned}$$

if no spatial derivatives of v_{ij} appear in the corrections and the Lagrangian, such that $\pi^{ij}(y) = \delta L/\delta v_{ij}(y) = \partial L(y)/\partial v_{ij}(y)$. In combination with (44), we have

$$\begin{aligned}&\frac{\delta L(x)}{\delta g_{ij}(x')}v_{ij}(x') + 2\beta_{|j}(x)\frac{\partial L(x)}{\partial v_{ij}(x)}\delta_{|i}(x,x')\\ &+2\beta\frac{\partial L(x)}{\partial v_{ij}(x)}\delta_{|ij}(x,x') - (x\leftrightarrow x') = 0\,.\end{aligned} \tag{48}$$

In cases of derivative expansions of non-local terms in v_{ij}, we use

$$\begin{aligned}&\frac{\delta L(x)}{\delta g_{ij}(x')}v_{ij}(x')\delta(x,x') + 2\beta_{|j}(x)\frac{\delta L(x)}{\delta v_{ij}(x')}\delta_{|i}(x,x')\\ &+2\beta\frac{\delta L(x)}{\delta v_{ij}(x')}\delta_{|ij}(x,x') - (x\leftrightarrow x') = 0\end{aligned} \tag{49}$$

and write

$$\frac{\delta L(x)}{\delta v_{ij}(x')} = \frac{\partial L(x)}{\partial v_{ij}(x')}\delta(x,x') + \frac{\partial L(x)}{\partial v_{ij|k}(x')}\delta_{|k}(x,x') + \cdots \tag{50}$$

1. *Expansion*

In spherical symmetry, it turned out to be useful to consider expansion coefficients by the momenta K_x and K_φ. As the next crucial step in solving the functional equation, we expand both L and β as series in powers of the normal change of the metric, v_{ij}:

$$L = \sum_{n=0}^{\infty} L^{i_1j_1\ldots i_nj_n}[g^{kl}]v_{i_1j_1}(x)\ldots v_{i_nj_n}(x) \tag{51}$$

$$\beta = \sum_{n=0}^{\infty} \beta^{i_1j_1\ldots i_nj_n}[g^{kl}]v_{i_1j_1}(x)\ldots v_{i_nj_n}(x) \tag{52}$$

assuming for now local functions without spatial derivatives. (See Sec. V D 4 for non-locality.) The expansion of β allows us to deal with inverse-triad corrections and local holonomy corrections at the same time. Holonomy corrections will then not appear as periodic functions such as $\sin(\delta K_\varphi)/\delta$ for K_φ in spherical symmetry, but as perturbative terms of a power series in K_φ. Such an expansion is more consistent with the perturbative nature of these higher-order corrections, which are expected in a similar form from higher-curvature terms or quantum back-reaction. Including all terms in a power series of $\sin(\delta K_\phi/\delta)$, even tiny ones at high orders, but ignoring quantum back-reaction would not be consistent. An expansion also makes it more clear how terms of higher order in v_{ij} can be combined with higher spatial derivatives of the metric.

We insert these expansions into (48) and first set $v_{ij}(x) = 0$ to obtain

$$\begin{aligned}&2L^{ij}(x)\beta^{\emptyset}_{|j}(x)\delta_{|i}(x,x') + 2L^{ij}(x)\beta^{\emptyset}(x)\delta_{|ij}(x,x')\\ &-(x\leftrightarrow x') = 0\,.\end{aligned} \tag{53}$$

We multiply by test functions $a(x)$ and $b(x')$ and integrate over x and x', drop total divergences and terms that vanish due to the symmetry of indices of $L^{ij}(x)$, cancel some other terms and are left with

$$\int \mathrm{d}x L^{ij}_{|j}(x)\beta^{\emptyset}(a_{|i}b - ab_{|i}) = 0\,. \tag{54}$$

Since a, b, $a_{|i}$ and $b_{|i}$ can be chosen independently, we conclude that $L^{ij}_{|j}(x)\beta^{\emptyset} = 0$. Note that $\beta^{\emptyset} \neq 0$ generically, so that we have

$$L^{ij}_{|j}(x) = 0\,. \tag{55}$$

We return to equation (48), do a functional differentiation with respect to $v_{kl}(z)$ and then set $v_{ij}(x)$ to zero everywhere. With the notation

$$\delta^{kl}_{ab}(x,z) = \frac{1}{2}(\delta^k_a\delta^l_b + \delta^l_a\delta^k_b)\,\delta(x,z) \tag{56}$$

we have

$$\begin{aligned}
&\frac{\delta L^{\emptyset}(x)}{\delta g_{kl}(x')}\delta(x',z) + 4L^{ijab}(x)\beta^{\emptyset}_{|j}\delta_{|i}(x,x')\delta^{kl}_{ab}(x,z) + 2L^{ij}(x)\delta_{|i}(x,x')\left(\beta^{ab}_{|j}\delta^{kl}_{ab}(x,z) + \beta^{ab}\delta^{kl}_{ab|j}(x,z)\right)\\
&+4L^{ijab}\delta^{kl}_{ab}(x,z)\delta_{|ij}(x,x') + 2L^{ij}(x)\beta^{ab}\delta^{kl}_{ab}\delta_{|ij}(x,x') - (x \leftrightarrow x') \qquad (57)\\
=\ & -\frac{\delta L^{\emptyset}(x')}{\delta g_{kl}(x)}\delta(x,z) + \left(4L^{ijkl}(x)\beta^{\emptyset}_{|j} + 2L^{ij}(x)\beta^{kl}_{|j}\right)\delta_{|i}(x,x')\delta(x,z) + 2L^{ij}(x)\beta^{kl}\delta_{|j}(x,z)\delta_{|i}(x,x')\\
&+\left(2L^{ij}(x)\beta^{kl} + 4L^{ijkl}(x)\beta^{\emptyset}\right)\delta_{|ij}(x,x')\delta(x,z) - (x \leftrightarrow x') = 0\,. \qquad (58)
\end{aligned}$$

We use

$$\begin{aligned}
2L^{ij}(x)\beta^{kl}\delta_{|j}(x,z)\delta_{|i}(x,x') \;=\; & \left(2L^{ij}(x)\beta^{kl}\delta(x,z)\delta_{|i}(x,x')\right)_{|j} - 2L^{ij}_{|j}(x)\beta^{kl}\delta(x,z)\delta_{|i}(x,x')\\
& - 2L^{ij}(x)\beta^{kl}_{|j}\delta(x,z)\delta_{|i}(x,x') - 2L^{ij}(x)\beta^{kl}\delta(x,z)\delta_{|ij}(x,x')\,,
\end{aligned} \tag{59}$$

drop the total divergence term in (59), and insert $L^{ij}_{|j}(x) = 0$ from (55):

$$\left(-\frac{\delta L^{\emptyset}(x')}{\delta g_{kl}(x)} + 4L^{ijkl}(x)\beta^{\emptyset}_{|j}\delta_{|i}(x,x') + 4L^{ijkl}\beta^{\emptyset}\delta_{|ij}(x,x')\right)\delta(x,z) - (x \leftrightarrow x') = 0. \tag{60}$$

This equation can be solved as in [3] where $\beta^{\emptyset} = 1$: define

$$A^{ij}(x,x') = \frac{\delta L^{\emptyset}(x)}{\delta g_{ij}(x')} - 4L^{ijkl}(x')\left(\beta^{\emptyset}_{|l}(x')\delta_{|k}(x',x) + \beta^{\emptyset}(x')\delta_{|kl}(x',x)\right) \tag{61}$$

and rewrite (60) as

$$A^{ij}(x,x')\delta(x',z) - A^{ij}(x',x)\delta(x,z) = 0\,. \tag{62}$$

Integrating over x', we find $A^{ij}(x,x'') = F^{ij}(x)\delta(x,x'')$ with $F^{ij}(x) = \int \mathrm{d}^3x' A^{ij}(x',x)$, a function of only one variable, and thus

$$\begin{aligned}
&\frac{\delta L^{\emptyset}(x)}{\delta g_{ij}(x')} = F^{ij}(x)\delta(x,x')\\
&+4L^{ijkl}(x')\left(\beta^{\emptyset}_{|l}(x')\delta_{|k}(x',x) + \beta^{\emptyset}(x')\delta_{|kl}(x',x)\right)\,.
\end{aligned} \tag{63}$$

2. *Coefficients*

As a spatial scalar density, $L^{\emptyset}$ can depend on the metric and its spatial derivatives only via the metric itself and suitable contractions of products of the spatial Riemann tensor. To second order in spatial derivatives,

$$L^{\emptyset}(x) = L^{\emptyset}(g_{ij}(x),\, {}^{(3)}R_{ij}(x))\,, \tag{64}$$

a fact, used in [3], that remains true in the deformed case with our assumption that the spatial part of the algebra stays classical. Define

$$\varphi^{ij} = \frac{\partial L^{\emptyset}(g_{kl},\, {}^{(3)}R_{kl})}{\partial g_{ij}} \qquad \Phi^{ij} = \frac{\partial L^{\emptyset}(g_{kl},\, {}^{(3)}R_{kl})}{\partial\, {}^{(3)}R_{ij}} \tag{65}$$

and write

$$\delta L^{\emptyset} = \left(\varphi^{ij} + \frac{1}{2}\,{}^{(3)}R^{i}{}_{kl}{}^{j}\Phi^{kl} + \frac{1}{4}\,{}^{(3)}R^{i}_{k}\Phi^{kj} + {}^{(3)}R^{j}_{k}\Phi^{ki}\right)\delta g_{ij}$$
$$+\frac{1}{4}\left(\Phi^{ik}g^{jl} + \Phi^{il}g^{jk} + \Phi^{jk}g^{il} + \Phi^{jl}g^{ik} - 2\Phi^{ij}g^{kl} - 2\Phi^{kl}g^{ij}\right)\delta g_{ij|kl}\,. \tag{66}$$

From (63), we also have

$$\delta L^{\emptyset} = \delta g_{ij}\left(F^{ij} + 4L^{ijkl}_{|lk}\beta^{\emptyset} + 4L^{ijkl}_{|l}\beta^{\emptyset}_{|k}\right)\delta g_{ij|k}\left(8L^{ijkl}_{|l}\beta^{\emptyset} + 4L^{ijkl}\beta^{\emptyset}_{|l}\right) + \delta g_{ij|kl}\left(4L^{ijkl}\beta^{\emptyset}\right)\,. \tag{67}$$

Comparing the various coefficients, we get

$$L^{ijkl}\beta^{\emptyset} = \frac{1}{16}\left(\Phi^{ik}g^{jl} + \Phi^{il}g^{jk} + \Phi^{jk}g^{il} + \Phi^{jl}g^{ik} - 2\Phi^{ij}g^{kl} - 2\Phi^{kl}g^{ij}\right) \tag{68}$$

$$2L^{ijkl}_{|l}\beta^{\emptyset} + L^{ijkl}\beta^{\emptyset}_{|l} = 0 \tag{69}$$

$$F^{ij} + 4L^{ijkl}_{|lk}\beta^{\emptyset} + 4L^{ijkl}_{|l}\beta^{\emptyset}_{|k} = \varphi^{ij} + \frac{1}{2}\,{}^{(3)}R^{i}{}_{kl}{}^{j}\Phi^{kl} + \frac{1}{4}\,{}^{(3)}R^{i}_{k}\Phi^{kj} + \frac{1}{4}\,{}^{(3)}R^{j}_{k}\Phi^{ki}\,. \tag{70}$$

Thus, $0 = 2L^{ijkl}_{|l}\beta^{\emptyset} + L^{ijkl}\beta^{\emptyset}_{|l} = -\beta^{\emptyset}_{|l}L^{ijkl} + 2(L^{ijkl}\beta^{\emptyset})_{|l}$. We compute each term using (68), and write

$$0 = -\frac{\beta^{\emptyset}_{|l}}{16\beta^{\emptyset}}\left(\Phi^{ik}g^{jl} + \Phi^{il}g^{jk} + \Phi^{jk}g^{il} + \Phi^{jl}g^{ik} - 2\Phi^{ij}g^{kl} - 2\Phi^{kl}g^{ij}\right)$$
$$+\frac{1}{8}\left(\Phi^{ik|j} + g^{jk}\Phi^{il}_{|l} + \Phi^{jk|i} + g^{ik}\Phi^{jl}_{|l} - 2\Phi^{ij|k} - 2\Phi^{kl}_{|l}g^{ij}\right)\,. \tag{71}$$

We contract this with g_{ij}, use $\delta^{i}_{i} = 3$, and denote Φ^{i}_{i} as Φ:

$$\frac{\beta^{\emptyset}_{|l}}{8\beta^{\emptyset}}\left(\Phi^{kl} + \Phi g^{kl}\right) - \frac{1}{4}\left(\Phi^{kl}_{|l} + \Phi^{ij|k}g_{ij}\right) = 0\,. \tag{72}$$

Note that $\Phi^{ij|k}g_{ij} = \Phi^{|k} = (\Phi g^{kl})_{|l}$. With $\Phi^{kl} + \Phi g^{kl}$ denoted as $\bar{\Phi}^{kl}$,

$$0 = \frac{\beta^{\emptyset}_{|l}}{8\beta^{\emptyset}}\bar{\Phi}^{kl} - \frac{1}{4}\bar{\Phi}^{kl}_{|l} = \frac{1}{4}\sqrt{|\beta^{\emptyset}|}\left(\frac{\beta^{\emptyset}_{|l}\mathrm{sgn}(\beta^{\emptyset})}{2|\beta^{\emptyset}|^{\frac{3}{2}}}\bar{\Phi}^{kl} - |\beta^{\emptyset}|^{-\frac{1}{2}}\bar{\Phi}^{kl}_{|l}\right) = -\frac{1}{4}\sqrt{|\beta^{\emptyset}|}\left(|\beta^{\emptyset}|^{-\frac{1}{2}}\bar{\Phi}^{kl}\right)_{|l}\,. \tag{73}$$

Again maintaining our assumption of an unmodified spatial structure, the only covariantly constant 2-tensors constructed from the metric and its derivatives up to second order are the metric itself and the spatial Einstein tensor. Noting the density weight one of $\bar{\Phi}^{kl}$, inherited from $L^{\emptyset}$, we conclude that

$$\frac{\bar{\Phi}^{kl}}{\sqrt{|\beta^{\emptyset}|}} = A\sqrt{\det g}\left({}^{(3)}R^{kl} - \frac{1}{2}\,{}^{(3)}Rg^{kl}\right) + B\sqrt{\det g}\,g^{kl} \tag{74}$$

where A and B are constants. This gives

$$\Phi^{kl} = A\sqrt{|\beta^{\emptyset}|\det g}\left({}^{(3)}R^{kl} - \frac{3}{8}\,{}^{(3)}Rg^{kl}\right) + \frac{B}{4}\sqrt{|\beta^{\emptyset}|\det g}\,g^{kl}\,. \tag{75}$$

Inserting this into (71), we find, after cancelling terms, that

$$\frac{A\sqrt{|\beta^{\emptyset}|\det g}}{8}\Big[{}^{(3)}R^{ik}g^{jl} + {}^{(3)}R^{il}g^{jk} + {}^{(3)}R^{jk}g^{li} + {}^{(3)}R^{jl}g^{ik} - 2\,{}^{(3)}R^{ij}g^{kl} - 2\,{}^{(3)}R^{kl}g^{ij}$$
$$-\frac{3}{8}\,{}^{(3)}R\left(2g^{ik}g^{jl} + 2g^{jk}g^{il} - 4g^{ij}g^{kl}\right)\Big]_{|l} = 0\,.$$

For this to be satisfied for general metrics, we must set $A = 0$. Writing $B = \frac{1}{4\pi G}$,

$$\Phi^{kl} = \frac{1}{16\pi G}\sqrt{|\beta^{\emptyset}|\det g} g^{kl} \,. \tag{76}$$

Then, from (68)

$$\begin{aligned} L^{ijkl} &= \frac{1}{16^2\pi G\beta^{\emptyset}}\left(\sqrt{|\beta^{\emptyset}|\det g} g^{ik}g^{jl} + \sqrt{|\beta^{\emptyset}|\det g} g^{il}g^{jk} + \sqrt{|\beta^{\emptyset}|\det g} g^{jk}g^{il}\right. \\ &\quad \left. + \sqrt{|\beta^{\emptyset}|\det g} g^{jl}g^{ik} - 2\sqrt{|\beta^{\emptyset}|\det g} g^{ij}g^{kl} - 2\sqrt{|\beta^{\emptyset}|\det g} g^{kl}g^{ij}\right) = \frac{\sqrt{\det g}\,\mathrm{sgn}\beta^{\emptyset}}{64\pi G\sqrt{|\beta^{\emptyset}|}}\left(g^{i(k}g^{l)j} - g^{ij}g^{kl}\right). \end{aligned} \tag{77}$$

We also have

$$\frac{\partial L^{\emptyset}(g_{kl}, {}^{(3)}R_{kl})}{\partial\, {}^{(3)}R_{ij}} = \Phi^{ij} = \frac{1}{16\pi G}\sqrt{|\beta^{\emptyset}|\det g} g^{ij} \tag{78}$$

from the definition (65). We integrate this to get

$$L^{\emptyset} = \frac{1}{16\pi G}\sqrt{\det g}\left(\sqrt{|\beta^{\emptyset}|}\, {}^{(3)}R + f(g)\right) \tag{79}$$

where, for a scalar density, $f(g) = -2\lambda$ must be a constant, the cosmological constant. (The previous equations do not determine $f(g)$ because it would follow from φ^{ij} according to (65), which by (70) is related to the free function F^{ij}.)

Combining (77) and (79), the regained Lagrangian up to second order is

$$L = \frac{\sqrt{\det g}}{16\pi G}\left(\frac{\mathrm{sgn}\beta^{\emptyset}}{\sqrt{|\beta^{\emptyset}|}}\frac{v_{ij}v^{ij} - v^i_i v^j_j}{4} + \sqrt{|\beta^{\emptyset}|}\, {}^{(3)}R - 2\lambda\right). \tag{80}$$

For $\beta^{\emptyset} = 1$, the classical Lagrangian is recovered with $v_{ij} = 2K_{ij}$ related to extrinsic curvature. But already to second order in derivatives, loop quantum gravity implies corrections to the Lagrangian from inverse-triad corrections with $\beta^{\emptyset} \neq 1$, a property that cannot be mimicked by any form of higher-curvature effective actions. Also holonomy corrections cannot provide a similar modification because they always come with higher powers of v_{ij}. Inverse-triad corrections can thus easily be distinguished from other quantum effects. (Holonomy corrections can provide similar modifications if the v_{ij} expansion is resummed; see Sec. VI C 1.)

The correction function $\beta^{\emptyset}[g_{ij}]$ relevant for these corrections must be scalar, which is not possible classically if only the metric can be used. For this reason, the full dynamics is more unique than the spherically symmetric one, where E^x is a scalar metric component without a density weight in the reduced model. In an effective formulation of quantum gravity, additional quantities become available that explicitly refer to properties of an underlying state, such as the discreteness scale in loop quantum gravity. It is then possible to construct nontrivial scalars of density weight zero by referring to the metric and state parameters, such as elementary fluxes [9].

Compared with the results in spherical symmetry, the full effective action is more unique, as already discussed. Other properties of the corrections are, however, very similar: The correction function β features in the same way in the curvature potential. Also the kinetic term is corrected in the same way, if we only note that a factor of $\sqrt{|\beta^{\emptyset}|}$ was obtained in spherical symmetry, where we used momenta K_x and K_φ instead of the normal change v_{ij} of the metric. If we substitute the normal changes $\delta H/\delta K_x$ and $\delta H/\delta K_\varphi$ for K_x and K_φ in spherical symmetry, we also obtain a kinetic term divided by $\sqrt{|\beta^{\emptyset}|}$. The sign of $\beta^{\emptyset}$ appears in different places in our expressions for spherical symmetry and the full theory, but the relative sign between the curvature and the kinetic terms is the same. The absolute placement of the sign is ambiguous because in the derivations it first appears in derivatives, for instance when we introduce B_1 after (35), or in (73).

3. Higher orders

To second order, Φ^{kl} determines both L^{ijkl} from (68) and $\partial L^{\emptyset}/\partial\, {}^{(3)}R_{ij}$ from (65), ensuring that time derivatives of g_{ij} and spatial Ricci contributions are combined to space-time covariant curvature terms. The same interplay is repeated for higher orders in the v-expansion, although with an increasing number of terms.

For the next order, as an example, we start again from (48) and gather all terms which are quadratic in $v_{ij}(x)$ and its derivatives.

$$\frac{\delta L^{ab}(x)}{\delta g_{ij}(x')} v_{ab}(x) v_{ij}(x') + 6L^{abcdij} v_{ab}(x) v_{cd}(x) \beta^{\emptyset}_{|j} \delta_{|i}(x,x') + 4L^{abij}(x) v_{ab}(x) \left(\beta^{ef} v_{ef}(x)\right)_{|j} \delta_{|i}(x,x')$$
$$+2L^{ij}(x) \left(\beta^{cdef} v_{cd}(x) v_{ef}(x)\right)_{|j} \delta_{|i}(x,x') + 6L^{abcdij} v_{ab}(x) v_{cd}(x) \beta^{\emptyset} \delta_{|ij}(x,x') + 4L^{abij} v_{ab}(x) \beta^{cd} v_{cd}(x) \delta_{|ij}(x,x')$$
$$+2L^{ij}(x) \beta^{cdef} v_{cd}(x) v_{ef}(x) \delta_{|ij}(x,x') - (x \leftrightarrow x') \,. \tag{81}$$

We multiply this by two test functions, $a(x)$ and $b(x')$ and integrate over x and x'. After integrating by parts, discarding total divergences, removing terms that disappear due to the symmetry and anti-symmetry of various indices, and using (55), we arrive at

$$\int \mathrm{d}x \mathrm{d}x' \left(\frac{\delta L^{ab}(x)}{\delta g_{ij}(x')} - \frac{\delta L^{ij}(x')}{\delta g_{ab}(x)}\right) v_{ab}(x) v_{ij}(x') a(x) b(x') - \int \mathrm{d}x \left(6L^{abcdij} v_{ab}(x) v_{cd}(x)\right)_{|j} \beta^{\emptyset} \left(a(x) b_{|i}(x) - a_{|i}(x) b(x)\right)$$
$$- \int \mathrm{d}x \left(4L^{abij} v_{ab}(x)\right)_{|j} \beta^{cd} v_{cd}(x) \left(a(x) b_{|i}(x) - a_{|i}(x) b(x)\right) = 0 \,. \tag{82}$$

Since $v_{ab}(x)$, $v_{ab|j}(x)$, $a(x)$, $b(x)$, $a_{|i}(x)$ and $b_{|i}(x)$ can all be varied independently, we arrive at the following three conditions: First, setting $\left(a(x) b_{|i}(x) - a_{|i}(x) b(x)\right) = 0$, we get

$$\left(\frac{\delta L^{ab}(x)}{\delta g_{ij}(x')} - \frac{\delta L^{ij}(x')}{\delta g_{ab}(x)}\right) v_{ab}(x) v_{ij}(x') a(x) b(x') = 0 \,. \tag{83}$$

Following the arguments in [3], we see that this eventually implies

$$\frac{\delta L^{ab}(x)}{\delta g_{ij}(x')} - \frac{\delta L^{ij}(x')}{\delta g_{ab}(x)} = 0. \tag{84}$$

This equation restricts the form of terms linear in v_{ij} in the action, which are absent anyway if the theory is time-reversal invariant. Then setting $v_{ab|j}(x) = 0$,

$$6L^{abcdij}_{|j} \beta^{\emptyset} + 4L^{ijab}_{|j} \beta^{cd} = 0 \,. \tag{85}$$

And finally:

$$12L^{abcdij} \beta^{\emptyset} + 4L^{abij} \beta^{cd} = 0. \tag{86}$$

We relabel indices, multiply (86) with $\beta^{\emptyset}_{|j}$ and use (69) to rewrite it.

$$\begin{aligned} & 12L^{ijklmn} \beta^{\emptyset} \beta^{\emptyset}_{|j} + 4L^{ijkl} \beta^{mn} \beta^{\emptyset}_{|j} \\ = \; & 12L^{ijklmn} \beta^{\emptyset} \beta^{\emptyset}_{|j} - 8L^{ijkl}_{|j} \beta^{mn} \beta^{\emptyset} \\ = \; & 12L^{ijklmn} \beta^{\emptyset} \beta^{\emptyset}_{|j} - 8L^{mnij}_{|j} \beta^{kl} \beta^{\emptyset} = 0 \,. \end{aligned} \tag{87}$$

(We use $L^{ijklmn} = L^{ijmnkl}$, referring to the definition in (51).) Using (85), we can write $24L^{ijklmn}_{|l} (\beta^{\emptyset})^2 + 24L^{ijklmn} \beta^{\emptyset} \beta^{\emptyset}_{|l} = 0$. Generically, $\beta^{\emptyset} \neq 0$, and so we have $\left(L^{ijklmn} \beta^{\emptyset}\right)_{|l} = 0$ solved by the classical covariantly constant quantities with the corresponding index structure, divided by $\beta^{\emptyset}$.

The third order in v_{ij} will therefore have terms with a factor of $1/\beta^{\emptyset}$ times corresponding orders possible for higher-curvature actions, while the quadratic order had a factor of $1/\sqrt{|\beta^{\emptyset}|}$, and the zeroth order a factor of $\sqrt{|\beta^{\emptyset}|}$. The same pattern is repeated at higher orders in the v-expansion: To order n in v_{ij}, we have terms as in higher-curvature actions but multiplied with $|\beta^{\emptyset}|^{(1-n)/2}$. To see this, we notice that Eq. (48), when expanded by powers of v_{ij}, has a first term which contains expansion coefficients of $L^{ij\cdots}$ two orders lower than the rest, which are all multiplied with $\beta^{\emptyset}$. If we use the equation to derive the L-coefficients by recurrence, we solve for a coefficient two orders higher by dividing by $\beta^{\emptyset}$. Starting with zeroth order in v_{ij} of magnitude $\sqrt{|\beta^{\emptyset}|}$ in the prefactor, the quoted orders follow. (If the corrected theory is not time-reversal invariant and odd orders appear in the v-expansion, the same powers of $|\beta^{\emptyset}|$ per order are obtained.)

4. *Non-locality*

So far, we have assumed only a local dependence on v_{ij}, with no spatial derivatives of v_{ij} that would otherwise be implied by a derivative expansion. In the classical case, locality follows from the relation of v_{ij} to extrinsic curvature, but it can easily be violated by some of the correction functions in quantum gravity.

In an effective action, non-locality usually makes itself noticeable in a derivative expansion of the fields. The basic equation (44) is valid also for non-local theories, without explicit terms with spatial derivatives of v_{ij}. However, (48) must be replaced by (49), and the general expansions (51) and (52) must also include terms with spatial derivatives of v_{ij}. We now define

$$L(x) = \sum_{n=0}^{\infty} \sum_{N_1,\ldots,N_n=0}^{\infty} L^{(i_1,j_1,k_1^{(1)},\ldots,k_1^{(N_1)}),\ldots,(i_n,j_n,k_n^{(1)},\ldots,k_n^{(N_n)})}[g_{ij}] v_{i_1 j_1|k_1^{(1)}\cdots k_1^{(N_1)}} \cdots v_{i_n j_n|k_n^{(1)}\cdots k_n^{(N_n)}} \quad (88)$$

$$\beta = \sum_{n=0}^{\infty} \sum_{N_1,\ldots,N_n=0}^{\infty} \beta^{(i_1,j_1,k_1^{(1)},\ldots,k_1^{(N_1)}),\ldots,(i_n,j_n,k_n^{(1)},\ldots,k_n^{(N_n)})}[g_{ij}] v_{i_1 j_1|k_1^{(1)}\cdots k_1^{(N_1)}} \cdots v_{i_n j_n|k_n^{(1)}\cdots k_n^{(N_n)}} \,. \quad (89)$$

Derivative terms in the expansion of β then require new terms in the Lagrangian that contain spatial derivatives. Going through the recurrence, an order n in the v-expansion again receives a coefficient of $|\beta^{\emptyset}|^{(1-n)/2}$.

In this context, we can distinguish between two expansions of the action, one by powers of v_{ij} and its spatial derivatives as in (88), and one by the total order of derivatives. The total order of derivatives is the crucial one for a comparison with higher-curvature terms in an effective action, which come arranged by the order of time and space derivatives. With v_{ij} related to the normal change of the metric, it counts as a derivative (by time) of order one. A term of $v_{i_1 j_1|k_1^{(1)}\cdots k_1^{(N_1)}}$ counts as a derivative of order $1 + N_1$, and therefore a general expansion term in (88) with coefficient $L^{(i_1,j_1,k_1^{(1)},\ldots,k_1^{(N_1)}),\ldots,(i_n,j_n,k_n^{(1)},\ldots,k_n^{(N_n)})}$ counts as a derivative of order $\sum_{i=1}^{n}(1+N_i) = n + \sum_{i=1}^{n} N_i$. Terms of the same v-order n, that is with the same number of factors of v_{ij} or its spatial derivatives, have different derivative orders of at least n. If we reorganize the expansion by derivative orders N, keeping track of $\beta^{\emptyset}$-factors that depend only on the v-order, we obtain effective-action terms of the schematic form

$$|\beta^{\emptyset}|^{(1-N)/2} v^N + |\beta^{\emptyset}|^{(2-N)/2} (v^{N-1} g' + v^{N-2} v') + |\beta^{\emptyset}|^{(3-N)/2} \left(v^{N-2}(g'' + (g')^2) + v^{N-3}(v'' + v'g') + v^{N-4}(v')^2 \right) + \cdots \,.$$

The highest power of $1/\sqrt{|\beta^{\emptyset}|}$ for a given derivative order is always obtained for the term v^N free of spatial derivatives. For small $\beta^{\emptyset}$, time derivatives in a derivative or curvature expansion are dominant.

VI. APPLICATIONS AND CONCLUSIONS

One of the main results of this paper, of general importance for loop quantum gravity, follows from the effective action (80), valid to second order in extrinsic curvature. Although we did allow for holonomy and higher-curvature corrections as well, only inverse-triad corrections are active at this order. This result is an independent confirmation, in addition to [1, 2, 18], that inverse-triad corrections can be much more significant than higher-curvature and holonomy corrections, both of which occur only at higher orders in v_{ij} and are of the tiny size $\ell_{\mathrm{P}}^2/\ell_{\mathcal{H}}^2$ throughout most of nearly isotropic cosmology with the Hubble distance $\ell_{\mathcal{H}}$. Our calculations show, for the first time, how different quantum effects in loop quantum gravity without any symmetry assumptions can be included all at once, but still show their own characteristic consequences. The complete correction function β in the constraint algebra may contain contributions from both inverse-triad and holonomy corrections, including non-local effects, but it is only the v-independent part $\beta^{\emptyset}$ which appears at second order of the effective action. This coefficient is affected by inverse-triad corrections, which therefore present the most important modification of the classical dynamics unless curvature is extremely large. Holonomy corrections, on the other hand, modify terms of higher order in v_{ij}; they mix with higher-curvature terms and can rarely be used in isolation. Moreover, U(1) calculations of inverse-triad correction functions are reliable because non-Abelian features would change merely the higher-v behavior.

The clear separation of some of the corrections allows us to discuss their cosmological consequences in very general terms.

A. Enhanced BKL scenario and the absence of singularities in consistent loop quantum gravity

All v_{ij}-terms in the effective action (80), to all orders, have at least one additional factor of $1/\beta^{\emptyset}$ compared with the spatial curvature term at zeroth order (or a factor of $1/\sqrt{|\beta^{\emptyset}|}$ if there are linear terms in v_{ij}, breaking time-reversal invariance). At higher orders, as shown in Secs. V D 3 and V D 4, v_{ij}-terms free of spatial derivatives have at least an additional factor of $1/\sqrt{|\beta^{\emptyset}|}$ compared to spatial-derivative terms of the same derivative order. When $\beta^{\emptyset}$ is very small, all spatial derivatives and curvature potentials are suppressed compared with the normal change of the metric in $v_{ij} = N^{-1}\{g_{ij}, H[N]\}$. Inverse-triad corrections, computed in Abelian models [32], imply that $\beta^{\emptyset}$ approaches zero for vanishing components of the densitized triad, right at classical singularities. As we approach such a singularity, quantum corrections become stronger, which could altogether stop the evolution down to smaller volumes. If this is the case, the singularity is resolved. However, such "bounces" have been difficult to generalize beyond the simple models in which they can be realized explicitly (see also Sec. VI C below), and therefore it is not guaranteed that vanishing components of the densitized triad can always be avoided. However, if such small values are approached, inverse-triad corrections become significant and suppress spatial derivatives. The evolution then follows a nearly homogeneous behavior of Bianchi-I type, for which singularity resolution in loop quantum cosmology can be shown in general terms by quantum hyperbolicity [52–55], based on properties of difference equations for wave functions. Even without symmetry assumptions and without restricting the class of quantum corrections included, the dynamics of loop quantum gravity is singularity-free. The same mechanism is hereby shown to apply in symmetric models [52, 54, 56, 57] and the full theory.

The concrete mechanism is reminiscent of the BKL scenario [58] in that spatial derivatives are suppressed and the dynamics becomes almost homogeneous near singularities. The present scenario, however, is much more general. We need not rely on details of the evolution because it is terms in the effective action itself that show the suppression. Moreover, the arguments are easily seen to be independent on what gauge, or spatial slicing in the classical setting, is chosen, because they make use of a consistent and anomaly-free theory exhibiting general covariance (in a deformed sense). Spatial terms are suppressed even in the $\{H, H\}$-algebra itself. This feature is also responsible for the covariance of the mechanism: if β is very small, normal deformations of hypersurfaces, governed by $\{H, H\}$ as in (4), do not generate spatial displacements from D. With the suppression by small β, normal deformations form a subalgebra of the full hypersurface-deformation algebra and can be considered in separation, eliminating the need of homogeneity assumptions.

B. Dispersion relations and causality

Our results show how consistent deformations of the type (4), for which several examples have been found in models of loop quantum gravity as recalled in Section II, affect the form of action principles reconstructed from them. From this perspective, the universal modification — irrespective of the precise form of the correction function β — is that a new coefficient β rescales time derivatives relative to spatial derivatives in matter terms as well as gravitational ones. The usual d'Alembertian $\Box = -\partial_t^2 + g^{ij}\partial_i\partial_j$ is replaced by $\Box_\beta := -\beta^{-1}\partial_t^2 + g^{ij}\partial_i\partial_j$. Dispersion relations and propagation speeds are then modified in a compatible way for matter and gravity, as shown explicitly in the special cases considered in [47]. (Counterterms in perturbative realizations of consistency lead to interesting new effects for non-propagating modes [17, 18].) In particular, while $\beta \neq 1$ implies that speeds of massless modes differ from the classical speed of light, they all propagate at the same speed as light in space-time according to deformed relativity. All massless excitations propagate with the velocity $\sqrt{\beta}$ times the classical speed of light for $\beta > 0$. If $\beta < 0$, which is possible for holonomy corrections, the d'Alembertian changes to a Euclidean-signature Laplacian, and all propagation ceases.

C. Signature change

Holonomy corrections cannot easily be analyzed in general terms because their mixing with higher-curvature corrections requires the latter to be derived in detail, too. In loop quantum gravity, however, the derivation of higher-curvature terms or their analog in quantum back-reaction remains incomplete. But there is one general property of holonomy corrections realized when they are large and near their maximum value. When this is the case, we must be careful with the v-expansions used. One consequence, fortunately, can be seen very generally.

1. *The high-density regime in models of loop quantum gravity*

In existing consistent examples, holonomy corrections always have the following form: A connection or extrinsic-curvature component in the classical Hamiltonian constraint is replaced by a bounded and periodic function of the same component (possibly depending also on the triad). For instance, in spherical symmetry we can consistently replace K_φ by $\delta^{-1}\sin(\delta K_\varphi)$ with some parameter δ [15], and in isotropic models we can replace the isotropic connection component c by $\delta^{-1}\sin(\delta c)$ [11]. The parameter δ may depend on triad components E^x or a if lattice-refinement is realized [59, 60]. When these bounded functions take their maximum value, at $\delta K_\varphi = \pi/2$ or $\delta c = \pi/2$, holonomy corrections are large and the Hamiltonian constraint ensures that we are at high energy densities if matter is present. As recalled in Sec. II, in the constraint algebra we obtain a deformation with correction function $\beta(K_\varphi) = \cos(2\delta K_\varphi)$ and $\beta(c) = \cos(2\delta c)$, respectively.

These functions are negative when $\sin(\delta c)/\delta$ is near its maximum as a function of c, continuing with the example of nearly isotropic cosmology. More precisely, a modified Hamiltonian constraint of the form

$$-\frac{3}{8\pi G\gamma^2\delta^2}\sin^2(\delta c)\sqrt{|p|} + H_{\mathrm{matter}} = H\,, \tag{90}$$

as commonly used in isotropic loop quantum cosmology, implies, using $\{c, p\} = 8\pi\gamma G/3$, Hamiltonian equations

$\dot{p}=\{p,H\}=(\gamma\delta)^{-1}\sin(2\delta c)\sqrt{|p|}$ and

$$\dot{c} = -\frac{\sin^2(\delta c)}{2\gamma\delta^2\sqrt{|p|}} - \frac{c\sin(2\delta c)}{\gamma\delta\sqrt{|p|}}\frac{\mathrm{d}\log\delta}{\mathrm{d}\log p} + \frac{2\sin^2\delta c}{\gamma\delta^2\sqrt{|p|}}\frac{\mathrm{d}\log\delta}{\mathrm{d}\log p} + \frac{8}{3}\pi\gamma G\frac{\partial H_{\mathrm{matter}}}{\partial p}\,.$$

(With $\partial H_{\mathrm{matter}}/\partial p = \frac{3}{2}a\partial H_{\mathrm{matter}}/\partial a^3 = -\frac{3}{2}aP$, the usual pressure contribution $-4\pi GP$ to acceleration follows.) We can combine these two equations to compute the acceleration of the scale factor,

$$\ddot{a} = -\cos(2\delta c)\frac{\sin^2\delta c}{2\gamma^2\delta^2\sqrt{|p|}} - \frac{2\sin^4\delta c}{\gamma^2\delta^2\sqrt{|p|}}\frac{\mathrm{d}\log\delta}{\mathrm{d}\log p} - 4\pi G\cos(2\delta c)aP\,. \tag{91}$$

To distinguish different types of inflation, it is also useful to rewrite the acceleration equation as an equation for the derivative of the Hubble parameter $\mathcal{H}$:

$$\dot{\mathcal{H}} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -\cos(2\delta c)\frac{3\sin^2\delta c}{2\gamma^2\delta^2|p|} - \frac{\sin^4\delta c}{\gamma^2\delta^2|p|}\left(1+2\frac{\mathrm{d}\log\delta}{\mathrm{d}\log p}\right) - 4\pi G\cos(2\delta c)P\,. \tag{92}$$

If we assume a power-law form $\delta(p)=|p|^x$ with $-1/2<x<0$ generically [59, 60], the gravitational contributions to $\ddot{a}$ are positive, implying inflation from quantum geometry, if $\sin^2\delta c > (2(1-2x))^{-1}$ ($\sin^2\delta c > 1/4$ or $\delta c > \pi/6$ for the limiting case $x=-1/2$ considered in [61]). We have super-inflation with $\dot{\mathcal{H}}>0$ if $\sin^2\delta c > 3/(4(1-x))$ ($\sin^2\delta c > 1/2$ or $\delta c > \pi/4$ for $x=-1/2$). In terms of densities, according to the modified constraint equation (90) showing that the energy density ρ is proportional to $\sin^2(\delta c)$, we have the maximum density ρ_{max} when $\sin^2\delta c = 1$, inflation for $\rho > \rho_{\mathrm{max}}/(2(1-2x))$ and super-inflation for $\rho > 3\rho_{\mathrm{max}}/(4(1-x))$. (For $x\neq -1/2$, ρ_{max} depends on the dynamical discreteness scale $a\delta$.) During super-inflation, we always have $\cos(2\delta c)<0$, and for $x=-1/2$, the super-inflationary regime $\sin^2(\delta c)>1/2$ is exactly the one with $\cos(2\delta c) = 1-2\sin^2(\delta c) < 0$. Our remarks about spherical symmetry in Sec. II B show that we generically have $\beta<0$ at high curvature: We saw that $\beta<0$ at the maximum of the first holonomy correction function f_1, which in the cosmological context would correspond to the maximum of $\sin^2(\delta c)$ in the Friedmann equation. No matter what form f_1 has, depending on quantization ambiguities, at its maximum we always have negative β.

Classically, there can be acceleration only with negative pressure of a suitable size. But with holonomy corrections, the trigonometrical factors can turn the sign of $\dot{\mathcal{H}}$, providing matter-independent acceleration from quantum geometry. The correction function β contains the same factor of $\cos(2\delta c)$ that appears in the acceleration equation. We have a negative correction function throughout the regime where holonomy effects make $\dot{\mathcal{H}}$ positive, which is in the purported super-inflationary regime. When $\dot{\mathcal{H}}$ is turned positive by holonomy effects, we therefore do not have space-time but rather (deformed) Euclidean space, with the derivatives of a taken by spacelike rather than timelike coordinates. There is no evolution in Euclidean space, and no super-inflation even if derivatives of $\mathcal{H}$ are positive. (For $x<0$, there is still a weak form of power-law inflation at the beginning of the Lorentzian expansion phase. However, the phase is too brief, with only a small number of e-foldings, for the usual consequences of inflation to be realized.)

It is of interest to see what an effective action for this Euclidean chunk of space may look like. In our derivation of effective actions, applied to such a regime of large curvature, we can no longer expand the correction function β in K_φ or c when δK_φ or δc is near $\pi/2$, but we can expand them in $2\delta\bar{K}_\varphi := 2\delta K_\varphi - \pi$ or $2\delta\bar{c} := 2\delta c - \pi$, writing $\cos(2\delta K_\varphi) = \cos(2\delta\bar{K}_\varphi + \pi) = -\cos(2\delta\bar{K}_\varphi)$ or $\cos(2\delta c) = \cos(2\delta\bar{c}+\pi) = -\cos(2\delta\bar{c})$. The new coefficients $\delta\bar{K}_\varphi$ or $\delta\bar{c}$ are small near maximum density, and we can expand the correction function as well as the Lagrangian by their powers. (For cosmological perturbations around spatially flat isotropic models, we would expand in $\bar{v}_{ij} := v_{ij} - \frac{1}{2}\delta^{-1}\pi\delta_{ij}$.) Resumming higher-curvature terms by making use of the small barred quantities, we obtain effective actions as before. The main consequence of holonomy corrections then appears even at leading order in the expansion, for $\beta^\emptyset$ in the new expansion takes the value $\beta^\emptyset=-1$. At the point of maximum density, where $\delta\bar{K}_\varphi = 0$ or $\delta\bar{c}=0$ and therefore $\beta=\beta^\emptyset=-1$, the gravitational action becomes classical, albeit of Euclidean signature. (From Sec. II B, argued in spherically symmetric models, we recall that β turning negative is a general feature near the maximum density of holonomy-modified systems, independently of quantization holonomies.)

2. *Euclidean space instead of holonomy-induced super-inflation*

Negative $\beta^\emptyset$, in all models studied consistently so far, are a necessary consequence of holonomy modifications in the high-density phases in which they may resolve singularities. With negative $\beta^\emptyset$, however, the dispersion relation is positive definite and the hypersurface-deformation algebra is of Euclidean signature, as seen in Sec. III B. These consequences are consistent with a formal transformation from positive to negative β in (4) by the replacements of N or t by iN or it, respectively. With a Euclidean action, the initial/boundary-value problem changes its form significantly and propagation in time no longer exists. Loop quantum gravity, in this way, provides a concrete mechanism for signature change.

In loop quantum cosmology, going through the Planck regime near the big bang does therefore not at all correspond to a bounce, as minisuperspace models are sometimes interpreted as suggesting [62]. The big bang is rather a transition from Euclidean 4-dimensional space to Lorentzian space-time which only appears dynamical in the homogeneous background. This observation shows some of the pitfalls and unexpected subtleties of minisuperspace models. We are also reminded that we have to be careful with gauge-fixings or deparameterization, which do not determine the constraint algebra and cannot show the consequences seen here (see e.g. [51]). One example for difficulties with deparameterization of cosmological evolution is realized in models with a positive cosmological constant [63]. The range of internal time provided by a free, massless scalar ϕ does not match with the range of proper time τ of observers, with τ diverging at large volume while ϕ changes in a finite range. Extending the internal-time evolution to all values of ϕ is then unphysical because no observer could see the extended space-time solution. The Euclidean phase found here provides another example, requiring us to bound the range of internal time ϕ also at small volume in loop quantum cosmology. Classically, we know the space-time structure and all we need to ensure for a good internal time is that its rate of change $\mathrm{d}\phi/\mathrm{d}\tau$ does not become zero. With a deformed notion of space-time structure, the derivatives in background equations of motion may not refer to time at all, and therefore ϕ cannot be called an internal "time" even if it keeps changing with the background coordinates. We can start our internal time ϕ only when space-time turns Lorentzian [91].

In addition to these cautionary remarks for some scenarios in loop quantum cosmology, the new picture of signature change also provides larger unity among the different scenarios for singularity resolution. The main mechanism [52] is based on properties of the underlying difference equations that appear with a loop quantization [64], with difference operators on minisuperspace. The resulting recurrence scheme of the wave function depending on an integer geometrical quantity, taking both signs thanks to orientation, allows one to evolve uniquely from one side of the classical singularity in minisuperspace to the other. With unique evolution, the singularity is resolved in this picture of quantum hyperbolicity making use of geometrical internal time. A scenario of less generality is realized for deparameterizable models sourced by a scalar field when its energy is almost all kinetic. Here, using the scalar as internal time, the minisuperspace evolution is non-singular with a minimum volume achieved at high density.

These pictures look inconsistent at first sight, with the oriented volume used as unbounded recurrence variable in the first one, but bouncing back from a small value in the second one. With the results of this paper we see that what is inconsistent is not the role of volume in the recurrence, but rather the interpretation of evolution as a smooth bounce. In both cases, a collapsing branch of shrinking volume is connected to an expanding branch of growing volume by a non-classical space-time region. In the first picture, based on a recurrence analysis of discrete wave equations, the non-classical part is modeled as a tunneling process of the wave function through small volume, while it becomes a Euclidean chunk of 4-dimensional space in the second picture. This scenario not only unifies different mechanisms of singularity resolution in loop quantum cosmology, it also shows an interesting and unexpected overlap with the tunneling aspects of [65] and the postulated signature change of [66].

3. *The question of cosmological initial values*

We arrive at several new possibilities for cosmological model building: Initial values can be posed only in the Lorentzian regime. Holonomy-induced super-inflation, as it appears in the background evolution in loop quantum cosmology at high density, is not realized; the corresponding background piece is not part of space-time but rather corresponds to a Euclidean chunk of 4-dimensional space. (Super-inflation from inverse-triad corrections [67, 68] has a positive β and could happen in the space-time part.) While the background equations, taken on their own, might be interpreted as implying super-inflationary evolution, they fail to provide any insight into the correct initial/boundary-value problem. Only an extension at least to perturbative inhomogeneity, without gauge fixing or deparameterization so as to have access to the off-shell constraint algebra, can provide this important input, and it shows the Euclidean nature. With the corresponding boundary-value instead of initial-value problem, even the background equations can no longer be interpreted as evolution equations in time.

The Euclidean nature of high-density regimes with holonomy corrections have several unanticipated consequences for initial values in cosmology. One cannot use this phase to evolve or generate structure, or to pose initial conditions within it, such as at the bounce of maximum density. Models making use of the super-inflationary phase to supply initial values, even if only for the background equations as suggested for instance in [69], are not consistent with quantum geometry. It becomes, however, very natural to pose initial values right at the boundary of Euclidean space, cutting off super-inflation. This procedure would be similar to the usual choice of initial values or an intial vacuum state before slow-roll inflation, but providing stronger justification of the choice.

There are several advantages. First, we can pose well-defined initial values in a non-singular regime. Classically, if we go back as far as possible to pose initial conditions close to what can be considered the beginning, we end up at the big-bang singularity. If there is a bounce [70], we end up far back at large volume in the preceding collapse branch. In the deformed solutions with holonomy corrections of loop quantum gravity, we end up at the non-singular beginning of the Lorentzian branch, a clearly distinguished and non-singular moment in time. Secondly, methods of Euclidean quantum gravity may be used to shed light on what initial conditions one should expect. These initial conditions would not be transferred from the collapse phase bordering the Euclidean chunk at its other end: In Euclidean 4-space we must choose boundary conditions for a well-posed formulation of partial differential equations for inhomogeneity. This boundary includes the initial-value slice of the expanding branch of the universe model and the final-value surface of the collapsing branch. Field values on these surfaces can be specified independently and freely for a complete set of Euclidean boundary conditions. We could, for instance, evolve the collapsing branch from its initial data to obtain field values at one piece of the Euclidean boundary. Boundary conditions will then be completed by choosing values on the rest, including the initial-data surface of the expanding branch. Therefore, the final values of the collapse do not determine initial values for expansion. There is no deterministic evolution across the Euclidean high-density phase [92]. Rather, the scenario describes a beginningless beginning, with a concrete physical realization of a distinguished initial-value surface. Although our scenario has cyclic features in that it combines collapsing and expanding branches, connected by Euclidean space not causally but at least as manifolds, we do not encounter the entropy problem. Entropy, like anything else, will simply not be transmitted through the Euclidean piece.

D. Additional modifications

Non-local corrections are possible in our formalism, extended from [3], but have not yet been realized explicitly in effective actions. We have identified additional difficulties which may prevent simple realizations of consistent deformations: gravity and matter terms in the constraints can no longer satisfy the hypersurface-deformation algebra independently. Instead, there must be delicate cancellations between matter and gravity Poisson brackets so as to ensure that the total constraints satisfy a consistently deformed algebra.

In addition to non-locality, modifications to the spatial part of the constraint algebra would prevent the steps followed here from going through. From the perspective of effective constraints, modifications to the spatial part may not seem likely because these constraints are formulated for fields on some manifold, which may not obey the classical geometry but nevertheless is a collection of points labeled, for the formulation of physical theories, by coordinates. The choice of coordinates cannot matter for the physics, and so there must be relabelling invariance. Such an invariance, in turn, leads very generally to the spatial part of the constraint algebra just based on properties of the Lie derivative [3].

Also from the point of view of full loop quantum gravity, modifications to the part of the constraint algebra involving the diffeomorphism constraint may not be called for. This constraint, unlike the Hamiltonian constraint, is implemented directly by its action on subsets in space (points or graphs) without any regularization or modification required to quantize it consistently. The final verdict on this question has not arrived, however, as shown by recent attempts to construct diffeomorphism constraint operators amenable to a closed operator algebras for the constraints [71].

The constraint algebra opens the way to specific results for space-time geometry in loop quantum gravity, extending some minisuperspace results to more general situations. A crucial open issue remains: deriving consistent deformations in more general terms than available now. Our results here do not provide new cases of consistent deformations, because we must assume consistency in order to employ our algebra. But the new methods do show how different terms in a consistently modified Hamiltonian constraint must be related to one another, as seen in conditions for dispersion relations and in the relations of v^n-terms to spatial metric derivatives. Thus, our methods help in finding new consistent models. But even for existing ones, the effective actions obtained provide new insights and several unexpected cosmological consequences.

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[85] In addition to $D[N^i]$ and $H[N]$, there are primary constraints given by the momenta of the non-dynamical N and N^i. Their algebra just mimics the canonical structure, not space-time structure, and can thus be ignored for the purposes of this article.

[86] Such deformations are similar in spirit to doubly special relativity [72–75], but the two different concepts are not straightforwardly related: Doubly special relativity deforms the Poincaré algebra non-linearly, with corrections depending on algebra generators such as the energy. In (4), the Poincaré algebra is affected as well by the subalgebra of the hypersurface-deformation algebra with linear N and N^i in Cartesian coordinates, but correction functions β depend on phase-space variables g_{ij} and π^{ij}, not on the constraints as algebra generators. In some backgrounds, a relationship can nevertheless be established, as will be discussed elsewhere.

[87] Sometimes in models of loop quantum gravity, higher-curvature actions have been used as an ansatz to compare with quantum-geometry corrections in restricted contexts [76, 77]. However, the gauge-fixings or complete reductions to homogeneity employed to formulate consistent equations in such a procedure leave too many ambiguities and prevent sufficient access to the gauge content of the theory. A large class of corrections in homogeneous or gauge-fixed models is possible which would be ruled out by a consistent extension to inhomogeneity; and for a given corrected version of a homogeneous model, many different action principles can be found by such an analysis. They would all yield the same homogeneous equations, but differ uncontrollably regarding the dynamics of inhomogeneities.

[88] Triad variables are subject not only to the diffeomorphism and Hamiltonian constraints but also to the Gauss constraint. However, the latter does not appear in the space-time constraint algebra and plays no role for choosing the space-time gauge. Moreover, in loop quantum gravity it can easily be implemented and solved without quantum corrections, directly using the classical form of internal rotations of phase-space variables. Therefore, there is no harm in ignoring the Gauss constraint, as we will do in this paper. In some calculations of deformed algebras, the Gauss constraint had been fixed, but one can easily see that different gauge fixings all produce the same physical results. Therefore, no anomaly or deformation arises from the Gauss constraint.

[89] We use U(1)-expressions $\exp(i\delta A)$ instead of SU(2)-holonomies in this equation, as is often realized in symmetric models. With non-Abelian SU(2)-holonomies in the full theory, the holonomies in Poisson brackets or commutators may not completely cancel depending on the ordering, leaving also a dependence on A. But the leading term in an A-expansion will remain unchanged which, as we will see later, is the crucial contribution from inverse-triad corrections.

[90] These local modifications are "point holonomies" [78] obtained by exponentiating the scalar pointwise, without integrations as in holonomies. Of similar nature are the modifications of the scalar sector proposed in [79], replacing the scalar momentum with bounded exponentials. No constraint analysis has been made for the latter case coupled to gravity, but from our results in Section V C it will become clear that it cannot correspond to an undeformed algebra.

[91] There can be several reasons for internal times to be defined for finite ranges only, making them local in nature. Quantization techniques for totally constrained systems often require global internal times, but with effective constraints [27, 28] there is a systematic framework for quantum evolution by local internal times [80–82].

[92] Some indications of non-deterministic evolution through bounces of loop quantum cosmology can be found already at the level of background evolution, where cosmic forgetfulness implies that not all moments of a pre-big bang state can be recovered after the big bang [83, 84].