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Gravitational- and Self- Coupling of Partially Massless Spin 2

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ABSTRACT

We show that higher spin systems specific to cosmological spaces are subject to the same problems as models with Poincaré limits. In particular, we analyse partially massless (PM) spin 2 and find that both its gravitational coupling and nonlinear extensions suffer from the usual [background- and self-coupling] difficulties: Consistent free field propagation does not extend beyond background Einstein geometries. Then, using conformal Weyl gravity (CG), which consists of relative ghost PM and graviton excitations, we find that avoiding graviton-ghosts restricts CG-generated PM self-couplings to the usual, safe, Noether current cubic ones.

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1 Introduction

The consistency difficulties of massless and massive higher spin fields in $d = 4$ are by now well-explored, both regarding their coupling to gravity and other fields as well as possible self-interactions. Our aim here is to investigate these problems for partially massless (PM) theories [1, 2], which have the novel feature that their (anti) de Sitter ((A)dS) higher spin representations have no direct Poincaré counterparts. For this we employ Weyl—conformal—gravity (CG) as a tool. Even though CG is physically unacceptable (being fourth derivative order, its physical excitations are relatively ghost-like) it can be safely used when one of its two, graviton and PM [3], components can be fixed, while studying the other [4].

We will begin by reviewing PM and then show that it precisely characterizes CG solutions that are not conformally Einstein spaces. We then explain, using recent mathematical tools, how CG can be safely exploited for our consistency analyses of PM. The first question:

what are the most general geometrical fixed backgrounds in which PM consistently propagates?

can then be answered—they are essentially restricted to Einstein spaces. The second consistency question:

can one define a self-interacting version of the free field, even in Einstein vacuum?

will then be addressed, yielding a minor triumph as well: only the usual cubic, abelian Noether current-field coupling is generated via CG. We conclude with speculations regarding PM's possible cosmological and formal uses.

2 Review of PM and its CG embedding

The PM tensor field $\varphi_{\mu\nu}$ dynamics are defined in any Einstein background by the action

$$= - \int \sqrt{-g} \left[\frac{1}{2} (\nabla_\rho \varphi_{\mu\nu})^2 - (\nabla^\nu \varphi_{\mu\nu})^2 + \nabla^\mu \varphi \nabla^\nu \varphi_{\mu\nu} - \frac{1}{2} (\nabla_\mu \varphi)^2 \right] \quad (1)$$

$$+ \varphi^{\mu\nu} W_{\rho\mu\nu\sigma} \varphi^{\rho\sigma} + \frac{2\Lambda}{3} \left[(\varphi_{\mu\nu})^2 - \frac{1}{4} \varphi^2 \right], \quad \varphi := \varphi^\rho_\rho, \quad (2)$$

and field equations

$$\Delta \varphi_{\mu\nu} - 2 \nabla_{(\mu} \nabla^{\rho} \varphi_{\nu)\rho} + g_{\mu\nu} \nabla^{\rho} \nabla^{\sigma} \varphi_{\rho\sigma} + \nabla_{\mu} \nabla_{\nu} \varphi - g_{\mu\nu} \Delta \varphi - 2 W_{\rho\mu\nu\sigma} \varphi^{\rho\sigma} - \frac{4}{3} \Lambda (\varphi_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \varphi) = 0,$$

where $W_{\mu\nu\rho\sigma}$ is the Weyl tensor. This system is invariant under a double derivative gauge transformation

$$\delta \varphi_{\mu\nu} = (\nabla_{\mu} \partial_{\nu} + \frac{\Lambda}{3} g_{\mu\nu}) \alpha(x), \quad (3)$$

which is the tuned sum of a metric fluctuation diffeomorphism (with parameter $\partial_{\mu} \alpha(x)$) and a conformal transformation. This system is a hybridization of strictly massless and normal massive, Fierz-Pauli, spin 2. Indeed, there are three varieties of spin 2 excitations in dS: massive, massless and PM [1, 2]. In dS, PM propagates lightlike, positive energy (inside the maximally accessible intrinsic horizon), helicity $\pm 2, \pm 1$ excitations in a unitary representation of the isometry group [5, 7, 6, 8]. This degree of freedom (DoF) count relies on the gauge invariance (3) and the divergence constraint $\nabla^{\mu} \varphi_{\mu\nu} = \nabla_{\nu} \varphi$ implied by integrability of (1).

Interactions of PM in four dimensions are particularly interesting because it is rigidly $SO(4, 2)$ conformally invariant [9], just like its vector Maxwell counterpart. In fact, PM can be coupled to charged matter fields [10] (see also [11]). [Forming non-abelian multiplets is still an open problem.] Instead, we will be concerned with its self interactions, whose cubic vertices were first given in [12] using a Stückelberg approach.¹

Since Weyl transformations underlie PM's invariances (see (3)), CG is a natural tool for studying its interactions. While CG always has six excitations, the detailed spectra are background-dependent. About flat space, it has two massless tensors and a photon with the same signature as one of them [15], while in constant curvature backgrounds there is still a (cosmological) graviton, but now the (tensor+photon) combination becomes the PM mode with helicities $(\pm 2, \pm 1)$. In each case, the two sets of modes are relatively ghost-like. The relative sign between PM's helicities depends on that of Λ : In AdS, one can truncate the solution space to just the unitary, massless graviton [3, 16, 17] (for related analysis of higher derivative theories

¹A general calculus of higher derivative PM cubic vertices was developed in [13]. Also, it has recently been suggested that a PM limit of putative massive gravity theories could be a candidate for an interacting PM theory [14].

see [18, 19]). The dS story is the more interesting one because we can truncate, leaving either mode unitary; keeping the unitary PM mode, is the relevant case here.

The CG action is

$$S[g] = \frac{1}{8} \int \sqrt{-g} W^{\mu\nu\rho\sigma} W_{\mu\nu\rho\sigma} = \frac{1}{4} \int \sqrt{-g} (R^{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2), \quad (4)$$

and its field equation is the vanishing of the Bach tensor,

$$B_{\mu\nu} := -\Delta P_{\mu\nu} + \nabla^\rho \nabla_{(\mu} P_{\nu)\rho} + W_{\rho\mu\nu\sigma} P^{\rho\sigma}, \quad P_{\mu\nu} := \frac{1}{2}(R_{\mu\nu} - \frac{1}{6} g_{\mu\nu} R).$$

The Schouten tensor $P_{\mu\nu}$ measures the difference between Riemann and Weyl tensors, $R_{\mu\nu\rho\sigma} - W_{\mu\nu\rho\sigma} = g_{\mu\rho} P_{\nu\sigma} - g_{\nu\rho} P_{\mu\sigma} + g_{\nu\sigma} P_{\mu\rho} - g_{\mu\sigma} P_{\nu\rho}$, and is a mainstay of conformal models in all dimensions: its variation is a pure (double) gradient,

$$\delta g_{\mu\nu} = 2\alpha g_{\mu\nu} \Rightarrow \delta P_{\mu\nu} = -\nabla_\mu \partial_\nu \alpha.$$

The Bach tensor $B_{\mu\nu}$ is, of course, invariant under this rescaling. For our purposes, it is more convenient to work with the cosmological Schouten tensor,

$$\varphi_{\mu\nu} := -P_{\mu\nu} + \frac{\Lambda}{6} g_{\mu\nu}, \quad (5)$$

in terms of which $B_{\mu\nu}$ reads

$$\begin{aligned} B_{\mu\nu}(g, \varphi) &= \Delta \varphi_{\mu\nu} - 2 \nabla_{(\mu} \nabla^\rho \varphi_{\nu)\rho} + g_{\mu\nu} \nabla^\rho \nabla^\sigma \varphi_{\rho\sigma} + \nabla_\mu \nabla_\nu \varphi - g_{\mu\nu} \Delta \varphi \\ &\quad - 2 W_{\rho\mu\nu\sigma} \varphi^{\rho\sigma} - \frac{4}{3} \Lambda (\varphi_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \varphi) + O(\varphi^2). \end{aligned} \quad (6)$$

Consider now configurations such that the metric is close to an Einstein one with cosmological constant Λ (it is important to note that the set of Bach flat, but non-Einstein metrics is non-empty, see [20]). Then, by (5), $\varphi_{\mu\nu}$ is a small excitation and its field equation (6) is precisely the PM one in this background, the Schouten tensor's Weyl transformation implying the PM gauge invariance (3). We have now recovered CG's (linearized) PM subsector by holding the metric constant (or in other words, setting the metric to a non-dynamical background field). This key fact motivates our use of CG as a probe of PM for two basic higher spin questions: How general are the geometries in which it can propagate consistently? Does CG provide a useful starting-point for studying possible self-interactions of PM?

We next answer the first question: we use CG to generate a list of increasingly general metrics, from dS to Einstein to Bach, and show that there is indeed a natural barrier—one that is much closer to Einstein than, as one might reasonably conjecture, to Bach.

3 PM in a background

It has been established that there exist Weyl invariant field equations enjoying a double derivative gauge invariance in Bach-flat backgrounds [22]. This result suggests that Bach-flat is the most general background supporting consistent (linear) PM propagation. In detail, the operator from scalars to trace-free symmetric tensors,

$$\mathbf{P}_{\mu\nu} := \nabla_{\{\mu} \partial_{\nu\}} + \mathbf{P}_{\{\mu\nu\}} ,$$

permits a factorization of the Bach tensor as

$$\begin{aligned} B_{\mu\nu} &= \mathbf{M}_{\mu\nu}^{\rho\sigma} \mathbf{P}_{\rho\sigma} , \\ \mathbf{M}_{\mu\nu}^{\rho\sigma} &:= \delta_{\{\mu}^{\rho} \delta_{\nu\}}^{\sigma} \Delta - \delta_{\{\mu}^{\rho} \nabla^{\sigma} \nabla_{\nu\}} - \frac{1}{3} \delta_{\{\mu}^{\rho} \nabla_{\nu\}} \nabla^{\sigma} - W^{\rho}{}_{\mu\nu}{}^{\sigma} . \end{aligned} \tag{7}$$

We observe that $\mathbf{M}_{\mu\nu}^{\rho\sigma}$ gives the non-linear answer to the question posed in the Introduction: characterizing Bach-flat metrics that are not conformally Einstein (the latter are characterized in [23] and [24]). We see that those require the range of $\mathbf{P}_{\mu\nu}$ to intersect the kernel of $\mathbf{M}_{\mu\nu}$; the operator $\mathbf{M}_{\mu\nu}$ is also conformally invariant and maps trace-free symmetric tensors to trace-free symmetric tensors. Physically, it implies that the field equation

$$\mathbf{M}_{\mu\nu}^{\rho\sigma} \tilde{\varphi}_{\rho\sigma} = \Delta \tilde{\varphi}_{\mu\nu} - \nabla^{\sigma} \nabla_{\{\mu} \tilde{\varphi}_{\nu\}\sigma} - \frac{1}{3} \nabla_{\{\mu} \nabla^{\sigma} \tilde{\varphi}_{\nu\}\sigma} - W^{\rho}{}_{\mu\nu}{}^{\sigma} \tilde{\varphi}_{\rho\sigma} = 0 , \tag{8}$$

for a trace-free symmetric tensor $\tilde{\varphi}_{\mu\nu} =: \varphi_{\{\mu\nu\}} = \varphi_{\mu\nu} - \frac{1}{4} g_{\mu\nu} \varphi$, enjoys the double derivative gauge invariance (and associated double derivative Bianchi identity)

$$\delta \tilde{\varphi}_{\mu\nu} = \mathbf{P}_{\mu\nu} \alpha = (\nabla_{\{\mu} \nabla_{\nu\}} + \tilde{\mathbf{P}}_{\mu\nu}) \alpha ,$$

in Bach-flat backgrounds. This was the motivation for our original conjecture that PM fields could propagate in them. We now proceed to disprove it and give necessary consistency conditions for PM-compatible backgrounds.

The Bach tensor, since it arises from a metric variational principle, is necessarily divergence-free, $\nabla^{\mu} \mathbf{M}_{\mu\nu}^{\rho\sigma} \mathbf{P}_{\rho\sigma} = 0$. However, it is neither true

that $\nabla^\mu \mathbf{M}_{\mu\nu}^{\rho\sigma} = 0$, nor even that $\nabla^\mu \mathbf{M}_{\mu\nu}^{\rho\sigma} = O(\nabla)$ (rather this operator is cubic in derivatives). But consistent PM propagation relies on a divergence constraint²; for a PM field equation (derived from an action) this requirement is precisely expressed by the condition $\nabla^\mu \mathbf{M}_{\mu\nu}^{\rho\sigma} = O(\nabla)$.

The failure of the field equation (8) to imply an appropriate divergence constraint does not yet rule out PM fields interacting with backgrounds more general than Einstein spaces, because we may still enlarge the space of field equation and gauge operators, $\mathbf{M}_{\mu\nu}^{\rho\sigma}$ and $\mathbf{P}_{\mu\nu}$ respectively, by relaxing their trace-free and conformal invariance properties. To test this, we make the following generalization

$$\begin{aligned}\mathbf{M}'_{\mu\nu}{}^{\rho\sigma} &= \mathbf{G}_{\mu\nu}^{\rho\sigma} - (\delta_{(\mu}^\rho \delta_{\nu)}^\sigma - g_{\mu\nu} g^{\rho\sigma}) \mathbf{P} + \alpha_1 \delta_{(\mu}^\rho \tilde{\mathbf{P}}_{\nu)}^\sigma + \alpha_2 (g_{\mu\nu} \tilde{\mathbf{P}}^{\rho\sigma} + \tilde{\mathbf{P}}_{\mu\nu} g^{\rho\sigma}) , \\ \mathbf{P}'_{\mu\nu} &= \nabla_\mu \partial_\nu + \frac{1}{2} \mathbf{P} g_{\mu\nu} + \beta \tilde{\mathbf{P}}_{\mu\nu} ,\end{aligned}$$

where the cosmological Einstein operator

$$\begin{aligned}\mathbf{G}_{\mu\nu}^{\rho\sigma} &:= (\delta_{(\mu}^\rho \delta_{\nu)}^\sigma - g_{\mu\nu} g^{\rho\sigma}) (\Delta - \mathbf{P}) - 2 \nabla_{(\mu} \nabla^{\rho} \delta_{\nu)}^\sigma + \nabla_{(\mu} \nabla_{\nu)} g^{\rho\sigma} + g_{\mu\nu} \nabla^\rho \nabla^\sigma \\ &\quad - 2 W^\rho{}_{\mu\nu}{}^\sigma - 8 \tilde{\mathbf{P}}_{\{\mu}^\rho \delta_{\nu\}}^\sigma - \frac{3}{2} g_{\mu\nu} \mathbf{P} g^{\rho\sigma} ,\end{aligned}\tag{9}$$

is identically conserved,

$$\nabla^\mu \mathbf{G}_{\mu\nu}^{\rho\sigma} = 0 ,$$

in Einstein backgrounds. The equation of motion of cosmological Einstein gravity linearized about an Einstein metric is $\mathbf{G}_{\mu\nu}^{\rho\sigma} \varphi_{\rho\sigma} = 0$.

The above ansatz is the most general one obeying the following requirements:

1. The operators $\mathbf{M}'_{\mu\nu}{}^{\rho\sigma}$ and $\mathbf{P}'_{\mu\nu}$ are second order in ∇ or derivatives on the metric $g_{\mu\nu}$.
2. The operator $\mathbf{M}'_{\mu\nu}{}^{\rho\sigma}$ is self-adjoint, to ensure the existence of an action principle.
3. The divergence $\nabla^\mu \mathbf{M}'_{\mu\nu}{}^{\rho\sigma}$ is an operator no more than linear in ∇ so that solutions of $\mathbf{M}'_{\mu\nu}{}^{\rho\sigma} \varphi_{\mu\nu} = 0$ obey a first order constraint.

²The DoF count for PM starts with ten off-shell fields $\varphi_{\mu\nu}$, minus four DoF thanks to the divergence constraint $\nabla^\mu \varphi_{\mu\nu} = \nabla_\nu \varphi$, minus two further DoF by the scalar gauge invariance, yielding a total of four on-shell excitations.

4. The operator product $\mathbf{M}'^{\rho\sigma}_{\mu\nu}\mathbf{P}'_{\rho\sigma}$ vanishes when $g_{\mu\nu}$ is an Einstein metric; this fixes their leading terms to be operators corresponding to the linear PM equation of motion (6) and its double derivative gauge invariance (3). The remaining freedom in the ansatz therefore depends only on the trace-free Schouten tensor $\tilde{\mathbf{P}}_{\mu\nu}$, since that quantity vanishes for Einstein metrics.

It remains to compute the product $\mathbf{M}'^{\rho\sigma}_{\mu\nu}\mathbf{P}'_{\rho\sigma}$. The result can be arranged as an expansion in the gradient operator ∇ . By construction, terms of order ∇^4 and ∇^3 necessarily vanish. Prefactors of the terms order ∇^2 only involve $\tilde{\mathbf{P}}_{\mu\nu}$ which we are now assuming to be non-vanishing, since we wish to investigate metrics that are *not* Einstein: we must choose the constants $(\alpha_1, \alpha_2, \beta)$ accordingly and find

$$\alpha_1 = 4 + 2\beta, \quad \alpha_2 = -\beta.$$

The analysis of terms order ∇ and lower is more complicated. First we consider the trace $g^{\mu\nu}\mathbf{M}'^{\rho\sigma}_{\mu\nu}\mathbf{P}'_{\rho\sigma}$ at order ∇ and find $3\beta(\nabla_\rho\mathbf{P})\nabla^\rho$. There are two possibilities, either $\beta = 0$ or the background metric has constant scalar curvature. Since the latter would rule out the PM conjecture in question, we choose $\beta = 0$. We then find $g^{\mu\nu}\mathbf{M}'^{\rho\sigma}_{\mu\nu}\mathbf{P}'_{\rho\sigma} = -3(\Delta\mathbf{P})$, which requires the scalar curvature to be harmonic, and hence also rules out the conjecture.

Having excluded Bach-flat backgrounds, we may still investigate whether some condition stronger than Bach-flat, but still less stringent than Einstein, could yield consistent propagation. The terms remaining at order ∇ in $\mathbf{M}'^{\rho\sigma}_{\mu\nu}\mathbf{P}'_{\rho\sigma}$ are

$$\beta g_{\mu\nu}(\nabla_\rho\mathbf{P})\nabla^\rho - (\beta - 2)(\nabla_{(\mu}\mathbf{P})\nabla_{\nu)} + 2(\beta - 1)(\nabla_\rho\mathbf{P}_{\mu\nu})\nabla^\rho - 2\beta(\nabla_{(\mu}\mathbf{P}_{\nu)\rho})\nabla^\rho;$$

clearly no choice of β removes all of them. Instead, we can restrict the background, one option being to Ricci-symmetric spaces, defined by $\nabla_\rho\mathbf{P}_{\mu\nu} = 0$. This condition is weaker than Einstein, but need not imply Bach-flat. However, even then we must cancel all terms in $\mathbf{M}'^{\rho\sigma}_{\mu\nu}\mathbf{P}'_{\rho\sigma}$ of order ∇^0 . In *general* backgrounds these are

$$\begin{aligned} & -\beta B_{\mu\nu} + 2\beta^2 \mathbf{P}^\rho_{(\mu}\mathbf{P}_{\nu)\rho} - \frac{1}{2}(\beta - 1)(\beta + 3) \mathbf{P}\mathbf{P}_{\mu\nu} - \frac{1}{2}(\beta - 2) \nabla_\mu\partial_\nu\mathbf{P} \\ & + g_{\mu\nu} \left[\frac{1}{2}(\beta - 2) \Delta\mathbf{P} - \beta(\beta + 1) \mathbf{P}_{\rho\sigma}\mathbf{P}^{\rho\sigma} + \frac{1}{8}(\beta + 2)(3\beta - 2) \mathbf{P}^2 \right]. \end{aligned}$$

Even for a Ricci-symmetric space (where the derivative terms drop), no choice of β removes all remaining terms quadratic in the Schouten tensor and its

trace. [We see no strong physical motivation to single out backgrounds with covariantly constant Einstein tensor subject to a further quadratic curvature constraint.] This last detour reassures us that no interesting, at best slightly more general than Einstein, backgrounds are allowed.

4 PM self-interaction?

We emphasize at the outset that the aim of this section is to study putative self-interacting extensions of PM solely within the context of the CG framework. That is, our results—which will face the usual stringent limitations on such extensions—strictly apply only to this framework, although they are suggestive, and the allowed nonlinearities are quite efficiently generated. We will need a version of the CG action that is more useful for our purposes, in which the PM field is clearly isolated. This is accomplished by CG’s “Ostrogradsky” second order formulation [21],

$$S[g, \varphi] = - \int \sqrt{-g} \left[\frac{\Lambda}{6} (R - 2\Lambda) + \varphi^{\mu\nu} \Gamma_{\mu\nu} + \varphi^{\mu\nu} \varphi_{\mu\nu} - \varphi^2 \right],$$

where $\Gamma_{\mu\nu} := G_{\mu\nu} + \Lambda g_{\mu\nu}$ is the cosmological Einstein tensor. Upon completing the square, we see that the auxiliary field becomes the cosmological Schouten tensor (5). To analyze the spectrum of the theory about an Einstein background $\bar{g}_{\mu\nu}$ with cosmological constant Λ , we linearize in metric perturbations $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$. Keeping terms quadratic in fluctuations and making the field redefinition

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \frac{6}{\Lambda} \varphi_{\mu\nu}, \quad (10)$$

yields the action (the metrics appearing in \mathbf{G} and \mathbf{F} are set to $\bar{g}_{\mu\nu}$)

$$S^{(2)}[h, \varphi] = -\frac{1}{4} \int \sqrt{-\bar{g}} \left[\frac{\Lambda}{6} h^{\mu\nu} \mathbf{G}_{\mu\nu}^{\rho\sigma} h_{\rho\sigma} - \frac{6}{\Lambda} \varphi^{\mu\nu} \left(\mathbf{G}_{\mu\nu}^{\rho\sigma} - \frac{2}{3} \Lambda \mathbf{F}_{\mu\nu}^{\rho\sigma} \right) \varphi_{\rho\sigma} \right]. \quad (11)$$

Here $-\mathbf{G}_{\mu\nu}^{\rho\sigma} h_{\rho\sigma}/2$ is the linearized cosmological Einstein tensor defined in (9) and all indices are moved by $\bar{g}_{\mu\nu}$. The Pauli–Fierz (PF) mass operator is defined as $\mathbf{F}_{\mu\nu}^{\rho\sigma} := \delta_\mu^\rho \delta_\nu^\sigma - g_{\mu\nu} g^{\rho\sigma}$, so the PM field equation is

$$\left(\mathbf{G}_{\mu\nu}^{\rho\sigma} - \frac{2}{3} \Lambda \mathbf{F}_{\mu\nu}^{\rho\sigma} \right) \varphi_{\rho\sigma} = 0.$$

Thus, the first term of (11) is linearized Einstein–Hilbert, while the terms with round brackets (the sum of the linearized gravity kinetic term and a Pauli–Fierz mass term tuned to the PM value $m^2 = 2\Lambda/3$) give the PM theory, all in an Einstein background. Hence the model describes the “difference” of massless and PM excitations. Moreover, integrating out (at linear level) the field $\varphi_{\mu\nu}$ appearing before the field redefinition (10), gives the fourth order equation

$$\mathbf{B}_{\mu\nu}^{\rho\sigma} h_{\rho\sigma} = 0, \quad \text{where} \quad \mathbf{B}_{\mu\nu}^{\rho\sigma} := \mathbf{G}_{\mu\nu}^{\alpha\beta} \mathbf{F}^{-1\gamma\delta}_{\alpha\beta} \mathbf{G}_{\gamma\delta}^{\rho\sigma} - \frac{2}{3} \Lambda \mathbf{G}_{\mu\nu}^{\rho\sigma},$$

for the original metric fluctuations. Indeed, $\mathbf{B}_{\mu\nu}^{\rho\sigma} h_{\rho\sigma}$ is the Bach tensor linearized about an Einstein background.

The relative sign of the two parts of the linearized action (11) reflects the unavoidable relative ghost structure. In particular, states with $\varphi_{\mu\nu} = 0$ constitute a unitary, massless spin $s = 2$ spectrum. When the cosmological constant is positive (dS), states with $h_{\mu\nu} = 0$ correspond to a unitary PM spectrum. We now proceed to study the latter truncation; a key step is to understand the model’s gauge structure. At linear level, the graviton $h_{\mu\nu}$ enjoys a linearized diffeomorphism symmetry³ $\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ while the PM field $\varphi_{\mu\nu}$ transforms according to the double derivative scalar variation (3); at linear level each field is inert under the other’s transformations. In fact, the PM gauge symmetry is inherited from the Weyl symmetry of CG. The full non-linear action (4) is invariant under both gauge transformations,

$$\begin{aligned} \delta g_{\mu\nu} &= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu + 2\alpha g_{\mu\nu}, \\ \delta \varphi_{\mu\nu} &= \mathcal{L}_\xi \varphi_{\mu\nu} + \left(\nabla_\mu \partial_\nu + \frac{\Lambda}{3} g_{\mu\nu} \right) \alpha. \end{aligned} \tag{12}$$

The metric transformation is now a sum of diffeomorphism and Weyl transformations as is the $\varphi_{\mu\nu}$ transformation: \mathcal{L}_ξ is the Lie derivative along the vector field ξ and the Weyl term follows from the transformation of the Schouten tensor (2).

³As an aside, we observe that the derivation of the linear PM model from Weyl invariant CG theory gives a novel proof of the $SO(4, 2)$ conformal invariance of PM excitations. (In fact, conformal invariance was the original rationale behind the PM model [1], and is enjoyed by all maximal depth, four-dimensional PM theories of generic spin [9].) In detail, whenever a field is coupled to the metric, maintaining Weyl invariance, then setting the metric to a background yields an action that enjoys any conformal isometries as symmetries. Thus the non-linear model generated by setting the metric in (4) to a background is guaranteed to enjoy this symmetry; since it holds order by order in φ , it is also a symmetry of linearized PM.

Without incurring the ghost problem of CG, we may search for some combination of fields that, when held to an appropriate background, yields a consistent truncation to a self-interacting PM model.⁴ We must now find the proper combination of fields to set to a background that yields the desired decoupling. At linear level, the answer to this requirement is given by the field redefinition (10). There, the choice for the metric fluctuations $h_{\mu\nu} = 0$ is respected by PM gauge transformations. This substitution in the linearized action (11) yields the free PM action in an Einstein background. Therefore we begin by positing a candidate for a non-linear version of the field redefinition (10) (that mixes $g_{\mu\nu}$ and $\varphi_{\mu\nu}$) such that a consistent PM theory results from holding the redefined metric to a suitable fixed value:

$$\begin{cases} g_{\mu\nu} & \rightarrow g_{\mu\nu} + \frac{6}{\Lambda} \varphi_{\mu\nu} \\ \varphi_{\mu\nu} & \rightarrow \varphi_{\mu\nu} . \end{cases} \quad (13)$$

[We could have allowed for further redefinitions of both fields, by adding (to each) initially arbitrary functions starting at second order, so as to preserve the linear choice (10), but in fact this would only affect quartic corrections, and we will, for good reason, stop at cubic order.] With this field redefinition, the CG action (4) reduces to that of a “matter” field $\varphi_{\mu\nu}$ coupled to a (dynamical) metric:

$$S[g, \varphi] = \int \sqrt{-g} \left[-\frac{\Lambda}{6} (R - 2\Lambda) + \frac{6}{\Lambda} \mathcal{L}_{\text{PM}}(\varphi, \nabla\varphi) \right] ,$$

where \mathcal{L}_{PM} is the candidate PM Lagrangian. Its $\varphi_{\mu\nu}$ dependence is highly non-linear, with self-interactions coming from re-expressing all the original metric dependence of the action (4) in terms of the shifted combination $g_{\mu\nu} + \frac{6}{\Lambda} \varphi_{\mu\nu}$. After making this expansion, we set $g_{\mu\nu}$ to any Einstein metric with cosmological constant Λ . This leaves us with the PM candidate

$$S_{\text{PM}}[\varphi] = \frac{6}{\Lambda} \int \sqrt{-g} \mathcal{L}_{\text{PM}}(\varphi, \nabla\varphi) ,$$

⁴Indeed, the converse version of this procedure can be applied to produce cosmological gravity from CG for the full, non-linear theory: Examining the gauge transformations (12), we see that the PM background $\varphi_{\mu\nu} = 0$ is preserved by diffeomorphisms but not Weyl transformations. Hence, setting $\varphi_{\mu\nu} = 0$ yields a diffeomorphism invariant theory; performing this substitution in the action (4) gives cosmological Einstein gravity.

to be computed as an expansion in $\varphi_{\mu\nu}$:

$$\mathcal{L}_{\text{PM}} = \frac{1}{4} \varphi^{\mu\nu} \left(\mathbf{G}_{\mu\nu}^{\rho\sigma} - \frac{2}{3} \Lambda \mathbf{F}_{\mu\nu}^{\rho\sigma} \right) \varphi_{\rho\sigma} + \sum_{n=3}^{\infty} \mathcal{L}_{\text{PM}}^{(n)}.$$

The absence of a term linear in $\varphi_{\mu\nu}$ follows from the linearized analysis and relies on the fact that $g_{\mu\nu}$ is now an Einstein metric.

Before presenting our explicit cubic vertices, let us show that there is no fully non-linear truncation of CG to an interacting PM theory. (This neither annuls consistency of the cubic vertices with respect to linearized gauge transformations, nor rules out any other ultimate theory of self-interacting PM fields.) To determine whether a truncation that takes $g_{\mu\nu}$ to be a fixed Einstein background is consistent, we must study the gauge invariances of the theory. The precise form of the underlying CG gauge transformations in terms of the redefined fields (13) is:

$$\begin{aligned} \delta g_{\mu\nu} &= \mathcal{L}_{\xi} g_{\mu\nu} - \frac{6}{\Lambda} \left[\nabla_{\mu} \partial_{\nu} + \frac{6}{\Lambda} \left[(g + \frac{6}{\Lambda} \varphi)^{-1} \right]^{\rho\sigma} \gamma_{\rho\mu\nu} \partial_{\sigma} \right] \alpha, \\ \delta \varphi_{\mu\nu} &= \mathcal{L}_{\xi} \varphi_{\mu\nu} + \left[\nabla_{\mu} \partial_{\nu} + \frac{\Lambda}{3} g_{\mu\nu} + \frac{6}{\Lambda} \left[(g + \frac{6}{\Lambda} \varphi)^{-1} \right]^{\rho\sigma} \gamma_{\rho\mu\nu} \partial_{\sigma} + 2 \varphi_{\mu\nu} \right] \alpha. \end{aligned}$$

Here we have denoted the Christoffel symbols of $\varphi_{\mu\nu}$, covariantized with respect to $g_{\mu\nu}$, by

$$\gamma_{\rho\mu\nu} := \frac{1}{2} (\nabla_{\mu} \varphi_{\nu\rho} + \nabla_{\nu} \varphi_{\mu\rho} - \nabla_{\rho} \varphi_{\mu\nu}).$$

Firstly observe that at leading order in φ , the choice of diffeomorphism parameter $\xi_{\mu} = 3 \partial_{\mu} \alpha / \Lambda$ cancels the Lie derivative term $\mathcal{L}_{\xi} g_{\mu\nu} = \nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu}$ against the double gradient of the scalar parameter α in the metric variation. This is just a restatement of our linear result that the dynamical metric can be decoupled (at that order), leaving linear PM. Consistency of the non-linear truncation requires that there exist a choice of ξ achieving this cancellation to all orders. This would determine the higher order terms in the variation of φ , leaving the PM action $S_{\text{PM}}[\varphi]$ invariant. To establish a no-go result, we need only show that already no choice of ξ achieves this cancellation for the next-to-leading order terms in φ in the metric variation. Focussing on the $\gamma_{\mu\nu}^{\rho} \partial_{\rho} \alpha$ part of $\delta g_{\mu\nu}$ that is linear in φ , we immediately see that it can never be written as $\nabla_{(\mu} X_{\nu)}$, for any X_{ν} even on PM-shell. This establishes our claimed no-go result for truncating CG to a PM theory beyond linear order.

Finally, we compute the cubic vertices, which, being guaranteed invariant under leading PM gauge transformations $\delta\varphi_{\mu\nu} = (\nabla_\mu\partial_\nu + \frac{\Lambda}{3}g_{\mu\nu})\alpha$, are candidate vertices for a putative non-linear self-interacting PM theory. The form of n -th order Lagrangian of the PM field determined by the field redefinition (13) can be obtained from the following correspondence,

$$\begin{aligned} \left(\frac{\Lambda}{6}\right)^{n+1}\sqrt{-g}\mathcal{L}_{\text{PM}}^{(n+2)} &= \frac{n+1}{(n+2)!}\varphi_{\mu\nu}\delta_{g|\varphi}^{n+1}[\sqrt{-g}\mathcal{G}^{\mu\nu}] \\ &+ \frac{\Lambda}{6}\frac{1}{n!}\delta_{g|\varphi}^n[\sqrt{-g}g^{\mu\nu}g^{\rho\sigma}](\varphi_{\mu\rho}\varphi_{\nu\sigma} - \varphi_{\mu\nu}\varphi_{\rho\sigma}). \end{aligned} \quad (14)$$

Here $\delta_{g|\varphi}^n$ signifies taking the n -th variation with respect to the metric and then replacing $\delta g_{\mu\nu}$ by $\varphi_{\mu\nu}$; the result is of order n in $\varphi_{\mu\nu}$. In the first line, we have used the fact that the first metric variation of the cosmological Einstein–Hilbert action produces the cosmological Einstein tensor $\mathcal{G}_{\mu\nu}$, which allows $(n+2)$ variations of that term to be combined with $(n+1)$ variations of the coupling of the cosmological Einstein tensor to the PM field in (4). If we evaluate the above interaction Lagrangians explicitly then, since they are given in terms of multiple variations of the Ricci tensor, the generic outcome for \mathcal{L}_{PM} is a two-derivative self-coupling of $\varphi_{\mu\nu}$, a curvature coupling and a potential for $\varphi_{\mu\nu}$. We also note that multiplying the original CG action (4) by the dimension-free combination $\Lambda^{-1}\kappa^{-2}$ of the cosmological and gravitational constants, and redefining the PM field $\varphi \rightarrow \Lambda\kappa\varphi$ gives, schematically, the canonically normalized action

$$S \sim \frac{1}{\kappa^2} \int (R - 2\Lambda) + \int [(\nabla\varphi)^2 + \Lambda\varphi^2] + \sum_{n=3}^{\infty} \kappa^{n-2} [\varphi^{n-2} \nabla\varphi \nabla\varphi + \Lambda\varphi^n].$$

Now, let us focus on computing the cubic part $\mathcal{L}_{\text{PM}}^{(3)}$ in (14). Note that since we work on an Einstein background, we may set $\mathcal{G}_{\mu\nu} = 0$ (when it is not varied); also, since we only quote the vertex up to a possible field redefinition, at this order we may use the linear PM field equation, which can be written as⁵ $\delta_{g|\varphi}\mathcal{G}_{\mu\nu} + \frac{\Lambda}{3}(\varphi_{\mu\nu} - g_{\mu\nu}\varphi) = 0$. Moreover, since the vertex is cubic in $\varphi_{\mu\nu}$, we may write

$$\frac{6}{\Lambda}T^{\mu\nu} := \frac{1}{3}\frac{1}{\sqrt{-g}}\frac{\delta S_{\text{PM}}^{(3)}}{\delta\varphi_{\mu\nu}}, \quad S_{\text{PM}}^{(3)} = \frac{6}{\Lambda}\int\sqrt{-g}\varphi_{\mu\nu}T^{\mu\nu}.$$

⁵Notice that the cubic vertex, therefore, schematically takes the form

$$S_{\text{PM}}^{(3)} = \delta_{g|\varphi}S_{\text{PM}}^{(2)} + \int\varphi^3,$$

where $S_{\text{PM}}^{(2)}$ is the leading order PM action and φ^3 denotes cubic potential terms in $\varphi_{\mu\nu}$.

By construction, $S_{\text{PM}}^{(3)}$ is invariant under the linear order PM gauge transformation (3) modulo the linear field equations. This guarantees that $T_{\mu\nu}$ obeys the Noether identity

$$(\nabla^\mu \nabla^\nu + \frac{\Lambda}{3} g^{\mu\nu}) T_{\mu\nu} \approx 0, \quad (15)$$

in an Einstein background; here \approx denotes equality modulo the linear PM field equations.

It remains to explicitly compute $T_{\mu\nu}$. In fact, the cubic vertex given by (14) at $n = 1$ is easily computed by hand. For the Noether form of the vertex, a computer aided computation [25] gives

$$\begin{aligned} T_{\mu\nu} \approx & \varphi^{\rho\sigma} \nabla_\rho \nabla_\sigma \varphi_{\mu\nu} + \frac{1}{2} \varphi_{\mu\nu} \Delta \varphi - \frac{4}{3} \varphi^{\rho\sigma} \nabla_{(\mu} \nabla_\rho \varphi_{\sigma|\nu)} - \varphi^\rho_{(\mu} \nabla_\nu) \nabla_\rho \varphi \\ & + \frac{2}{3} \varphi^{\rho\sigma} \nabla_\mu \nabla_\nu \varphi_{\rho\sigma} + \frac{1}{6} \varphi \nabla_\mu \nabla_\nu \varphi + \frac{1}{6} g_{\mu\nu} (\varphi^{\rho\sigma} \nabla_\rho \nabla_\sigma \varphi - \varphi \Delta \varphi) \\ & + \nabla^\rho \varphi (\frac{3}{2} \nabla_\rho \varphi_{\mu\nu} - \frac{2}{3} \nabla_{(\mu} \varphi_{\nu)\rho}) - \frac{1}{3} \nabla^\rho \varphi^\sigma_\mu \nabla_\rho \varphi_{\sigma\nu} - \nabla^\rho \varphi^\sigma_{(\mu} \nabla_\sigma \varphi_{\nu)\rho} \\ & + \frac{2}{3} \nabla_{(\mu} \varphi^{\rho\sigma} \nabla_\rho \varphi_{\nu)\sigma} + \frac{1}{6} \nabla_\mu \varphi^{\rho\sigma} \nabla_\nu \varphi_{\rho\sigma} - \frac{1}{3} \nabla_\mu \varphi \nabla_\nu \varphi \\ & - g_{\mu\nu} (\frac{5}{12} \nabla^\rho \varphi^{\sigma\tau} \nabla_\rho \varphi_{\sigma\tau} - \frac{1}{2} \nabla^\rho \varphi^{\sigma\tau} \nabla_\sigma \varphi_{\rho\tau} + \frac{1}{12} \nabla^\rho \varphi \nabla_\rho \varphi) \\ & - \Lambda (\frac{1}{18} \varphi \varphi_{\mu\nu} + \frac{5}{9} \varphi^\rho_\mu \varphi_{\nu\rho}) + \Lambda g_{\mu\nu} (\frac{11}{36} \varphi^{\rho\sigma} \varphi_{\rho\sigma} - \frac{1}{36} \varphi^2) \\ & - \frac{2}{3} W^{\rho\tau}{}_{(\mu}{}^\sigma \varphi_{\nu)\tau} \varphi_{\rho\sigma} - \frac{2}{3} W^\rho{}_{(\mu\nu)}{}^\sigma \varphi^\tau_\rho \varphi_{\tau\sigma} - \frac{1}{3} g_{\mu\nu} W^{\rho\tau\kappa\sigma} \varphi_{\rho\sigma} \varphi_{\tau\kappa}. \end{aligned}$$

As a check, we verified that this $T_{\mu\nu}$ obeys the Noether identity (15) for constant curvature backgrounds (vanishing Weyl tensor).

As stated at the start of this Section, our cubic results were obtained entirely within the CG framework. However, their consistency is independent of their origin, since they are of course disjoint from any higher-order problems. Indeed, the vertex $S_{\text{PM}}^{(3)}$ was constructed by a Stückelberg method in [12], where it was also shown that two-derivative PM self-interactions exist only for $d = 4$, which dovetails perfectly with their CG origin uncovered here. These results also fit with the recent work of [13] where all consistent cubic interactions (not necessarily two-derivative ones) involving PM fields of generic spin were considered. There it was shown that for generic dimensions there are only two PM self-couplings involving at most four and six derivatives respectively. However, precisely in four dimensions, the Gauß-Bonnet identity reduces the maximal four-derivative coupling to a two-derivative one.⁶

⁶In fact, for constant curvature backgrounds, the Cotton-like tensor [10]

$$F_{\mu\nu}{}^\rho := \nabla_\mu \varphi_\nu{}^\rho - \nabla_\nu \varphi_\mu{}^\rho$$

5 Conclusions

We have used $d = 4$ conformal Weyl gravity as a tool to explore the extent of the usual higher spin constraints on PM self-and gravitational- couplings. We concluded that these obstructions were indeed present here as well: first, no backgrounds more general than Einstein were permitted for PM's propagation. Then, we exploited the truncation of CG to PM in a fixed geometry to find what ghost-free self-couplings, if any, might be permitted within the CG framework. Although relative ghost-like graviton modes could be removed at linear order leaving (consistent) linear PM, in contrast to the PM truncation of CG to cosmological gravity [3], the gauge structure of CG does not allow the graviton truncation to continue to higher orders. An old problem (one that already occurs in similar attempts at extending other higher-spin) has struck again: despite the possibility of a lowest order invariant cubic self-interaction (expressed as the coupling of the quadratic Noether current maintaining the initial Abelian invariance to the field amplitude), self-coupling inconsistencies set in at quartic order. CG underlies cosmological Einstein gravity but it does not truncate to a non-linear “PM general relativity”. Despite the results achieved here, we should emphasize that they merely begin to reflect CG's potential to explore (A)dS models' physical content in a direct way. The underlying CG technology is clearly capable of yielding far more insight.

No-go theorems are notorious for their loopholes. Spin $(2,3/2)$ -gravity and supergravity theories circumvent just such higher-spin pitfalls [26] while for (towers of) massive higher spins, string theory provides presumably consistent interactions; infinite towers of massless higher spins can also be written in (A)dS backgrounds [27]. Nonetheless, our results relying on CG as the underpinning of PM self-interactions seem quite robust; they agree with the claim of [12] that it is impossible to proceed beyond cubic order for the two-derivative PM theory.

One interesting feature of CG is that the PM field can be consistently turned off, leaving cosmological Einstein gravity (at least classically). In

is invariant under PM gauge transformations (3). (Strictly this version of the Cotton tensor is not the metric one, because the PM field is not the Schouten tensor, although in the underlying CG setting this is in fact the case.) Therefore any quartic derivative order, cubic vertex of type $\int (\nabla F) F F$ is PM invariant. In four dimensions, it should be possible to employ the Gauß–Bonnet identity to write this as a manifestly invariant cubic vertex quadratic in derivatives.

other words, without additional matter couplings, choosing initial conditions such that $\varphi_{\mu\nu}$ is zero at some initial time, it will remain trivial while the metric $g_{\mu\nu}$ can realize any Einstein solution [3]. This suggests the converse truncation: a situation where the PM field $\varphi_{\mu\nu}$ is not strictly zero but rather nearly zero in some arbitrarily large time interval $t_i \ll t_f$. Cosmology would then have approximate Einstein behavior for that epoch, while in the region $t \ll t_i$ or $t \gg t_f$, non-Einstein solutions could emerge. (The consequences for cosmological expansion with a partially conserved symmetric two index boundary operator were also considered in [28].) CG could then be used to generate transitions from a dS inflationary behavior of the cosmic scale factor to one controlled by PM modes. Ghosts and loss of stability at early and late times may even be a useful/acceptable feature in this scenario.

A separate speculation is that gravity-like, or even self-interacting PM-like models for higher $s > 2$, might be achievable by studying higher-spin versions of CG. Indeed, interacting conformally invariant higher-spin models that can be viewed as analogs of CG do exist [29, 30]. Perhaps a higher spin version of our approach could be fruitfully applied to them.

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