Lorentz invariance in heavy particle effective theories

Johannes Heinonen, Richard J. Hill, and Mikhail P. Solon

Phys. Rev. D **86**, 094020 — Published 15 November 2012

DOI: 10.1103/PhysRevD.86.094020

EFI Preprint 12-16

Lorentz invariance in heavy particle effective theories

Johannes Heinonen, Richard J. Hill and Mikhail P. Solon

Enrico Fermi Institute and Department of Physics
The University of Chicago, Chicago, Illinois, 60637, USA

E-mail: heinonen,richardhill,mpsolon@uchicago.edu

Abstract

Employing induced representations of the Lorentz group (Wigner's little group construction), formalism for constructing heavy particle effective Lagrangians is developed, and Lagrangian constraints enforcing Lorentz invariance of the S matrix are derived. The relationship between Lorentz invariance and reparameterization invariance is established and it is shown why a standard ansatz for implementing reparameterization invariance in heavy fermion effective Lagrangians breaks down at order $1/M^4$. Formalism for fields of arbitrary spin and for self-conjugate fields is presented, and the extension to effective theories of massless fields is discussed.

1 Introduction

Heavy particle effective field theories find a wide range of applications in particle, nuclear and atomic physics [1–5]. Recent investigations demand high orders in the $1/M$ expansion (see e.g. [6, 7]), and involve construction of effective theories for which a simple underlying ultraviolet completion is unknown, or unspecified (see e.g. [8] and references therein). To avoid a proliferation of undetermined constants, and to enable efficient computations, it is important to recognize that many Wilson coefficients are linked by Lorentz invariance to coefficients appearing at lower orders. This may be viewed in analogy to the constraints imposed by enforcing invariance under broken chiral symmetries in low-energy chiral effective field theories. The procedure for implementing such chiral symmetry constraints, via the formalism of nonlinear realizations, is well known [9, 10]. It is our aim here to bring similar clarity to the implementation of Lorentz invariance in heavy particle effective field theories, and to provide a practical and systematic implementation of Lorentz invariance constraints suitable for arbitrary orders in phenomenological applications.

When the heavy particle is fundamental, we may derive the effective theory Lagrangian by introducing a field redefinition in the full theory. For example, in terms of an arbitrary (spacetime independent) time-like unit vector v^μ, the decomposition of a quark field $Q(x)$ of mass M,

$$Q(x) = e^{-iMv\cdot x}\left[h_v(x) + H_v(x)\right] , \tag{1.1}$$

with $\not v\, h_v = h_v$ and $\not v\, H_v = -H_v$, defines an effective heavy quark field $h_v(x)$, and after integrating out the antiparticle field $H_v(x)$, we arrive at the effective Lagrangian for a heavy quark. Invariance of observables under small changes of v, so-called "reparameterization invariance", enforces certain constraints on the coefficients of the effective Lagrangian [11]. These constraints are consistent with the requirements of Lorentz invariance, e.g. as imposed by matching effective theory S matrix elements to Lorentz-invariant full theory S matrix elements. However, this construction raises several questions. Is reparameterization invariance a sufficient condition for Lorentz invariance? How do we derive a reparameterization transformation law without first constructing the underlying theory and explicitly integrating out degrees of freedom? For applications involving a composite particle such as the proton, or hypothetical new particles that may not be fundamental, we cannot in an obvious way introduce v as a parameter inside of a field redefinition. What is the significance of v in such cases? What is the general method for constructing a Lorentz invariant heavy particle effective field theory?

In this paper we present the formalism of induced representations of the Lorentz group (Wigner's "little group" construction [12]) for application to field transformation laws. The parameter v enters as an arbitrary reference vector in the little group construction. The relationship between Lorentz invariance and reparameterization invariance is stated precisely, and a class of allowable reparameterization transformations is obtained. We find that a standard ansatz for implementing reparameterization invariance breaks down starting at order $1/M^4$. We explain this subtlety and its resolution.

A large literature exists on topics relating to reparameterization invariance, especially as applied to heavy quark Lagrangians [11, 13–19]. We aim to present a conceptually clear statement of the constraints imposed by Lorentz invariance, and of the relationship between

Lorentz invariance and reparameterization invariance. At a practical level, we derive explicit field transformation laws that can be consistently used to build Lorentz invariant Lagrangians to arbitrary order in $1/M$.

The remainder of the paper is structured as follows. In Section 2 we briefly review the construction of Lorentz invariant field theories based on finite dimensional representations of the Lorentz group. In Section 3 we introduce the formalism of induced representations and investigate the necessary conditions for a Lorentz invariant S matrix. Section 4 establishes the connection between Lorentz invariance and reparameterization invariance. A subtlety in the identification of allowable reparameterization transformations is explained, and a correct solution to the invariance equation (4.17) is found for applications to $1/M^4$ heavy fermion Lagrangians. Section 5 provides a brief overview of the analogous framework for effective theories describing energetic massless particles. Section 6 concludes with a discussion. Appendix A presents formalism in covariant notation for arbitrary spin particles and for self-conjugate fields. Appendix B describes the solution of the invariance equation for the construction of invariant operators to arbitrary order in $1/M$.

2 Finite dimensional representations of the Lorentz algebra

The standard method for constructing Lorentz invariant Lagrangians postulates the field transformation law

$$\phi_a(x) \to M(\Lambda)_{ab}\phi_b(\Lambda^{-1}x)\,, \tag{2.1}$$

where $M(\Lambda)$ is a finite dimensional (coordinate-independent and, in general, non-unitary) representation of the Lorentz group. In infinitesimal form, including also spacetime translations $\phi(x) \to \phi(x-a)$, we have

$$\delta\phi = i(a_0 h - \boldsymbol{a}\cdot\boldsymbol{p} - \boldsymbol{\theta}\cdot\boldsymbol{j} + \boldsymbol{\eta}\cdot\boldsymbol{k})\phi\,, \tag{2.2}$$

where $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are infinitesimal rotation and boost parameters, and the generators of the Poincaré group acting on fields are[1]

$$h = i\partial_t\,, \tag{2.3a}$$

$$\boldsymbol{p} = -i\boldsymbol{\partial}\,, \tag{2.3b}$$

$$\boldsymbol{j} = \boldsymbol{r}\times\boldsymbol{p} + \boldsymbol{\Sigma}\,, \tag{2.3c}$$

$$\boldsymbol{k} = \boldsymbol{r}h - t\boldsymbol{p} \pm i\boldsymbol{\Sigma}\,, \tag{2.3d}$$

with Σ^i the $(2s+1)$-dimensional matrix generators of the spin-s representation of rotations (e.g. for spin-1/2 Weyl fermions, $\boldsymbol{\Sigma} = \boldsymbol{\sigma}/2$ with σ^i the Pauli matrices). Using (2.1) it is straightforward to construct Lorentz invariant actions, and correspondingly to prove Lorentz invariance of the S matrix. Let us briefly review this procedure.[2]

[1]We use bold letters for euclidean three-vectors, e.g. $\boldsymbol{\partial} = (\boldsymbol{\partial}^i) = (\boldsymbol{\partial}_i) = (\partial_x, \partial_y, \partial_z)$.

[2]For a pedagogical discussion, see [20].

Recall the Poincaré algebra for generators of time translations H, space translations P^i, rotations J^i, and boosts K^i:

$$[H, P^i] = 0\,, \tag{2.4a}$$

$$[H, J^i] = 0\,, \tag{2.4b}$$

$$[P^i, P^j] = 0\,, \tag{2.4c}$$

$$[J^i, P^j] = i\epsilon^{ijk}P^k\,, \tag{2.4d}$$

$$[J^i, J^j] = i\epsilon^{ijk}J^k\,, \tag{2.4e}$$

$$[J^i, K^j] = i\epsilon^{ijk}K^k\,, \tag{2.4f}$$

$$[H, K^i] = -iP^i\,. \tag{2.4g}$$

$$[P^i, K^j] = -iH\delta^{ij}\,, \tag{2.4h}$$

$$[K^i, K^j] = -i\epsilon^{ijk}J^k\,. \tag{2.4i}$$

Having built a Lagrangian that is invariant under (2.2), we may construct the corresponding conserved charges. Using (2.3), we find in canonical quantization that these charges obey the commutation relations (2.4).

Lorentz invariance of the S matrix demands that the free-particle charges, denoted by H_0, $\mathbf{P}_0$, $\mathbf{J}_0$, $\mathbf{K}_0$, commute with the scattering operator, $S = \lim_{T\to\infty}\Omega(T)^\dagger\Omega(-T)$, where $\Omega(T) = e^{iHT}e^{-iH_0T}$. We assume that momentum and angular momentum operators for the interacting theory are unchanged from the free theory and furthermore demand translational and rotational invariance of the interaction

$$\boldsymbol{P} = \boldsymbol{P}_0\,, \quad \boldsymbol{J} = \boldsymbol{J}_0\,, \quad [H - H_0, \boldsymbol{P}_0] = [H - H_0, \boldsymbol{J}_0] = 0\,. \tag{2.5}$$

Then $[\boldsymbol{P}_0, S] = [\boldsymbol{J}_0, S] = 0$, and by the definition of S also $[H_0, S] = 0$. Finally, if one can show (2.4g) and that an asymptotic smoothness condition for $\Delta\boldsymbol{K} = \boldsymbol{K} - \boldsymbol{K}_0$ is obeyed, it follows that

$$\begin{aligned} [\boldsymbol{K}_0, S] &= \lim_{T\to\infty}[\boldsymbol{K}_0, \Omega(T)^\dagger\Omega(-T)] \\ &= \lim_{T\to\infty}\Big\{ -[e^{iH_0T}\Delta\boldsymbol{K}e^{-iH_0T}]\Omega(T)^\dagger\Omega(-T) + \Omega(T)^\dagger\Omega(-T)[e^{-iH_0T}\Delta\boldsymbol{K}e^{iH_0T}]\Big\} = 0\,, \end{aligned} \tag{2.6}$$

completing the proof of the Lorentz invariance of the S-matrix. For later application, we note that of the commutation relations involving $\boldsymbol{K}$, it is only necessary to show the relation (2.4g); relations (2.4f), (2.4h) and (2.4i) are not required to complete the proof.[3]

[3] In fact, these relations *do* follow from the observation that having proven Lorentz invariance of the S matrix, it can be shown that H, $\boldsymbol{P}$, $\boldsymbol{J}$ and $\boldsymbol{K}$ are related to their free counterparts by the similarity transformation $\Omega(\pm\infty)$ [20].

3 Effective field theory and the little group

The field transformation law (2.1), based on finite dimensional representations of the Lorentz group, is not suitable for heavy particle effective field theories. For example, the associated irreducible representations of the Lorentz group are chiral, in conflict with the low-energy limit of a parity conserving theory such as QED or QCD. Let us consider instead the class of infinite dimensional induced representations. We first review their appearance in transformations of physical states, and then apply them as transformations acting on fields.

3.1 Little group formalism

Consider Lorentz transformations acting on the Hilbert space of physical states for a spin-s particle of mass M. These transformations are implemented by an induced representation [12]. In terms of a fixed timelike reference vector v^μ (we assume $v^2 = 1$), define the associated "little group" as the subgroup of Lorentz transformations leaving v invariant, $\Lambda v = v$. The little group for massive particles is isomorphic to $SO(3)$, the group of rotations. Let $L(p)$ denote a standard Lorentz transformation taking Mv to p, yielding a (momentum-dependent) mapping of the Lorentz group into the little group,

$$\Lambda \to W(\Lambda, p) = L(\Lambda p)^{-1} \Lambda L(p) \,. \tag{3.1}$$

We may define physical states to transform schematically as

$$|p, m\rangle \to U(\Lambda, p)|p, m\rangle = \sum_{m'=-s}^{s} D_{m'\,m}[W(\Lambda, p)]|\Lambda p, m'\rangle \,, \tag{3.2}$$

where $p^0 = \sqrt{M^2 + \boldsymbol{p}^2}$, and $D(W)$ is a spin-s representation matrix for rotations. A representation for the little group thus induces a representation for the full Lorentz group.

A convenient choice for the standard Lorentz transformation is $L(p) = \Lambda(p/M, v)$, where $\Lambda(w, v)$ denotes the generalized rotation in the plane of the unit vectors v and w such that $\Lambda(w, v)v = w$. This matrix is given by $\Lambda(w, v) = \exp[-i\theta \mathcal{J}_{\alpha\beta} w^\alpha v^\beta]$, with the Lorentz generators $\mathcal{J}_{\alpha\beta}$ defined in Eq. (A.2) and the angle θ chosen appropriately [11]. In the vector and spinor representations we have, respectively

$$\Lambda(w, v)^\mu{}_\nu = g^\mu{}_\nu - \frac{1}{1 + v\cdot w}\left(w^\mu w_\nu + v^\mu v_\nu\right) + w^\mu v_\nu - v^\mu w_\nu + \frac{v \cdot w}{1 + v\cdot w}\left(w^\mu v_\nu + v^\mu w_\nu\right) \,, \tag{3.3a}$$

$$\Lambda_{\frac{1}{2}}(w, v) = \frac{1 + \not{w}\,\not{v}}{\sqrt{2(1 + v\cdot w)}} \,. \tag{3.3b}$$

It is straightforward to verify that for elements of the little group, i.e. "rotations" with $\mathcal{R}v = v$, this choice of $L(p)$ implies

$$W(\mathcal{R}, p) = \mathcal{R} \,, \tag{3.4}$$

a property that greatly simplifies the construction of invariant Lagrangians, cf. Sections 3.3, 4.1 and 4.2 below. Other choices of $L(p)$ do not share this property. For example, suppose that we introduce a spacelike vector s^μ with $s^2 = -1$. Then we may define $L'(p) = R(p)B(p)$, with $B(p)$ a boost taking Mv^μ to $MB(p)^\mu{}_\nu v^\nu = (v\cdot p)v^\mu + \sqrt{(v\cdot p)^2 - M^2}s^\mu$, and $R(p)$ a rotation taking $MB(p)^\mu{}_\nu v^\nu$ to p^μ. Such an $L'(p)$ provides a simple interpretation of $U[L(p)]|Mv, m\rangle$ in terms of helicity eigenstates (note that the spacelike vector is required to define a direction for helicity decomposition), but this consideration is secondary to the simplicity of (3.4) for our present purposes.

The remaining independent Lorentz generators represent "boosts" that shift v. They can be chosen as $\mathcal{B}(q) = \Lambda(v - q/M, v)$ with $(v - q/M)^2 = 1$. The appearance of the $1/M$ factor in $v - q/M$ will be explained in Section 3.3 below. For an infinitesimal momentum q, which obeys $v\cdot q = \mathcal{O}(q^2)$, these boosts are given by

$$\mathcal{B}(q)^\mu{}_\nu = g^\mu{}_\nu + \frac{v^\mu q_\nu - q^\mu v_\nu}{M} + \mathcal{O}(q^2)\,, \tag{3.5a}$$

$$\mathcal{B}_{\frac{1}{2}}(q) = 1 - \frac{\not{q}\not{v}}{2M} + \mathcal{O}(q^2)\,. \tag{3.5b}$$

For the transformation (3.2), we find

$$W(\mathcal{B}(q), p) = 1 - \frac{i}{2}\left[\frac{1}{M(M + v\cdot p)}(q^\alpha p_\perp^\beta - p_\perp^\alpha q^\beta)\right]\mathcal{J}_{\alpha\beta} + \mathcal{O}(q^2), \tag{3.6}$$

where for any four-vector k we define $k_\perp^\mu \equiv k^\mu - (v\cdot k)v^\mu$.

3.2 Field transformation law and Lorentz invariance

In place of (2.1) let us postulate the transformation law for free massive fields,

$$\phi_a(x) \to D[W(\Lambda, i\partial)]_{ab}\phi_b(\Lambda^{-1}x)\,. \tag{3.7}$$

For notational simplicity consider the special choice $v = (1, 0, 0, 0)$. Equation (3.7) together with Eq. (3.6) corresponds to replacing the boost generator (2.3d) by[4]

$$\boldsymbol{k} = \boldsymbol{r}h - t\boldsymbol{p} \pm i\frac{\boldsymbol{\Sigma}\times\boldsymbol{\partial}}{M + \sqrt{M^2 - \boldsymbol{\partial}^2}}\,. \tag{3.8}$$

The generators (2.3a)-(2.3c) together with (3.8) will satisfy the Poincaré algebra when acting on fields satisfying

$$i\partial_t\phi = \pm\sqrt{M^2 - \boldsymbol{\partial}^2}\phi\,. \tag{3.9}$$

It follows that the conserved charges derived from a free field Lagrangian invariant under (3.7) will satisfy (2.4).

In contrast to (2.1), transformation (3.7) acts on the field coordinates, spoiling gauge invariance. To include gauge interactions, we promote the partial derivatives in (3.7) to covariant derivatives $D_\mu = \partial_\mu - igA_\mu^A t^A \equiv \partial_\mu - igA_\mu$,

$$\phi_a(x) \to D[W(\Lambda, iD)]_{ab}\phi_b(\Lambda^{-1}x)\,, \tag{3.10}$$

[4]For spin-1/2 particles, (3.8) may also be obtained by performing a Foldy-Wouthuysen transformation on Eq. (2.3d) [21, 22].

and correspondingly the infinitesimal generators become

$$h = i\partial_t \,, \tag{3.11a}$$

$$\boldsymbol{p} = -i\boldsymbol{\partial} \,, \tag{3.11b}$$

$$\boldsymbol{j} = \boldsymbol{r}\times\boldsymbol{p} + \boldsymbol{\Sigma} \,, \tag{3.11c}$$

$$\boldsymbol{k} = \boldsymbol{r}h - t\boldsymbol{p} \pm i\frac{\boldsymbol{\Sigma}\times\boldsymbol{D}}{M+\sqrt{M^2-\boldsymbol{D}^2}} + \mathcal{O}(g) \,. \tag{3.11d}$$

In the expansion of $\boldsymbol{D}/(M+\sqrt{M^2-\boldsymbol{D}^2})$ we assume a choice of ordering for the covariant derivatives. The $\mathcal{O}(g)$ terms in $\boldsymbol{k}$ denote field strength-dependent corrections that vanish for the non-interacting theory (i.e. $g \to 0$). Such $\mathcal{O}(g)$ terms can be introduced so that the resulting invariant Lagrangian is in "canonical form", i.e. where the only time derivative acting on ϕ appears in the leading term,

$$\mathcal{L} = \bar{\phi}(iD_t + \dots)\phi \,. \tag{3.12}$$

The existence of suitable field strength-dependent terms, ensuring a boost generator $\boldsymbol{k}$ which yields a non-zero invariant Lagrangian, is implied by the all-orders construction in Section 4 and Appendix B. The explicit form of these corrections is not required for the following argument.

Although the field-dependent generators (3.11) do not obey simple commutation relations, we may nevertheless show that the S matrix derived from the resulting invariant action is Lorentz invariant (and hence that the conserved charges in the interacting theory satisfy the Poincare algebra). To see this, we assume as before the relations (2.5). Relation (2.4g) is satisfied if the explicit time dependence of the conserved charge $\boldsymbol{K}$ satisfies $\partial\boldsymbol{K}/\partial t = -\boldsymbol{P}$, so that

$$0 = \frac{d}{dt}\boldsymbol{K} = \frac{\partial}{\partial t}\boldsymbol{K} + i[H,\boldsymbol{K}] = -\boldsymbol{P} + i[H,\boldsymbol{K}] \,. \tag{3.13}$$

The fact that $\partial\boldsymbol{K}/\partial t = -\boldsymbol{P}$ follows from the assumed form of the infinitesimal generators (3.11). For the boost $\phi \to (1+i\boldsymbol{\eta}\cdot\boldsymbol{k})\phi$, we find the conserved charge[5]

$$\boldsymbol{K} = \sum_\phi i\int d^3x \frac{\delta\mathcal{L}}{\delta\dot{\phi}}\,\boldsymbol{k}\,\phi + \dots = \sum_\phi i\int d^3x \frac{\delta\mathcal{L}}{\delta\dot{\phi}}\,[-t\boldsymbol{p}]\,\phi + \dots = -t\boldsymbol{P} + \dots \,. \tag{3.14}$$

Here the important point is that the remaining terms have no *explicit* time dependence, so that (3.13) follows.

Let us close this section with two comments. First, the choice $v = (1,0,0,0)$ is not essential to the argument. The generators for arbitrary v can be obtained by a coordinate change using a boost which takes $(1,0,0,0)$ to v. While the resulting explicit expressions for rotation and boost generators become more complicated, the demonstration of Lorentz invariance is not essentially changed. Second, having specified an ordering for covariant derivatives appearing

[5]The first ellipsis in (3.14) includes possible contributions from a surface term in $\delta\mathcal{L}$, which do not affect the term with explicit t dependence in (3.14).

in the boost generator $\boldsymbol{k}$, additional field strength-dependent corrections are determined at each order in $1/M$ by enforcing that the resulting invariant Lagrangian is in canonical form. We illustrate this with an explicit example in the following subsection. The existence of such a generator is implied by the analysis of Section 4 and Appendix B.

3.3 $1/M$ expansion and Lagrangian constraints

To enable the $1/M$ expansion we extract the rest mass by the field redefinition,

$$\phi(x) = e^{-iMt}\phi'(x)\,. \tag{3.15}$$

In phenomenological applications it is also convenient to work with non-relativistic field normalization

$$\phi'(x) = \left(\frac{M^2}{M^2 - \boldsymbol{D}^2}\right)^{\frac{1}{4}} \phi''(x)\,. \tag{3.16}$$

We enforce invariance under (3.11a), (3.11b) and (3.11c) by ensuring translational invariance (no explicit factors of x^μ) and rotational invariance. For the boost transformation (3.11d) we use $\boldsymbol{\eta} = -\boldsymbol{q}/M$ in (2.2) to preserve the power counting $D_t = \mathcal{O}(1/M)$ in (3.18). This explains the appearance of $1/M$ in (3.5). The resulting $1/M$ expansion becomes[6]

$$\phi'' \to e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\left\{1 + \frac{i\boldsymbol{q}\cdot\boldsymbol{D}}{2M^2} + \frac{i\boldsymbol{q}\cdot\boldsymbol{D}\boldsymbol{D}^2}{4M^4} - \frac{\boldsymbol{\Sigma}\times\boldsymbol{q}\cdot\boldsymbol{D}}{2M^2}\left[1 + \frac{\boldsymbol{D}^2}{4M^2}\right] + \mathcal{O}(g, 1/M^5)\right\}\phi''\,. \tag{3.17}$$

Gauge fields are assumed to transform as usual, in the vector representation of the Lorentz group. Combined with derivatives acting on the transformed coordinate in (3.17), we have

$$D_t \to D_t + \frac{1}{M}\boldsymbol{q}\cdot\boldsymbol{D}\,, \quad \boldsymbol{D} \to \boldsymbol{D} + \frac{1}{M}\boldsymbol{q}D_t\,. \tag{3.18}$$

To illustrate the constraints, consider the canonical form of the abelian gauged heavy spin-1/2 fermion effective Lagrangian (i.e., NRQED) through $\mathcal{O}(1/M^3)$. Identifying $\phi'' = \psi$ as a two-component spinor and setting $g = -e$ we obtain [14, 23]

$$\begin{aligned}\mathcal{L} = \psi^\dagger\Big\{ & iD_t + c_2\frac{\boldsymbol{D}^2}{2M} + c_4\frac{\boldsymbol{D}^4}{8M^3} + c_F e\frac{\boldsymbol{\sigma}\cdot\boldsymbol{B}}{2M} + c_D e\frac{[\boldsymbol{\partial}\cdot\boldsymbol{E}]}{8M^2} + ic_S e\frac{\boldsymbol{\sigma}\cdot(\boldsymbol{D}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{D})}{8M^2} \\ & + c_{W1}e\frac{\{\boldsymbol{D}^2, \boldsymbol{\sigma}\cdot\boldsymbol{B}\}}{8M^3} - c_{W2}e\frac{D^i\boldsymbol{\sigma}\cdot\boldsymbol{B}D^i}{4M^3} + c_{p'p}e\frac{\boldsymbol{\sigma}\cdot\boldsymbol{D}\boldsymbol{B}\cdot\boldsymbol{D} + \boldsymbol{D}\cdot\boldsymbol{B}\boldsymbol{\sigma}\cdot\boldsymbol{D}}{8M^3} \\ & + ic_M e\frac{\{\boldsymbol{D}^i, [\boldsymbol{\partial}\times\boldsymbol{B}]^i\}}{8M^3} + c_{A1}e^2\frac{\boldsymbol{B}^2 - \boldsymbol{E}^2}{8M^3} - c_{A2}e^2\frac{\boldsymbol{E}^2}{16M^3} + \mathcal{O}(1/M^4)\Big\}\psi\,. \end{aligned} \tag{3.19}$$

Here we have defined $E^i = (-i/e)[D_t, D^i]$, $\epsilon^{ijk}B^k \equiv (i/e)[D^i, D^j]$. Under (3.17), a straightforward computation yields

$$\delta\mathcal{L} = \frac{1}{M}\delta\mathcal{L}_1 + \frac{1}{M^2}\delta\mathcal{L}_2 + \frac{1}{M^3}\delta\mathcal{L}_3 + \dots\,, \tag{3.20}$$

[6]For notational clarity we leave the coordinate change $x \to x' = \mathcal{B}^{-1}x$ implicit and suppress primes on coordinates and derivatives in (3.17) and (3.18).

where using $\boldsymbol{\Sigma} = \boldsymbol{\sigma}/2$ in (3.17),

$$\delta\mathcal{L}_1 = \psi^\dagger \left[(1-c_2) i\boldsymbol{q}\cdot\boldsymbol{D}\right]\psi\,, \tag{3.21a}$$

$$\delta\mathcal{L}_2 = \psi^\dagger \left[-\frac{1}{2}(1-c_2)\{\boldsymbol{q}\cdot\boldsymbol{D}, D_t\} + \frac{e}{4}(1-2c_F+c_S)\boldsymbol{\sigma}\times\boldsymbol{q}\cdot\boldsymbol{E}\right]\psi\,, \tag{3.21b}$$

$$\begin{aligned}\delta\mathcal{L}_3 = \psi^\dagger \Bigg[&\frac{e}{8}c_D[D_t, \boldsymbol{q}\cdot\boldsymbol{E}] + \frac{e}{8}\left(c_F - c_D + 2c_M\right)\boldsymbol{q}\cdot[\boldsymbol{\partial}\times\boldsymbol{B}] + \frac{i}{4}\left(c_2-c_4\right)\{\boldsymbol{q}\cdot\boldsymbol{D}, \boldsymbol{D}^2\} \\ &+ \frac{ie}{8}c_S\{D_t, \boldsymbol{\sigma}\times\boldsymbol{q}\cdot\boldsymbol{E}\} + \frac{ie}{8}\left(c_2+2c_F-c_S-2c_{W1}+2c_{W2}\right)\{\boldsymbol{q}\cdot\boldsymbol{D}, \boldsymbol{\sigma}\cdot\boldsymbol{B}\} \\ &+ \frac{ie}{8}\left(-c_2+c_F-c_{p'p}\right)\{\boldsymbol{\sigma}\cdot\boldsymbol{D}, \boldsymbol{q}\cdot\boldsymbol{B}\} + \frac{ie}{8}\left(-c_F+c_S-c_{p'p}\right)\boldsymbol{q}\cdot\boldsymbol{\sigma}(\boldsymbol{D}\cdot\boldsymbol{B}+\boldsymbol{B}\cdot\boldsymbol{D})\Bigg]\psi\,.\end{aligned} \tag{3.21c}$$

From $\delta\mathcal{L}_1$ and $\delta\mathcal{L}_2$, we find

$$c_2 = 1\,, \quad c_S = 2c_F - 1\,. \tag{3.22}$$

The variation $\delta\mathcal{L}_3$ is equivalent to zero upon a field strength-dependent modification of the boost transformation (3.17),

$$\psi(x) \to e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\left\{1 + \frac{i\boldsymbol{q}\cdot\boldsymbol{D}}{2M^2} - \frac{\boldsymbol{\sigma}\times\boldsymbol{q}\cdot\boldsymbol{D}}{4M^2} + \frac{ic_D}{8M^3}e\boldsymbol{q}\cdot\boldsymbol{E} + \frac{c_S}{8M^3}e\boldsymbol{q}\cdot\boldsymbol{\sigma}\times\boldsymbol{E} + \mathcal{O}\left(\frac{1}{M^4}\right)\right\}\psi(\mathcal{B}^{-1}x)\,, \tag{3.23}$$

and upon enforcing the constraints [7, 14]

$$c_4 = 1\,, \quad 2c_M = c_D - c_F\,, \quad c_{W2} = c_{W1} - 1\,, \quad c_{p'p} = c_F - 1\,. \tag{3.24}$$

The computation of the complete Lagrangian at $\mathcal{O}(1/M^4)$ is presented in [24].

4 Reparameterization invariance and invariant operators

While in practice it may be convenient to enforce Lorentz invariance only after expanding the Lagrangian in a series of rotationally-invariant, but not Lorentz invariant, operators, it is interesting to consider formalism that permits an explicitly Lorentz-invariant construction. This formalism also addresses the question of existence of a suitable boost generator, extending (3.23) to arbitrary order in $1/M$.

This section begins by introducing covariant notation that can either be used in place of the $v = (1, 0, 0, 0)$ formalism above, or used to construct manifestly invariant operators. The relation between Lorentz invariance and reparameterization invariance is then demonstrated, and a general discussion of the invariant operator method is presented. In particular, we derive the necessary invariance equation (4.17) and present the solution to order $1/M^3$. A systematic, all-orders solution of the invariance equation is given in Appendix B.

4.1 Covariant notation

The formalism of Appendix A allows us to straightforwardly extend the discussion to a general reference vector v and to arbitrary spin. Consider a term in the Lagrangian of the schematic form

$$\bar{\phi}_v \Big\{ \cdots v^\mu \cdots D^\mu \cdots \gamma^\mu \cdots \Big\} \phi_v \,, \tag{4.1}$$

where indices are contracted with $g_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$. Invariance under generalized rotations of such a term in the action follows using the field transformation (3.4),

$$\phi_v(x) \to \mathcal{R}\phi_v(x') \,, \tag{4.2}$$

where $x' \equiv \mathcal{R}^{-1}x$. The transformation of the derivative and the gauge field are as usual,

$$\partial^\mu \to \partial^\mu = \mathcal{R}^\mu{}_\nu \partial'^\nu \,, \quad A^\mu \to \mathcal{R}^\mu{}_\nu A^\nu(x') \,. \tag{4.3}$$

If the Lagrangian is already constructed such that all vector and spinor indices are contracted in (4.1), we can easily see that the Lagrangian is invariant under generalized rotations using the identities

$$v^\mu = \mathcal{R}^\mu{}_\nu v^\nu \,, \qquad \gamma^\mu = \mathcal{R}_{\frac{1}{2}} \left(\mathcal{R}^\mu{}_\nu \gamma^\nu \right) \mathcal{R}_{\frac{1}{2}}^{-1} \,. \tag{4.4}$$

According to (3.6), the infinitesimal boosts are implemented by

$$\phi_v(x) \to W(\mathcal{B}, iD)\phi_v(x') \,, \tag{4.5}$$

where $x' \equiv \mathcal{B}^{-1}x$, together with the transformation of the derivative and gauge field,

$$\partial^\mu \to \partial^\mu = \mathcal{B}^\mu{}_\nu \partial'^\nu \,, \quad A^\mu(x) \to \mathcal{B}^\mu{}_\nu A^\nu(x') \,. \tag{4.6}$$

We may proceed as in Section 3.3 above to construct invariant combinations of Lagrangian interactions of the form (4.1), order by order in $1/M$.

As an explicit example, let us focus presently on the phenomenologically important one-heavy particle sector of a spin-1/2 theory. To enable the $1/M$ expansion and convert to non-relativistic normalization, we introduce the field redefinition as in (3.15) and (3.16),

$$\psi_v(x) = e^{-iMv\cdot x} N(v, iD)\psi'_v(x) \,, \quad N(v, iD) = \left(\frac{M^2}{M^2 + D_\perp^2} \right)^{\frac{1}{4}} \,. \tag{4.7}$$

The boost transformation (4.5) becomes

$$\psi'_v \to e^{iq\cdot x} \tilde{W}_{\frac{1}{2}}(\mathcal{B}, iD + Mv)\psi'_v \,, \tag{4.8}$$

where

$$\tilde{W}(\mathcal{B}, iD + Mv) = N(v + q/M, iD - q)^{-1} W(\mathcal{B}, iD + Mv) N(v, iD) \,. \tag{4.9}$$

The $1/M$ expansion of this transformation is the extension to arbitrary v, for spin-1/2, of the previous (3.17):

$$\psi'_v \to e^{iq\cdot x} \bigg\{ 1 + \frac{iq\cdot D_\perp}{2M^2} - \frac{iq\cdot D_\perp D_\perp^2}{4M^4} + \frac{1}{4M^2}\sigma_{\alpha\beta} q^\alpha D_\perp^\beta \left[1 - \frac{D_\perp^2}{4M^2} \right] + \mathcal{O}(g, 1/M^5) \bigg\} \psi'_v \,. \tag{4.10}$$

Similarly, we find the extension to arbitrary v of the transformations (3.18)

$$v \cdot D \to v \cdot D + \frac{1}{M} q \cdot D_\perp \,, \quad D_\perp^\mu \to D_\perp^\mu - \frac{1}{M} q^\mu (v \cdot D). \tag{4.11}$$

Using these transformations one can build an invariant Lagrangian, which (in the abelian case) is equivalent to the extension of the Lagrangian (3.19) to arbitrary v with the same constraints (3.22) and (3.24).

4.2 Reparameterization invariance

We can reformulate the transformation law for generalized boosts by using the identities,

$$v^\mu = \mathcal{B}^\mu{}_\nu (\mathcal{B}^{-1})^\nu{}_\rho v^\rho \equiv \mathcal{B}^\mu{}_\nu w^\nu \,, \quad \gamma^\mu = \mathcal{B}_{\frac{1}{2}} \left(\mathcal{B}^\mu{}_\nu \gamma^\nu \right) \mathcal{B}_{\frac{1}{2}}^{-1} \,. \tag{4.12}$$

In place of (4.5) and (4.6) the transformation of any operator of the form (4.1) is identical to the transformation obtained by the substitutions

$$v \to w = v + q/M \,, \quad \phi_v \to \phi_w \equiv \mathcal{B}^{-1} W(\mathcal{B}, iD_\mu) \phi_v \,, \tag{4.13}$$

with no transformation of the coordinate and gauge field. The rules (4.13), with suitable choice for W, may be identified with the rules obtained by enforcing "reparameterization invariance" [11]. However, we emphasize that from the present perspective, we are not changing the reference vector v, but simply noticing the equivalence of (4.5) and (4.6) on the one hand, and (4.13) on the other hand, when acting on operators of the form (4.1).

4.3 Invariant operator method

It is not obvious that a non-zero Lagrangian, invariant under (4.5) and (4.6) to arbitrary order, will exist. For example, in (3.20) invariance relies on the possibility to enforce $\delta \mathcal{L}_n = 0$ by modifying the boost generator as in (3.23) and enforcing relations as in (3.22) and (3.24). It is not evident that this procedure can be extended to arbitrary order. We present here a method of constructing operators that are manifestly invariant under a particular choice of boost generator, to arbitrary order in $1/M$. The details of the construction are given in Appendix B.

The embedding of the little group into constrained representations of the full Lorentz group (cf. Appendix A) provides a framework for constructing explicitly invariant operators. Suppose that we find an operator $\Gamma(v, iD)$ such that

$$\Gamma(\Lambda^{-1} v, iD) \Lambda^{-1} W(\Lambda, iD) = \Gamma(v, iD) \,. \tag{4.14}$$

when acting on fields ϕ_v obeying the appropriate constraints, as given in Appendix A (e.g. $\not{v}\, \phi_v = \phi_v$ for spin-1/2). It follows from the rules (4.13) that the combination

$$\Phi_v \equiv \Gamma(v, iD) \phi_v \tag{4.15}$$

is invariant under the reparameterization implementation (4.13) of generalized boosts. Provided that invariance under generalized rotations (4.2)-(4.4) is maintained, we may build operators that are explicitly invariant. For example, in the spin-1/2 case

$$\bar\Psi_v i\not{D}\Psi_v\,, \quad \bar\Psi_v\Psi_v\,, \quad \bar\Psi_v i\sigma^{\mu\nu}[D_\mu, D_\nu]\Psi_v\,, \tag{4.16}$$

are invariant. Note that because of Eq. (3.4) the only constraints on $\Gamma(v, iD)$ from Eq. (4.14) come from boosts $\Lambda = \mathcal{B}$.

Applying field redefinitions as in (4.7), the condition (4.14) for Γ becomes

$$\Gamma(v + q/M, iD - q)\mathcal{B}^{-1}\tilde W(\mathcal{B}, iD + Mv) = \Gamma(v, iD)\,. \tag{4.17}$$

We will refer to (4.17) as the "invariance equation". Provided that such a $\Gamma(v, iD)$ can be found, the field

$$\Phi'_v(x) \equiv \Gamma(v, iD)\phi'_v(x) \tag{4.18}$$

obeys a simple transformation law under the reparameterization implementation of generalized boosts (4.13),

$$\Phi'_v \to \Phi'_w \equiv e^{iq\cdot x}\Phi'_v\,. \tag{4.19}$$

Noting that $e^{-iq\cdot x}(iD^\mu + Mw^\mu)e^{iq\cdot x} = iD^\mu + Mv^\mu$, invariant operators may thus be built from contractions of polynomials of γ^μ and $v^\mu + iD^\mu/M$, between $\bar\Phi'_v$ and Φ'_v. For example in the spin-1/2 case,

$$\bar\Psi'_v(i\not{D} + M\not{v})\Psi'_v\,, \quad \bar\Psi'_v\Psi'_v\,, \quad \bar\Psi'_v i\sigma^{\mu\nu}[D_\mu, D_\nu]\Psi'_v\,, \tag{4.20}$$

are invariant.

4.4 Solution for $\Gamma(v, iD)$

The key element of the invariant operator construction is a solution of the invariance equation (4.17). Without loss of generality, let us set $N(v, iD) = 1$; the solution for general N can then be obtained by $\Gamma(v, iD) \to \Gamma(v, iD)N(v, iD)^{-1}$. The method presented can be easily extended to arbitrary spin, but for illustration we focus on the one-heavy particle sector of a spin-1/2 theory.

In order to obtain a solution in closed form for the free theory, and to make contact with previous work, it is convenient to take the free theory limit for $W_{\frac12}(\mathcal{B}, i\partial + Mv)$ of the form [11]

$$\begin{aligned} W_{\frac12}(\mathcal{B}, i\partial + Mv) &= \mathcal{B}_{\frac12}\Lambda_{\frac12}(\hat{\mathcal{V}}_{\rm free}, v + q/M)^{-1}\Lambda_{\frac12}(\hat{\mathcal{V}}_{\rm free}, v) \\ &= 1 + \frac{1}{4M^2}\sigma_\perp^{\mu\nu}q_\mu\partial_\nu\left[1 - \frac{iv\cdot\partial}{M} + \frac{1}{M^2}\left((iv\cdot\partial)^2 - \frac14(i\partial_\perp)^2\right)\right] + \mathcal{O}(1/M^5)\,, \end{aligned} \tag{4.21}$$

where $\Lambda_{\frac12}(u, v)$ was defined in (3.3), $\mathcal{V}^\mu_{\rm free} \equiv v^\mu + i\partial^\mu/M$ and $\hat{\mathcal{V}}^\mu_{\rm free} \equiv \mathcal{V}^\mu_{\rm free}/|\mathcal{V}_{\rm free}|$. We have also used that $\not{v}\,\psi_v = \psi_v$. Inspection of (4.17) shows that an all-orders solution can be written for Γ in the non-interacting theory,

$$\begin{aligned} \Gamma(v, i\partial) = \Lambda_{\frac12}(\hat{\mathcal{V}}_{\rm free}, v) &= 1 + \frac{i\not{\partial}_\perp}{2M} + \frac{1}{M^2}\left[-\frac18(i\partial_\perp)^2 - \frac12 i\not{\partial}_\perp iv\cdot\partial\right] \\ &+ \frac{1}{M^3}\left[\frac14(i\partial_\perp)^2 iv\cdot\partial + \frac{i\not{\partial}_\perp}{2}\left(-\frac38(i\partial_\perp)^2 + (iv\cdot\partial)^2\right)\right] + \mathcal{O}(1/M^4)\,. \end{aligned} \tag{4.22}$$

In the interacting theory it turns out that one cannot simply replace ∂ by D in (4.22) to obtain a solution for $\Gamma(v, iD)$. It is instead necessary to add specific field strength dependent terms, first to W (as in (4.23) and (B.2a) below) in order to satisfy consistency conditions, and then to Γ in order to solve the invariance equation (4.17). The computations of Appendix B show that a solution for $\Gamma(v, iD)$ will exist if we specify

$$W_{\frac{1}{2}}(\mathcal{B}, iD + Mv) = 1 + \frac{1}{4M^2}\sigma^{\perp}_{\mu\nu} q^\mu D^\nu_\perp \left(1 - \frac{iv\cdot D}{M}\right) + \mathcal{O}(1/M^4)\,, \tag{4.23}$$

with (4.23) reducing to (4.21) at $g = 0$. Let us proceed through $\mathcal{O}(1/M^3)$, writing

$$\Gamma = 1 + \frac{1}{M}\Gamma^{(1)} + \frac{1}{M^2}\Gamma^{(2)} + \frac{1}{M^3}\Gamma^{(3)} + \dots\,, \tag{4.24}$$

and deriving a solution to the invariance equation (4.17) order by order in $1/M$. In Appendix B we present a systematic construction that extends the solution to arbitrary order.

Modulo terms that vanish when acting on ψ_v with $\not{v}\,\psi_v = \psi_v$, we find

$$\Gamma^{(1)} = \frac{1}{2} i\not{D}_\perp\,. \tag{4.25a}$$

$$\Gamma^{(2)} = -\frac{1}{8}(iD_\perp)^2 - \frac{1}{2} i\not{D}_\perp iv\cdot D + gA\sigma^{\mu\nu}G_{\mu\nu} + gB\gamma^\mu v^\nu G_{\mu\nu}\,. \tag{4.25b}$$

$$\Gamma^{(3)} = \frac{1}{4}(iD_\perp)^2 iv\cdot D + \frac{i\not{D}_\perp}{2}\left[-\frac{3}{8}(iD_\perp)^2 + (iv\cdot D)^2\right] - \frac{g}{8}G_{\mu\nu}v^\mu D^\nu_\perp - \frac{g}{16}\sigma^{\mu\nu}_\perp G_{\mu\nu} i\not{D}_\perp\,, \tag{4.25c}$$

where we define $[iD_\mu, iD_\nu] = igG_{\mu\nu}$. Starting at order $1/M^2$ the solution is not unique. However, since we will consider arbitrary factors of $\mathcal{V}^\mu \equiv v^\mu + iD^\mu/M$ when constructing invariant operators, we can set $A = B = 0$ by considering instead of Γ, the operator Γ' given by

$$\Gamma(v, iD) = (1 - iA\sigma_{\mu\nu}[\mathcal{V}^\mu, \mathcal{V}^\nu] - iB\gamma_\mu\mathcal{V}_\nu[\mathcal{V}^\mu, \mathcal{V}^\nu] + \dots)\,\Gamma'(v, iD)\,. \tag{4.26}$$

Similarly, we have absorbed additional $1/M^3$ terms in (4.25c). The remaining terms in (4.25) have free derivatives D_μ acting to the right, and cannot be removed as in (4.26).

A complete basis of bilinears required through order $1/M^3$ is

$$\mathcal{L} = \bar{\Psi}_v\Big\{M(\not{\mathcal{V}} - 1) - a_F g\frac{\sigma^{\mu\nu}G_{\mu\nu}}{4M} + ia_D g\frac{\{\mathcal{V}_\mu, [M\mathcal{V}_\nu, G^{\mu\nu}]\}}{16M^2} - a_{W1} g\frac{[M\mathcal{V}^\alpha, [M\mathcal{V}_\alpha, \sigma^{\mu\nu}G_{\mu\nu}]]}{16M^3}$$
$$+ a_{A1} g^2\frac{G_{\mu\nu}G^{\mu\nu}}{16M^3} + a_{A2} g^2\frac{\mathcal{V}_\alpha G^{\mu\alpha}G_{\mu\beta}\mathcal{V}^\beta}{16M^3}\Big\}\Psi_v\,. \tag{4.27}$$

Performing field redefinitions to arrive at canonical form, we recover the result (3.19) with constraints (3.22) and (3.24). The computation at $\mathcal{O}(1/M^4)$ is presented in [24]. We may perform a similar computation for heavy vector particles (or particles of arbitrary spin), and/or enforce constraints appropriate to self-conjugate fields (cf. Appendix A).

The passage from (4.22) to (4.25) is not as simple as previously envisaged [11, 14], and careful attention must be paid to the interplay of Lorentz and gauge symmetry. The computations in Appendix B show that an arbitrary "covariantization" of (4.21) does *not* solve the invariance equation (4.17). The covariant little group element $W(\mathcal{B}, iD+Mv)$ must satisfy consistency conditions for a solution to exist, and specific field strength dependent terms, such as those appearing in (4.25c), are necessary in order that $\Gamma(v, iD)$ satisfy the resulting invariance equation (4.17). These considerations have previously been overlooked [11, 14]. For example, a naive covariantization of Eq. (4.22),

$$\Gamma^{\rm naive}(v,iD) = 1 + \frac{i\not{D}_\perp}{2M} + \frac{1}{M^2}\left[-\frac{1}{8}(iD_\perp)^2 - \frac{1}{2}i\not{D}_\perp iv\cdot D\right] \\ + \frac{1}{M^3}\left[\frac{1}{4}(iD_\perp)^2 iv\cdot D + \frac{i\not{D}_\perp}{2}\left(-\frac{3}{8}(iD_\perp)^2 + (iv\cdot D)^2\right)\right] + \mathcal{O}(1/M^4)\,, \tag{4.28}$$

is not a solution to the invariance equation.

The necessity for such additional field strength dependent terms can also be seen from the fact that the right hand side of (4.28) would imply a transformation $\psi_v \to \psi_w = \Gamma^{\rm naive}(w,iD)^{-1}e^{iq\cdot x}\Gamma^{\rm naive}(v,iD)\psi_v$ that takes ψ_v outside of the assumed representation space, with $\not{v}\,\psi_v = \psi_v$. In the heavy fermion Lagrangian, the effects of these field-strength dependent terms appear first[7] at order $\mathcal{O}(1/M^4)$, where omission of the final term in (4.25c) would lead to incorrect $1/M^4$ Lagrangian coefficient relations [24].

Before closing this section, let us summarize the value of the invariant operator method. Appendix B shows that we can find a suitable covariantization of $W(\mathcal{B}, i\partial + Mv)$ that allows solution of the invariance equation for $\Gamma(v, iD)$ to any order in $1/M$. Hence this method proves the existence of a covariantized boost operator and a non-zero, Lorentz invariant Lagrangian to arbitrary order. We may proceed in either of two ways to construct invariant Lagrangians. Firstly, we may proceed as in (4.27), where we construct manifestly invariant interactions through some fixed order in $1/M$; to achieve canonical form we must then perform field redefinitions. Alternatively, we may proceed as in (3.19) (or its generalization to arbitrary v), armed with the knowledge that a suitable boost generator as in (3.23) can be reconstructed order by order.

5 Effective field theories for massless particles

Although our primary focus has been on the constraints imposed by Lorentz invariance in heavy particle effective field theories, it is interesting to consider the applications of other Lorentz representations. Recall that for physical states, representations of the Lorentz group fall into distinct classes, depending on the nature of p^0 and p^2. For example, in our heavy particle applications we considered the little group for $p^0 > 0$ and $p^2 = M^2 > 0$.

Consider now the case $p^0 > 0$ and $p^2 = 0$. This applies to the collinear sector of soft-collinear effective theory [25–29]. The little group in this case is isomorphic to $E(2)$, the

[7] When building invariant fermion bilinears, the leading terms involve $iv\cdot D$ multiplying $1/M$ corrections appearing in $\Gamma(v, iD)$. Since such terms are eliminated in going to canonical form, nontrivial effects of the $1/M^3$ corrections to $\Gamma(v, iD)$ appear first at order $1/M^4$.

euclidean group of rotations and translations in two dimensions. In analogy to the construction in Section 3.1, let us consider the little group defined by the invariant vector $E\,n$, where $n^2 = 0$ and E is a reference energy. In order to define the induced representation, let us also introduce a timelike unit vector v with $v^2 = 1$.[8] Given n and v we may define an additional lightlike vector,

$$\bar{n}^\mu = \frac{1}{n\cdot v}\left(2v^\mu - \frac{n^\mu}{n\cdot v}\right)\,, \tag{5.1}$$

satisfying $\bar{n}^2 = 0$ and $n\cdot\bar{n} = 2$. In this section (*only*) we define perpendicular components $p_\perp$ with respect to n and $\bar{n}$. We also define vectors p_+ and p_- along the n and $\bar{n}$ directions respectively,

$$p^\mu \equiv \frac{\bar{n}\cdot p}{2}n^\mu + \frac{n\cdot p}{2}\bar{n}^\mu + p_\perp^\mu \equiv p_+^\mu + p_-^\mu + p_\perp^\mu\,. \tag{5.2}$$

With this notation let us define a standard Lorentz transformation taking En to p as

$$L(p) = L_{\bar{\mathcal{S}}}(p)L_{\mathcal{B}}(p)\,, \tag{5.3}$$

where $L_{\mathcal{B}}$ is a boost that takes En^μ to p_+^μ and $L_{\bar{\mathcal{S}}}$ is a parabolic Lorentz transformation taking p_+^μ to p^μ. They are given by

$$L_{\mathcal{B}}(p)^\mu{}_\nu = g^\mu_\nu + \frac{1}{2}\left(\frac{\bar{n}\cdot p}{2E} - 1\right)n^\mu\bar{n}_\nu + \frac{1}{2}\left(\frac{2E}{\bar{n}\cdot p} - 1\right)\bar{n}^\mu n_\nu\,, \tag{5.4a}$$

$$L_{\bar{\mathcal{S}}}(p)^\mu{}_\nu = g^\mu_\nu + \frac{1}{\bar{n}\cdot p}\left(p_\perp^\mu\bar{n}_\nu - \bar{n}^\mu p_{\perp\nu}\right) - \frac{p_\perp^2}{2(\bar{n}\cdot p)^2}\bar{n}^\mu\bar{n}_\nu\,. \tag{5.4b}$$

The choice (5.3) for $L(p)$ is convenient due to the resulting simplicity of $W(\Lambda, p)$. The space of physical states generated by $U[L(p)]|En^\mu,\sigma\rangle$ is sufficient to describe particles of a given helicity with non-vanishing $\bar{n}\cdot p$.

It is straightforward to compute the little group element corresponding to arbitrary Lorentz transformations according to (3.1). The six independent Lorentz transformations can be grouped into four classes. First, there is the one-parameter group of rotations $\mathcal{R}$ that keep n and $\bar{n}$ fixed. Second, there is the two-parameter group of parabolic Lorentz transformations $\mathcal{S}$ that keep n fixed but change $\bar{n}$. These two classes form the little group of n. Third, there is the one-parameter group of boosts $\mathcal{B}$ in the n direction that change n and $\bar{n}$. Fourth, there is the two-parameter group of parabolic transformations $\bar{\mathcal{S}}$ that keep $\bar{n}$ fixed but change n. In infinitesimal form these transformations are given by

$$\mathcal{R}(\theta)^\mu{}_\nu = g^\mu_\nu + \theta\epsilon^\mu{}_{\nu\rho\sigma}n^\rho\bar{n}^\sigma + \mathcal{O}(\theta^2) \qquad [n\to n, \bar{n}\to\bar{n}]\,, \tag{5.5a}$$

$$\mathcal{S}(\alpha)^\mu{}_\nu = g^\mu_\nu + \frac{\alpha^\mu n_\nu - n^\mu\alpha_\nu}{2} + \mathcal{O}(\alpha^2) \qquad [n\to n, \bar{n}\to\bar{n}+\alpha]\,, \tag{5.5b}$$

$$\mathcal{B}(\eta)^\mu{}_\nu = g^\mu_\nu + \eta\frac{n^\mu\bar{n}_\nu - \bar{n}^\mu n_\nu}{2} + \mathcal{O}(\eta^2) \qquad [n\to(1+\eta)n, \bar{n}\to(1-\eta)\bar{n}]\,, \tag{5.5c}$$

[8] In applications to heavy quark processes the vector v is naturally identified with the reference vector for the heavy quark field.

$$\bar{\mathcal{S}}(\beta)^{\mu}{}_{\nu} = g^{\mu}_{\nu} + \frac{\beta^{\mu}\bar{n}_{\nu} - \bar{n}^{\mu}\beta_{\nu}}{2} + \mathcal{O}(\beta^2) \qquad [n \to n+\beta, \bar{n} \to \bar{n}]\,, \tag{5.5d}$$

where $\alpha^{\mu} = \alpha^{\mu}_{\perp}$ and $\beta^{\mu} = \beta^{\mu}_{\perp}$. Note that physical states must transform trivially under $\mathcal{S}$ to avoid continuous internal degrees of freedom and that little group elements can be parameterized as (e.g. see [20])

$$W(\Lambda, p) = \mathcal{S}[\tilde{\alpha}(\Lambda, p)]\mathcal{R}[\tilde{\theta}(\Lambda, p)]\,. \tag{5.6}$$

We find that the mapping (3.1) with $L(p)$ chosen as in Eq. (5.3) takes the little group rotation $\mathcal{R}(\theta)$ into itself. Of the remaining three cases only the little group elements $\mathcal{S}(\alpha)$ have a non-trivial mapping

$$\tilde{\theta}[\mathcal{R}(\theta), p] = \theta\,, \qquad \tilde{\alpha}[\mathcal{R}(\theta), p]^{\mu} = 0\,, \tag{5.7a}$$

$$\tilde{\theta}[\mathcal{S}(\alpha), p] = -\frac{1}{2(\bar{n}\cdot p)}\epsilon_{\mu\nu\rho\sigma}\alpha^{\mu}p^{\nu}_{\perp}n^{\rho}\bar{n}^{\sigma}\,, \qquad \tilde{\alpha}[\mathcal{S}(\alpha), p]^{\mu} = \frac{E}{\bar{n}\cdot p}\alpha^{\mu}\,, \tag{5.7b}$$

$$\tilde{\theta}[\mathcal{B}(\eta), p] = 0\,, \qquad \tilde{\alpha}[\mathcal{B}(\eta), p]^{\mu} = 0\,, \tag{5.7c}$$

$$\tilde{\theta}[\bar{\mathcal{S}}(\beta), p] = 0\,, \qquad \tilde{\alpha}[\bar{\mathcal{S}}(\beta), p]^{\mu} = 0\,. \tag{5.7d}$$

The result (5.6) with little group parameters (5.7) defines the transformation law for particle states. As in the timelike case, we postulate the field transformation law,

$$\phi_a(x) \to D[W(\Lambda, iD_{\mu})]_{ab}\phi_b(\Lambda^{-1}x)\,, \tag{5.8}$$

where now $D(W)$ refers to a representation of the $E(2)$ little group.

We focus on the representation appropriate to a massless spin-1/2 particle,

$$D[\mathcal{S}(\tilde{\alpha})\mathcal{R}(\tilde{\theta})] = \exp[i\tilde{\theta}/2]\,, \tag{5.9}$$

and embed this representation into a Dirac spinor representation ψ_n of the Lorentz group. A trivial action of $\mathcal{S}$ on this field is equivalent to the constraint

$$\not{n}\,\psi_n = 0\,. \tag{5.10}$$

The transformation law,

$$\psi_n(x) \to \left(1 + \frac{i}{4}\omega(\Lambda, iD)_{\mu\nu}\sigma^{\mu\nu}\right)\psi_n(\Lambda^{-1}x)\,, \tag{5.11}$$

with $\omega_{\mu\nu}(\Lambda, iD)$ obtained from (5.7) and ψ_n satisfying (5.10), reduces to (5.9).

Similar to the timelike case, we may investigate general conditions under which (5.11) leads to a Lorentz invariant theory. We note that for terms in the fermion Lagrangian of the form

$$\bar{\psi}_n\Big\{\cdots n^{\mu}\ldots\bar{n}^{\mu}\cdots D^{\mu}\cdots\gamma^{\mu}\cdots\Big\}\psi_n\,, \tag{5.12}$$

we may recast invariance under (5.11) as a collection of "reparameterization" transformations acting on n and $\bar{n}$, cf Section 4.2. In particular, invariance under rotations $\mathcal{R}(\theta)$ is ensured by writing a naively covariant Lagrangian in terms of the constrained field ψ_n, as in (5.12). Transformations $\bar{\mathcal{S}}(\beta)$, $\mathcal{S}(\alpha)$ and $\mathcal{B}(\eta)$ translate to the "type-I", "type-II" and "type-III" transformations considered in [30]. A more detailed discussion of the lightlike case, involving a rigorous discussion of Lorentz invariance, and the inclusion of multiple momentum modes and multiple gauge symmetries, is beyond the scope of the present paper and is left to future work.

6 Summary

The usual procedure of implementing Lorentz invariance via finite dimensional representations of the Lorentz group is insufficient for application to heavy particle effective theories. We have adapted the formalism of induced representations for application to heavy particle field transformation laws. Returning to the questions posed in the Introduction, we see that the parameter v enters as an arbitrary reference vector in the effective theory construction. Rules identifiable with "reparameterization invariance" (4.13) are obtained by a rewriting of the transformation law for generalized boosts, and the class of reparameterization transformations consistent with Lorentz and gauge invariance is identified through a systematic solution of the invariance equation (4.17). While an explicit construction such as in (1.1) must map into this framework, it is not necessary to refer to a specific underlying ultraviolet completion, or to explicitly integrate out degrees of freedom when deriving these transformation laws.

Let us compare our formalism to previous work. A naive ansatz for implementing Lorentz invariance via reparameterization invariance breaks down for $\Gamma(v, iD)$ starting at order $1/M^3$, corresponding to new effects at order $1/M^4$ in the canonical Lagrangian. The transformation law defined by $W(\Lambda, iD)$ is corrected at order $1/M^4$. These subtleties were not treated in the classic work of Luke and Manohar [11, 14], and the ansatz proposed there would lead to inconsistencies at the orders in $1/M$ specified above. Brambilla et al. [17, 18], who studied Lorentz invariance constraints in nonrelativistic QCD (NRQCD), recognized that Wilson-coefficient dependent corrections to $W(\Lambda)$ must be included when deriving an invariant Lagrangian in canonical form. In Refs. [17, 18] the constraints of Lorentz invariance are derived (through order $1/M^2$ in the one-heavy-particle sector) at the level of canonically quantized charges, a procedure that becomes increasingly cumbersome at high orders in the $1/M$ expansion. In Section 3 we have used general properties of commutators of the S matrix with conserved charges to derive constraints at the Lagrangian level that implement Lorentz invariance for heavy particle effective theories in canonical form. In Section 4 we have derived consistent reparameterization transformations that allow solution to the invariance equation (4.17), and hence the construction of manifestly invariant Lagrangians to arbitrary order.

We demonstrated the application of our formalism in the case of NRQED (i.e., the parity and time-reversal symmetric theory of a heavy spin-1/2 particle coupled to an abelian gauge field). At a practical level, the main results for building heavy fermion Lagrangians are contained in (3.23), or for the invariant operator method, in (4.24) and (4.25). The NRQED Lagrangian is computed at $\mathcal{O}(1/M^4)$ in [24].

We note that a choice must be made between a canonical form of the Lagrangian with somewhat complicated boost generator, versus a simpler form of the boost generator with non-canonical Lagrangian. In practical computations, it is typically easier to choose the former approach. We remark that a regularization scheme that breaks Lorentz symmetry must be accompanied by counterterms that reinstate the symmetry.[9] Renormalization of the Lagrangian in canonical form should be defined in such a way that non-canonical terms are not generated.

The heavy particle limit considered here assumes a single large mass scale. Interesting complications can arise when this is not the case, e.g. in the phenomenology of heavy baryons in low-energy processes involving pions, Δ excitations and electroweak gauge interactions. Numerically large coefficients appearing in the m_π/m_N expansion limit the usefulness of the heavy particle expansion unless certain formally suppressed terms are "resummed", introducing nontrivial power counting and renormalization issues [31–36]. While it may be possible to embed a given heavy particle theory into a larger structure, this does not lessen the importance of understanding Lorentz invariance in the low energy limit.[10]

The formalism presented here can be applied to straightforwardly construct heavy particle Lagrangians of arbitrary spin. It can also be easily extended to include multiple heavy particle fields, and other relativistic degrees of freedom beyond the abelian gauge fields considered here. As described in Section 5 the extension to effective field theories for massless particles involves induced representations for little group isomorphic to $E(2)$, the euclidean transformations in two dimensions. A rigorous analysis along these lines may help clarify several outstanding issues in SCET, ranging from the appearance of new momentum modes, to the interplay of ultraviolet regulators and factorization [37–39]. It may be interesting to investigate the application of the little group corresponding to a spacelike reference vector, $s^2 = -1$ (cf. our $v^2 = 1$ and $n^2 = 0$ cases), and to explore embeddings into nonlinear realizations with fictitious Goldstone fields [40].

Acknowledgments

We thank G. Lee and G. Paz for collaboration on explicit computations with the $1/M^4$ NRQED Lagrangian that helped reveal the subtleties in reparameterization invariance at this order, and for comments on the manuscript. Work supported by NSF Grant 0855039 and DOE Grant DE-FG02-90ER-40560. J.H. is supported by the Fermi McCormick fellowship.

A Extension to arbitrary spin and self-conjugate fields

Although the explicit results in this paper are focused on spin-1/2 fields transforming under an abelian (i.e. complex) gauge group, the formalism extends straightforwardly to fields of arbitrary spin or to self-conjugate fields. In section A.1 we describe the formalism for embedding arbitrary spin representations within products of Dirac spinor and Lorentz vector representations of the Lorentz group. For a related discussion see e.g. [41]. Section A.2 describes the constraints imposed on the effective theory deriving from self-conjugate fields. For a related discussion see e.g. [8].

[9] We thank T. Becher for a discussion on this point.

[10] This may be viewed in analogy to embedding nonlinear sigma models into linear sigma models with extra degrees of freedom.

A.1 Higher spin representations

Irreducible higher spin representations can be built using products of the Dirac spinor and vector representations

$$\psi_v \to \Lambda_{\frac{1}{2}} \psi_v \,, \quad Z_v^\alpha \to \Lambda^\alpha{}_\beta Z_v^\beta \,, \tag{A.1}$$

where $\Lambda = D(W)$ is a little group element as in Section 3.1, i.e., $\Lambda v = v$. The corresponding generators for these two representations are given by

$$\mathcal{J}_{\frac{1}{2}}^{\alpha\beta} = \frac{1}{2}\sigma^{\alpha\beta} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta] \,, \qquad (\mathcal{J}^{\alpha\beta})_{\mu\nu} = i(g^\alpha{}_\mu g^\beta{}_\nu - g^\beta{}_\mu g^\alpha{}_\nu). \tag{A.2}$$

We enforce a maximal set of constraints to isolate the appropriate irreducible representation.

Integer spin: For integer spin $s = n$, consider the totally symmetric and traceless tensor $Z_v^{\mu_1 \dots \mu_n}$, which has $(n+1)^2$ degrees of freedom. Imposing

$$v_{\mu_1} Z_v^{\mu_1 \dots \mu_n} = 0 \tag{A.3}$$

yields n^2 additional constraints, leaving us with $2n+1 = 2s+1$ degrees of freedom as desired. Under Lorentz transformations this field transforms as

$$Z_v^{\mu_1 \dots \mu_n} \to \Lambda^{\mu_1}{}_{\nu_1} \dots \Lambda^{\mu_n}{}_{\nu_n} Z_v^{\nu_1 \dots \nu_n} \,. \tag{A.4}$$

Using $\Lambda^T g \Lambda = g$ and $\Lambda v = v$, it is easy to see that symmetry, tracelessness and the constraint (A.3) are preserved by this transformation.

Half-integer spin: For half-integer spin $s = n + 1/2$, consider the spinor-tensor $\psi_v^{\mu_1, \mu_2, \dots, \mu_n}$, which is totally symmetric in the indices $\mu_1 \dots \mu_n$ and therefore has $2(n+1)(n+2)(n+3)/3$ degrees of freedom. We impose the constraints [11]

$$\not{v}\, \psi_v^{\mu_1 \dots \mu_n} = \psi_v^{\mu_1 \dots \mu_n} \,, \quad \gamma_{\mu_1} \psi_v^{\mu_1 \dots \mu_n} = 0. \tag{A.5}$$

The second constraint yields $n(n+1)(n+5)/3$ equations, while the first projects a four-component spinor onto a two-dimensional subspace, reducing the degrees of freedom by $1/2$. In total $2(n+1) = 2s+1$ degrees of freedom remain. Under Lorentz transformations this field transforms as

$$\psi_v^{\mu_1 \dots \mu_n} \to \Lambda^{\mu_1}{}_{\nu_1} \dots \Lambda^{\mu_n}{}_{\nu_n} \Lambda_{\frac{1}{2}} \psi_v^{\nu_1 \dots \nu_n} \,. \tag{A.6}$$

This is symmetric in $\mu_1 \dots \mu_n$. That equations (A.5) are preserved follows immediately from $\Lambda v = v$ and $\Lambda_{\frac{1}{2}}^{-1} \gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu \gamma^\nu$.

[11] Note that the second constraint implies $g_{\mu\nu} \psi_v^{\mu\nu\mu_3 \dots \mu_n} = 0$ and, furthermore, is equivalent to imposing $v_{\mu_1} \psi_v^{\mu_1 \dots \mu_n} = 0$ and $\epsilon_{\nu\alpha\beta\mu_1} v^\nu \sigma^{\alpha\beta} \psi_v^{\mu_1 \dots \mu_n} = 0$

A.2 Self-conjugate fields

The self-conjugacy of $SU(2)$ implies that for any field $\phi(x)$ transforming as in (2.3) or (3.8) with the plus sign, the field

$$\phi^c(x) = S\phi^*(x)\,, \tag{A.7}$$

transforms as in (2.3) or (3.8) with the minus sign. Here S is the $(2s+1)\times(2s+1)$ similarity transformation for the spin-s representation of $SU(2)$, such that $(-\Sigma^i)^* = S\Sigma^i S^{-1}$. In covariant language, this translates to the simultaneous transformations

$$\phi_v(x) \to \phi_v^c(x)\,, \quad v^\mu \to -v^\mu\,. \tag{A.8}$$

In terms of the irreducible representations constructed in Section A.1, the field transformation in (A.8) reads[12]

$$Z_v^{\mu_1\dots\mu_s} \to (Z_v^c)^{\mu_1\dots\mu_s} = (Z_v^{\mu_1\dots\mu_s})^*\,, \quad \psi_v^{\mu_1\dots\mu_s} \to (\psi_v^c)^{\mu_1\dots\mu_s} = \mathcal{C}(\psi_v^{\mu_1\dots\mu_s})^*\,, \tag{A.9}$$

for integer spin and half-integer spin fields, respectively. The charge conjugation matrix $\mathcal{C}$ acts on the spinor index of ψ_v. It is symmetric and unitary, and obeys $\mathcal{C}^\dagger\gamma^\mu\mathcal{C} = -\gamma^{\mu *}$. The parity (A.8) arises if the effective theory is describing a full theory of a self-conjugate field (necessarily transforming in a real representation of a gauge group). For example, the effective theory field for a real scalar $\varphi = \varphi^*$ can be obtained via

$$\varphi(x) = e^{-iMv\cdot x}\varphi_v(x)/\sqrt{M} = e^{iMv\cdot x}\varphi_v^*(x)/\sqrt{M} = \varphi^*(x)\,. \tag{A.10}$$

Similarly, the effective theory for a Majorana fermion represented by a Dirac spinor $\psi_M = \psi_M^c$ can be obtained via

$$\psi_M = \sqrt{2}e^{-iMv\cdot x}(h_v + H_v) = \sqrt{2}e^{iMv\cdot x}(h_v^c + H_v^c) = \psi_M^c\,, \tag{A.11}$$

where $\not{v}\, h_v = h_v$ and $\not{v}\, H_v = -H_v$.

It follows from (A.8) that the allowed operators $\bar{\phi}_v\mathcal{O}(v)\phi_v$ in the Lagrangian representing a self-conjugate field can be chosen such that

$$\mathcal{O}(v) = \mathcal{C}\mathcal{O}(-v)^*\mathcal{C}^\dagger. \tag{A.12}$$

Since we are often interested in constructing the Lagrangian in canonical form, i.e., without higher $iv\cdot D$ derivatives acting on ϕ_v, it is important to ask whether this condition is preserved by the requisite field redefinitions. By a similar reasoning to above, operators of the form $\bar{\phi}_v[iv\cdot DX(v) + X^\dagger(v)iv\cdot D]\phi_v$ appearing in the Lagrangian must be such that $X(v) = \mathcal{C}X(-v)^*\mathcal{C}^\dagger$. Hence field redefinitions of the form $\phi_v \to [1 - X(v)]\,\phi_v$ achieve canonical form of the Lagrangian while preserving (A.12).

[12] We here choose a basis such that $S = 1$ for vectors.

B Solution to the invariance equation

Section 4.4 describes the solution of the invariance equation (4.17) for the function $\Gamma(v, iD)$ in the free theory. The solution in the interacting theory is not simply obtained from the free one by replacing ∂ with D. Here we present a method of solution that is valid to any order in $1/M$. Since we use $\Gamma(v, iD)$ to construct the invariant Lagrangian, the existence of a solution for $\Gamma(v, iD)$ proves that a non-zero Lagrangian exists at any order in $1/M$. First, we will construct the general solution in section B.1 and then explicitly apply this construction to the spin 1/2 theory up to order $1/M^3$ in section B.2.

B.1 Series solution for Γ

Recall the equation (4.17) for Γ required to build explicitly invariant operators,

$$\Gamma(v + q/M, iD - q)\mathcal{B}^{-1}W(\mathcal{B}, iD + Mv) = \Gamma(v, iD)\,, \tag{B.1}$$

where to first order in q we have $\mathcal{B}^{-1}v = v + q/M$. Let us expand in orders of $1/M$ and define

$$X \equiv \mathcal{B}^{-1}W = 1 + q^\mu X_\mu = 1 + q^\mu \left[\frac{1}{M}X^{(1)}_\mu + \frac{1}{M^2}X^{(2)}_\mu + \dots\right]\,, \tag{B.2a}$$

$$\Gamma = 1 + \frac{1}{M}\Gamma^{(1)} + \frac{1}{M^2}\Gamma^{(2)} + \dots\,. \tag{B.2b}$$

We note that the variation in Γ arises from the variations in v and in iD,

$$\delta\Gamma = \Gamma(v + q/M, iD - q) - \Gamma(v, iD) = q^\mu\left(-\frac{\partial}{\partial iD^\mu}\Gamma + \frac{1}{M}\frac{\partial}{\partial v^\mu}\Gamma\right)\,. \tag{B.3}$$

Equating orders in $1/M$, we find

$$\frac{\partial}{\partial iD^\mu}\Gamma^{(n)} = \frac{\partial}{\partial v^\mu}\Gamma^{(n-1)} + \Gamma^{(n-1)}X^{(1)}_\mu + \Gamma^{(n-2)}X^{(2)}_\mu + \dots + \Gamma^{(0)}X^{(n)}_\mu \equiv Y^{(n)}_\mu\,, \tag{B.4}$$

where we define $\Gamma^{(0)} = 1$. Note that Eq. (B.4) is understood to be contracted with q^μ so that pieces proportional to v^μ should be dropped. We can solve this equation for $\Gamma^{(n)}$ obtaining

$$\begin{aligned}\Gamma^{(n)} &= \sum_{m=1}^{n}\frac{(-1)^{m-1}}{m!}iD_\perp^{\mu_1}iD_\perp^{\mu_2}\dots iD_\perp^{\mu_m}\frac{\partial}{\partial iD^{\mu_1}}\frac{\partial}{\partial iD^{\mu_2}}\cdots\frac{\partial}{\partial iD^{\mu_{m-1}}}Y^{(n)}_{\mu_m}\\ &= iD_\perp^\mu Y^{(n)}_\mu - \frac{1}{2!}iD_\perp^\mu iD_\perp^\nu\frac{\partial}{\partial iD^\mu}Y^{(n)}_\nu + \dots\,,\end{aligned} \tag{B.5}$$

provided that at each order, the $Y^{(n)}$ derived from the already determined $\Gamma^{(1)}, \dots, \Gamma^{(n-1)}$ satisfy[13]

$$\frac{\partial}{\partial iD^{[\nu}}Y^{(n)}_{\mu]} = 0\,, \tag{B.6}$$

[13] This is the analog of $\vec{\nabla}\times\vec{E} = \vec{0}$ for the existence of a solution ϕ of $\vec{\nabla}\phi = \vec{E}$ in electrostatics.

where $A^{[\mu}B^{\nu]} = (A^\mu B^\nu - A^\nu B^\mu)/2$ denotes antisymmetrization. Using the definition of $Y^{(n)}$ we can show that this imposes constraints on $X^{(n)}$, for $n \geq 2$,

$$\frac{\partial}{\partial iD^{[\nu}} X^{(n)}_{\mu]} = -\frac{\partial}{\partial v^{[\mu}} X^{(n-1)}_{\nu]} + X^{(n-1)}_{[\mu} X^{(1)}_{\nu]} + X^{(n-2)}_{[\mu} X^{(2)}_{\nu]} + \cdots + X^{(1)}_{[\mu} X^{(n-1)}_{\nu]} \equiv Z^{(n)}_{\mu\nu} \,. \quad \text{(B.7)}$$

For Eq. (B.7) to have a solution, a consistency condition on $Z^{(n)}_{\mu\nu}$ requires that[14]

$$0 = v_\sigma \epsilon^{\mu\nu\rho\sigma} \frac{\partial}{\partial iD^\rho} Z^{(n)}_{\mu\nu} \,. \quad \text{(B.8)}$$

We can show by induction that Eq. (B.7) can be solved at each order. Since $X^{(1)}$ is dimensionless, it cannot depend on iD; hence $Z^{(2)}$ from (B.7) is also independent of iD and solves (B.8). Now assume that we have constructed solutions $X^{(n)}$ to Eq. (B.7) for $n = 1, ..., N-1$ (necessarily obeying the constraint (B.8)). Application of the Jacobi identity shows that the constraint (B.8) is then obeyed for $n = N$ and a solution to Eq. (B.7) can be found for $n = N$.

Let us find a solution to Eq. (B.7) that reduces to a given $X_{\rm free}$ for the non-interacting theory (e.g., $X_{\rm free} = \mathcal{B}^{-1} W$ from (4.21)). First, note that the existence of the free case solution given in (4.22) implies that the $X^{(n)}$ defined in the free case from (4.21) must obey the constraint (B.7). Let us define naively covariantized quantities $\hat{X}^{(n)} = X^{(n)}_{\rm free}\Big|_{\partial \to D}$, with a definite ordering prescription, e.g. as in (4.28), and define $\hat{Z}^{(n)}$ by

$$\hat{Z}^{(n)}_{\mu\nu} \equiv \frac{\partial}{\partial iD^{[\nu}} \hat{X}^{(n)}_{\mu]} \,. \quad \text{(B.9)}$$

A straightforward calculation then shows that (B.7) is solved by

$$X^{(n)}_\mu = \hat{X}^{(n)}_\mu + 2 \sum_{m=1}^{n-1} \frac{(-1)^m}{(m+1)!} iD^{\nu_1}_\perp \cdots iD^{\nu_m}_\perp \frac{\partial}{\partial iD^{\nu_1}} \cdots \frac{\partial}{\partial iD^{\nu_{m-1}}} \left(Z^{(n)}_{\nu_m \mu} - \hat{Z}^{(n)}_{\nu_m \mu} \right) \,. \quad \text{(B.10)}$$

In the free case we have $Z^{(n)} = \hat{Z}^{(n)}$ and $X^{(n)}$ reduces to the free case solution. Having found a suitable $X^{(n)}$ satisfying (B.7) we may then proceed to build $\Gamma^{(n)}$ satisfying (B.4), and hence Γ satisfying (4.17).

Note that $Z^{(n)}$ has mass dimension $n-2$ so that $n = 4$ is the first order at which field strength dependent terms can cause $Z^{(n)} \neq \hat{Z}^{(n)}$. Correspondingly, our choice (B.10) ensures that field-strength dependent corrections to $X^{(n)} - \hat{X}^{(n)}$ can first appear at order $n = 4$. This can be explicitly seen in the solution for the spin 1/2 theory in the next section.

B.2 Explicit solution for Γ in the spin 1/2 theory

To illustrate, let us calculate Γ for the spin 1/2 theory. Consider the free solution (4.21),

$$X_\mu(v, i\partial) \;=\; \frac{1}{2M}\gamma^\perp_\mu + \frac{1}{4M^2}\sigma^\perp_{\mu\nu}\partial^\nu \left[1 - \frac{iv\cdot\partial}{M} + \frac{1}{M^2}\left((iv\cdot\partial)^2 - \frac{1}{4}(i\partial_\perp)^2 \right) + \dots \right], \quad \text{(B.11)}$$

[14] This is the analog of $\vec{\nabla}\cdot\vec{B} = 0$ for the existence of a solution $\vec{A}$ of $\vec{\nabla}\times\vec{A} = \vec{B}$ in magnetostatics.

and the arbitrary covariantization,

$$\hat{X}_\mu(v,iD) = \frac{1}{2M}\gamma_\mu^\perp + \frac{1}{4M^2}\sigma_{\mu\nu}^\perp D^\nu \left[1 - \frac{iv\cdot D}{M} + \frac{1}{M^2}\left((iv\cdot D)^2 - \frac{1}{4}(iD_\perp)^2\right) + \ldots\right]. \quad \text{(B.12)}$$

A corresponding solution for Γ in the free theory is displayed in (4.22). Now let us follow the construction of the previous section order by order.

Order $1/M$**:** First, we determine,

$$Y_\mu^{(1)} = X_\mu^{(1)} = \hat{X}_\mu^{(1)} = \frac{\gamma_\mu^\perp}{2}\,. \quad \text{(B.13)}$$

This function clearly satisfies Eq. (B.6) so that we may solve for

$$\Gamma^{(1)} = \frac{1}{2}i\not{D}_\perp\,. \quad \text{(B.14)}$$

Order $1/M^2$**:** Continuing to the next order, we evaluate

$$Z_{\mu\nu}^{(2)} = -\frac{i}{4}\sigma_{\mu\nu}^\perp = \hat{Z}_{\mu\nu}^{(2)}\,, \quad \text{(B.15a)}$$

$$X_\mu^{(2)} = \frac{1}{4}\sigma_{\mu\nu}^\perp D^\nu = \hat{X}_\mu^{(2)}\,, \quad \text{(B.15b)}$$

$$Y_\mu^{(2)} = -\frac{1}{2}\gamma_\mu^\perp iv\cdot D - \frac{1}{4}iD_\mu^\perp\,. \quad \text{(B.15c)}$$

Solving for $\Gamma^{(2)}$ yields

$$\Gamma^{(2)} = -\frac{1}{8}(iD_\perp)^2 - \frac{1}{2}i\not{D}_\perp iv\cdot D\,. \quad \text{(B.16)}$$

Order $1/M^3$**:** At the next order, we find

$$Z_{\mu\nu}^{(3)} = \frac{i}{4}\sigma_{\mu\nu}^\perp iv\cdot D = \hat{Z}_{\mu\nu}^{(3)}\,, \quad \text{(B.17a)}$$

$$X_\mu^{(3)} = -\frac{1}{4}\sigma_{\mu\nu}^\perp D^\nu iv\cdot D = \hat{X}_\mu^{(3)}\,, \quad \text{(B.17b)}$$

$$Y_\mu^{(3)} = \frac{1}{2}\gamma_\mu^\perp(iv\cdot D)^2 + \frac{3}{8}iD_\mu^\perp iv\cdot D + \frac{1}{8}iv\cdot D iD_\mu^\perp - \frac{1}{2}i\not{D}_\perp iD_\mu^\perp - \frac{1}{16}(iD_\perp)^2\gamma_\mu^\perp + \frac{1}{8}i\not{D}_\perp\sigma_{\mu\nu}^\perp D^\nu\,. \quad \text{(B.17c)}$$

After some manipulations, the resulting $\Gamma^{(3)}$ is

$$\Gamma^{(3)} = \frac{1}{4}(iD_\perp)^2 iv\cdot D + \frac{i\not{D}_\perp}{2}\left[-\frac{3}{8}i\not{D}_\perp(iD_\perp)^2 + (iv\cdot D)^2\right] - \frac{g}{8}v^\alpha G_{\alpha\beta}D_\perp^\beta - \frac{g}{16}\sigma_{\alpha\beta}^\perp G^{\alpha\beta}i\not{D}_\perp + \frac{g}{8}\left[i\gamma_\perp^\beta\sigma_\perp^{\mu\alpha}[D_\mu, G_{\beta\alpha}] - v^\alpha[D_\perp^\mu, G_{\alpha\mu}] - [D_\perp^\mu, G_{\mu\beta}^\perp]\gamma_\perp^\beta\right]. \quad \text{(B.18)}$$

Order $1/M^4$**:** Continuing to higher order we find

$$Z^{(4)}_{\mu\nu} = \hat{Z}^{(4)}_{\mu\nu} + \frac{g}{32}\left(-iG^{\perp}_{\mu\nu} + \sigma^{\perp}_{\mu\sigma}G^{\perp\sigma}_{\nu} - \sigma^{\perp}_{\nu\sigma}G^{\perp\sigma}_{\mu}\right) , \tag{B.19a}$$

$$X^{(4)}_{\mu} = \sigma^{\perp}_{\mu\nu}D^{\nu}\left[\frac{1}{4}(iv\cdot D)^2 - \frac{1}{16}(iD_{\perp})^2\right] + \frac{g}{32}iD^{\nu}_{\perp}\left(-iG^{\perp}_{\mu\nu} + \sigma^{\perp}_{\mu\sigma}G^{\perp\sigma}_{\nu} - \sigma^{\perp}_{\nu\sigma}G^{\perp\sigma}_{\mu}\right) . \tag{B.19b}$$

Note that $X^{(4)}_{\mu}$ differs from the trial solution $\hat{X}^{(4)}_{\mu}$. We may continue in this manner to construct $Y^{(4)}_{\mu}$ and $\Gamma^{(4)}$.

References

[1] W. E. Caswell and G. P. Lepage, Phys. Lett. B **167**, 437 (1986).

[2] N. Isgur and M. B. Wise, Phys. Lett. B **232**, 113 (1989).

[3] E. Eichten and B. R. Hill, Phys. Lett. B **234**, 511 (1990).

[4] For reviews of heavy quark effective theory see: M. Neubert, Phys. Rept. **245**, 259 (1994). A. V. Manohar and M. B. Wise, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **10**, 1 (2000).

[5] E. Epelbaum, H. -W. Hammer and U. -G. Meissner, Rev. Mod. Phys. **81**, 1773 (2009).

[6] N. Brambilla, E. Mereghetti and A. Vairo, Phys. Rev. D **79**, 074002 (2009) [Erratum-ibid. D **83**, 079904 (2011)].

[7] R. J. Hill and G. Paz, Phys. Rev. Lett. **107**, 160402 (2011).

[8] R. J. Hill and M. P. Solon, Phys. Lett. B **707**, 539 (2012).

[9] S. R. Coleman, J. Wess and B. Zumino, Phys. Rev. **177**, 2239 (1969).

[10] C. G. Callan, Jr., S. R. Coleman, J. Wess and B. Zumino, Phys. Rev. **177**, 2247 (1969).

[11] M. E. Luke and A. V. Manohar, Phys. Lett. B **286**, 348 (1992).

[12] E. P. Wigner, Annals Math. **40**, 149 (1939) [Nucl. Phys. Proc. Suppl. **6**, 9 (1989)].

[13] W. Kilian and T. Ohl, Phys. Rev. D **50**, 4649 (1994).

[14] A. V. Manohar, Phys. Rev. D **56**, 230 (1997).

[15] R. Sundrum, Phys. Rev. D **57**, 331 (1998).

[16] M. Finkemeier, H. Georgi and M. McIrvin, Phys. Rev. D **55**, 6933 (1997).

[17] N. Brambilla, D. Gromes and A. Vairo, Phys. Rev. D **64**, 076010 (2001) [hep-ph/0104068].

[18] N. Brambilla, D. Gromes and A. Vairo, Phys. Lett. B **576**, 314 (2003).

[19] K. Kopp and T. Okui, Phys. Rev. D **84**, 093007 (2011).

[20] S. Weinberg, Cambridge, UK: Univ. Pr. (1995) 609 p

[21] Leslie L. Foldy, Phys. Rev. A **102**, 568 (1956).

[22] B. Long and V. Lensky, Phys. Rev. C **83**, 045206 (2011).

[23] T. Kinoshita and M. Nio, Phys. Rev. D **53**, 4909 (1996).

[24] R.J. Hill, G. Lee, G. Paz and M.P. Solon, "The NRQED Lagrangian at order $1/M^4$", in preparation.

[25] C. W. Bauer, S. Fleming and M. E. Luke, Phys. Rev. D **63**, 014006 (2001).

[26] C. W. Bauer, S. Fleming, D. Pirjol and I. W. Stewart, Phys. Rev. D **63**, 114020 (2001).

[27] J. Chay and C. Kim, Phys. Rev. D **65**, 114016 (2002).

[28] M. Beneke, A. P. Chapovsky, M. Diehl and T. Feldmann, Nucl. Phys. B **643**, 431 (2002).

[29] R. J. Hill and M. Neubert, Nucl. Phys. B **657**, 229 (2003).

[30] A. V. Manohar, T. Mehen, D. Pirjol and I. W. Stewart, Phys. Lett. B **539**, 59 (2002).

[31] J. Gasser, M. E. Sainio and A. Svarc, Nucl. Phys. B **307**, 779 (1988).

[32] E. E. Jenkins and A. V. Manohar, Phys. Lett. B **255**, 558 (1991).

[33] T. Becher and H. Leutwyler, Eur. Phys. J. C **9**, 643 (1999).

[34] T. Fuchs, J. Gegelia, G. Japaridze and S. Scherer, Phys. Rev. D **68**, 056005 (2003).

[35] V. Pascalutsa and M. Vanderhaeghen, Phys. Lett. B **636**, 31 (2006).

[36] R. J. Hill, Phys. Rev. D **81**, 013008 (2010).

[37] T. Becher, R. J. Hill and M. Neubert, Phys. Rev. D **69**, 054017 (2004).

[38] T. Becher and M. Neubert, Eur. Phys. J. C **71**, 1665 (2011).

[39] J.-y. Chiu, A. Jain, D. Neill and I. Z. Rothstein, Phys. Rev. Lett. **108**, 151601 (2012).

[40] D. V. Volkov, Fiz. Elem. Chast. Atom. Yadra **4**, 3 (1973). V. I. Ogievetsky, "Nonlinear realizations of internal and space-time symmetries,", in *Proceedings of the X-th winter school of theoretical physics in Karpacz*, 227 (1974).

[41] A. F. Falk, Nucl. Phys. B **378**, 79 (1992).