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The Pomeranchuk instability, in which an isotropic Fermi surface distorts and becomes anisotropic due to strong interactions, is a possible mechanism for the growing number of experimental systems which display transport properties that differ along the x and y axes. We show here that the gauge-gravity duality can be used to describe such an instability in fermionic systems. Our holographic model consists of fermions in a background which describes the causal propagation of a massive neutral spin-two field in an asymptotically AdS spacetime. The Fermi surfaces in the boundary theory distort spontaneously and become anisotropic once the neutral massive spin-two field develops a normalizable mode in the bulk. Analysis of the fermionic correlators reveals that the low-lying fermionic excitations are non-Fermi liquid-like both before and after the Fermi surface shape distortion. Further, the spectral weight along the Fermi surface is angularly dependent and can be made to vanish along certain directions.

I. INTRODUCTION

To a surprising extent, the electronic properties of most metals are well described by Landau Fermi liquid theory. The key building blocks of this theory are quasi-particles and a Fermi surface, the surface in momentum space that demarcates filled from empty states. For a system of weakly interacting electrons, the shape of the Fermi surface is isotropic. On the other hand, if the interactions among the quasi-particles are strong enough, other possibilities might emerge. Pomeranchuk [1] showed that forward scattering among quasi-particles with non-trivial angular momentum and spin structure can lead to an instability of the ground state towards forming an anisotropic Fermi surface. For example, the simplest form of a two-dimensional nematic Fermi liquid can be thought of, in the continuum limit, as a realization of the Pomeranchuk instability in the angular momentum $l = 2$ and spin $s = 0$ particle-hole channel whereby a circular Fermi surface spontaneously distorts and becomes elliptical with a symmetry of a quadrupole (d-wave). Such an instability, depicted in Figure 1(a), is often referred to in the literature as the quadrupolar Pomeranchuk instability[1]. In practice, however, the instability breaks the discrete point group symmetry of the underlying lattice. As such, systems undergoing a Pomeranchuk instability exhibit manifestly distinct transport properties along the different crystal axes. Experimentally several instances of anisotropic transport in strongly correlated electronic systems have been reported [2–5] and occupy much of the current focus in correlated electron matter. Although it is unclear at present whether the
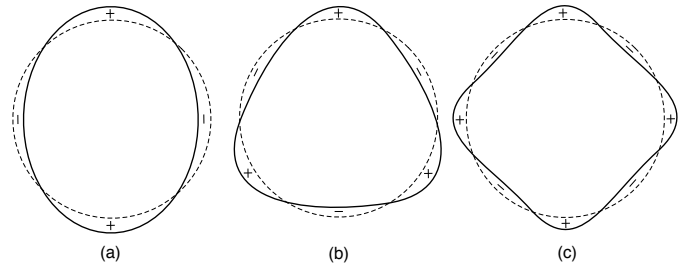

FIG. 1. (a) Examples of the Pomeranchuk instability in (a) $l = 2$, (b) $l = 3$ and (c) $l = 4$ channels. The $+$ and $-$ signs on the anisotropic Fermi surfaces show the momentum gain and loss compared to the isotropic Fermi surfaces (which are shown above by the dashed lines).

Pomeranchuk mechanism is the only root cause[2], it nevertheless offers a framework where such anisotropies can be characterized in detail. So far, much of the work in this direction has focused on reaching the nematic phases of correlated electron matter, via the Pomeranchuk instability, in a weakly interacting Fermi fluid [9, 10]. A natural question to ask is whether a nematic phase could be approached, via a Pomeranchuk instability, from a non-Fermi liquid phase. To answer the question, one has to address the origin of the Pomeranchuk instability beyond the Fermi liquid framework. Unlike Fermi liquids, there is no notion of stable quasi-particles in non-Fermi liquids. Nevertheless, non-Fermi liquids have sharp Fermi surfaces. Consequently, a Pomeranchuk instability, viewed simply as a spontaneous shape distortion of the underlying Fermi surface, is still a theoretical possibility.

We explore here how such an instability can be studied using the gauge-gravity duality, hereafter referred to as holography. The duality, which maps certain d-dimensional strongly coupled quantum field theories to

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[1] The instabilities in other channels, including the antisymmetric spin channel, are also interesting. (Note that there is no Pomeranchuk instability in the $l = 1$, $s = 0$ channel.) In this paper, however, we only focus on the quadrupolar Pomeranchuk instability and the resulting nematic phase.

[2] Nematic phases could also be obtained by the melting of stripe phases; see [7, 8] and references therein.

$(d + 1)$-dimensional semiclassical gravitational theories in asymptotically AdS spacetimes, is a promising tool in modeling some features of strongly correlated condensed matter systems. For reviews on the applications of holography for condensed matter physics, see [11–17]. We take the system to be $d = 2+1$ dimensional, as this is the most interesting case experimentally [2–5]. Also, we consider the case where the broken phase is a nematic (non-)Fermi liquid with the symmetry of a quadrupole. In other words, we only explore the possibility of a quadrupolar Pomeranchuk instability. Such an instability is characterized by an order parameter which is a neutral symmetric traceless tensor $\mathcal{O}_{ij}$ where i and j denote spatial indices. Thus, in order to use holography to realize the transition to such a phase in the boundary theory, one has to turn on a neutral massive spin-two field $\varphi_{\mu\nu}$ in the bulk and show that, under some circumstances, it develops a non-trivial normalizable profile in the bulk radial direction. From the point of view of the physics in the bulk, there is a serious hurdle that one needs to overcome in order to successfully describe the propagation of a massive spin-two field, which has to do with the existence of extra unwanted degrees of freedom (ghosts) and superluminal modes. Eliminating ghosts and superluminal modes within a theory that also allows for the condensation of the operator dual to the massive spin-two field is the first step. In addition, a coupling of the correct type must be introduced in the bulk between the massive spin-two field and the fermions in order to bring about the shape distortion of the Fermi surface in the boundary. We present here a minimal model which can address each of these problem, and show that the isotropic Fermi surface of the boundary non-Fermi liquid gets spontaneously distorted and becomes elliptical. We also find that the broken (nematic) phase is a non-Fermi liquid. We interpret this transition as a holographic realization of the quadrupolar Pomeranchuk instability in non-Fermi liquids. Nonetheless, a disclaimer is necessary here. The background we study cannot be extended consistently to $T = 0$ on the count that so doing would require the back reaction of the gauge field. The metric that results would be non-Einstein and there is no causal Lagrangian for a spin-two field in a curved background that is not Einstein. Within the background we consider, the vacuum expectation value of the spin-two field actually grows as the temperature is increased. Consequently, there is strictly no instability in the traditional sense in the context of spontaneous symmetry breaking. Hence, our usage of the term instability is not entirely warranted. Nonetheless, the program we outline here does provide a consistent holographic framework to address how to engineer shape distortions of the underlying Fermi surface. Hence, it serves as a valid first step in realizing the physics of the Pomeranchuk instability.

The outline of this paper is as follows. Section II sets up the Lagrangian for the neutral massive spin-two field. Included here will be an explicit mechanism for obtaining a normalizable profile for the bulk spin-two field. Section III contains the fermonic degrees of freedom as well as the results illustrating that the low-lying excitations are non-Fermi liquid in nature as well as the shape distortion of the underlying Fermi surface.

II. BACKGROUND

Since we want to explore the possibility of a transition from a non-Fermi liquid to a nematic Fermi liquid phase in the boundary, the first step is to construct a background which consistently incorporates a neutral massive spin-two field $\varphi_{\mu\nu}$. This field is dual to the operator whose vacuum expectation value (vev) is the order parameter for the nematic phase.

A. BGP Lagrangian

In flat $(d+1)$-dimensional Minkowski spacetime, Fierz and Pauli suggested [18] a Lagrangian for a free neutral massive spin-two field which is quadratic in derivatives and correctly describes the propagating degrees of freedom for such a field. Generalizing to curved spacetimes, Buchbinder, Gitman and Pershin [19], following [20], proposed a simple Lagrangian (quadratic in the massive spin-two field, and also quadratic in derivatives) which describes the causal propagation of the correct number of degrees of freedom of a neutral massive spin-two field in a fixed Einstein background in arbitrary dimensions. In four bulk dimensions, their Lagrangian for $\varphi_{\mu\nu}$, denoted by $\mathcal{L}_{\mathrm{BGP}}$, reads

$$\frac{\mathcal{L}_{\mathrm{BGP}}}{\sqrt{-g}} = \frac{1}{4}\Big\{ - \nabla_\mu \varphi_{\nu\rho}\nabla^\mu\varphi^{\nu\rho} + \nabla_\mu\varphi\nabla^\mu\varphi$$
$$+ 2\nabla_\mu\varphi_{\nu\rho}\nabla^\rho\varphi^{\nu\mu} - 2\nabla^\mu\varphi_{\mu\nu}\nabla^\nu\varphi \qquad (1)$$
$$- m^2\big(\varphi_{\mu\nu}\varphi^{\mu\nu} - \varphi^2\big) + \frac{R}{2}\,\varphi_{\mu\nu}\varphi^{\mu\nu} - \frac{R}{4}\,\varphi^2\Big\}.$$

Here R is the Ricci scalar. Also, we have defined $\varphi = \varphi_\mu^\mu$. Several comments are warranted here. First, it is crucial that the fixed background satisfies the Einstein relation (which, in four bulk dimensions, reads) $R_{\mu\nu} = \Lambda g_{\mu\nu}$, with Λ being the cosmological constant, otherwise some constraint equations become dynamical which would give rise to extra propagating degrees of freedom for $\varphi_{\mu\nu}$. In our context this restriction simply implies that one should ignore the backreaction of $\varphi_{\mu\nu}$, as well as other bulk fields that will be introduced later, on the metric of the fixed Einstein background. Second, in the flat spacetime limit, the above Lagrangian becomes the Fierz-Pauli Lagrangian [18]. Third, the massless limit of the BGP Lagrangian, $i.e.$ $m^2 = 0$, has an emergent gauge symmetry (diffeomorphism) which is unique to the graviton: $\varphi_{\mu\nu} \to \varphi_{\mu\nu} + \nabla_{(\mu}\xi_{\nu)}$ with ξ_μ being an infinitesimal vector. Indeed, as pointed out in [19], this symmetry has been used to settle the ambiguity associated with the definition

of the mass of $\varphi_{\mu\nu}$. Fourth, for $m^2 = R/6$ (here $d+1 = 4$) the above Lagrangian has another emergent gauge symmetry [21]: $\varphi_{\mu\nu} \to \varphi_{\mu\nu} + (\nabla_\mu \nabla_\nu + g_{\mu\nu} R/12)\epsilon$ with ϵ being an infinitesimal scalar gauge parameter. In the following, we always assume that $m^2 \neq 0$ and $m^2 \neq R/6$. Excluding these two special values for the mass, the equations of motion and constraints obtained from (1) are given by [19]

$$0 = \left(\nabla^2 - m^2\right)\varphi_{\mu\nu} + 2R^\rho{}_\mu{}^\sigma{}_\nu \varphi_{\rho\sigma}, \tag{2}$$

$$0 = \nabla^\mu \varphi_{\mu\nu}, \tag{3}$$

$$0 = \varphi, \tag{4}$$

$$0 = \dot{\varphi}, \tag{5}$$

$$0 = g^{00}\nabla_0\nabla_i\varphi^i{}_\nu - g^{0i}\nabla_0\nabla_i\varphi^0{}_\nu - g^{0i}\nabla_i\nabla_0\varphi^0{}_\nu$$
$$- g^{ij}\nabla_i\nabla_j\varphi^0{}_\nu - 2R^{\rho\,0\,\sigma}{}_\nu\varphi_{\rho\sigma} + m^2\varphi^0{}_\nu. \tag{6}$$

The above expressions give the correct number of propagating degrees of freedom for a massive spin-two field in (3+1)-dimensions. Such a field transforms in the five-dimensional irreducible representation of SO(3), with SO(3) being the little group of the Lorentz group SO(3,1).

As we alluded to earlier, in order to satisfy the constraint $R_{\mu\nu} = \Lambda g_{\mu\nu}$, we will work in a regime where matter fields do not backreact on the bulk geometry. So, we take the Schwarzschild AdS$_4$ black hole as our background geometry ($\Lambda = -3$)

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$
$$= r^2\left(-f(r)\,dt^2 + dx^2 + dy^2\right) + \frac{dr^2}{r^2 f(r)}, \tag{7}$$

where $f(r) = 1 - (r_0^3/r^3)$ with r_0 being the horizon radius, given by the largest real root of $f(r_0) = 0$. Note that we are working in units where we set the curvature radius L of AdS$_4$ equal to unity. Also, note that in the coordinates we have chosen above, the asymptotic boundary of the spacetime is at $r \to \infty$. The temperature of the black hole is given by $T = 3r_0/(4\pi)$.

To solve for $\varphi_{\mu\nu}$, we only consider the configuration where $\varphi_{\mu\nu} = \varphi_{\mu\nu}(r)$ for all $\mu, \nu \in \{t, r, x, y\}$. Analyzing the equations (2)–(6), one can show [22, 23] that it is consistent to put $\varphi_{\mu\nu}(r)$ in the form

$$\varphi_{\mu\nu}(r) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \varphi_{ij}(r) \end{pmatrix}, \tag{8}$$

where $\varphi_{ij}(r)$, with $i, j = x, y$, is a symmetric traceless two-by-two matrix. Defining the "director" $\vec{\varphi}(r) = \varphi_{xx}(r) + i\varphi_{xy}(r)$, under a rotation θ in the (x,y)-plane, $\vec{\varphi}(r)$ transforms as

$$\vec{\varphi}(r) \to e^{2i\theta}\vec{\varphi}(r). \tag{9}$$

One can then choose a particular value of θ to set either $\varphi_{xx}(r)$ or $\varphi_{xy}(r)$ equal to zero, or make them equal.

Suppose now there exists a (2+1)-dimensional field theory dual to the gravitational setup under consideration here. Focusing for the moment on the bulk neutral massive spin-two field $\varphi_{\mu\nu}$, holography tells us that it should be dual to some operator in the boundary theory. Since $\varphi_{\mu\nu}$ is a bulk field with spin greater than zero, its number of components does not naively match the number of components of the dual operator. But, as explained in [23], since $\varphi_{\mu\nu}$ can be put in the form (8), it is φ_{ij} which sources a neutral symmetric traceless operator $\mathcal{O}_{ij}$ in the boundary theory. (The vev of the operator $\mathcal{O}_{ij}$ determines the nematicity of the (2+1)-dimensional boundary theory.) More concretely, equation (2) implies that $\varphi_{ij}(r)$ takes the following form near the boundary as $r \to \infty$

$$\varphi_{ij}(r) = A_{ij}\, r^{\Delta-1}\left(1 + \cdots\right) + B_{ij}\, r^{2-\Delta}\left(1 + \cdots\right), \tag{10}$$

with A_{ij} and B_{ij} being symmetric traceless tensors. A_{ij} is the source for the boundary theory operator $\mathcal{O}_{ij}$ while B_{ij} is proportional to its vev, $\langle \mathcal{O}_{ij} \rangle$. In (10), Δ denotes the UV scaling dimension of $\mathcal{O}_{ij}$, which is related to the mass of the bulk field $\varphi_{\mu\nu}$ via

$$\Delta = \frac{3}{2} + \sqrt{\frac{9}{4} + m^2}. \tag{11}$$

Note that for $m = 0$, i.e. when $\varphi_{\mu\nu}$ is the graviton, equation (11) yields $\Delta = 3$ which is the dimension of $T_{\mu\nu}$, the energy-momentum tensor operator, of the (2+1)-dimensional boundary theory. There is a Breitenlohner-Freedman bound [24] for the propagation of $\varphi_{\mu\nu}$ in an asymptotically AdS$_4$ geometry which reads $m^2 \geq 0$ [23, 25, 26]. Equation (11) then implies that $\Delta \geq 3$. In our discussions in this paper, we will always take the mass squared of $\varphi_{\mu\nu}$ in the asymptotic AdS$_4$ region of the geometry to satisfy the condition $m^2 > 0$. Note that the condition $m^2 \neq R/6 = -2$ is then trivially satisfied.

Given the ansatz (8), we are interested in solutions in which $\varphi_{\mu\nu}(r)$, or rather $\varphi_{ij}(r)$, is regular near the horizon and normalizable as $r \to \infty$. More concretely, near the horizon, we look for a solution of the equation (2) which is regular as $r \to r_0$, namely

$$\varphi_{ij}(r) = a_{ij} + b_{ij}(r - r_0) + c_{ij}(r - r_0)^2 + \cdots, \tag{12}$$

where the coefficient tensors b_{ij}, c_{ij}, $\cdots$ are all determined in terms of a_{ij}. In the boundary as $r \to \infty$, we demand $\varphi_{ij}(r)$ to be normalizable

$$\varphi_{ij}(r) = B_{ij}\, r^{2-\Delta}\left(1 + \cdots\right). \tag{13}$$

If a solution with the above two boundary conditions exists, then the boundary theory would be in a phase where the neutral symmetric traceless operator $\mathcal{O}_{ij}$ (which is the operator dual to φ_{ij}) spontaneously condenses. It is easy to show that given the set of equations (2)–(6), such a solution does not exist. As we will explain in the next section, one way forward is to minimally modify the BGP

Lagrangian (1) such that the new action yields the correct number of causal propagating degrees of freedom for $\varphi_{\mu\nu}$, and also permits the aforementioned spontaneous condensation to occur.

B. Modifying the BGP Lagrangian

Undoubtedly, there are many ways to modify the BGP Lagrangian to facilitate spontaneous condensation of the operator dual to $\varphi_{\mu\nu}$. Instead of categorizing all such modifications, we take perhaps the simplest possibility. Gubser [27] has shown that coupling a neutral scalar field to the square of the Weyl tensor leads to a scalar hair on an asymptotically flat Schwarzschild black hole. This coupling is of interest here because the square of the Weyl tensor vanishes at the boundary ($r \to \infty$) and hence does not affect the scaling dimension of the dual boundary operator. To this end, we modify the BGP Lagrangian (1) as follows

$$\mathcal{L}_\varphi = \mathcal{L}_{\text{BGP}} + \frac{\ell^2}{4}\sqrt{-g}\, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\left(\varphi_{\gamma\delta}\varphi^{\gamma\delta} - \varphi^2\right), \quad (14)$$

with $C_{\mu\nu\rho\sigma}$ the Weyl tensor. In the Appendix, we show that the above modification to the BGP Lagrangian, though not unique, offers a valid description of the neutral massive spin-two field $\varphi_{\mu\nu}$, meaning that $\varphi_{\mu\nu}$ still propagates causally in a fixed Einstein spacetime with the correct number of degrees of freedom. In what follows, we show that the equations of motion for $\varphi_{\mu\nu}$ obtained from the new Lagrangian now admits non-trivial normalizable solutions (whose near horizon and asymptotic boundary behaviors are given by (12) and (13), respectively).

C. Normalizable Solution

From the Appendix, the relevant equation for determining whether there exists a normalizable solution for $\varphi_{\mu\nu}$ is

$$\left(\nabla^2 - m^2 + \ell^2 C_{\gamma\delta\rho\sigma}C^{\gamma\delta\rho\sigma}\right)\varphi_{\mu\nu} + 2R^\rho{}_\mu{}^\sigma{}_\nu\varphi_{\rho\sigma} = 0. \quad (15)$$

Note that the background metric must again satisfy the Einstein relation. Since we have taken the spacetime to be the Schwarzschild AdS$_4$ black hole, the square of the Weyl tensor is $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = 12(r_0/r)^6$. One can easily show that, given the constraints presented in the Appendix and the equation (15), it is again consistent to put $\varphi_{\mu\nu}$ in the form given in (8). Notice that as shown above, the square of the Weyl tensor evaluated on the background vanishes as $r \to \infty$. This then implies that the relationship between the asymptotic mass m of φ_{ij} and the UV dimension Δ of the dual operator $\mathcal{O}_{ij}$ is still given by (11). Also, the mass of φ_{ij} in the asymptotic AdS$_4$ region satisfies the same Breitenlohner-Freedman

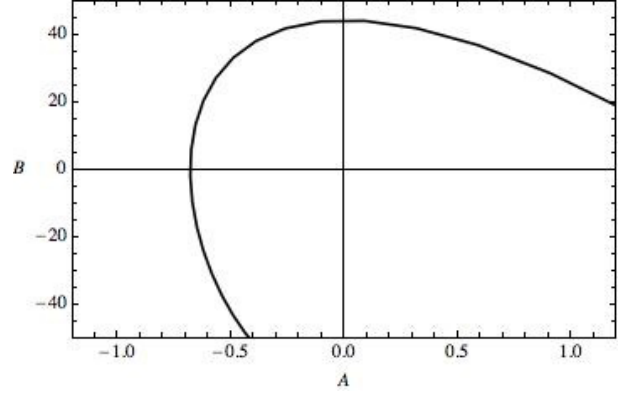FIG. 2. A plot of B_{ij} verse A_{ij} for $\ell^2 = 1.0866$ and $m^2 = 1/100$. The values of B_{ij} and A_{ij} have been rescaled so that they are dimensionless. The gauge in (9) has been used to set $\varphi_{xx} = \varphi_{xy}$.

bound as before. Thus, regardless of the value of the coupling ℓ^2, the asymptotic AdS$_4$ region of the background remains stable.

As Figure 2 illustrates, there exists a normalizable solution with $A_{ij} = 0$ and $B_{ij} \neq 0$. (We have used the gauge in (9) to set $\varphi_{xx} = \varphi_{xy}$.) The holographic interpretation of this normalizable solution is that $\mathcal{O}_{ij}$ has spontaneously condensed in the boundary theory. While it may be possible to obtain a normalizable solution for $\varphi_{\mu\nu}$ using an alternative mechanism, we believe that the details of how the dual operator condenses in the boundary theory should be irrelevant to the Pomeranchuk instability as we discuss in the next section.

D. Maxwell's Lagrangian

Since we will be considering the boundary theory at a finite chemical potential μ for a U(1) charge, we need to introduce in the bulk an Abelian gauge field A_μ with the Lagrangian

$$\mathcal{L}_{\text{M}} = -\frac{1}{4}\sqrt{-g}\, F_{\mu\nu}F^{\mu\nu}. \quad (16)$$

The equations of motion for A_μ is then

$$\nabla^\mu F_{\mu\nu} = 0. \quad (17)$$

We will only be interested in the case where A_t is non-zero and, moreover, depends only on the radial coordinate r. So, we take the ansatz $A_\mu = A_t(r)\delta_{\mu 0}$. Given our assumption that A_μ does not back react on the metric (7), the solution to (17) is easily obtained to be

$$A = \mu\left(1 - \frac{r_0}{r}\right)dt, \quad (18)$$

where the boundary conditions at the horizon and the asymptotic boundary have been fixed by demanding $A_t(r_0) = 0$ and $A_t(r \to \infty) = \mu$.

III. FERMIONS

In this section, we study the shape distortion of the Fermi surface as a result of the condensation of the boundary theory operator $\mathcal{O}_{ij}$. We start by introducing a bulk spinor field ψ with mass m_ψ and charge q which is dual to a boundary theory fermionic operator $\mathcal{O}_\psi$ with a UV scaling dimension Δ_ψ and charge q. Note that the bulk spinor ψ is a four-component Dirac spinor while the dual operator $\mathcal{O}_\psi$ is a two-component Dirac spinor. In this paper, we take $m_\psi \in [0, \frac{1}{2})$ and choose the so-called "conventional quantization" where the scaling dimension of the fermionic operator is related to the mass of the bulk spinor field through $\Delta_\psi = \frac{3}{2} + m_\psi$.

A. Fermion Lagrangian and Equation of Motion

In our discussions below, the bulk spinor field ψ is treated as a probe where its backreaction on the background metric, gauge field and neutral massive spin-two field is ignored. The Dirac Lagrangian,

$$\mathcal{L}_{\text{Dirac}} = -\sqrt{-g}\, i\bar{\psi}(\slashed{D} - m)\psi, \quad (19)$$

describes the bulk spinor field, where $\bar{\psi} = \psi^\dagger\, \Gamma^{\underline{t}}$ and $\slashed{D} = e^\mu_c \Gamma^c \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab} - iqA_\mu\right)$ and e^μ_a and $\omega_\mu{}^{ab}$ being respectively the vielbein and the spin connection. Also, $a, b, \cdots = \{\underline{t}, \underline{x}, \underline{y}, \underline{r}\}$ denote the tangent space indices, $\Gamma^{\underline{t}}, \Gamma^{\underline{x}}, \Gamma^{\underline{y}}, \Gamma^{\underline{r}}$ are the Dirac matrices satisfying the Clifford algebra $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$ and $\Gamma_{ab} = \frac{1}{2}[\Gamma_a, \Gamma_b]$. The overall negative sign in (19) ensures that the bulk action, once holographically renormalized, gives rise to a positive spectral density for the fermionic operator $\mathcal{O}_\psi$. In (19), $g_{\mu\nu}$ and A_μ are the background metric and the background gauge field whose expressions are given in (7) and (18), respectively.

In the probe limit that we are considering here the Dirac Lagrangian (19) is not capable of holographically realizing how the spectral function of the boundary theory fermionic operator $\mathcal{O}_\psi$ is affected in the presence the order parameter $\langle\mathcal{O}_{ij}\rangle$. To do so, one has to explicitly introduce bulk couplings between the spinor field ψ and the neutral massive spin-two field $\varphi_{\mu\nu}$. A similar problem arises in the context of holographic d-wave superconductivity [22, 23] although in that case one has to deal with coupling the bulk spinor to a charged massive spin-two field. Since our goal here is to capture the leading order effects on the retarded two-point function of the fermionic operator due to the presence of the order parameter, we consider those couplings in the bulk which are quadratic in the spinor field and have relatively low mass dimension. Furthermore, among the aforementioned couplings, we will only be interested in those which will potentially give rise to anisotropic features in the fermion spectral function. In other words, we ignore the terms which would just modify the mass of the spinor field. Leav-

ing aside those terms which would vanish once evaluated on the background, we focus on the leading-order[3] non-trivial coupling (which can potentially contribute asymmetric features to the fermionic spectral function)

$$\mathcal{L}_{\text{int}} = -i\lambda\sqrt{-g}\,\varphi_{\mu\nu}\bar{\psi}\,\Gamma^\mu D^\nu\psi, \quad (20)$$

with λ real. Thus, the bulk fermion Lagrangian that we study is $\mathcal{L}_\psi = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{int}}$.

The Dirac equation following from $\mathcal{L}_\psi$ takes the form

$$(\slashed{D} - m + \lambda\varphi_{\mu\nu}\Gamma^\mu D^\nu)\psi = 0. \quad (21)$$

To solve the above equation, we go to momentum space by Fourier transforming $\psi(r, x^\mu) \sim e^{ik.x}\psi(r, k^\mu)$, where $k^\mu = (\omega, \vec{k})$. It is expedient to choose a basis for the Dirac matrices as follows

$$\Gamma^{\underline{r}} - - = \begin{pmatrix} -\sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \Gamma^{\underline{t}} = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix},$$

$$\Gamma^{\underline{x}} = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad \Gamma^{\underline{y}} = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}. \quad (22)$$

It is also convenient to remove the spin connection from the Dirac operator $\slashed{D}$ by rescaling ψ according to $\psi \to (-gg^{rr})^{-1/4}\psi$. Splitting the new (rescaled) spinor ψ according to $\psi^T = (\psi_1, \psi_2)$, where $\psi_1(r; \omega, \vec{k})$ and $\psi_2(r; \omega, \vec{k})$ are two-component spinors, and given the ansatz for the background form of $\varphi_{\mu\nu}$ where the only non-vanishing components are $\varphi_{xx} = -\varphi_{yy}$ and $\varphi_{xy} = \varphi_{yx}$, the equation (21) results in the following two coupled differential equations for ψ_1 and ψ_2

$$\left(\tilde{D}_r - i\sigma_2\sqrt{g^{xx}}\,\tilde{k}_x\right)\psi_1 + i\sigma_2\sqrt{g^{xx}}\,\tilde{k}_y\psi_2 = 0, \quad (23)$$

$$\left(\tilde{D}_r + i\sigma_2\sqrt{g^{xx}}\,\tilde{k}_x\right)\psi_2 + i\sigma_2\sqrt{g^{xx}}\,\tilde{k}_y\psi_1 = 0, \quad (24)$$

where we have defined

$$\tilde{D}_r = -\sigma_3\sqrt{g^{rr}}\,\partial_r + \sigma_1\sqrt{-g^{tt}}\,(\omega + qA_2) - m_\psi, \quad (25)$$

$$\tilde{k}_x = k_x\,(1 + \lambda g^{xx}\varphi_{xx}) + \lambda g^{xx}\varphi_{xy}k_y, \quad (26)$$

$$\tilde{k}_y = k_y\,(1 - \lambda g^{xx}\varphi_{xx}) + \lambda g^{xx}\varphi_{xy}k_x. \quad (27)$$

For $\lambda = 0$, or when $\varphi_{xx} = \varphi_{xy} = 0$, the Dirac equation (21) has been studied extensively in the literature (although mainly on the Reissner-Nordström AdS$_4$ black hole background), following the work of [28–31]. These studies show that in these cases there are symmetrical Fermi surfaces in the boundary theory whose underlying excitations can either be Fermi or non-Fermi liquid-like. On the other hand, when φ_{xx} develops a normalizable

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[3] There is also a coupling of the form $\lambda_5\sqrt{-g}\,\varphi_{\mu\nu}\bar{\psi}\,\Gamma^5\Gamma^\mu D^\nu\psi$ that breaks parity in the boundary theory when the operator dual to $\varphi_{\mu\nu}$ condenses. We will not consider such a coupling in this paper.

mode in the bulk, $\tilde{k}_x$ and $\tilde{k}_y$ will differ thereby giving rise to asymmetrical features in the fermionic correlators of the boundary theory. This can be shown explicitly.

To compute the retarded correlator of the boundary theory operator $\mathcal{O}_\psi$, one needs to solve the equations (23) and (24) with in-falling boundary condition at the horizon [32] and read off the source and the expectation value of $\mathcal{O}_\psi$ from the asymptotic expansion of ψ_α ($\alpha = 1, 2$) following the prescription of [33]. Indeed, choosing in-falling boundary conditions for ψ_α near the horizon, the leading-order asymptotic ($r \to \infty$) behavior of ψ_α takes the form $\psi_\alpha(r \to \infty) = (B_\alpha\, r^{-m_\psi}, A_\alpha r^{m_\psi})^T$, where the two-component spinor $A = (A_1, A_2)^T$ sources the boundary theory fermionic operator $\mathcal{O}_\psi$, while $B = (B_1, B_2)^T$ gives the vev of the operator. The spinor B is related to the spinor A through $B = \mathcal{S}A$ from which the retarded Green function of the operator $\mathcal{O}_\psi$ take the form [33]

$$G_R(\omega, \vec{k}) = -i\mathcal{S}\gamma^t. \tag{28}$$

Note that in our basis of gamma matrices $\gamma^t = i\sigma_1$. Up to a numerical constant, the spectral function of the operator $\mathcal{O}_\psi$ is given by $\rho(\omega, \vec{k}) = \mathrm{Tr}\,\mathrm{Im}G_R(\omega, \vec{k})$.

B. Fermi Surfaces and Low-Energy Excitations

To obtain the retarded Green function (28) of the fermionic operator $\mathcal{O}_\psi$, one should numerically solve the equations (23) and (24), or the flow equations obtained from them, with in-falling boundary condition at the horizon. We do this computation at finite temperature mainly because our background, in which there is a normalizable solution for the neutral massive spin-two field, is not, strictly speaking, valid at zero temperature. The Femi momentum $\vec{k}_F$ appears as a pole in $\mathrm{Re}\,G_R(\omega, \vec{k})$, although at finite temperature such poles, instead of being the delta function, become broadened.

Carrying out the numerical computation, we find multiple Fermi surfaces for the parameters chosen. To demonstrate proof of concept, it is sufficient to focus on the first Fermi surface, where $|\vec{k}_F|$ (measured compared to the effective chemical potential $q\mu$) has the smallest value. In generating the plots shown in this section, we first used the gauge (9) to set $\varphi_{xx} = \varphi_{xy}$ and then numerically solved the Dirac equations for a very small but non-zero frequency such that the delta functions at the Fermi surface are broadened.

First, consider the case where the background value of $\varphi_{\mu\nu}(r)$ identically vanishes in the bulk. In other words the boundary theory is in the symmetry-unbroken phase $\langle \mathcal{O}_{ij} \rangle = 0$. Not surprisingly, the Fermi surface obtained in this phase is isotropic, as is evident from Figure 3(a). To investigate the nature of the excitations, we computed the quasiparticle dispersion relation by focusing on the peak in the spectral function. In a Fermi liquid, $\omega(k) = k_\perp^z$ with $k_\perp = k - k_F$ and $z = 1$. Figure 4 (see
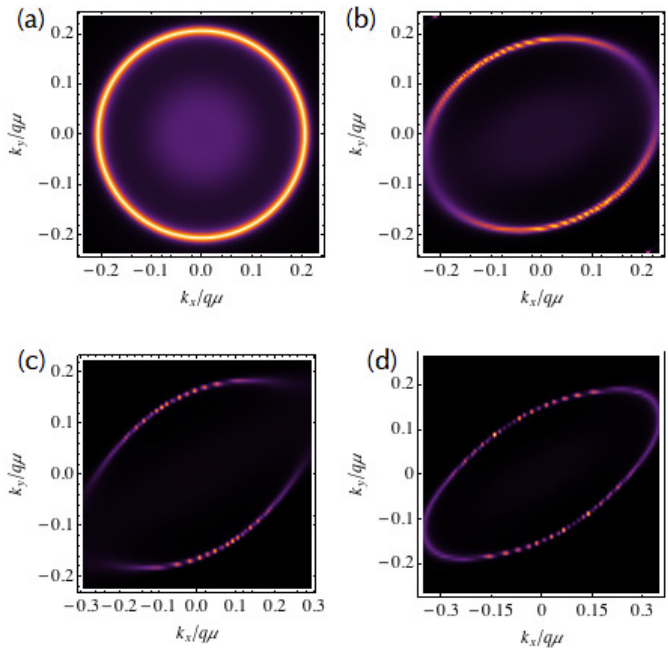

FIG. 3. Density plots of the spectral function of the fermionic operator $\mathcal{O}_\psi$ in the boundary field theory. a) Spectral density when the boundary field theory is in the unbroken phase b) Spectral density in the broken phase showing an elliptic Fermi surface where we set $\lambda = -0.4$, c) spectral density in the broken phase where $\lambda = -0.95$ showing a suppression of the spectral weight and d) spectral density, also in the broken phase, when $\lambda = -1.4$.

open circles) reveals that in the unbroken phase of the boundary theory where the Fermi surface is circular, the dynamical exponent $z = 0.64$ which indicates that the low-lying excitations form a non-Fermi liquid. Importantly then, any subsequent breaking of rotational symmetry of the Fermi surface will be from a non-Fermi liquid state, one of the key hurdles in describing the strong correlations in the experimental systems.

Focusing now on the interesting phase where the neutral symmetric traceless operator $\mathcal{O}_{ij}$ has spontaneously condensed, we find three significant features in the spectral function of the fermionic operator $\mathcal{O}_\psi$. First, the rotational symmetry of the Fermi surface is broken. As we alluded to earlier, such behavior is expected given the form of $\tilde{k}_x$ and $\tilde{k}_y$ at each r-slicing in the bulk. Second, the spectral weight is angularly dependent. For intermediate values of the coupling λ, a gap-like feature opens at the two end points of the major axis. This highly non-trivial behavior is not a generic feature of traditional treatments of the Pomeranchuk instability [8] but can be understood within a WKB approximation to the spectral function as outlined in the next subsection. The diminished spectral density reappears once λ exceeds a critical value as shown in Figure 3(d). Finally, the excitations near the Fermi surface remain non-Fermi liquid-like. The solid circles and triangles in Figure 4 show that the dispersions along k_x and k_y deviate from linearity with
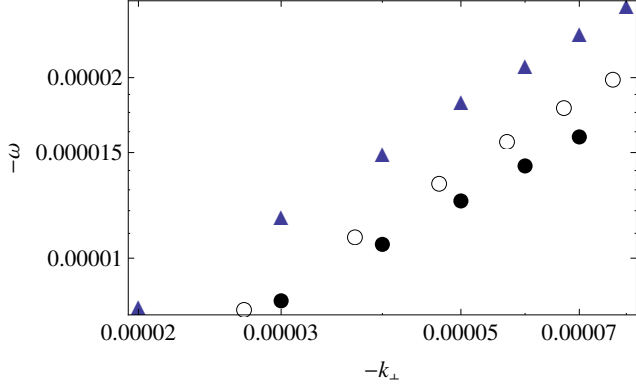
FIG. 4. The dispersion relation, $\omega(k) \propto k_\perp^z$, of quasi-particles around k_F plotted on a log-log scale. The three cases correspond to a) open circles for the unbroken phase of the boundary theory with a circular Fermi surface where we find $z = 0.64$, b) solid circles for the dispersion relation (in the broken phase) along x-axis where we find $z = 0.74$ with $\lambda = -0.4$ and c) solid triangles for the dispersion relation along y-axis (in the broken phase) where $z = 0.84$ with $\lambda = -0.4$.

exponents of $z = 0.74$ and $z = 0.84$, respectively. These values are, of course, parameter dependent.

C. WKB Analysis of Spectral Function

In this section we employ a WKB approximation to uncloak uncloak the origin of the momentum-dependent spectral weight in Figure 3. Such an approximation has been used recently in [34] for the semi-analytic analysis of spectral functions in the electron star background [35]. Setting $k_x = k \cos\theta$ and $k_y = k \sin\theta$, we define new momentum coordinates

$$\tilde{k}'_x = \sqrt{\tilde{k}_x^2 + \tilde{k}_y^2} \qquad \tilde{k}'_y = 0, \qquad (29)$$

which are written in the so-called "lab" frame. Note that the expressions for $\tilde{k}_x$ and $\tilde{k}_y$ have been defined in (26) and (27), respectively. With these new momenta and for a particular θ and k, the original Dirac equations (23) and (24) give rise to two decoupled equations

$$\left(\tilde{D}_r - i\sigma_2 \sqrt{g^{xx}} \, \tilde{k}'_x \right) \psi_1 = 0, \qquad (30)$$

$$\left(\tilde{D}_r + i\sigma_2 \sqrt{g^{xx}} \, \tilde{k}'_x \right) \psi_2 = 0. \qquad (31)$$

Any anisotropy that arises now in the spectral function must be tied to $\tilde{k}'_x$. To confirm this, we construct a contour plot of $\tilde{k}'_x$ at the horizon where the anisotropy is largest. The plots in Figure 5 show clearly that for a fixed value of k, $\tilde{k}'_x$ is minimum at $\theta = \pi/8$ and $9\pi/8$ which is consistent with the values of θ where the spectral density is a minimum.

Indeed, one can develop a better understanding on how $\tilde{k}'_x$ affects the spectral function by performing a WKB
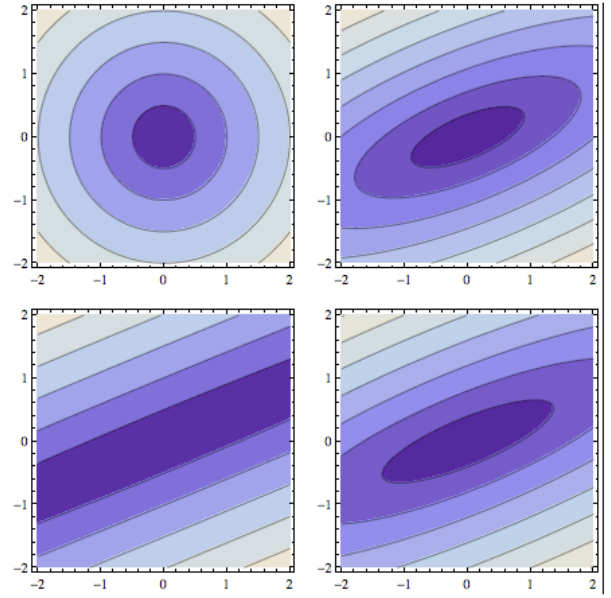

FIG. 5. Contour plot of $\tilde{k}'_x$ for a) $\lambda = 0$, b) $\lambda = -0.4$, c) $\lambda = -0.95$ and d) $\lambda = -1.4$. The plots have been generated with $m = 0$, $\omega = 0$ and $q\mu = 17$.

analysis of the Dirac equation. In order to make the notation less cluttered, we will replace $\tilde{k}'_x$ by k in what follows. In order to apply the WKB approximation, we will take the limit where k, m, ω and q are large, but their ratio remains constant. To this end, it is convenient to rescale all of these quantities with a factor γ such that $k = \gamma \hat{k}$, $m_\psi = \gamma \hat{m}$, $q = \gamma \hat{q}$ and $\omega = \gamma \hat{\omega}$, where γ is taken to be large (compared to one).

Central to the WKB approximation is the construction of a Schrödinger-like equation for each of the components of the two-component spinor

$$\psi_1 = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}.$$

The same treatment applies equally to ψ_2 with the replacement of $k \to -k$. To leading order in γ, the second-order differential equation for Φ_1 takes on the form

$$\Phi_1'' = \gamma^2 g_{rr} \Big\{ \hat{m}^2 + g^{tt} \left[\hat{\omega} + \mu \hat{q} \left(1 - r_0/r \right) \right]^2$$
$$+ g^{xx} \hat{k}^2 \Big\} \Phi_1, \qquad (32)$$

with

$$\Phi_2 = \frac{\left(-\sqrt{g^{rr}} \partial_r - m_\psi \right) \Phi_1}{-\sqrt{-g^{tt}}(\omega + qA_t) + \sqrt{g^{xx}}k}. \qquad (33)$$

In the context of a Schrödinger equation, the right-hand side of equation (32) can be regarded as the zero-energy potential

$$V = \gamma^2 g_{rr} \Big\{ \hat{m}^2 + g^{tt} \left[\hat{\omega} + \mu \hat{q}(1 - r_0/r) \right]^2 + g^{ii} \hat{k}^2 \Big\}. \quad (34)$$
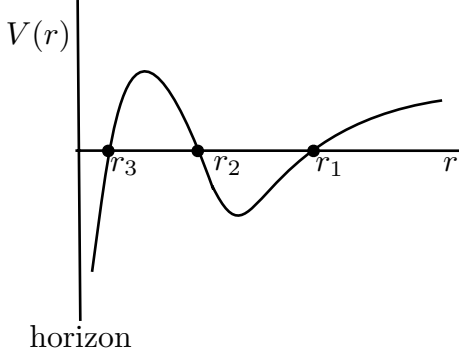
FIG. 6. Potential V in the Schrödinger equation as a function of the radial coordinate r. The classical turning points as measured from the horizon are denoted by r_3, r_2, and r_1.

It is the turning points of this potential that governs the physics of the WKB approximation. In the near-horizon region, the potential is approximated by

$$
V(r \to r_0) = \frac{L^2\gamma^2}{3r_0^3(r-r_0)}\left[L^2\hat{k}^2 + \hat{m}^2r_0^2 - \hat{\omega}^2\frac{L^2r_0}{3(r-r_0)}\right]
+ \cdots \tag{35}
$$

If $\hat{\omega}$ is non-zero, the near horizon potential is dominated by the term proportional to $\hat{\omega}^2$ and is negative for real positive $\hat{\omega}$. Close to the boundary, the leading order term of the potential

$$
V(r \to \infty) = \frac{\gamma^2 L^2 \hat{m}^2}{r^2} + \cdots , \tag{36}
$$

is always positive. Between these two regions, there can be at most three turning points which we denote by r_1, r_2 and r_3, with r_1 being the turning point closest to the boundary as depicted in Figure 6. For the parameter range of interest here, all three turning points will enter the matching conditions as we describe below.

What remains is a simple matching procedure to determine the form of Φ_1 in each of the regions that bracket the turning points. This procedure, detailed previously in [34], utilizes the functions

$$
X = \int_{r_3}^{r_2} dr\sqrt{V} + \log 2, \tag{37}
$$

$$
Y = \int_{r_2}^{r_1} dr\sqrt{-V} + \frac{\pi}{2}, \tag{38}
$$

which directly enter the retarded Green function

$$
G_R(\omega,k) \propto \frac{i\mathcal{G}}{2}\lim_{r\to\infty} r^{2mL}\exp\left(-2\int_{r_1}^{r} dr'\sqrt{V}\right) \tag{39}
$$

where

$$
\mathcal{G} = \frac{\cosh(X+iY) + \sinh(X-iY)}{\cosh(X-iY) - \sinh(X+iY)}. \tag{40}
$$

The poles of the retarded Green function only come from $\mathcal{G}$ and are given by $Y = \pi n - ie^{-2X} + \cdots$ where we have included the imaginary part up to the leading order only. Since we are interested in the non-analytic dependence of the Green function on frequency and momentum, we will focus on the behavior in the vicinity of the pole where $\mathcal{G}$ can be written as

$$
\mathcal{G} = \sum_n \frac{i}{Y(\omega,k) - \pi n + ie^{-2X(\omega,k)}}. \tag{41}
$$

Close to $\omega = 0$ and $k = k_{\mathrm{F}}^{(n)}$, we can expand[4] $Y(\omega,k)$ as $Y\left(0, k_{\mathrm{F}}^{(n)}\right) + \omega\partial_\omega Y\left(0, k_{\mathrm{F}}^{(n)}\right) + (k - k_{\mathrm{F}}^{(n)})\partial_k Y\left(0, k_{\mathrm{F}}^{(n)}\right) + \cdots$. As a result of this expansion, the retarded Green function takes the form

$$
G_R(\omega,k) \propto \sum_n \frac{-c_n e^{-2a_n}}{\omega + v_n(k - k_{\mathrm{F}}^{(n)}) + ic_n e^{-2X(\omega,k)}} \tag{42}
$$

$$
\propto \sum_n \frac{-\left[\omega + v_n(k - k_{\mathrm{F}}^{(n)})\right]c_n e^{-2a_n} + ic_n^2 e^{-2a_n}e^{-2X}}{\left[\omega + v_n(k - k_{\mathrm{F}}^{(n)})\right]^2 + c_n^2 e^{-4X}},
$$

where

$$
v_n = \frac{\partial_k Y\left(0, k_{\mathrm{F}}^{(n)}\right)}{\partial_\omega Y\left(0, k_{\mathrm{F}}^{(n)}\right)},
$$

$$
c_n = \frac{1}{\partial_\omega Y\left(0, k_{\mathrm{F}}^{(n)}\right)}, \tag{43}
$$

$$
a_n = \int_{r_1}^{\infty} dr\sqrt{V\left(0, k_{\mathrm{F}}^{(n)}\right)},
$$

which is essentially the equation (48) of [34].

Close to the Fermi surface, where the first term in the denominator of the second equation in (42)) can be dropped[5], the spectral function $A(\omega,k)$ can be extracted from the imaginary part of the retarded Green function. One then obtains

$$
A(\omega,k) \propto e^{-2a_n+2X}. \tag{44}
$$

We can now understand the anisotropy of spectral function from the dependence of a_n and X on the area enclosed by the potential integrated between the turning points r_2 and r_3[6] and between r_1 and ∞ which will determine a_n. In fact, for $m = 0$, $r_1 \to \infty$, a_n therefore vanishes and the angular dependence of the spectral function will be dominated by X alone. Close to the horizon,

————

[4] Note that in equation (34) we have written the potential in terms of the hatted quantities $\hat{\omega}$, $\hat{k}$, however, in doing the expansion of $Y(\omega,k)$, the factors of γ cancel and we can drop the hat on ω and k.

[5] We have assumed that in the limit $\omega \to 0$, ω vanishes faster than $e^{-2X(\omega,k)}$ which is true in the range of parameters in which we are interested.
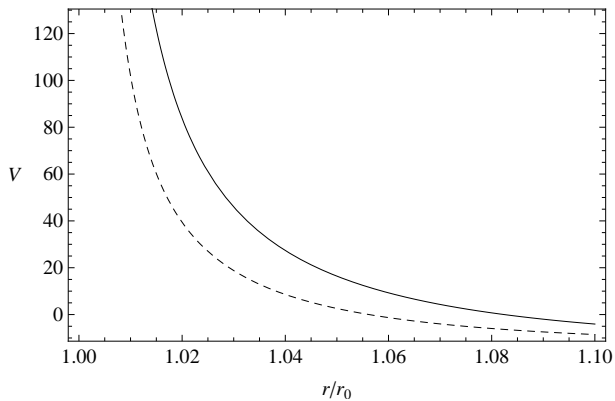
[6] This will define the quantity X.

FIG. 7. Effective WKB potential for $\lambda = -0.4$, $m = 0$ at the first Fermi surface. The solid line corresponds to $\theta = 0$ and the dashed line to $\theta = \pi/8$.

the first turning point of the potential, r_3, is roughly given by

$$r_3 \approx r_0 + \frac{\omega^2 L^2 r_0}{3\left[m^2 r_0^2 + k(r_0)^2 L^2\right]}. \tag{45}$$

From this we can see that in the limit $\omega \to 0$, that is, close to the Fermi surface, $r_3 \to r_0$.

In our analysis of the spectral function, where we have set $m = 0$, only X determines the spectral function. To illustrate the angular dependence, we plot the near-horizon effective WKB Schrodinger potential for $\lambda = -0.4$, $m = 0$ at first Fermi surface ($n = 0$) in Figure 7 (corresponding to Figure 3(b)). The dashed line corresponds to the potential along $\theta = \pi/8$, while the solid line corresponds to $\theta = 0$. It is now obvious that r_2 is larger for $\theta = 0$ than for $\theta = \pi/8$ and the potential is always larger for $\theta = 0$ than $\theta = \pi/8$ in the range of r shown. Hence X and spectral weight are larger along $\theta = 0$ than $\theta = \pi/8$, thereby explaining our key finding that the spectral weight is angularly dependent. In principle, the WKB approximation works only for large n, nevertheless, we see that the result qualitatively agrees with our numerical results even for small n.

IV. DISCUSSION

We have shown here how holography can be used to model a shape distortion of the underlying Fermi surface in a non-Fermi liquid through the condensation of a neutral symmetric traceless operator which is dual to a neutral massive spin-two field in the bulk. An open challenging problem remains: Is the condensation of the boundary theory operator $\mathcal{O}_{ij}$ possible starting from a bulk geometry which is not an Einstein manifold? In other words, what is the backreacted background that allows for the ghost-free and causal propagation of a neutral massive spin-two field. These extensions are extremely

desirable since they would not only enable an analysis at zero-temperature but would also cure at finite temperature some of the peculiar thermodynamical properties of the conduced phase in the probe limit. In addition, is the angular dependent spectral weight a generic feature of holographic non-Fermi liquids with (partially) broken rotational symmetry? If so, then holography would have provided a key ingredient missing from most condensed matter analyses of the Pomeranchuk instability, namely a vanishing spectral weight for a finite range of momenta and hence a pseudogap as a function of frequency. Also, how does the resistivity tensor behave in the two different directions? In our probe analysis, where the backreaction of the massive spin-two field on the metric is ignored, the resistivity tensor is not sensitive to the anisotropy of the system when the boundary theory is in the broken phase. Nevertheless, one might be able to see the anisotropic contribution of the fermions to the resistivity through a fermion loop computation in the bulk similar to the analysis of [36].

APPENDIX

In this appendix, we demonstrate that adding a coupling of the form $\ell^2 C_{\gamma\delta\rho\sigma} C^{\gamma\delta\rho\sigma}(\varphi_{\mu\nu}\varphi^{\mu\nu} - \varphi^2)$ to the BGP Lagrangian, given that the background spacetime is an Einstein manifold, does not induce any extra degrees of freedom for the neutral massive spin-two field $\varphi_{\mu\nu}$ nor violate its causal propagation. Our discussion below closely follows that of [19], where it has been shown that neutral massive spin-two field in an Einstein spacetime, such as the AdS$_{d+1}$ Schwarzschild black hole, has the correct number of degrees of freedom and propagates causally. In our discussion below, we keep the number of dimensions of the bulk spacetime (denoted by $d+1$) arbitrary.

Consider the following Lagrangian [20]

$$\begin{aligned}\mathcal{L} = \frac{1}{4}\Big\{ &-\nabla_\mu \varphi_{\nu\rho}\nabla^\mu \varphi^{\nu\rho} + \nabla_\mu\varphi\nabla^\mu\varphi - 2\nabla^\mu\varphi_{\mu\nu}\nabla^\nu\varphi \\ &+ 2\nabla_\mu\varphi_{\nu\rho}\nabla^\rho\varphi^{\nu\mu} - m_0^2(\varphi_{\mu\nu}\varphi^{\mu\nu} - \varphi^2) \\ &+ 2a_1 R\,\varphi_{\mu\nu}\varphi^{\mu\nu} + 2a_2 R\,\varphi^2 + 2a_3 R^{\mu\lambda\nu\rho}\varphi_{\mu\nu}\varphi_{\lambda\rho} \\ &+ 2a_4 R^{\mu\nu}\varphi_{\mu\lambda}\varphi_\nu^\lambda + 2a_5 R^{\mu\nu}\varphi_{\mu\nu}\varphi \Big\}, \tag{46}\end{aligned}$$

where $\varphi = \varphi^\mu_\mu$. The above Lagrangian is the most general two-derivative action for a neutral massive spin-two field $\varphi_{\mu\nu}$ (up to the quadratic order in $\varphi_{\mu\nu}$, though) in a curved background. The coefficients in the first two rows of (46) are fixed because in the flat spacetime limit, one demands the Lagrangian to go over to the Fierz-Pauli Lagrangian [18], which is known to have the correct number of causally-propagating degrees of freedom for a neutral massive spin-two field. Indeed, in order for a massive spin-two field (represented by a symmetric tensor $\varphi_{\mu\nu}$) to have the correct number of propagating degrees of freedom in a $(d+1)$-dimensional spacetime, one needs $2(d+2)$ constraint equations to kill the extra degrees of freedom.

Now, add to the above Lagrangian an additional coupling of the form $\ell^2 C^2(\varphi_{\mu\nu}\varphi^{\mu\nu} - \varphi^2)$ and define the Lagrangian

$$\mathcal{L}_\varphi = \mathcal{L} + \frac{1}{4}\ell^2 C^2 \left[\varphi_{\mu\nu}\varphi^{\mu\nu} - \varphi^2\right], \qquad (47)$$

where, in order to make the expressions less cluttered, we defined $C^2 = C_{\gamma\delta\rho\sigma}C^{\gamma\delta\rho\sigma}$. For $\ell = 0$, the analysis has been performed in [19], where, in order to get rid of the extra degrees of freedom of $\varphi_{\mu\nu}$, one obtains a one-parameter family of solutions for the coefficients $a_1, \cdots, a_5$ (labeled by an arbitrary real parameter ξ) which can be represented as

$$a_1 = \frac{\xi}{d+1}, \quad a_2 = \frac{1-2\xi}{2d+2}, \quad a_3 = a_4 = a_5 = 0, \quad (48)$$

provided that the background metric $g_{\mu\nu}$ satisfies the Einstein relation $R_{\mu\nu} = 2\Lambda g_{\mu\nu}/(d-1)$, and $m_0^2 \neq 2(\xi - 1)/(d+1)$. Defining $m^2 = m_0^2 + 2(1-\xi)R/(d+1)$ and given the above solution for a_i's, the Lagrangian (46) takes the form given in the expression (1), the so-called BGP Lagrangian.

For $\ell \neq 0$, one can easily obtain the constraint equations following [19]. Basically, constraints come from the equations of motion and their derivatives which do not contain two time derivatives of spin-two field $\ddot{\varphi}_{\mu\nu}$. The equations of motion read

$$\tilde{E}_{\mu\nu} = E_{\mu\nu} + \ell^2 C^2 \left(\varphi_{\mu\nu} - g_{\mu\nu}\varphi\right) = 0, \qquad (49)$$

where $E_{\mu\nu}$ denotes the equations of motion for the $\ell = 0$ case. The explicit form of $E_{\mu\nu}$ is given in [19]. Since $E_{\mu 0}$ does not involve second time-derivatives of $\varphi_{\mu\nu}$, and obviously the new term in (49) does not involve any second time-derivative of $\varphi_{\mu\nu}$ either, the equations $\tilde{E}_{\mu 0} = 0$ actually give us $d+1$ (primary) constraints.

In order to obtain the secondary constraints, we take the covariant derivative of the equations of motion

$$\nabla^\mu \tilde{E}_{\mu\nu} = 0. \qquad (50)$$

Again, neither $\nabla^\mu E_{\mu\nu}$ nor $\nabla^\mu \left[C^2(\varphi_{\mu\nu}\varphi^{\mu\nu} - \varphi^2)\right]$ involves two time-derivatives of the spin-two field. Thus, the equations (50) give us $d+1$ secondary constraints. Following [19], the equations (50) should contain $\dot{\varphi}_{\mu 0}$ through a rank d matrix in order for their derivatives to define d accelerations $\ddot{\varphi}_{i0}$:

$$E_{\mu 0}{}^{,\mu} = \mathcal{A}\,\dot{\varphi}_{00} + \mathcal{B}^j\dot{\varphi}_{j0} + \cdots,$$
$$E_{\mu i}{}^{,\mu} = \mathcal{C}_i\,\dot{\varphi}_{00} + \mathcal{D}_i^j\dot{\varphi}_{j0} + \cdots, \qquad (51)$$
$$\text{rank}\,\hat{\Phi}_\mu^\nu \equiv \text{rank}\left\|\begin{matrix}\mathcal{A} & \mathcal{B}^j \\ \mathcal{C}_i & \mathcal{D}_i^j\end{matrix}\right\| = d.$$

The explicit form for $\hat{\Phi}_\mu^\nu$ when $\ell = 0$ appears in [19]. The additional non-vanishing contributions to $\mathcal{A}$, $\mathcal{B}^j$, $\mathcal{C}_i$ and $\mathcal{D}_j^i$ are given by

$$\delta\mathcal{B}^j = -\ell^2 C^2 g^{j0}, \qquad \delta\mathcal{D}_j^i = \ell^2 C^2 g^{00}\delta_j^i. \qquad (52)$$

Hence, in order for the rank of $\hat{\Phi}_\mu^\nu$ to be d, we require $m_0^2 - \ell^2 C^2 + 2(1-\xi)R/(d+1) \neq 0$, where ξ is arbitrary parameter. We now have $2d+2$ constraints.

One of the remaining constraints can be obtained from a linear combination of the equations of motion and the primary and secondary constraints:

$$\frac{m_0^2 - \ell^2 C^2}{d-1}\tilde{E}_\mu^\mu + \nabla^\mu\nabla^\nu\tilde{E}_{\mu\nu} + \frac{2(1-\xi)}{(d+1)(d-1)}R\tilde{E}_\mu^\mu$$
$$= \frac{\varphi}{d-1}\left[\frac{2(1-\xi)}{d+1}R + m_0^2 - \ell^2 C^2\right]$$
$$\times \left[\frac{d+1-2\xi d}{d+1}R + (m_0^2 - \ell^2 C^2)d\right] \qquad (53)$$
$$+ 2\ell^2(\nabla^\nu C^2)(\nabla^\mu\varphi_{\mu\nu} - \nabla_\nu\varphi)$$
$$+ \ell^2(\nabla^\mu\nabla^\nu C^2)(\varphi_{\mu\nu} - g_{\mu\nu}\varphi) \approx 0.$$

This equation can be used as a constraint since it does not contain $\ddot{\varphi}_{\mu\nu}$. It contains $\dot{\varphi}_{\mu\nu}$ which can be removed by the equations $\nabla^\mu\tilde{E}_{\mu\nu} = 0$. After such a removal, the time-derivative of (53) does not contain $\ddot{\varphi}_{\mu\nu}$ and can be used as the last constraint. We then have in total $2d+4$ constraints and hence the correct number of propagating degrees of freedom.

Following the discussion in [19], it can be seen easily that the characteristic matrix remains unchanged for $\ell \neq 0$, due to the fact that the Weyl squared coupling does not affect any of the terms with two time-derivatives. So, the equations remain hyperbolic and causal even for the $\ell \neq 0$ case. The resulting equations of motion are

$$\left(\nabla^2 - m^2 + \ell^2 C^2\right)\varphi_{\mu\nu} + 2R^\rho{}_\mu{}^\sigma{}_\nu\varphi_{\rho\sigma} = 0. \qquad (54)$$

where, as we have defined before, $m^2 = m_0^2 + 2(1 - \xi)R/(d+1)$.

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