From fixed points to the fifth dimension

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FROM FIXED POINTS TO THE FIFTH DIMENSION

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Abstract

4D Lorentzian conformal field theory (CFT) is mapped into 5D anti-de Sitter spacetime (AdS), from the viewpoint of "geometrizing" conformal current algebra. A large-N expansion of the CFT is shown to lead to (infinitely many) weakly coupled AdS particles, in one-to-one correspondence with minimal-color-singlet CFT primary operators. If all but a finite number of "protected" primary operators have very large scaling dimensions, it is shown that there exists a low-AdS-curvature effective field theory regime for the corresponding finite set of AdS particles. Effective 5D gauge theory and General Relativity on AdS are derived in this way from the most robust examples of protected CFT primaries, Noether currents of global symmetries and the energy-momentum tensor. Witten's prescription for computing CFT local operator correlators within the AdS dual is derived. The main new contribution is the derivation of 5D locality of AdS couplings. This is accomplished by studying a confining IR-deformation of the CFT in the large-N "planar" approximation, where the discrete spectrum and existence of an S-matrix allow the constraints of unitarity and crossing symmetry to be solved (in standard fashion) by a tree-level expansion in terms of 4D local "glueball" couplings. When the deformation is carefully removed, this 4D locality (with plausible assumptions specifying its precise nature) combines with the restored conformal symmetry to yield 5D AdS locality. The sense in which AdS/CFT duality illustrates the possibility of emergent relativity, and the special role of strong coupling, are briefly discussed. Care is taken to conclude each step with well-defined mathematical expressions and convergent integrals.

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1 Introduction

Every child knows that things which are further away are really just smaller. It is only grown-ups who think this an illusion. After all, a distant object looks smaller in every detail, in principle all the way down to its atomic structure. And yet, the grown-ups point out, the Bohr radius is a constant of Nature.

But perhaps it is the grown-ups who are under the illusion. Physics may indeed play out on a flat screen, with an illusion of "depth" created by shrinking or expanding mutable 2D "atoms", conspiring to fake 3D atoms of fixed size but varying distance from us. Minimally, this requires the "holographic" [1] screen physics to be self-similar, so that structures ("atoms") on one length scale can be faithfully reproduced on a different scale. The best understood version of such self-similarity is the scale invariance enjoyed by local relativistic quantum field theories at fixed points of their renormalization group flow. A weakly coupled screen theory could not pull off such a grand "deception", so we deduce that the screen theory has to be at a strongly coupled fixed point.

If one imagines such a quantum field theoretic screen, then the fact that scale transformations are Lorentz-scalar implies that characteristic time scales get re-scaled along with characteristic spatial features. This property would necessarily create a peculiar illusion of depth: clocks which are further away would tick more rapidly. Those trapped in the illusion might ascribe this effect to gravitational time-dilation in a curved $(3+1)D$ spacetime. Indeed, if the screen physics is $(2+1)D$ Poincare and scale invariant, the unique $(3+1)D$ geometry realising these symmetries, with gravitational red-shift as a function of depth, is anti-de Sitter (AdS),

$$ds^2_{AdS_4} = \frac{R^2_{AdS}}{z^2}(dt^2 - dx^2 - dy^2 - dz^2). \tag{1.1}$$

R_{AdS} is a constant radius of curvature, possibly so large that spacetime appears approximately $(3+1)$-Minkowski for "practical" purposes.

For a truly seamless plot along these lines, there would have to be new symmetries that put the illusory dimension of "depth" on par with the flat dimensions of the screen, that transform one into the others. These symmetries clearly lie outside $(2+1)D$ Poincare and scale symmetry. We can see what is required: the $z \leftrightarrow t, z \leftrightarrow x, z \leftrightarrow y$ symmetries of AdS_4, generalizing Lorentz symmetries of Minkowski$_4$, are (infinitesimally)

$$\begin{aligned} \delta_{x^{\hat{\mu}}-z} z &= -2x^{\hat{\mu}} z \\ \delta_{x^{\hat{\mu}}-z} x^{\hat{\nu}} &= (x^{\hat{\alpha}} x_{\hat{\alpha}} - z^2)\eta^{\hat{\mu}\hat{\nu}} - 2x^{\hat{\mu}} x^{\hat{\nu}}, \end{aligned} \tag{1.2}$$

where the hatted indices run over the screen dimensions, t, x, y. Remarkably, candidates for playing this role do emerge at renormalization-group fixed points of quantum field theory, along with scale symmetry.[1] These are the special conformal transformations, a kind of $x^{\hat{\mu}}$-dependent

[1] This is a strong conjecture in more than two spacetime dimensions. See the discussion in Ref. [2].

dilatation. Together with $(2+1)D$ Poincare and scale invariance they realize full conformal invariance on the screen, so the screen theory is a conformal field theory (CFT).

The fact that $(3+1)D$ spacetime contains gravity may seem problematic because the notions of strongly-coupled quantum field theory that we used above to think about the screen physics are those of rigid ($(2+1)D$) spacetime. (We do not yet know if a strong *and* gravitational fixed point (or self-similar) theory self-consistently exists.[2]) But we can hope that strongly-coupled quantum field theory on a *rigid* screen gives rise to the illusion of a *dynamical* $(3+1)D$ spacetime (with an AdS ground state). At first, this seems like asking too much, naively contradicting the Weinberg-Witten theorem [4], which famously finds that a quantum field theory with a standard local, conserved energy-momentum tensor cannot contain a massless spin-2 "graviton" in the spectrum. But in the present case, the graviton resides in a *different* spacetime, one dimension higher than that of the screen quantum field theory, and the Lorentz representation-theoretic analysis of Ref. [4] fails to apply.

In this way, we have arrived at a daring, almost far-fetched, plot, pulling the magic of quantum gravity and emergent dimensions out of "mere" quantum field theory. The "AdS/CFT correspondence" is the conjecture that this plot can, in fact, be theoretically realized [5] [6] [7]. More generally, it claims that there exist strongly-coupled d-dimensional "screen" CFTs, for various d, that project "holograms" that are weakly-coupled $(d+1)$-dimensional quantum gravities on low-curvature AdS backgrounds.[3] The correspondence claimed is so perfect that it is in the end physically meaningless to take sides, to say that the CFT is "real" and the AdS theory an "illusion". We say instead simply that each theory is "dual" to the other.

The best studied example of this type is the CFT of strongly-coupled $\mathcal{N}=4$ supersymmetric Yang-Mills theory in $(3+1)D$ (with many gauge colors), dual to Type IIB string theory on an $AdS_{4+1} \times S^5$ background (stabilized by a large Ramond-Ramond flux). While still at the level of a conjecture because the strong CFT dynamics are not fully soluble, there is strong evidence based on exploiting the high degree of supersymmetry, as well as the original arguments based on D-brane constructions. There is, however, a strong suspicion that AdS/CFT duality transcends these particular considerations, and that there is a general AdS/CFT grammar that is less conjecture and more "theorem". In this approach, any CFT_d is dual to *some* AdS_{d+1} theory, but one wants to *derive* certain broad CFT features that guarantee a "useful" AdS theory, one with a semi-classical General Relativity regime and a few light particle species, inside a large AdS radius of curvature. Some of the requisite "input" strong-coupling CFT properties might be a matter of conjecture, but their translation into AdS could be on surer footing.

Many of the central insights for such a robust AdS/CFT translation already exist in the literature, chiefly the importance of a large gap in the spectrum of CFT scaling dimensions to a

[2]But see Ref. [3] for a review of work in this direction.

[3]More precisely, AdS_{d+1} may be just one factor in a product spacetime of even higher dimensionality.

large general relativistic effective field theory regime on the AdS side. See Ref. [8], for example. In the present paper, these insights are fit into a continuous narrative, starting from a CFT with broadly stated properties and then deducing the existence of an AdS mapping, including AdS effective gravity + gauge theory, and AdS "Witten diagrams" [7] dual to correlators of local CFT operators. Where there are gaps in the literature to the main logic of the CFT $\longrightarrow$ AdS construction, these are filled. The paper can serve as a review of AdS/CFT foundations, with a somewhat anti-historical slant. In particular, supersymmetry, string theory, D-branes, and the specifics of $\mathcal{N} = 4$ supersymmetric Yang-Mills, play little role in the discussion, although they can then be used to flesh out the well-known examples from the basic AdS/CFT grammar derived here. In this sense, standard reviews provide important complementary treatments [9]. The large-N_{color} expansion for the CFT does play an important role in this paper, but ultimately this may itself be an unnecessary scaffolding, as discussed briefly in subsection 5.5.

For the sake of familiarity with 4D quantum field theory, the case of $CFT_{3+1} \longrightarrow AdS_{4+1}$ is presented, though only minor modifications are required for other dimensionalities, such as the more visualizable case of $CFT_{2+1} \longrightarrow AdS_{3+1}$ in the story above. For the same reason, the CFT is taken to live on Minkowski spacetime, yielding a dual on the Poincare patch of AdS (reviewed in [9]), rather than the case of a CFT on a three-sphere (plus time) which is often considered so as to yield a dual on global AdS. The presentation is almost completely in Lorentzian signature rather than the technically simpler Euclidean signature, so as to be conceptually clearest. Early Lorentzian AdS/CFT work can be found in Refs. [10] [11]. More recent work can be found in Ref. [12] and references therein. Our approach emphasizes that the AdS/CFT correspondence equates *states* of the CFT with *states* in AdS, as they evolve in time. The more abstract identification of correlators of local CFT operators with AdS Witten diagrams is then derived from this core result. In this connection, there is a mild concession to Euclidean signature in subsection 6.5, in favor of technical simplicity, to short-circuit a longer Lorentzian discussion, but even here some physical pointers precede it.

In the story told above, one of the qualitative puzzles that emerges is why, if "depth" is a mere illusion, can one not just reach out and touch objects that only seem to be very far away. Two objects separated only in the depth dimension of the AdS illusion correspond to a big object right on top of a small object on the CFT screen, so why can they not directly interact? Yet, locality of couplings in all the dimensions of AdS is an essential part of the illusion. One does not expect to interact directly with a distant object. Showing this has been a central challenge for the $CFT \rightarrow AdS$ plot, which we address in this paper. Earlier progress in this direction, and a sharp framing of the question, appear in Ref. [8]. Naively, the locality of quantum field theory *on* the screen, plus conformal invariance "rotating" the screen dimensions into the "depth" dimension, should imply locality in all the AdS dimensions. The difficulty is that CFT couplings are local at the level of its elementary fields, say "quarks" and "gluons" of a strongly-coupled gauge theory with large-N_{color} structure, while the "particles" of

AdS correspond to gauge-invariant color-singlet multi-quark/gluon CFT states, for which the constraints of locality are opaque. For example, in a CFT such, necessarily extended, multi-quark/gluon states have no characteristic length scale on which one can have even an infrared notion of locality. The strategy employed in this paper is to deform the CFT so that the result is asymptotically conformal in the UV, but confining in the IR, below some characteristic confinement scale. One can then exploit the excellent understanding of locality we have for color-singlet states within large-N *confining* gauge theories (reviewed in Refs. [13]): in the leading planar approximation, minimal-color-singlet mesons and glueballs have tree-level local couplings. It is a remarkable feature of this approximation, that this locality necessarily holds at all energies. In particular, in our case, locality holds in the far UV where the deformation can be ignored, and we are asymptotically in the undeformed CFT. Combining this result with UV-asymptotic conformal invariance yields full locality on the AdS side.

The physical reason why this passage through the deformed CFT is useful is this. The locality properties of confining large-N gauge theories are deduced by careful consideration of the meson/glueball S-matrix, realized as LSZ-like limits of correlators of gauge-invariant local operators, and by exploiting the simple form taken by the constraints of unitarity and crossing symmetry. In a CFT, without a gap in the spectrum, one cannot tune momenta in this fashion to be nearly on-shell for some exclusive hadron state, and off-shell for others. Instead any (timelike) momentum through a local operator is always exactly on-shell with respect to some physical state, and only infinitesimally off shell for others. But if the IR-deformed CFT is confining, then one has embedded the CFT in a confining theory with S-matrix, where one recovers conformal invariance at short distances. Therefore in thought experiments, one can aim exclusive confined hadrons sent in from infinity, so that they collide in a small spacetime region where conformal invariance holds to good approximation, and the scattering products emerge from this region, resolve themselves back to confined hadrons, and propagate out to infinity. In this paper, we show how the locality results for large-N confining theories can thereby extrapolate to CFTs. In a similar spirit, Ref. [11] considered a "regulated" AdS.

There are simple and plausible technical assumptions going into our derivation of AdS-locality, that spell out the precise nature of "4D locality" above. In principle, the above type of thought experiments only determine that meson/glueball interaction vertices are *analytic* in 4D momenta, whereas we will assume that the vertices for a fixed set of "hadrons" coming into a vertex are in fact *polynomial* in momenta. While taken as an input assumption we will motivate it by tying it to the good initial value problem enjoyed by the CFT.

There are several motivations for trying to work out the AdS/CFT plot as carefully and broadly as possible:

AdS quantum gravity (and gauge theory), with a classical and low curvature regime, shares qualitative features and mysteries in common with our own universe. The CFT dual is its

"DNA", a complete blueprint which we can partially decode, but whose very existence already changes our world view. We obviously would like to know what are the central features of this DNA that cause it to unfold into such an AdS dual, and what features are inessential details. This exercise is then the first step in generalizing further, to understand the holographic encoding of gravitating spacetimes even closer to our own, say those with Big Bang initial conditions.

The AdS/CFT correspondence can be deformed to give dualities between strongly-coupled non-conformal field theories and higher-dimensional non-AdS gravity and gauge theory. These deformations often have the effect of compactifying the AdS space, such as when the deformed CFT is confining [14] [15] [16] [17]. In this way, a variety of strongly-coupled quantum field theories can be partially "solved" in terms of a weakly coupled general relativistic dual effective field theory. The strong coupling has gone into assembling the higher-dimensional degrees of freedom, which then have only weakly-coupled ($1/N$) residual interactions. Of course, it is the entire UV-complete quantum gravity theory on the (deformed) AdS side that is dual to a (deformed) CFT. If one instead starts with a (deformed) AdS *effective* field theory for which a UV completion is unknown, then it is dual to a set of robust dynamical assumptions about a possible strongly coupled (deformed) CFT. The effective field theory self-consistency on the AdS side translates into self-consistency of the dynamical assumptions being made about the CFT dynamics. But only proof of the existence of a UV completion of the AdS effective theory can imply proof of existence of a CFT with these dynamical properties. This is a seemingly weak position, but it is often the position we are in, in phenomenologically-oriented research, when we suspect that strong dynamics is at work. Fitting the phenomenological considerations to an AdS-side effective field theory provides a powerful kind of rapid reconnaisance of the strong dynamics features and interconnections. The generality with which we understand AdS/CFT translates into the generality of this kind of "effective CFT" [18] tool.

The AdS/CFT correspondence demonstrates the power of strong coupling to produce a diverse range of emergent phenomena: extra dimensions, general relativity, gauge theory. Even the pre-requisite of conformal invariance can itself be an emergent phenomenon, if it is the result of a quantum field theory flowing in the IR to a renormalization group fixed point. One can take it even a step further. Continuum quantum field theory and the underlying special relativity may themselves be emergent. It is well-understood that continuum field theory and spatial rotational invariance can readily emerge as the long-wavelength limit of discrete systems, such as lattice theories. Such continuum field theories can be further enhanced to have emergent special relativity in the IR, but at weak coupling this is a very delicate affair. Here too strong coupling can help, allowing a fundamentally non-relativistic theory to robustly and rapidly flow in the IR towards Lorentz invariance [19]. See Section 9 for a discussion.

In this way, one may have a sequence of emergent phenomena: strongly-coupled discrete quantum system $\rightarrow$ continuum quantum field theory $\rightarrow$ Special Relativistic field theory $\rightarrow$

CFT → AdS General Relativity + gauge theory. It is obviously important to understand the robustness of each of these steps. In this way, one can hope to use weakly coupled AdS effective field theories, strongly constrained by powerful local symmetries, to capture the IR properties of the far less symmetric strongly-coupled systems found in condensed matter physics. See Refs. [20] for reviews.

Reading in reverse, one might well suspect that our own Universe has a discrete but strongly-interacting "DNA". Emergent relativity, even general relativity, need not be as perfect as fundamental relativity. There may be long-range defects. AdS/CFT allows us to probe these possibilities, as illustrated in Ref. [21].

2 Conformal Field Theory

The defining notion of conformal symmetry is given by its action on 4D Minkowski spacetime. One can define conformal transformations as general coordinate transformations that take the Minkowski metric in its standard form, $\eta_{\mu\nu} \equiv \text{diag}(1, -1, -1, -1)$, to $f(x)\eta_{\mu\nu}$, where $f(x)$ is a function of spacetime. For a rapid review of conformal transformations, see Section 1 of Ref. [22]. The generators of conformal symmetry include the usual (infinitesimal) Poincare transformations (with $f(x) = 1$), as well as infinitesimal dilatations, S, and infinitesimal special conformal transformations, K_μ:

$$\begin{aligned} \delta_S x^\mu &= x^\mu \\ \delta_{K_\nu} x^\mu &= x^2 \delta_\nu{}^\mu - 2x_\nu x^\mu. \end{aligned} \tag{2.1}$$

These infinitesimal conformal transformations are well-defined on Minkowski spacetime and define a closed Lie algebra. However, the full conformal *group* connects finite points in Minkowski spacetime to points at infinity. This subtlety will not concern us in this paper, where we work mostly at the level of the conformal algebra.

2.1 Hermitian conformal generators

CFTs are relativistic local quantum field theories on Minkowski spacetime which are invariant under conformal symmetry. See Refs. [23] for reviews. More precisely, the generators of the conformal algebra are realized as hermitian operators on Hilbert space that annihilate the CFT vacuum state. The usual Poincare algebra of hermitian operators,

$$\begin{aligned} [J_{\mu\nu}, P_\rho] &= -i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \\ [J_{\mu\nu}, J_{\rho\sigma}] &= -i\eta_{\mu\rho} J_{\nu\sigma} \pm \text{permutations} \\ [P_\mu, P_\nu] &= 0, \end{aligned} \tag{2.2}$$

is supplemented by

$$
\begin{aligned}
[S, K_\mu] &= iK_\mu \\
[S, P_\mu] &= -iP_\mu \\
[J_{\mu\nu}, K_\rho] &= -i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \\
[S, J_{\mu\nu}] &= 0 \\
[P_\mu, K_\nu] &= 2iJ_{\mu\nu} - 2i\eta_{\mu\nu}S. \qquad (2.3)
\end{aligned}
$$

This Lie algebra is isomorphic to that of $SO(4,2)$, the Lorentz transformations $J_{\mu\nu}$ generating the $SO(3,1)$ subgroup, while the remaining generators form $J_{\mu 4} = \frac{1}{2}(K_\mu - P_\mu)$, $J_{\mu 5} = \frac{1}{2}(K_\mu + P_\mu)$, $J_{54} = S$, in an obvious notation.

2.2 Local operators

The conformal algebra acts linearly on local operators $\mathcal{O}(x)$ by commutation. Irreducible representations are labelled by *primary* operators, $\mathcal{O}_n(x)$, themselves in irreducible Lorentz representations. For example, primary operators can be Lorentz scalar, vector, spinor, tensor, and so on. Primary operators transform according to [24]

$$
\begin{aligned}
[P_\mu, \mathcal{O}_n(x)] &= i\partial_\mu \mathcal{O}_n(x) \\
[J_{\mu\nu}, \mathcal{O}_n^\alpha(x)] &= [i(x_\mu\partial_\nu - x_\nu\partial_\mu)\delta^\alpha_{\ \beta} + \Sigma^{n\ \alpha}_{\mu\nu\beta}]\mathcal{O}_n^\beta \\
[S, \mathcal{O}_n(x)] &= -i(\Delta_n + x.\partial)\mathcal{O}_n(x) \\
[K_\mu, \mathcal{O}_n(x)] &= -i(x^2\partial_\mu - 2x_\mu x.\partial - 2x_\mu\Delta_n)\mathcal{O}_n(x) - 2x^\nu\Sigma^n_{\mu\nu}\mathcal{O}_n(x), \qquad (2.4)
\end{aligned}
$$

where α, β are indices for the Lorentz representation of $\mathcal{O}_n$, $\Sigma^n_{\mu\nu}$ are Lorentz transformation matrices (in α, β) for this Lorentz representation, and Δ_n is the *primary scaling dimension* (and canonical dimension[4]).

All other local operators can be expressed as derivatives ("descendents") of primary operators,

$$
\partial_{\mu_1} \ldots \partial_{\mu_k}\mathcal{O}_n(x). \qquad (2.5)
$$

Their conformal transformations follow by differentiation of Eq. (2.4), and in particular they have scaling dimension $\Delta_n + k$. Since such differentiation arises from repeated commutation with translation operators, P_μ, all these descendents are in the same conformal representation as $\mathcal{O}_n$, and indeed together they span the irreducible conformal representation of local operators labelled by n.

[4]For convenience we will assume that any scaling operator has been multiplied by a suitable power of the renormalization scale to make its canonical dimension equal to its scaling dimension.

2.3 Lightning derivation

For completeness, here is a brief review of the above statements. The first three of Eqs. (2.4) are the straightforward expression of Poincare and scale symmetry in the basis of scaling operators, with only the last of Eqs. (2.4) being subtle.

First note that commutation with S simplifies at $x = 0$,

$$[S, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0), \tag{2.6}$$

where $\mathcal{O}$ is a scaling operator with scale dimension Δ. Eqs. (2.3) then imply that K_μ and P_μ act as lowering and raising operators for scaling dimension,

$$\begin{aligned} [S, [K_\mu, \mathcal{O}(0)]] &= -i(\Delta - 1)[K_\mu, \mathcal{O}(0)] \\ [S, [P_\mu, \mathcal{O}(0)]] &= -i(\Delta + 1)[P_\mu, \mathcal{O}(0)]. \end{aligned} \tag{2.7}$$

The raising of dimension by commuting with translations is clearly just the process of taking derivatives of $\mathcal{O}$ at $x = 0$, and can be done repeatedly without bound. However, repeated lowering of scale dimension must stop at some point because there is a lower bound on how small scaling dimensions can be in a unitary CFT, known as the "unitarity bound" [25], which depends on the Lorentz representation of the scaling operator.

Here, we settle for a crude argument, based on scale symmetry, for why scale dimensions are bounded below. For simplicity, focus on a Lorentz-scalar scaling operator, $\mathcal{O}$, with scaling dimension (and canonical dimension) Δ. Its two-point function has a spectral decomposition given by inserting a complete set of states of invariant mass m and total spatial momentum $\vec{p}$,

$$\begin{aligned} \langle 0|\mathcal{O}(x)\mathcal{O}(0)|0\rangle &= \int dm^2 \frac{d^3\vec{p}}{(2\pi)^3\sqrt{\vec{p}^2 + m^2}} e^{-ip.x} |\langle 0|\mathcal{O}(0)|\vec{p}, m\rangle|^2 \\ &\sim \int dm^2 \frac{d^3\vec{p}}{(2\pi)^3\sqrt{\vec{p}^2 + m^2}} e^{-ip.x} m^{2\Delta - 4}, \end{aligned} \tag{2.8}$$

where the last line follows from dimensional analysis, Lorentz invariance, and the fact that there is no intrinsic scale in our scale-invariant theory. Notice that for non-coincident points, $x \neq 0$, this two-point function should be well-defined (once the composite operator $\mathcal{O}$ itself is renormalized). Yet the small-m behavior of the spectral decomposition is $\sim \int dm^2 m^{2\Delta-4}$, which diverges unless $\Delta > 1$. Notice that this is an IR, not UV, divergence, that cannot simply be "renormalized" away. It is therefore unphysical. There is one exceptional case, $\Delta = 1$, where the dimensional analysis allows $m^{2\Delta-4}$ in the last line of Eq. (2.8) to be replaced by $\delta(m^2)$, without any divergence. Therefore, we conclude that there is a lower bound, $\Delta \geq 1$.

For other Lorentz representations of operators, the essential point is the same: for sufficiently low scaling dimension, the spectral decomposition is IR divergent and the operator (correlators at non-coincident points) become ill-defined. The bounds using only scale invariance are not as tight as the unitarity bounds exploiting conformal invariance in general, but they make the point we want: repeated lowering of scale dimension with K_μ must always terminate.

We conclude that within each conformal representation, n, of local operators at $x = 0$ there must be one (Lorentz multiplet) scaling operator, $\mathcal{O}_n(0)$, which cannot be further lowered,

$$[K_\mu, \mathcal{O}_n(0)] = 0. \tag{2.9}$$

$\mathcal{O}_n$ is said to be "primary". Since the action of $J_{\mu\nu}$ and S takes (the Lorentz multiplet) $\mathcal{O}_n(0)$ to itself, only translations ("raising") via P_μ connect the primary operator to other (Lorentz multiplet) scaling operators in the same conformal representation. That is, the primary $\mathcal{O}_n(0)$ and its derivatives, or "descendents", at $x = 0$ span the irreducible conformal representation, n. Of course this representation is infinite-dimensional since the process of raising, differentiation, does not terminate.

The remaining question is how $[K_\mu, \mathcal{O}_n(x)]$ can be represented for general $x \neq 0$ and primary $\mathcal{O}_n$. The result should be some local operator at x which is in the same conformal representation as $\mathcal{O}_n(x)$. By translation invariance and our results at $x = 0$, the result must be a linear combination of (multiple) derivatives of $\mathcal{O}_n(x)$, which we can write as some differential operator acting on the primary operator,

$$[K_\mu, \mathcal{O}_n(x)] = \mathcal{D}_n(x)\mathcal{O}_n(x). \tag{2.10}$$

If the primary is in a non-trivial Lorentz representation, then $\mathcal{D}_n$ is implicitly matrix-valued in this representation space. The coefficients of (multiple) derivatives can in general be functions of x, which is what is denoted by the x-dependence of $\mathcal{D}_n(x)$. To solve for $\mathcal{D}_n(x)$, we start with the general Jacobi identity,

$$[P_\mu, [K_\nu, \mathcal{O}_n(x)] - [[P_\mu, K_\nu], \mathcal{O}_n(x)] + [K_\nu, [\mathcal{O}_n(x), P_\mu]] = 0, \tag{2.11}$$

which by the last of Eqs. (2.3), Eq. (2.10), and the first of Eqs. (2.4) translates into

$$\begin{aligned} & i\mathcal{D}_n(x)\partial_\mu\mathcal{O}_n(x) + 2(x_\mu\partial_\nu x_\nu\partial_\mu)\mathcal{O}_n(x) - 2i\Sigma^n_{\mu\nu}\mathcal{O}_n(x) + 2\eta_{\mu\nu}(\Delta_n + x.\partial)\mathcal{O}_n(x) \\ & -i\partial_\mu(\mathcal{D}_n(x)\mathcal{O}_n(x)) = 0. \end{aligned} \tag{2.12}$$

From this, it follows that

$$\partial_\mu\mathcal{D}_n(x) = 2i(x_\nu\partial_\mu - x_\mu\partial_\nu) - 2\Sigma^n_{\mu\nu} - 2i\eta_{\mu\nu}(\Delta_n + x.\partial). \tag{2.13}$$

It is straightforward to check that this is solved by

$$\mathcal{D}_n(x) \;\; = \;\; -i(x^2\partial_\mu - 2x_\mu x.\partial - 2x_\mu\Delta_n) - 2x^\nu\Sigma^n_{\mu\nu}, \tag{2.14}$$

where the integration constant vanishes, $\mathcal{D}_n(0) = 0$, because $[K, \mathcal{O}_n(0)] = 0$ by definition of "primary".

We have arrived at the result

$$[K_\mu, \mathcal{O}_n(x)] \;\; = \;\; -i(x^2\partial_\mu - 2x_\mu x.\partial - 2x_\mu\Delta_n)\mathcal{O}_n(x) - 2x^\nu\Sigma^n_{\mu\nu}\mathcal{O}_n(x). \tag{2.15}$$

3 Geometrizing Conformal Field Theory

While conformal invariance provides a powerful constraint on quantum field theory, the transformation laws are somewhat opaque at first viewing. Ideally, we would like some way of "geometrizing" them and making them more intuitive. A rough analogy is what happens in supersymmetric field theory where the supersymmetry transformations between component fields are quite complicated. But one can formally extend Minkowski spacetime to *superspace*, whose "isometries" contain the supersymmetry algebra. Different spacetime fields related by supersymmetry then unify into a single field on superspace. Such a "superfield" transforms simply, according to its geometric status on superspace. Similarly, the approach of geometrizing conformal symmetry leads to the extension of ordinary 4D Minkowski spacetime to (the Poincare patch of) AdS_5. Our motivations and approach in this section are similar in spirit to Refs. [26] [11] [27] [28].

We begin (and proceed until Section 7) with the simplest kind of conformal representation of local operators, namely one where the primary operator is a single Lorentz-scalar $\mathcal{O}(x)$. A scalar field has simple spacetime transformations, namely the spacetime argument of the field alone transforms, $x \to x'$. Indeed, the first two of Eqs. (2.4) show that $\mathcal{O}$ is a scalar field in this sense (for $\Sigma_{\mu\nu} = 0$) under infinitesimal Poincare transformations. However the latter two of Eqs. (2.4) show that this is *not* the case for dilatations and special conformal transformations: they are not captured purely by $x \to x'$, there are extra terms depending on the scale dimension, Δ. Let us try to remedy this.

3.1 Geometrizing dilatations

We first focus on just dilitations, neglecting special conformal transformations. There is a simple trick for making dilatations act only on coordinates, by introducing a fictitious fifth-dimensional coordinate, $w > 0$, and defining a 5D "field",

$$\phi(x, w) \equiv w^{\Delta}\mathcal{O}(x). \tag{3.1}$$

Obviously, the transformation law of $\mathcal{O}$ is thereby re-expressed as

$$i[S, \phi(x, w)] = (x^{\mu}\partial_{\mu} + w\partial_{w})\phi(x, w). \tag{3.2}$$

In this way, all dilatations and Poincare transformations (which form a closed subalgebra of the conformal algebra) are realized on 5D spacetime,

$$\begin{aligned} \delta_{J_{\mu\nu}} x^\rho &= \delta_\mu{}^\rho x_\nu - \delta_\nu{}^\rho x_\mu \\ \delta_{J_{\mu\nu}} w &= 0 \\ \delta_S x^\mu &= x^\mu \\ \delta_S w &= w \\ \delta_{P_\mu} x^\rho &= \delta_\mu{}^\rho \\ \delta_{P_\mu} w &= 0. \end{aligned} \tag{3.3}$$

and $\phi(x, w)$ transforms simply as a scalar field with respect these. To "geometrize" this symmetry we must identify it with isometries of some 5D spacetime geometry. It is straightforward to see that the unique (Lorentzian $(4+1)$D) geometry with isometries given by Eq. (3.3) is that of AdS_5:

$$ds^2_{AdS} = \frac{R^2_{AdS}}{w^2}(\eta_{\mu\nu}dx^\mu dx^\nu - dw^2), \tag{3.4}$$

where $\eta_{\mu\nu}dx^\mu dx^\nu$ denotes the usual 4D Minkowski metric, R_{AdS} is a constant radius of curvature, and $w > 0$. From now on we will work in $R_{AdS} \equiv 1$ units. R_{AdS}-dependence can be recovered by dimensional analysis. Eq. (3.4) describes only the "Poincare patch" of AdS_5. (See the reviews of Ref. [9].) We will discuss later in this section why we are naturally restricted to this patch, starting from CFT in Minkowski spacetime.

Notice that this spacetime has the same causal structure (null geodesics) as 5D Minkowski spacetime,

$$ds^2_{5DMink.} = \eta_{\mu\nu}dx^\mu dx^\nu - dw^2. \tag{3.5}$$

The restriction to $w > 0$ means that physics in AdS_5 is causally equivalent to physics on *half* of 5D Minkowski spacetime, $w > 0$. That is we are doing physics on a spacetime with a boundary at $w = 0$, on which boundary conditions will have to be stipulated. Particles moving at light speed can propagate from this boundary to points in the interior in finite time, even though the boundary is infinitely far away in proper distance. The boundary of AdS_5 will play an important role in the AdS/CFT correspondence to follow.

3.2 Mismatch in AdS/CFT conformal transformations

It is straightforward to check that, although we have only demanded isometries corresponding to scale and Poincare symmetry, the 5D spacetime isometry algebra is "accidentally" larger, encompassing infinitesimal special conformal transformations as well, but taking the 5D incarnation

$$\begin{aligned} \delta_{K_\nu} x^\mu &= (x^2 - w^2)\delta_\nu{}^\mu - 2x_\nu x^\mu \\ \delta_{K_\nu} w &= -2x_\nu w. \end{aligned} \tag{3.6}$$

The algebra of 5D isometries, Eqs. (3.3, 3.6), is readily checked to be isomorphic to the conformal algebra.[5]

With this 5D realization of the full conformal symmetries, we will try to promote our scalar primary operator $\mathcal{O}(x)$ into a 5D AdS scalar field, $\phi(x, w)$, such that only the coordinates transform under any of the conformal transformations, according to Eqs. (3.3, 3.6). Eq. (3.1) was constructed so as to accomplish this for scale symmetry, but the special conformal transformations do *not* match between their CFT and AdS forms (Eqs. (2.4) and (3.6)):

$$\begin{aligned} \delta^{CFT}_{K_\mu}\phi(x,w) &= iw^\Delta[K_\mu, \mathcal{O}(x)] \\ &= w^\Delta(-2x_\mu x.\partial + x^2\partial_\mu - 2x_\mu\Delta)\mathcal{O}(x) \\ &= (-2x_\mu x.\partial + x^2\partial_\mu - 2x_\mu w\partial_w)\phi(x,w), \end{aligned} \tag{3.7}$$

compared with

$$\begin{aligned} \delta^{AdS}_{K_\mu}\phi(x,w) &= (\delta^{AdS}_{K_\mu}x^\nu)\partial_\nu\phi + (\delta^{AdS}_{K_\mu}w)\partial_w\phi \\ &= (-2x_\mu x.\partial + (x^2 - w^2)\partial_\mu - 2x_\mu w\partial_w)\phi(x,w). \end{aligned} \tag{3.8}$$

As can be seen these do not match in the w^2 term. Therefore, as it stands, Eq. (3.1) does not define an AdS scalar field.

3.3 AdS/CFT matching of conformal transformations

To try to improve our construction of ϕ, note that as we approach the AdS boundary, $w \to 0$, the discrepancy discussed above disappears. So let us retain Eq. (3.1) as only the limiting behavior near the boundary,

$$\phi(x,w) \underset{w\to 0}{\longrightarrow} w^\Delta\mathcal{O}(x). \tag{3.9}$$

We shall see that there is then a unique way of extending ϕ to the interior of AdS, so that ϕ is a properly transforming scalar field under all the isometries of AdS.

To see this, let us assume we have such a ϕ already in hand, and deduce its properties. The 5D AdS d'Alambertian operator,

$$\Box_5 \equiv \frac{1}{\sqrt{G}}\partial_M\sqrt{G}G^{MN}(w)\partial_N = w^2\Box_4 - w^5\partial_w\frac{1}{w^3}\partial_w, \tag{3.10}$$

(where G_{MN} is the AdS metric and $M, N = \mu, w$) is an AdS-invariant hermitian operator acting on AdS scalar fields, so we can always choose to decompose our ϕ in an eigenbasis of $\Box_5$. Eigenfunctions of $-\Box_5$ satisfy an AdS Klein-Gordon equation

$$-\Box_5\phi = m_5^2\phi, \tag{3.11}$$

[5]Indeed the conformal group is isomorphic to $SO(4,2)$, and AdS_5 can be realized as the (covering space of the) hyperboloid, $X_0^2 + X_5^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = R^2_{AdS}$, manifestly symmetric under $SO(4,2)$, where X_M transforms in the fundamental representation.

for some eigenvalue m_5^2. One can easily separate variables and solves such equations, but let us just focus on the behavior of eigenfunctions near the boundary of AdS, $w \to 0$. For this purpose, note that the $\Box_4$ term is subdominant as $w \to 0$. Therefore eigenfunctions satisfy

$$w^5 \partial_w \frac{1}{w^3} \partial^w \phi \underset{w \to 0}{\longrightarrow} m_5^2 \phi. \tag{3.12}$$

This gives the near-boundary solution

$$\phi(x, w) \underset{w \to 0}{\longrightarrow} w^{2 \pm \sqrt{4 + m_5^2}}. \tag{3.13}$$

Comparing with Eq. (3.9), we see that the AdS-scalar field we are trying to construct from $\mathcal{O}(x)$ must be a pure eigenfunction, satisfying the AdS Klein-Gordon equation (3.11), with

$$m_5^2 = \Delta(\Delta - 4). \tag{3.14}$$

We have arrived at a unique prescription for how to construct $\phi(x, w)$: solve the Klein-Gordon equation Eq. (3.11) with AdS mass-squared, Eq. (3.14), subject to the boundary asymptotics, Eq. (3.9). Conformal transformations of $\mathcal{O}(x)$, Eq. (2.4), match AdS conformal transformations of boundary conditions Eq. (3.9) (since the w^2 term discrepancy in special conformal transformations pointed out in the last subsection becomes negligible as $w \to 0$). Since the AdS Klein-Gordon equation is invariant under AdS isometries, a conformal transformation of the AdS boundary conditions induces an AdS symmetry transformation of the solution $\phi(x, w)$ everywhere in AdS. In this way conformal transformations on $\mathcal{O}$ induce AdS symmetry transformations on ϕ as a scalar field.

3.4 The direct approach

The above logic is perfectly correct, but may seem a little slick on first reading. It is therefore useful to see a more blow-by-blow account of the same result. It is efficient to work in 4D momentum space, but remain in position space in the fifth dimension. It is then straightforward to see that Eq. (3.1) can be generalized while retaining the feature that ϕ is a scalar field under dilatations and Poincare transformations:

$$\phi(p_\mu, w) \equiv k(p^2 w^2) w^\Delta \mathcal{O}(p), \tag{3.15}$$

where k is an arbitrary function. Clearly, with this generalization, ϕ remains a scalar under 4D Poincare symmetry, and is also invariant under

$$\begin{aligned} \delta_S p_\mu &= -p_\mu \\ \delta_S w &= w. \end{aligned} \tag{3.16}$$

We will choose k by demanding that special conformal transformations, K_μ, match up between the CFT version on $\mathcal{O}$, Eq. (2.4), and its action on ϕ as an AdS scalar, Eq. (3.8).

Given that this was already successful for Eq. (3.1) for *small* w, we take Eq. (3.1) as our small w limit of Eq. (3.15),

$$k(0) = \text{constant}. \tag{3.17}$$

We have generalized in an obvious way by letting the constant be arbitrary (rather than unity as in Eq. (3.1), for later convenience. The passage to 4D momentum space of Eqs. (2.4, 3.8), follows from the usual

$$\begin{aligned} \partial_\mu &\equiv -ip_\mu \\ x^\mu &\equiv i\partial_{p_\mu}. \end{aligned} \tag{3.18}$$

After a little algebra, one finds from Eq. (3.8) and Eq. (2.4)

$$\begin{aligned} \delta^{AdS}_{K_\mu}\phi(p,w) &\equiv \{-(w^2+\partial_p^2)p_\mu + 2\partial_{p_\mu}\partial_p.p - 2w\partial_w\partial_{p_\mu}\}\{k(p^2w^2)w^\Delta\mathcal{O}(p)\} \\ &= ik(p^2w^2)w^\Delta[K^{CFT}_\mu,\mathcal{O}(p)] - p_\mu w^{\Delta+2}\mathcal{O}(p)\{4w^2p^2k'' + 4(\Delta-1)k' + k\} \\ &\equiv i[K^{CFT}_\mu,\phi(p,w)] - p_\mu w^{\Delta+2}\mathcal{O}(p)\{4w^2p^2k'' + 4(\Delta-1)k' + k\}, \end{aligned} \tag{3.19}$$

where primes indicate differentiation of k with respect to its argument. The required condition for agreement betweeen AdS and CFT representations of special conformal transformations is therefore

$$4w^2p^2k'' + 4(\Delta-1)k' + k = 0. \tag{3.20}$$

This is precisely equivalent to the AdS Klein-Gordon equation,

$$-\Box_5\phi = \Delta(\Delta-4)\phi, \tag{3.21}$$

and Eq. (3.17) is equivalent to the AdS boundary condition of Eq. (3.9).

Eq. (3.20) is straightforwardly massaged into a Bessel equation, with boundary condition Eq. (3.17). The solution is given by

$$k(p^2w^2) = (p^2w^2)^{1-\Delta/2}J_{\Delta-2}(\sqrt{p^2w^2}). \tag{3.22}$$

Eq. (3.15) then reads

$$\phi(p,w) = w^2(p^2)^{1-\Delta/2}J_{\Delta-2}(\sqrt{p^2w^2})\mathcal{O}(p). \tag{3.23}$$

3.5 Obstruction to AdS/CFT at level of operators

Although we have realized conformal symmetry in geometric terms, Eq. (3.23) is problematic as an operator equation. In general we should be able to probe this equation for arbitrary p_μ, timelike or spacelike. Both will appear when we Fourier transform back to define $\phi(x,w)$. In such a Fourier integral over p, there is no problem for large timelike p, where $J \sim \cos(\sqrt{p^2w^2} - \text{constant})/(p^2w^2)^{1/4}$, but for large spacelike p the oscillatory behavior continues to an exponential growth, $J \sim e^{\sqrt{-p^2w^2}}/(-p^2w^2)^{1/4}$. Because of this the Fourier transform is ill-defined. If we were to simply neglect the spacelike Fourier components, we would not faithfully translate the local operator $\mathcal{O}$ into AdS. See also Ref. [27].

3.6 Construction of AdS/CFT at level of states

Fortunately, we will not need a full AdS scalar field *operator* in general. Essentially, we will be able to proceed with a *state* in the CFT which transforms as an AdS scalar field, which follows by acting with the above construction on the vacuum state:

$$|p, w\rangle \equiv k(p^2w^2)w^{\Delta}\mathcal{O}(p)|0\rangle = w^2(p^2)^{1-\Delta/2}J_{\Delta-2}(\sqrt{p^2w^2})\mathcal{O}(p)|0\rangle. \tag{3.24}$$

The reason this is safe is that $\mathcal{O}$ acting on the vacuum can only create physical states, which have timelike 4-momenta with positive energy. Therefore the right-hand side automatically vanishes for spacelike p, and we can Fourier transform to position space without difficulty:

$$\begin{aligned}
|x, w\rangle_n &\equiv \int d^4x' w^{\Delta_n} \int \frac{d^4p}{(2\pi)^4} e^{ip.(x-x')} k_n(p^2w^2)\mathcal{O}_n(x')|0\rangle \\
&= \int d^4x' w^{\Delta_n} \int dm^2 \int \frac{d^3\vec{p}}{(2\pi)^4 2\sqrt{\vec{p}^2+m^2}} e^{ip.(x-x')} k_n(m^2w^2)\mathcal{O}_n(x')|0\rangle \\
&= \int d^4x' w^2 \int dm^2 \int \frac{d^3\vec{p}}{(2\pi)^4 2\sqrt{\vec{p}^2+m^2}} m^{2-\Delta_n} e^{ip.(x-x')} J_{\Delta_n-2}(mw)\mathcal{O}_n(x')|0\rangle.
\end{aligned} \tag{3.25}$$

We have used the standard identity between integration measures,

$$\int \frac{d^4p}{(2\pi)^4}\theta(p^2)\theta(p_0)... = \int dm^2 \frac{d^3\vec{p}}{(2\pi)^4 2\sqrt{\vec{p}^2+m^2}}..., \tag{3.26}$$

where

$$p_0 \equiv \sqrt{\vec{p}^2+m^2}, \tag{3.27}$$

taking advantage of the positivity of p_0, p^2 for physical states. The large-m rapidly oscillating asymptotics $J(mw) \sim \cos(mw + \text{constant})/\sqrt{mw}$ ensures convergence of the m^2 integral. (For example, the Fourier transform of $f(x) = |x|^{\alpha}$ is finite, and shares the same asymptotics as the m-integral.)

By construction, the CFT states, $|x, w\rangle_n$, transform simply under conformal symmetry by having (x, w) transform as points in AdS_5, and are related by the AdS Klein-Gordon equation,

$$(\Box_5 + \Delta_n(\Delta_n - 4))|x, w\rangle_n = 0. \tag{3.28}$$

3.7 Interpreting AdS/CFT degrees of freedom

The introduction of the “fifth dimension”, w, cannot be just an algebraic trick; it represents a degree of freedom, and we should understand in what sense. We will settle for an intuitive but non-rigorous accounting. It will provide useful perspective but not be an essential part of the technical derivation. See Ref. [29] for a different, more precise, counting of AdS/CFT states.

We will see that there are really three equivalent desciptions that we are juggling. The first is simply given by CFT states in the Hilbert space (independent of time). The second is given by the $|t = 0, \vec{x}, w\rangle_n$ states in AdS, in which "w" tracks the size of CFT states, but in a way that simplifies the action of special conformal transformations. The third description, used en route from the first to the second, is given by using *time*, or more precisely *age*, as a way of keeping track of the size of CFT states. A good analogy for this last description is given by the way we describe a child. We can always say, "my daughter is three feet tall". That is a very direct statement of the child's state. But we frequently use a different description: "my daughter is a three-year-old". Here, we have used the time it takes to grow a child three feet tall to describe the child *right now*.

Let us start with Eq. (3.25), which is a complete set of superpositions of CFT states of the form,

$$\mathcal{O}(x)|0\rangle \equiv e^{iH_{CFT}t}\mathcal{O}(\vec{x}, 0)|0\rangle, \tag{3.29}$$

in a way that geometrizes conformal symmetry considerations. Note that even $|0, \vec{x}, w\rangle_n$ are superpositions of CFT states of the above form at different times, t. At $t = 0$, the usual Heisenberg operators are just Schrodinger operators. The Schrodinger operator acting on the vacuum, $\mathcal{O}(\vec{x}, 0)|0\rangle$, is just a point-like disturbance of the vacuum at the point $\vec{x}$. Time evolution, given by $e^{iH_{CFT}t}$, results in the spread of the disturbance to a finite size, maximally of radius t, given causality. (Whether t is positive or negative is immaterial). In other words, an experiment localized outside the ball of radius t about $\vec{x}$ will be unable to distinguish such a state from the pure CFT vacuum $|0\rangle$. Let us call such Schrodinger states which are indistinguishable from the vacuum outside some finite ball,"finite-radius states". We see that any local Heisenberg operator acting on the vacuum is necessarily of this type.

While we know that time evolution will in general cause a point-like disturbance to grow to finite size, this does not by itself tell us the precise nature of that growth. But scale symmetry gives more information in this case. By spatial translation invariance we might as well focus on a local disturbance originating at $\vec{x} = \vec{0}$. Let us apply a finite dilatation, by a factor $\lambda > 0$, to the time-evolved disturbance,

$$\lambda^{iS}e^{iH_{CFT}t}\mathcal{O}(\vec{0}, 0)|0\rangle = \lambda^{\Delta}e^{iH_{CFT}\lambda t}\mathcal{O}(\vec{0}, 0)|0\rangle, \tag{3.30}$$

following from $\mathcal{O}$ being a (primary) scaling operator of scale dimension Δ. In other words, time evolution of this local disturbance is essentially rescaling of the disturbance. This is just the moral of the three-year-old. Note that it is only this simple for states created by a scaling operator. In a general superposition of such states, following from a general local operator, different factors of λ^{Δ} change the superposition upon rescaling.

In this way, we see that if we take a snap-shot of the set of states that can be created by the Heisenberg operator $\mathcal{O}$ on the vacuum, they are finite-radius states with a spatial center $\vec{x}$, some particular size, and a "shape" consisting of all other scale invariant properties of their Schrodinger wavefunctional. Different primary operators acting on the vacuum will correspond to different "shapes", but all will have a center $\vec{x}$ and an overall size. As a matter of counting, it is the size degree of freedom of these states that is encoded in the fifth dimension w, in the subtle manner of Eq. (3.25). Objects which are really smaller in the CFT appear to be "further back" in the fifth dimension (at smaller w).

As time proceeds, finite-radius states evolve among themselves. As we saw, in the language of $\mathcal{O}|0\rangle$ they simply grow, while in the language of $|x, w\rangle$ they evolve according to the AdS Klein-Gordon equation with $m_5^2 = \Delta(\Delta - 4)$. It is in this sense, that the apparent "extra" degree of freedom of the fifth dimension is compensated by the fact that the $|x, w\rangle$ are constrained by the Klein-Gordon equation, while there is no such contraint in the direct CFT language of $\mathcal{O}(x)|0\rangle$. This reflects the so-called on-shell/off-shell aspect of the AdS/CFT correspondence.

Finite-radius states clearly span an interesting subspace of field theory Hilbert space, which is closed under time evolution. We have seen that states created on the vacuum by a *single* local operator are among the finite-radius states. But, at first, it might appear that finite-radius states contain other possibilities. For example, $\mathcal{O}(x)\mathcal{O}(x')|0\rangle$ is also clearly a finite radius state. However in a scale-symmetric theory, this state, and all finite-radius states, can indeed be expressed as a single local operator acting on the vacuum. This is because given a finite-radius state we can act on it with a dilatation so as to "shrink" it to infinitesimal size. The shrunk state is now an infinitesimal disturbance of the vacuum. In other words, it is the result of some local operator acting on the vacuum, an operator which can then be expanded as a linear combination of primary scaling operators. For example,

$$\lambda^{iS}\mathcal{O}(x)\mathcal{O}(x')|0\rangle = \lambda^{2\Delta}\mathcal{O}(\lambda x)\mathcal{O}(\lambda x')|0\rangle, \tag{3.31}$$

and as $\lambda \to 0$ the two operators on the right-hand side approach each other at the origin, and can therefore be replaced by their OPE. The mapping, in this sense, between finite-radius states and local operators is a reflection of the *state-operator map* of CFTs, made precise in the Euclidean field theory formulation.[6]

Eq. (3.25) defines a map for every scalar primary operator $\mathcal{O}_n$ of the CFT to an AdS-valued state $|x, w\rangle_n$, with AdS mass-squared $m_{5,n}^2 = \Delta_n(\Delta_n - 4)$. We will later show that this extends to non-scalar operators as well. In this way, we map all finite-radius states of the CFT to AdS states. More precisely, we have mapped onto states in the *Poincare patch* of AdS. This restriction to just the AdS Poincare patch reflects our restriction in the CFT to just finite-radius states on 4D Minkowski spacetime. When the CFT is formulated on a spatial 3-sphere plus time, one instead obtains an AdS/CFT mapping to the entirety of AdS spacetime [26]. This "complete coverage" reflects the fact that on the finite 3-sphere, *all* CFT states are necessarily "finite radius" states.

We do not repeatedly return to state these qualifications in what follows. In essence, we have mapped CFT states to AdS states in a manner that faithfully realizes the conformal

[6] This is usually derived in two dimensions, but extends straighforwardly to higher dimensions. See Ref. [30] for a review.

symmetries as AdS isometries. The CFT is therefore *some* AdS theory in disguise: CFT Hilbert space carries a representation of the AdS isometries. The question becomes, what AdS theory?

3.8 Free AdS field equations do not imply free AdS dynamics

The fact that the AdS states constructed, Eq. (3.25), satisfy free field equations in AdS, in no way implies that the AdS theory is a theory of free particles. This issue arose in Ref. [26]. For example, in standard QED one may have a state with 4D invariant mass 1 MeV and charge $-e$, consisting of a single electron interacting with a (changing) number of photons. Such a state would satisfy a "free" Klein-Gordon equation with invariant mass 1 MeV, preserved by energy-momentum conservation, but of course the elementary particles within this state are interacting. What is unfamiliar is that the AdS_5 mass-squared spectrum is discrete, not continuous, matching the discrete set of primary operators (and scaling operators) in a CFT. For example, in free field theory in Minkowski spacetime a state consisting of two identical massless particles can realize any positive invariant mass-squared. However, in AdS free field theory such a two-particle (or more generally multi-particle) state can only take discrete values, as explained in the next section. It is in this sense that AdS curvature is sometimes said to effectively act as a "box", even though AdS space is not compact.

Nevertheless, there is a limit in which the CFT really does simplify such that the $|x, w\rangle$ become free particles in AdS. This is the large-N_{color} limit. Before studying that, let us first see what the "target", free (scalar) field theory on AdS_5 looks like.

4 Free AdS Scalar Field Theory

In this section, we review some basics of free quantum field theory in AdS_5, without any reference to a CFT connection.

4.1 Separation of variables

The free scalar action on AdS is given by

$$\begin{aligned} S &= \frac{1}{2}\int d^4x dw \sqrt{G}\{G^{MN}\partial_M\phi\partial_N\phi - m_5^2\phi^2\} \\ &= \frac{1}{2}\int d^4x dw \{\frac{1}{w^3}\eta^{MN}\partial_M\phi\partial_N\phi - \frac{m_5^2}{w^5}\phi^2\}, \end{aligned} \tag{4.1}$$

where $G_{MN}(w)$ is the AdS metric, corresponding to Eq. (3.4). Integrating with respect to w by parts (not worrying about the AdS boundary term, momentarily) and changing field variables to

$$\phi(x, w) \equiv w^{3/2}\hat{\phi}(x, w), \tag{4.2}$$

the action takes the form

$$S = -\frac{1}{2}\int d^4x dw \hat{\phi}\{\Box_4 - \partial_w^2 + \frac{(15/4 + m_5^2)}{w^2}\}\hat{\phi} \tag{4.3}$$

If we can diagonalize the hermitian differential operator,

$$-\partial_w^2 + \frac{(15/4 + m_5^2)}{w^2}, \tag{4.4}$$

we will be able to separate x and w variables, and write the action as a sum of purely 4D free field modes, with 4D mass-squareds given by the eigenvalues of Eq. (4.4). In other words, we will have achieved a "Kaluza-Klein" decomposition of the free 5D field ϕ into many 4D component free fields.

This diagonalization again involves Bessel functions (not coincidentally),

$$\{-\partial_w^2 + \frac{(15/4 + m_5^2)}{w^2}\}\{(mw)^{1/2} J_{\pm\sqrt{4+m_5^2}}(mw)\} = m^2\{(mw)^{1/2} J_{\pm\sqrt{4+m_5^2}}(mw)\}, \quad m > 0, \tag{4.5}$$

as can be straightforwardly checked by massaging this equation into Bessel form.

4.2 Boundary conditions and complete basis of eigenfunctions

We can now be careful about the boundary term in the integration by parts above, by noting the near-boundary behavior of these eigenfunctions,

$$(mw)^{1/2} J_{\pm\sqrt{4+m_5^2}}(mw) \underset{w\to 0}{\longrightarrow} \text{constant } (mw)^{1/2\pm\sqrt{4+m_5^2}}. \tag{4.6}$$

Therefore if we expand $\hat{\phi}$ in terms of a general linear combination of $(mw)^{1/2} J_{+\sqrt{4+m_5^2}}(mw)$ and $(mw)^{1/2} J_{-\sqrt{4+m_5^2}}(mw)$, it is the latter term which would dominate for $w \sim 0$, in which case throwing out the boundary term of the action in the integration by parts we performed above is illegal. But the boundary term vanishes if we choose only the positive root for the eigenfunctions, and as long as the square-root is real,

$$m_5^2 > -4. \tag{4.7}$$

We will proceed by taking these conditions, one a boundary condition and the other a restriction on mass, to hold in constructing AdS field theory. Eq. (4.7) is the Breitenlohner-Freedman bound (if one includes the more delicate possibility of $m_5^2 = -4$, which we avoid in this paper for simplicity) [31].

The restriction to just the positive-root eigenfunctions $(mw)^{1/2} J_{+\sqrt{4+m_5^2}}(mw)$ provides a complete and orthonormal basis for functions on the half-line $w > 0$, captured by the standard Hankel-transform (also known as the Bessel-Fourier transform):

$$\begin{aligned}
\int_0^\infty dw\ (mw)^{1/2} J_{\sqrt{4+m_5^2}}(mw)(m'w)^{1/2} J_{\sqrt{4+m_5^2}}(m'w) &= \delta(m - m') \\
\int_0^\infty dm\ (mw)^{1/2} J_{\sqrt{4+m_5^2}}(mw)(mw')^{1/2} J_{\sqrt{4+m_5^2}}(mw') &= \delta(w - w').
\end{aligned} \tag{4.8}$$

4.3 “Kaluza-Klein” decomposition into 4D modes

We can use this basis to expand $\hat{\phi}$ and hence ϕ,

$$\begin{aligned} \hat{\phi}(x,w) &= \int_0^\infty dm\ \chi_m(x)(mw)^{1/2} J_{\sqrt{4+m_5^2}}(mw) \\ \phi(x,w) &= w^{3/2} \int_0^\infty dm\ \chi_m(x)(mw)^{1/2} J_{\sqrt{4+m_5^2}}(mw), \end{aligned} \tag{4.9}$$

and re-write the action,

$$S = -\frac{1}{2} \int d^4x \int_0^\infty dm\ \chi_m \{\partial_\mu \partial^\mu + m^2\} \chi_m. \tag{4.10}$$

In this form, we see that we have a continuum of component 4D free fields, $\chi_m(x)$, with 4D masses, m.

To quantize ϕ as an AdS scalar free field, we must quantize the $\chi_m(x)$ as free scalar fields in 4D Minkowski spacetime,

$$\chi_m(x) \equiv \int \frac{d^3\vec{p}}{(2\pi)^3 (2\sqrt{\vec{p}^2+m^2})^{1/2}} \{a_m^\dagger(\vec{p}) e^{i\sqrt{\vec{p}^2+m^2}t - i\vec{p}.\vec{x}} + a_m(\vec{p}) e^{-i\sqrt{\vec{p}^2+m^2}t + i\vec{p}.\vec{x}}\}, \tag{4.11}$$

but with the *continuum* normalization,

$$\begin{aligned} [a_m(\vec{p}), a_{m'}^\dagger(\vec{q})] &= \delta(m-m')(2\pi)^3 \delta^3(\vec{p}-\vec{q}) \\ [a_m^\dagger(\vec{p}), a_{m'}^\dagger(\vec{q})] &= 0 \\ [a_m(\vec{p}), a_{m'}(\vec{q})] &= 0. \end{aligned} \tag{4.12}$$

4.4 AdS Feynman propagator

The free-field ϕ propagator is then given by

$$\begin{aligned} &\langle 0|T\phi(x,w)\phi(0,w')|0\rangle \\ &= (ww')^2 \int dm \int dm' (mm')^{1/2} J_{\sqrt{4+m_5^2}}(mw) J_{\sqrt{4+m_5^2}}(m'w') \\ &\quad \times \{\theta(t)\langle 0|\chi_m(x)\chi_{m'}(0)|0\rangle + \theta(-t)\langle 0|\chi_m(0)\chi_{m'}(x)|0\rangle\} \\ &= (ww')^2 \int dm \int dm' (mm')^{1/2} J_{\sqrt{4+m_5^2}}(mw) J_{\sqrt{4+m_5^2}}(m'w')\ \delta(m-m') G_m(x) \\ &= (ww')^2 \int dm\ m\ J_{\sqrt{4+m_5^2}}(mw) J_{\sqrt{4+m_5^2}}(mw') G_m(x), \end{aligned} \tag{4.13}$$

where $G_m(x)$ is the standard Feynman propagator in 4D Minkowski spacetime for a scalar field of mass m,

$$G_m(x) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip.x}. \tag{4.14}$$

Note that the m-integral converges for large m because of the oscillatory Bessel asymptotics,

$$J_{\sqrt{4+m_5^2}}(\xi) \underset{\xi\to\infty}{\longrightarrow} (\frac{2}{\pi\xi})^{1/2}\cos(\xi - \frac{\pi}{2}\sqrt{4+m_5^2} - \pi/4). \quad (4.15)$$

It is straightforward, using Eq. (4.8), to show that the ϕ propagator is the "inverse" of the AdS Klein-Gordon operator in the usual sense:

$$(\Box_{5,(x,w)} + m_5^2)\langle 0|T\phi(x,w)\phi(0,w')|0\rangle \;=\; -i\frac{\delta^4(x)\delta(w-w')}{\sqrt{G}}. \quad (4.16)$$

That is,

$$\langle 0|T\phi(x,w)\phi(x',w')|0\rangle = (\frac{i}{-\Box_5 - m_5^2 + i\epsilon})_{|(x,w),(x',w')}. \quad (4.17)$$

The appearance of "$i\epsilon$" is due to the time-ordering, which we can see as follows. Indeed, an $i\epsilon$ is there in Eq. (4.13), inside the 4D Feynman propagator $G_m(x) \equiv i/(-\partial^2 - m^2 + i\epsilon)$. We see that there is a single combination "$-\partial^2 + i\epsilon$" which appears together there. Since $-\partial^2$ originates from

$$\Box_{5,(x,w)} + m_5^2 = w^2\partial^2 - w^5\partial_w\frac{1}{w^3}\partial_w + m_5^2, \quad (4.18)$$

it follows that the $i\epsilon$ appears only in the combination $-\Box_{5,(x,w)} + i\epsilon$, hence Eq. (4.17). Because $\Box_{AdS_5,(x,w)} + m_5^2 - i\epsilon$ is invariant under AdS isometries, even including the $i\epsilon$, it follows that its inverse, the *time-ordered* ϕ propagator, is also invariant. That is,

$$\langle 0|T\phi(x+\delta x, w+\delta w)\phi(x'+\delta x', w'+\delta w')|0\rangle = \langle 0|T\phi(x,w)\phi(x',w')|0\rangle, \quad (4.19)$$

where $(\delta x, \delta w)$ correspond to any of the infinitesimal isometry transformations of Eqs. (3.3, 3.6).[7]

4.5 Discreteness of multi-particle mass spectrum

Let us turn to the properties of a single free AdS particle state under dilatations, realized in AdS as $x \to \lambda x$ *and* $w \to \lambda w$. Such a state is given by the free field operator acting on the vacuum,

$$\phi(x,w)|0\rangle \equiv w^{3/2}\int dm(mw)^{1/2}J_{\sqrt{4+m_5^2}}(mw)\int\frac{d^3\vec{p}}{(2\pi)^3(2\sqrt{\vec{p}^2+m^2})^{1/2}}e^{i\sqrt{\vec{p}^2+m^2}t - i\vec{p}.\vec{x}}a_m^\dagger(\vec{p})|0\rangle$$
$$\to \lambda^2 w^{3/2}\int dm(mw)^{1/2}J_{\sqrt{4+m_5^2}}(\lambda mw)\int\frac{d^3\vec{p}}{(2\pi)^3(2\sqrt{\vec{p}^2+m^2})^{1/2}}e^{i\sqrt{\vec{p}^2+m^2}\lambda t - i\lambda\vec{p}.\vec{x}}a_m^\dagger(\vec{p})|0\rangle. \quad (4.20)$$

[7]This would be totally obvious given that ϕ is an AdS scalar field, except for the time-ordering subtlety. But we have shown this prescription to be completely equivalent to the $i\epsilon$ prescription, which is manifestly AdS-invariant.

Given the (convergent) series expansion of the Bessel function,

$$J_{\sqrt{4+m_5^2}}(\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{\sqrt{4+m_5^2}+2k}}{2^{\sqrt{4+m_5^2}+2n}\Gamma(k+1+\sqrt{4+m_5^2})}, \tag{4.21}$$

and the usual series expansion of the exponential in $e^{i\sqrt{\vec{p}^2+m^2}\lambda t - i\lambda\vec{p}.\vec{x}}$, we see that this state is a superposition of dilatation eigenstates with discrete eigenvalues of the form $\sqrt{4+m_5^2}+2+k$, where k is a non-negative integer.

This is straightforwardly generalized to an AdS two-free-particle state, where we see that the dilatation eigenvalues must be the sum of two possible one-particle dilatation eigenvalues. That is, the dilatation eigenvalues of the two-particle state are also a discrete set, of the form $\sqrt{4+m_5^2}+\sqrt{4+m_5'^2}+4+k$, where k is a non-negative integer. Any AdS state, including this two-particle state, can be decomposed as a superposition of states with definite AdS-invariant mass-squareds, M_5^2. By essentially the above logic, a state (with however many free particles) with AdS-invariant mass-squared M_5^2, must be a superposition of dilatation eigenstates with a discrete set of possible eigenvalues, $\sqrt{4+M_5^2}+2+k$. We thereby conclude that the two-free-particle state must be a superposition of states with AdS-invariant mass-squareds M_5^2, satisfying

$$\sqrt{4+M^2_{5,2-particle}} = \sqrt{4+m_5^2} + \sqrt{4+m_5'^2} + k, \quad k \text{ arbitrary integer.} \tag{4.22}$$

In particular, this discrete set of $M^2_{5,2-particle}$ verifies the claim made in the last section: unlike Minkowski spacetime, in AdS one cannot obtain a continuum of invariant mass-squareds by simply considering multi-particle states. If this were not the case, any AdS/CFT correspondence would be puzzling since the scaling dimensions (dilatation eigenvalues) of local operators in a 4D CFT is discrete, since the local scaling operators form a discrete set.

5 1/N

Let us return to the AdS/CFT plot. Our AdS fields, constructed from composite CFT operators, are not automatically free fields. A free (AdS) field only has two-point connected correlators, whereas our AdS fields are (superpositions of) composite CFT operators, which in general have multi-point connected correlators, *even if the CFT itself is free.* Instead, for our AdS construction to be free requires the limit of a new small parameter. The classic example of such a small parameter is $1/N$ in a large-N_{color} gauge theory structure for the CFT, where all the "elementary" fields of the CFT are in adjoint representation of the gauge group. Then, all local (gauge-invariant) CFT operators can be written as products of the subset of local *single-color-trace* operators, $\mathcal{O}_n(x)$, whose correlators have a simple N-scaling (see Ref. [13] for reviews):

$$\langle 0|T\mathcal{O}_{n_1}(x_1)....\mathcal{O}_{n_k}(x_k)|0\rangle_{\text{connected}} \sim \frac{1}{N^{k-2}}. \tag{5.1}$$

Here, the operators have been suitably normalized with a power of N so that the two-point function is order one. Scale symmetry precludes one-point functions, so $k \geq 2$. In particular, in the $N = \infty$ limit, only two-point connected correlators survive, just as required.

In what follows, it does not matter that the CFT literally has a large-N gauge structure, but rather that the CFT is at least "$1/N$-like", in that there is some small parameter, and a preferred subset of operators in terms of which all local operators are products, which for convenience we will continue to call "$1/N$" and "single-trace" respectively, with scaling given by Eq. (5.1).

5.1 $N = \infty \ \equiv$ (infinitely many) AdS free fields

By the above scaling, once we set $N = \infty$, all correlators factorize into products of just two-point functions of single-trace operators,

$$\langle 0|\mathcal{O}_{n_1}(x_1)\mathcal{O}_{n_2}(x_2)|0\rangle. \tag{5.2}$$

Consequently, conformal invariance implies that single-trace operators transform among themselves. Therefore, single-trace operators can be decomposed into single-trace *primary* operators and their single-trace descendents. We continue by letting $\mathcal{O}_n$ denote just the *single-trace primary* operators. We again restrict to Lorentz-scalar $\mathcal{O}_n$ for now. Conformal invariance can be used to diagonalize their correlators,

$$\langle 0|\mathcal{O}_{n_1}(x_1)\mathcal{O}_{n_2}(x_2)|0\rangle \propto \delta_{n_1 n_2}. \tag{5.3}$$

This follows by noting that

$$\begin{aligned}\langle 0|\mathcal{O}_{n_1}(x_1)K_\mu\mathcal{O}_{n_2}(x_2)|0\rangle &= -i(x_2^2\partial_{2,\mu} - 2x_{2,\mu}x_2.\partial_2 - 2x_{2,\mu}\Delta_{n_2})\langle 0|\mathcal{O}_{n_1}(x_1)\mathcal{O}_{n_2}(x_2)|0\rangle \\ &= i(x_1^2\partial_{1,\mu} - 2x_{1,\mu}x_1.\partial_1 - 2x_{1,\mu}\Delta_{n_1})\langle 0|\mathcal{O}_{n_1}(x_1)\mathcal{O}_{n_2}(x_2)|0\rangle, \end{aligned} \tag{5.4}$$

where we have commuted K_μ forwards in the right-hand side of the first line and backwards on the second line. By translation invariance, $\langle 0|\mathcal{O}_{n_1}(x_1)\mathcal{O}_{n_2}(x_2)|0\rangle$ is a function of $x_1 - x_2$, so that the last equality implies that $(\Delta_{n_1} - \Delta_{n_2})\langle 0|\mathcal{O}_{n_1}(x_1)\mathcal{O}_{n_2}(x_2)|0\rangle = 0$. That is, non-trivial correlators require $\Delta_{n_1} = \Delta_{n_2}$. One can straightforwardly further diagonalize primaries with degenerate Δ_n so that Eq. (5.3) holds.

The spacetime dependence is determined by inserting between the operators a resolution of the identity in terms of a complete set of states, $|\vec{p}, m, \alpha\rangle$, of spatial momentum $\vec{p}$, invariant 4D mass m, and any other label/feature α, as well as Eq. (3.26):

$$\begin{aligned}\langle 0|\mathcal{O}_{n_1}(x_1)\mathcal{O}_{n_2}(x_2)|0\rangle &= \sum_\alpha \int dm^2 \int \frac{d^3\vec{p}}{(2\pi)^4 2\sqrt{\vec{p}^2 + m^2}} \langle 0|\mathcal{O}_{n_1}(x_1)|\vec{p}, m; \alpha\rangle\langle\vec{p}, m; \alpha|\mathcal{O}_{n_2}(x_2)|0\rangle \\ &= \delta_{n_1 n_2}\sum_\alpha \int dm^2 \int \frac{d^3\vec{p}}{(2\pi)^4 2\sqrt{\vec{p}^2 + m^2}} e^{ip.(x_1 - x_2)}|\langle 0|\mathcal{O}_{n_1}(0)|\vec{p}, m; \alpha\rangle|^2 \\ &\propto \delta_{n_1 n_2}\int dm^2 \int \frac{d^3\vec{p}}{(2\pi)^3 2\sqrt{\vec{p}^2 + m^2}} e^{ip.(x_1 - x_2)} m^{2\Delta_{n_1} - 4}. \end{aligned} \tag{5.5}$$

The matrix element in the second line must be a 4D Lorentz invariant since we are only considering Lorentz-scalar $\mathcal{O}_n$ for now, and therefore it is actually independent of $\vec{p}$. After summing over any α, it must scale as $m^{2\Delta_{n_1}-4}$ simply by dimensional analysis, since there is no intrinsic scale in the CFT. The proportionality constant in the last line will define the normalization of the operator, which we leave open for now. We only consider non-coincident points, $x_1 \neq x_2$, so that these expressions are well-defined and convergent by virtue of the rapidly oscillating phase factor for large m or $\vec{p}$.

Because two-point correlators are the only connected correlators to survive at $N = \infty$, we see that local single-trace operators always appear in the combination $\langle 0|\mathcal{O}(x)...$ or $...\mathcal{O}(x)|0\rangle$. This means that at $N = \infty$ we can return to the operator form of AdS/CFT map, Eq. (3.23), and define AdS-scalar field *operators* associated to each scalar primary single-trace operator [26],

$$\phi_n(p, w) = w^2 (p^2)^{1-\Delta_n/2} J_{\Delta_n - 2}(\sqrt{p^2 w^2}) \mathcal{O}_n(p), \tag{5.6}$$

which can then be Fourier transformed to $\phi_n(x, w)$. Recall, that such a construction failed in general because the Fourier integral was ill-defined for spacelike p, due to the exponential growth of the Bessel function in that regime, and that $\mathcal{O}$ in a general theory and correlator has support at both spacelike and timelike momenta. However, at $N = \infty$ the fact that $\mathcal{O}_n(x)$ always appears acting on the vacuum (bra or ket) implies that only timelike momenta can appear, namely the momenta of physical states interpolated by $\mathcal{O}_n$ on the vacuum.

Using the identity of Eq. (3.26) we can explicitly project onto only timelike momenta and positive energy, knowing now that spacelike momenta cannot appear within correlators, and explicitly write the Fourier transform to convert $\phi_n(p, w)$ to $\phi_n(x, w)$:

$$\phi_n(x, w) = \int d^4x' w^2 \int dm^2 \int \frac{d^3\vec{p}}{(2\pi)^4 2\sqrt{\vec{p}^2 + m^2}} m^{2-\Delta_n} e^{ip.(x-x')} J_{\Delta_n - 2}(mw) \mathcal{O}_n(x'). \tag{5.7}$$

By construction $\phi_n(x)$ transforms under conformal symmetry as an AdS-scalar field, and satisfies the AdS Klein-Gordon equation,

$$\begin{aligned} -\Box_5 \phi_n &= m_{5,n}^2 \phi_n \\ m_{5,n}^2 &= \Delta_n(\Delta_n - 4). \end{aligned} \tag{5.8}$$

We immediately see that the only time-ordered connected correlator of such AdS field operators that does not vanish is the two-point correlator, since this is true of $\mathcal{O}_n$, and it is given by

$$\langle 0|T\phi_n(x,w)\phi_{n'}(0,w')|0\rangle$$

$$\begin{aligned}
&= \int d^4x' w^2 \int dm^2 \int \frac{d^3\vec{p}}{(2\pi)^4 2\sqrt{\vec{p}^2+m^2}} m^{2-\Delta_n} e^{ip.(x-x')} J_{\Delta_n-2}(mw) \\
&\quad \times \int d^4x'' w'^2 \int dm'^2 \int \frac{d^3\vec{q}}{(2\pi)^4 2\sqrt{\vec{q}^2+m'^2}} m'^{(2-\Delta_{n'})} e^{-iq.x''} J_{\Delta_{n'}-2}(m'w') \\
&\quad \times \{\theta(t)\langle 0|\mathcal{O}_n(x')\mathcal{O}_{n'}(x'')|0\rangle + \theta(-t)\langle 0|\mathcal{O}_{n'}(x'')\mathcal{O}_n(x')|0\rangle\} \\
&\propto \delta_{nn'}(ww')^2 \int dm^2 \int \frac{d^3\vec{p}}{(2\pi)^3 2\sqrt{\vec{p}^2+m^2}} J_{\Delta_n-2}(mw) J_{\Delta_{n'}-2}(m'w')\{\theta(t)e^{-ip.x} + \theta(-t)e^{ip.x}\} \\
&\propto \delta_{nn'}(ww')^2 \int dm\ m J_{\Delta_n-2}(mw) J_{\Delta_{n'}-2}(mw') G_m(x). \qquad (5.9)
\end{aligned}$$

The second equality follows by plugging in Eq. (5.5) and doing the x' and x'' integrals. We have arrived at the free particle AdS scalar propagator for mass $m^2_{5,n_1} = \Delta_n(\Delta_n - 4)$, up to a normalization constant to be fixed later. Therefore at $N = \infty$, arbitrary ϕ_n correlators satisfy a Wick Theorem where they factorize into products of free AdS propagators. In other words, the CFT at $N = \infty$ defines a *free* AdS field theory, but with a discrete infinity of fields. (Again, we have restricted to Lorentz-scalar fields/operators for now.)

Our job now is to expand away from the $N = \infty$ limit, and understand the general structure of $k-$point correlators at leading non-vanishing order in $1/N$, namely the *planar limit*. We will see that it is precisely given by a set of tree diagrams in AdS with local AdS vertices.

5.2 A confining deformation in the planar limit

We will accomplish this task by connecting it to the more familiar large-N expansion of *confining* theories. Let us imagine that one of the scalar single-trace primary operators $\mathcal{O}$ has dimension $2 < \Delta < 4$, so that it can be used as an IR-relevant deformation of the CFT:

$$\mathcal{L}_{CFT} \rightarrow \mathcal{L}_{CFT} + \sigma^{4-\Delta}\mathcal{O}. \qquad (5.10)$$

The dimensionful coupling constant of the deformation has been expressed as a power of a mass parameter σ. If other single-trace operators are irrelevant, this deformation does not introduce any new divergence. (This implies that all other local operators are irrelevant because multi-trace operators have scaling dimension equal to the sum of their single-trace factors, up to order $1/N$ corrections. In an expansion in the deformation to k-th order, a new divergence would have to take the form $(\sigma^{4-\Delta})^k \Lambda^d_{\rm cutoff}\mathcal{O}'$, where $d \geq 0$ corresponds to some degree of divergence, $k \geq 1$ integer, and $\mathcal{O}'$ is the form of the local divergence. This is impossible by dimensional analysis.) The deformation represents a *soft* breaking of conformal symmetry. Far above σ the deformed theory behaves like the undeformed CFT. But near σ and below, conformal symmetry is badly broken in the deformed theory. We will assume that this leads to confinement in the IR.[8] See the reviews [9] for examples and earlier discussion of such deformations in the AdS/CFT context. It is also possible that a relevant deformation does not lead to confinement, but instead, for example, to a new CFT. We will not consider such a case here. Therefore the deformed theory is a large-N confining theory for which the standard leading $1/N$ expansion, or planar limit, follows. We can recover the undeformed CFT by taking the limit $\sigma \to 0$. The results derived in the end will not depend on σ, which can therefore be seen as merely a convenient intermediate IR regularization of our thinking.

In position space, we see two qualitatively different regimes. Correlators, $\langle 0|T\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \ldots \mathcal{O}_k(x_k)|0\rangle$, in the deformed theory will very closely approximate those of the undeformed CFT if all length scales are small compared to the confinement scale, $|x_1^\mu|, ..., |x_k^\mu| \ll \sigma^{-1}$. But the confining character of the deformed theory will be apparent once we consider length scales, $|x_1^\mu|, ..., |x_k^\mu| \gg \sigma^{-1}$. For example, the confining theory will have a "glueball" spectrum with characteristic σ-scale splittings. Localized and separated wavepackets in $x_1, ..., x_k$ with sizes of order $1/\sigma$ can be chosen to produce, scatter, and detect specific glueball states. By contrast, with wavepackets restricted to sizes $\ll 1/\sigma$, the operators $\mathcal{O}$ will necessarily have the momenta to produce or aborb many different glueball states. In this sense, the deformed CFT allows us to get "outside" the CFT and to probe it with a finer scalpel, as the correlators can be tuned to put exclusive glueball states nearly on-shell.

5.2.1 Locality of glueball couplings

The general analysis of all such confined glueball scattering processes in the planar limit is well known and independent of UV behavior. See the reviews in Ref. [13]. The general conclusion is that glueball scattering is given by *tree-level* diagrams specified by a (4D) action,

$$S_{\text{glueball}} = \int d^4x \{ -\frac{1}{2} \sum_j \chi_j (\Box_j + m_j^2) \chi_j + \mathcal{L}_{int}(\chi(x), \partial) \}, \tag{5.11}$$

where the χ_j are a discretely *infinite* set of confined glueball fields with 4D masses m_j, and some spins (which we suppress). The glueball interactions, $\mathcal{L}_{int}$, are given by *local* products of k glueball fields and derivatives, with dimensions balanced by (possibly negative) powers of σ, and with dimensionless coefficients of order $1/N^{k-2}$.

We will make the further simplifying technical assumption that the number of derivatives in such vertices is finite, that is that the vertex for a *given set* of incoming glueball fields is *polynomial* in derivatives (4D momentum), rather than an infinite series. This assumption is quite plausuble from the viewpoint of matching the good initial value problem of the CFT gauge

[8]The central feature of confinement being assumed is that the physical Hilbert space (of the deformed CFT) is spanned by multi-"glueball" states, for a discrete (4D) mass spectrum of glueballs, each being a *color-singlet* composite particle made of the gauge theory quarks and gluons.

theory. Naively, this should translate into a glueball action containing only up to first time derivatives of glueballs fields. However, a finite number of higher time-derivative interactions can be tolerated since they can be reduced to the more canonical form by applying the equations of motion (field redefinitions) order by order in $1/N$. Indeed general couplings of higher-spin glueballs require several time derivatives on a single field as part of constructing a Lorentz invariant vertex. For example $\chi^{\mu\nu\rho}\chi_1\partial_\mu\partial_\nu\partial_\rho\chi_2$ couples a spin-3 glueball field χ to spinless fields, χ_1, χ_2. One can remove all higher time derivatives of χ_2 by its leading equation of motion, but not without losing the manifestly Lorentz invariant form. We proceed by assuming any finite number of time derivatives for a fixed set of glueball fields into a vertex. Lorentz invariance equates this to our assumption above, that vertices are polynomial in all components of momenta.

With this understanding, Eq. (5.11) will be our key departure point for showing 5D locality when we take the $\sigma \to 0$ limit to go to the undeformed conformal theory. Given its central importance we clarify and illustrate its meaning below, although we refer to (especially the second of) Refs. [13] for fuller discussion.

The locality of the glueball couplings in the planar limit, namely their analyticity in momentum (which we are further plausibly assuming to be polynomial), is at first sight surprising because it is true at *all* momentum scales, not just those below σ. And yet, for momenta much larger than the confinement scale σ we typically expect to encounter non-analytic form-factors in momenta.[9] The consistency of these two statements is enforced in a very special way in the planar limit. Form-factors in relativistic quantum field theory amplitudes can in general be cut to reveal on-shell intermediate states, in our case states made out of confined glueballs. If such an intermediate state contains more than one glueball, then by color confinement it is a non-minimal color singlet, and one can use this to show that the color flow of the full amplitude is non-planar and subdominant in $1/N$ to planar amplitudes. Therefore at planar level, all form factors are associated with single-glueball cuts, that is they are made from single-glueball propagators (momentum poles).

The subtlety and range of behavior expected from gauge theory form-factors cannot be captured by a *finite* sum over species of glueball propagators, and indeed it is just this fact that is used to deduce that the number of glueball species is infinite. Given the good high energy behavior of gauge theory (in our case a UV CFT), these form-factors from infinite sums can be deduced to cut off the bad high-energy behavior normally expected in amplitudes involving higher-spin particles (among the excited glueballs) or glueballs with derivative couplings (suppressed by σ). Even though the $1/N$ approach is deductive and does not give us the explicit construction of these remarkable properties, the classic existence proof is provided by the

[9]From the spacetime viewpoint, the confined glueballs have sizes set by σ^{-1}, whereas polynomials in derivatives/momenta appear to translate into glueball interactions within just an infinitesimal neighbourhood of a point. Non-analytic form-factors would spread these interactions out over large finite regions of spacetime.

Veneziano amplitude [32] (further discussed in Ref. [33]).

But these form factors are not directly visible in Eq. (5.11), precisely because it is the *one-particle irreducible* effective action of the glueballs in the planar limit, so that all one-particle exchanges have been amputated and only the vertices of such exchanges are retained. When one uses this action to form amplitudes, glueball propagators are used to connect vertices into trees, and in any particular channel (for example, $\chi_1 + \chi_2 \to \chi_3 + \chi_4$) form-factors reappear with infinite-species sums over glueball exchanges.

The remarkable property that all (confining) gauge theory amplitudes can be re-expressed in terms of confined glueballs, even for momenta far above the confinement scale, is often referred to as "quark-hadron duality", and in the planar limit it takes its most striking form. It is ultimately a consequence of unitarity.

5.2.2 Example of quark-hadron duality in planar limit

Let us illustrate some of the above points by deriving them in a very simple example, namely the spectral decomposition of the two-point correlator of a minimal color-singlet local scalar operator of the deformed CFT,

$$\langle 0|T\{\mathcal{O}(x)\mathcal{O}(0)\}|0\rangle = i\int \frac{d^4p}{(2\pi)^4} e^{-ip.x} \sum_j \frac{\langle 0|\mathcal{O}(0)|j\rangle\langle j|\mathcal{O}(0)|0\rangle}{p^2 - m_j^2 + i\epsilon}. \tag{5.12}$$

In general the intermediate states would consist of a complete set of all on-shell physical states, but the planar limit restricts us to just minimal color-singlet states, which by confinement must be single glueballs. The sum is therefore over all single-glueball species.

Now, if $|x^\mu| \ll \sigma^{-1}$ then we are at distances far smaller than the length scale at which the CFT deformation becomes important and confines, $\sim \sigma^{-1}$. At these short distances the correlator on the left-hand side should be that of the undeformed CFT to excellent approximation. And yet, the right-hand side evidently expresses this correlator as a sum of confined glueball poles. If the glueball sum were over a *finite* number of species, then the small x scaling would be dominated by large p^2 and would be given by $\sim 1/x^2$. But this would conflict with the undeformed CFT if $\mathcal{O}$ were a scaling operator with any other scaling dimension than one. Therefore in general, to reproduce fractional powers of x that can appear in correlators, we see that there must be *infinitely* many glueball species.

There is no paradox in the statement that single-glueball exchanges can reproduce correlators at distances much smaller than the confinement scale, where the CFT is approximately undeformed. All it requires is that the single-glueball states form a *complete basis* for expanding minimal-color singlet collections of CFT "gluons", not that the CFT have any "foreknowledge" of the confinement to come at larger distances, σ^{-1}. $x\sigma$ can be taken smaller and smaller without any breakdown in the above relation, resulting in the correlator being given more and more closely by the underformed CFT.

Notice that one can think of the "Feynman vertex" for the (source of the) operator to couple to glueball-j to be given by the (rather trivial) polynomial in momentum, $\langle 0|\mathcal{O}(0)|j\rangle$. And yet the "form-factor" given by the two-point correlator is much subtler, as the result of the infinite sum over single-glueball exchanges with these couplings. In addition to these vertices, one can also have standard "seagull" type vertices, bilinear in the source of the operator, which are polynomial in momentum. These do not appear in the above analysis because they do not contribute for non-zero x (their Fourier transform being a distribution about $x = 0$), but again they illustrate the general feature that vertices are polynomial in momenta.

Finally, let us contrast the locality of the planar limit with the more general "effective locality" at long distances compared to the confinement scale, such as is familiar from the chiral Lagrangian of light pions. Let us Fourier transform the spectral representation above to momentum space and expand for small momentum,

$$\int d^4x \; e^{ip.x} \langle 0|T\{\mathcal{O}(x)\mathcal{O}(0)\}|0\rangle = i \sum_j \frac{\langle 0|\mathcal{O}(0)|j\rangle\langle j|\mathcal{O}(0)|0\rangle}{p^2 - m_j^2 + i\epsilon} + \text{seagulls} - \text{terms}$$
$$\sim \quad -i \sum_j \langle 0|\mathcal{O}(0)|j\rangle\langle j|\mathcal{O}(0)|0\rangle(\frac{1}{m_j^2} + \frac{p^2}{m_j^4} + \frac{p^4}{m_j^6} + \frac{p^6}{m_j^8} + ...) \; + \text{seagull} - \text{terms}. \quad (5.13)$$

We see that for low momentum we actually expand the glueball propagators and thereby end up with a "derivative expansion" or "low momentum expansion" that in principle is an infinite series (non-polynomial), which we can truncate if we work to a fixed precision in p^2/σ^2. But for large p we can and must use the full spectral decomposition and keep the full glueball poles. This illustrates why the locality of the planar limit is more powerful than (and should not be confused with) a long-distance derivative expansion.

5.2.3 Translation to AdS and $\sigma \to 0$

Let us first compare the exact $N = \infty$ limit of the deformed CFT with the undeformed CFT. The confining deformed CFT is now given by free glueballs,

$$S_{glueballs} \underset{N=\infty}{=} -\frac{1}{2} \int d^4x \sum_j \chi_j(\Box_j + m_j^2)\chi_j, \quad (5.14)$$

while we have shown that the undeformed CFT at $N = \infty$ is equivalent to a free AdS theory with action,

$$S_{AdS} \underset{N=\infty}{=} \frac{1}{2} \sum_n \int d^4x dw \sqrt{G}\{G^{MN}\partial_M\phi_n\partial_N\phi_n - m_{5,n}^2\phi_n^2\}$$
$$= \sum_n \int dm \int d^4x \{-\frac{1}{2}\chi_m^{(n)}(\Box_4 + m^2)\chi_m^{(n)}, \quad (5.15)$$

where we recall Eq. (4.10) in the second line. We see that the infinite but discrete set of glueballs, χ_j, with a mass gap set by σ becomes a continuous spectrum of 4D fields, $\chi_m^{(n)}$, as $\sigma \to 0$.

Now let us consider the planar limit. As noted, what is striking in Eq. (5.11) of the deformed theory is that the interactions are precisely local, that is, polynomial in glueball fields and their spacetime derivatives, regardless of the scales being probed. In particular, this locality remains at distances much shorter than $1/\sigma$, or equivalently, for fixed distances with $\sigma \to 0$. This means the undeformed CFT in the planar limit must be equivalent to a tree-level theory defined by the effective action,

$$S \underset{\text{planar limit}}{\equiv} \int d^4x \{-\frac{1}{2}\sum_n \int dm \chi_m^{(n)}(\Box_4 + m^2)\chi_m^{(n)} + \mathcal{L}_{int}(\chi(x), \partial)\}, \tag{5.16}$$

where $\mathcal{L}_{int}(\chi(x), \partial)$ is local in x, in that it is made from polynomials in any of the fields $\chi_m^{(n)}(x)$ and their x-derivatives. This is just the structure of the confined glueball theory when we can no longer resolve the $\mathcal{O}(\sigma)$ glueball splittings. For example a possible interaction term could be

$$\mathcal{L}(\chi(x), \partial) \supset \int dm_1 \int dm_2 \int dm_3 \; g_{n_1 n_2 n_3}(m_1, m_2, m_3) \; \chi_{m_1}^{(n_1)} \partial_\mu \chi_{m_2}^{(n_2)} \partial^\mu \chi_{m_3}^{(n_3)}, \tag{5.17}$$

where $g_{n_1 n_2 n_3}(m_1, m_2, m_3)$ is a coupling function of 4D masses. This generalizes the discrete glueball interactions in a confining theory, such as

$$\mathcal{L}_{glueball}(\chi(x), \partial) \supset \sum_{i,j,k} g_{ijk} \; \chi_i \partial_\mu \chi_j \partial^\mu \chi_k. \tag{5.18}$$

Eq. (5.16) describes the planar limit of the CFT in precisely the same sense that Eq. (5.11) describes the confining deformation.

One subtlety can be seen in an interaction such as Eq. (5.17), where since we are in the undeformed CFT, the coupling function must scale as $1/m^{3/2}$ by dimensional analysis and the absence of intrinsic scales. (Note that the continuum normalization of the χ_m gives them engineering dimension 1/2.) This appears singular for very low χ_m modes, $m \to 0$. But from subsection 4.2 and the Bessel asymptotics we see that these $m \to 0$ modes are supported in AdS at $w \to \infty$. Finite-w wavepackets in AdS will not encounter such singularities. But there is a residue of this "bad" behavior as $w \to \infty$, which we defer discussing until subsections 6.4 and 6.5.

Perturbative expansions in general, build interaction terms out of the free-field creation and destruction operators. In the confining glueball theory these are contained in the χ_j, while in the undeformed limit these are contained in the χ_m, Eqs. (4.11, 4.12). By the orthonormality relations of Eq. (4.8), one can rewrite the CFT action in the large-N limit in terms of the free field construction of Eq. (4.9), so that Eq. (5.16) takes the form

$$S \;=\; \sum_n \frac{1}{2} \int d^4x dw \sqrt{G} \{G^{MN} \partial_M \phi_n \partial_N \phi_n - m_{5,n}^2 \phi_n^2\} + \int d^4x \mathcal{L}_{int}(\phi, \partial_\mu). \tag{5.19}$$

Again, the locality of interactions in terms of $\chi_m(x)$ implies x-locality of interactions of $\phi(x, w)$, in that $\mathcal{L}_{int}$ consists of polynomials in $\phi(x, \;)$ and their x-derivatives. (There is also the input of 4D Poincare symmetry so that there is no explicit x-dependence in $\mathcal{L}_{int}$.)

5.3 Locality in the fifth dimension

We still have no indication that these interaction terms are local in the fifth dimension, w. For example, a typical interaction,

$$\mathcal{L}_{int}(\phi, \partial_\mu) \supset \int dw_1 \int dw_2 \int dw_3 \; g(w_1, w_2, w_3) \int d^4x \; \phi(x, w_1) \partial_\mu \phi(x, w_2) \partial^\mu \phi(x, w_3), \quad (5.20)$$

might have coupling function g with support where w_1, w_2, w_3 have *finite* separations. Let us show that this is not possible, by using conformal invariance which has been restored at $\sigma = 0$. That is, x-locality along with conformal invariance implies full 5D locality in both x and w.

By its construction, the free field operator $\phi(x, w)$, which then appears in the interaction terms of Eq. (5.19), transforms under conformal symmetry as an AdS scalar field. The simplest and most insightful way to apply the constraint of conformal invariance is to exponentiate infinitesimal special conformal transformations to a *finite* one, given by

$$\begin{aligned} x^\mu &\rightarrow \frac{x^\mu + a^\mu (x^2 - w^2)}{1 + 2a.x + a^2(x^2 - w^2)} \\ w &\rightarrow \frac{w}{1 + 2a.x + a^2(x^2 - w^2)}. \end{aligned} \quad (5.21)$$

One can check straightforwardly that this leaves ds^2_{AdS} invariant. Such *finite* conformal transformations become ill-defined when the above denominator vanishes, so like 4D Minkowski spacetime, the Poincare patch of AdS does not carry a representation of the full conformal *group*. Nevertheless, we can consider the action of such finite transformations for small but finite transformation parameters a^μ, acting on a small but finite patch of x^μ around the origin, such that the denominator is dominated by 1 and does not vanish. (Alternately, but more clumsily, we could proceed with repeated infinitesimal conformal transformations.) Since ϕ is an AdS scalar, conformal invariance implies that the action should be invariant when the 5D spacetime argument of ϕ transforms as above.

Now consider any interaction term in $\mathcal{L}_{int}(\phi, \partial_\mu)$ made out of a product of k fields and possible x-derivatives. By x-locality it must be a superposition of terms of the form

$$\phi(x + \text{infinitesimal}, w_1)\phi(x + \text{infinitesimal}, w_2)...\phi(x + \text{infinitesimal}, w_k), \quad (5.22)$$

where the infinitesimal corrections to x come from possible x-derivatives, whereas we have to consider the possibility that the w arguments have finite separations. Under conformal transformations of the form of Eq. (5.21) the above ϕ product transforms as

$$\begin{aligned}
&\phi(x + \text{infinitesimal}, w_1)\phi(x + \text{infinitesimal}, w_2)...\phi(x + \text{infinitesimal}, w_k) \to \\
&\phi(\frac{x^\mu + a^\mu(x^2 - w_1^2)}{1 + 2a.x + a^2(x^2 - w_1^2)} + \text{infinitesimal}, \frac{w_1}{1 + 2a.x + a^2(x^2 - w_1^2)} + \text{infinitesimal}) \\
&\times \phi(\frac{x^\mu + a^\mu(x^2 - w_2^2)}{1 + 2a.x + a^2(x^2 - w_2^2)} + \text{infinitesimal}, \frac{w_2}{1 + 2a.x + a^2(x^2 - w_2^2)} + \text{infinitesimal})... \\
&\times \phi(\frac{x^\mu + a^\mu(x^2 - w_k^2)}{1 + 2a.x + a^2(x^2 - w_k^2)} + \text{infinitesimal}, \frac{w_k}{1 + 2a.x + a^2(x^2 - w_k^2)} + \text{infinitesimal}). \quad (5.23)
\end{aligned}$$

In particular, the transformation has resulted in the x arguments shifting to finite separations, unless all of $w_1, ..., w_k$ are infinitesimally close together. Therefore conformal invariance of the CFT is incompatible with the x-locality of the large-N expansion in Eq. (5.19) *unless* the ϕ interaction terms are also w-local. Note that this derivation has made use of our technical assumption that, for fixed set of fields coming into a vertex, the vertex is a polynomial in x-derivatives. If it were a more general analytic function of x derivatives, say involving the Taylor operator $e^{\xi\partial_x}$, we could not conclude that the vertex only relates points in an infinitesimal x-neighbourhood as we have done.

We conclude that the planar limit of the undeformed CFT is given by tree-level diagrams in AdS obtained from the 5D local effective action,

$$S_{AdS} = \int d^4x dw \sqrt{G} \{\sum_n \frac{1}{2}(G^{MN}\partial_M\phi_n\partial_N\phi_n - m_{5,n}^2\phi_n^2) + \mathcal{L}_{int}(\phi(x,w), \partial_\mu, \partial_w, w)\} \quad (5.24)$$

where $\mathcal{L}_{int}(\phi(x,w), \partial_\mu, \partial_w, w)$ is a polynomial in ϕ fields and their $x-$ and $w-$derivatives, evaluated at the same 5D point (x, w). The polynomial coefficients may be w-dependent, but not x-dependent, by 4D Poincare invariance. But now that we know we have to write a 5D local action for a tree-level theory whose conformal invariance is realized as AdS isometries, and under which ϕ is a scalar field, we have a standard formalism for forming invariants in terms of ϕ fields, the AdS metric $G_{MN}(w)$, and covariant derivatives. $\mathcal{L}_{int}(\phi(x,w), \partial_\mu, \partial_w, G_{MN}(w))$ must be a local AdS invariant density formed in this way.

That is, the CFT in the planar limit has mapped to a tree-level expansion of a local AdS field theory, but with a discrete infinity of fields, corresponding to the discrete infinity of single-trace primary operators at $N = \infty$. Notice that we have *not* given the explicit construction of AdS field operators, appearing in S_{AdS} above, in terms of CFT operators, although such a mapping must exist. Rather, we have *deduced* the existence of the local AdS theory, using a variety of observations. This is very much in keeping with the large-N expansion approach of confining theories: one deduces that there *exists* a local glueball theory dual to the the large-N gauge theory, but it is far more difficult in general to explicitly construct the various glueball field operators in terms of gauge theory operators. Fortunately, we will not strictly need such an explicit construction; the locality, symmetries and a large AdS mass-gap (CFT conformal dimension gap) are sufficient to make S_{AdS} predictive.

More constructive approaches relating AdS fields to CFT (local and non-local) operators are discussed in Refs. [26] [34] [35] [27] [36] [37] [38], and provide additional insight into the nature of AdS/CFT.

5.4 AdS effective field theory

Just knowing we have a tree-level AdS theory is not very predictive when there are an infinite number of fields. But let us suppose that there is a large gap, $\Delta_{gap} \gg 1$, in the spectrum of scaling dimensions of single-trace primaries at $N = \infty$, such that a finite number of scaling dimensions are order one, while the rest are $\geq \Delta_{gap}$. (See also the earlier formulation of Ref. [28].) This translates into AdS as the statement that a finite number of fields have $m_5^2 \sim \mathcal{O}(1)$, while the rest have $m_5^2 \geq \Delta_{gap}(\Delta_{gap} - 4) \sim \Delta_{gap}^2 \gg 1$. In that case, we can imagine integrating out the high-mass AdS states to yield the local AdS effective field theory describing the finite number of low-mass particles. There is then a large hierarchy between the curvature of AdS spacetime and the cutoff of this effective field theory. In this regime the physics is approximately that of 5D Minkowksi spacetime. We arrive at this effective theory as follows.

Because we have introduced a second formal large parameter beyond N, namely Δ_{gap}, we must make precise the nature of the large Δ_{gap} asymptotics. We will assume that in the large-N planar limit, that up to the standard factors of N, CFT single-trace correlators are order one in large Δ_{gap}, that is they do not grow with Δ_{gap}. In the AdS description, this translates to tree amplitudes which do not grow with large Δ_{gap}.

To see the central issues in integrating out heavy 5D fields in arriving at 5D effective field theory, imagine for simplicity that just one AdS scalar field ϕ is light $m_{\phi,5}^2 R_{AdS}^2 \sim \mathcal{O}(1)$, while all other AdS fields, collectively denoted ψ, are heavy, $m_{\psi,5}^2 R_{AdS}^2 > \Delta_{gap}^2$. (That is, there is a single low-dimension single-trace primary scalar operator in the CFT.) Also for simplicity we imagine that ϕ is odd under a $\mathbf{Z}_2$ discrete symmetry. Then the full 5D action related to ϕ has the schematic form,

$$\begin{aligned} S &= \int d^4x dw \sqrt{G} \{ \frac{1}{2} G^{MN} \partial_M \phi \partial_N \phi - \frac{1}{2} m_5^2 \phi^2 \\ &\quad - \frac{\Delta_{gap}^{1/2} . p(R_{AdS} D/\Delta_{gap})}{N R_{AdS}^{1/2}} \phi \phi \psi \ - \ \frac{R_{AdS} . q(R_{AdS} D/\Delta_{gap})}{N^2 \Delta_{gap}} \phi^4 + ... \ \}, \end{aligned} \tag{5.25}$$

where p, q are polynomials (polynomial by our earlier assumption of subsection 5.2.1) in AdS-covariant derivatives, D, acting on any of the ϕ appearing in the vertices. The ψ's in the vertices are those which are even under the discrete symmetry.

The polynomial coefficients of $(R_{AdS} D/\Delta_{gap})^k$ in p and q are of order one for large Δ_{gap} in order to satisfy our assumption above that amplitudes do not grow with Δ_{gap}. For example, ψ decay into ϕ pairs has a rate given crudely by

$$\frac{\Gamma}{m_\psi} \sim \frac{1}{N^2} p^2 (\sim \mathcal{O}(1)), \tag{5.26}$$

where the 5D momentum-transfers represented by the derivatives in p are set by the decaying ψ mass $\sim \Delta_{gap}/R_{AdS}$. This and similar processes involving ψ imply that p is order one for order one arguments. For $\phi - \phi$ scattering for Δ_{gap}/R_{AdS} momentum transfers, but below ψ thresholds, the amplitudes (made dimensionless in units of the momentum transfer) scale crudely as $\sim [\mathcal{O}(1).p^2(\sim \mathcal{O}(1)) + q(\sim \mathcal{O}(1))]/N^2$. The first term comes from ψ exchange with propagator $\sim R_{AdS}^2/\Delta_{gap}^2$. Our assumption that amplitudes do not grow with Δ_{gap} implies that q is order one for order one arguments. Given p, q are polynomials which are order one for order one arguments, the coefficients of each of their monomials must be order one, as stated above. Notice that this conclusion follows from the polynomial nature of p, q and would not be true for a more general analytic function. For example, the function $f(x) \equiv 1 + e^{-\xi x^2}$ is order one for order one real x, even for very large parameter $\xi > 0$. Different powers of x in its power series expansion have coefficients which are not order one, but are given by powers of large ξ. See Ref. [39] for an alternative discussion of locality and AdS effective field theory in which the need for polynomial behavior is stressed. (The author is grateful to the authors of Ref. [39] for pointing out the issue to him.)

We now integrate out the heavy ψ to get the effective theory of just ϕ for low momentum transfers, of the form

$$\begin{aligned} S &= \int d^4x dw \sqrt{G} \{ \frac{1}{2} G^{MN} \partial_M \phi \partial_N \phi - \frac{1}{2} m_5^2 \phi^2 \\ &- \frac{R_{AdS}}{\Delta_{gap} N^2} (p(R_{AdS} D/\Delta_{gap})\phi\phi) \sum_k \frac{(R_{AdS} D)^{2k}}{\Delta_{gap}^{2k}} (p(R_{AdS} D/\Delta_{gap})\phi\phi) \\ &- \frac{R_{AdS}.q(R_{AdS} D/\Delta_{gap})}{N^2 \Delta_{gap}} \phi^4 + ... \}, \end{aligned} \tag{5.27}$$

where the new derivative terms come from tree-level exchange of ψ and expanding its propagator about its mass. Because of this, effective vertices are not polynomial in D, but rather an infinite series. We see that the typical quartic interaction term in the derivative expansion is of the form $\sim R_{AdS}^{k+1} D^k \phi^4/(\Delta_{gap}^{k+1} N^2)$, with order one coefficients. That is, the dominant low-energy behavior (below Δ_{gap}/R_{AdS}) is given by retaining just the fewest derivatives in the interaction,

$$S = \int d^4x dw \sqrt{G} \{ \frac{1}{2} G^{MN} \partial_M \phi \partial_N \phi - \frac{1}{2} m_5^2 \phi^2 - \frac{1}{M} \phi^4 \}, \quad M \sim \mathcal{O}(N^2 \Delta_{gap}/R_{AdS}). \tag{5.28}$$

This is now a predictive effective theory.

It is important to note that standard effective field theory expectations can break down in the presence of large redshifts, such as exist in AdS. It is only those AdS states which are localized in spacetime well within a single AdS radius of curvature and also are low-enough energy states to not excite the heavy particles, $m_\psi > \Delta_{gap}/R_{AdS}$, that have an approximate 5D Minkowski spacetime effective theory regime. The dual CFT description of such AdS states localized inside a radius of curvature is not easy to *explicitly* construct within the CFT description, even given a suitable CFT with large Δ_{gap}. But it must exist, given its existence in the AdS description. In the next section, we will see that the subtlety of large AdS redshifts does complicate the derivation of CFT correlators of local operators, the more standard probe of CFT physics.

The classic example of $\mathcal{N} = 4$ supersymmetric Yang-Mills CFT does not quite satisfy the minimal version of the large-gap criterion described above. Instead there are an infinite number of single-trace operators with protected scaling dimensions which start at order one and grow without a large gap, corresponding to an infinite number of AdS particles with masses starting at order one and growing without a large gap. Fortunately, this spectrum is consistent with identifying these infinite towers of AdS_5 particles as (parts of) a *finite* number of Kaluza-Klein towers in the decomposition of a finite number of ten-dimensional massless particles in $AdS_5 \times S^5$ down to AdS_5. It is *conjectured* that all the unprotected single-trace scaling dimensions, not dual to these ten-dimensional massless fields, are very large. Then, there is in fact an effective field theory description: not an AdS_5 effective theory, but rather an $AdS_5 \times S^5$ 10D effective theory of massless fields (ten-dimensional IIB supergravity).

5.5 Is $N \gg 1$ necessary ?

The $1/N$ expansion has been useful in getting to the result that the CFT has a regime described by Eq. (5.28). But note that this describes 5D Minkowski local effective field theory over the large N*-independent* range of energies/momenta between $1/R_{AdS}$ and Δ_{gap}/R_{AdS}, the upper cutoff set by the heavy ψ which have been integrated out. N*-dependence* appears in the effective theory's non-renormalizable coupling $1/M \sim \mathcal{O}(R_{AdS}/(\Delta_{gap}N^2))$, ensuring the maximal dimensionless coupling strength at the UV cutoff of $\sim 1/N^2$. Therefore, if we imagine reducing N to be order one, we run into strong coupling in the effective theory right at the cutoff, but we appear to have a weakly coupled effective field theory at lower energies. This is very much the way the non-renormalizable effective chiral lagrangian theory of real world pions behaves. It is therefore possible that large N is merely a useful theoretical scaffolding to get us going, but not strictly necessary for a weakly coupled AdS dual description. Similarly, the deformation of the CFT we used to match onto the standard results of confining $1/N$ theories also does not seem to be necessary in the final result. The effective theory makes sense even if no such *relevant* deformation exists.

The only crucial ingredient is the appearance of the large scaling dimension gap, Δ_{gap}, in the CFT, which corresponds to the AdS description having a finite number of light particles and then a large 5D (or even higher-dimensional as discussed at the end of the last subsection) mass gap (compared to $1/R_{AdS}$). This requires strong coupling on the CFT side. (It is straightforward to check that a weakly coupled theory has no sizeable gap, even for $N \gg 1$.) In the best understood case of $\mathcal{N} = 4$ SUSY Yang-Mills however, such a "super-strong" coupling *requires* large-N because of S-duality, but perhaps there exists a CFT with small N and yet strong enough coupling to lead to Eq. (5.28).

6 CFT Correlators of Local Operators

Here, we will add source terms for local operators of the CFT, to thereby define the generating functional of their correlators. We only consider such correlators of operators at *non-coincident* spacetime points. We derive the map of such CFT correlators to Witten diagrams in AdS_5, with restrictions which we explain. We begin with the full set of AdS fields and tree-level interactions which are dual to the CFT planar limit. The validity and use of AdS effective field theory in conjunction with Witten diagrams will be discussed in subsections 6.4 and 6.5.

6.1 $N = \infty$

In the exact $N = \infty$ limit, we add sources for single-trace primary operators $\mathcal{O}_n$,

$$S_{CFT} \to S_{CFT} + \int d^4x j_n(x) \mathcal{O}_n(x). \tag{6.1}$$

We can realize the source terms in terms of the free AdS fields of Eq. (5.6),

$$S_{CFT} \to S_{CFT} + \int d^4x j_n(x) \lim_{w \to 0} \frac{\phi_n(x,w)}{w^{\Delta_n}}, \tag{6.2}$$

where we have used the near-boundary behavior, Eq. (3.9), or equivalently the small-argument Bessel asymptotics of Eq. (5.6). We have been sloppy about an overall (n-dependent) constant in this source term matching because it can simply be absorbed into the normalization of the CFT operators $\mathcal{O}_n$. From here on, it is convenient to simply take Eq. (6.2), for canonically normalized ϕ_n, as defining our CFT operator normalization. Then, applying the central $N = \infty$ result, Eq. (5.15),

$$S_{CFT} \to -\frac{1}{2} \int d^4x \int dw \sqrt{G} \phi_n (\Box_5 + m_{5,n}^2) \phi_n + \int d^4x j_n(x) \lim_{w \to 0} \frac{\phi_n(x,w)}{w^{\Delta_n}}. \tag{6.3}$$

Of course, this is not to be interpreted as saying the CFT and AdS *actions* are the same, but rather that Eqs. (6.1, 6.3) yield the same correlation functions sourced by j_n.

Since our 5D spectrum is assumed to lie above the Breitenlohner-Freedman bound, $m_5^2 > -4$, there is no subtlety at the AdS boundary $w = 0$ regarding integrating by parts. It is therefore straightforward to integrate out ϕ_n to arrive at the generating functional of connected correlators of single-trace primaries at $N = \infty$,

$$W[j]_{N=\infty} = \frac{i}{2} \int d^4x \int d^4x' \sum_n \lim_{w,w' \to 0} \frac{1}{w^{\Delta_n} w'^{\Delta_n}} j_n(x) \langle 0 | T \phi_n(x,w) \phi_n(x',w') | 0 \rangle j_n(x'). \tag{6.4}$$

(Recall, the propagator inverts the Klein-Gordon operator according to Eq. (4.17)). It is useful to rederive this in the 4D decomposition into $\chi_m(x)$ fields, using Eq.(4.9), Eq. (4.6) and Eq. (5.15),

$$S_{CFT} \to \sum_n \int d^4x \int dm \{ -\frac{1}{2} \chi_m^{(n)} (\Box_4 + m^2) \chi_m^{(n)} + c_n j_n(x) m^{\Delta_n - 3/2} \chi_m^{(n)}(x) \}, \tag{6.5}$$

where c_n is a constant. We can integrate out the free 4D χ_m fields in standard fashion to get the generator of connected correlators,

$$W[j] = \frac{i}{2} \sum_n c_n^2 \int d^4x \int d^4x' \int dm \; m^{2\Delta_n - 3} j_n(x) G_m(x - x') j_n(x'). \tag{6.6}$$

This is precisely equivalent to Eq. (6.4), again by the Bessel asymptotics, Eq. (4.6). This version makes clear the relation to the confining deformation, where we must get a discrete sum over single-glueball 4D propagators. The sum becomes an integral over 4D states interpolated by local operators when the deformation is removed, with a spectral density imposed by conformal symmetry.

Notice that as long as we only use $W[j]$ for correlators at *non-coincident points* in Eq. (6.6) (or equivalently Eq. (6.4)) the oscillatory behavior in $G_m(x - x')$ ensures the convergence of the m integral.

6.2 "Witten Diagrams" for $N \gg 1$

Now let us consider finite but large N. In general, the single-trace primary operators of $N = \infty$ need no longer be primary for finite N. Instead, such operators receive $1/N$ corrections in order to remain primary. We will therefore refer to a new set of primary operators at finite N, related to the single-trace primaries of $N = \infty$, by

$$\mathcal{O}_n^{(N<\infty)} = \mathcal{O}_n^{(N=\infty)} + \text{order } 1/N, \tag{6.7}$$

where the $1/N$ corrections can include multi-trace operators. It is important to note, however, that

$$\Delta_n^{(N<\infty)} = \Delta_n^{(N=\infty)}, \tag{6.8}$$

in the planar limit. This is clear from the equivalence of the planar CFT to a tree-level 5D AdS theory, where the Δ_n correspond to 5D particle masses. $1/N$ corrections to the $\Delta_n^{(N=\infty)}$ then correspond to self-energy corrections in AdS. Self-energy corrections are necessarily loop effects in AdS, and therefore outside the planar limit. We conclude that the corrected primary operators, $\mathcal{O}_n^{(N<\infty)}$, have uncorrected scaling dimensions.

For now, let us simply assume that the translation of source terms, of the form $j_n(x)\mathcal{O}_n^{(N<\infty)}(x)$, into the AdS description remains as in Eq. (6.2). That is, the only $1/N$ corrections are in replacing $\mathcal{O}^{(N=\infty)}$ with $\mathcal{O}^{(N<\infty)}$ as discussed above. With this assumption, and the central result that the planar approximation to the CFT is given by a tree-level expansion of a 5D AdS theory, with a local invariant 5D action of AdS fields $\phi_n(x, w)$, we arrive at

$$S_{CFT} + \int d^4x j_n(x)\mathcal{O}_n^{(N<\infty)}(x) \;\; \rightarrow \;\; \int d^4x \int dw \sqrt{G} \{ -\frac{1}{2}\phi_n(\Box_5 + m_{5,n}^2)\phi_n + \mathcal{L}_{int}(\phi, \partial_{\mu,w}, G_{MN}) + \int d^4x \; j_n(x) \; \underset{w\to 0}{\text{limit}} \frac{\phi_n(x, w)}{w^{\Delta_n}} \}. \tag{6.9}$$

Again, this is not to be interpreted as equality of CFT and AdS actions, but rather as saying both sides define the same generating functional in j_n, with the planar expansion of the CFT and tree expansion of the AdS side. For now, we will take this as a plausible guess, and proceed to formally evaluate the associated AdS tree expansion. In subsection 6.6 we show that the results can be ill-defined for larger Δ_n because of the high degree of concentration of the sources to the boundary, and we will have to restrict which n get sources. In subsection 6.7 we will understand this breakdown more physically. Taking this into account, we will finally prove our assumption that the source terms in Eq. (6.9) indeed match up between CFT and AdS. We defer the subtleties so that we can more rapidly converge on the "big picture" and discussions in the literature.

Eq. (6.9) leads straightforwardly to the Witten-diagram expansion [7],

$$W[j] = W_{N=\infty}[j] + \text{connected tree diagrams}, \tag{6.10}$$

with interaction vertices taken from $\sqrt{G}\mathcal{L}_{int}$ and with AdS propagators, $\langle 0|T\phi(x,w)\phi(x',w')|0\rangle$, on internal lines. External lines connect to sources as usual, but we must take the same boundary limit that appears in the source Lagrangian. That is, the external lines are given by

$$\int d^4x' K_n(x-x',w) j_n(x'), \tag{6.11}$$

where K_n is the "bulk-boundary" propagator,

$$\begin{aligned} K_n(x-x',w) &\equiv \operatorname*{limit}_{w'\to 0} \frac{\langle 0|T\phi_n(x,w)\phi_n(x',w')|0\rangle}{w'^{\Delta_n}} \\ &= \frac{w^2}{2^{\Delta_n-2}\Gamma(\Delta_n-1)} \int dm\; m^{\Delta_n-1} J_{\Delta_n-2}(mw) G_m(x-x'). \end{aligned} \tag{6.12}$$

The second line follows from the leading term of Eq. (4.21).

We can also compactly re-express the two-point correlators, $W_{N=\infty}[j]$ of Eq. (6.4), in terms of the bulk-boundary propagator,

$$W[j]_{N=\infty} = \frac{i}{2}\int d^4x \int d^4x' \operatorname*{limit}_{w\to 0}\frac{1}{w^{\Delta_n}} j_n(x) K_n(x-x',w) j_n(x'). \tag{6.13}$$

6.3 Equivalence to standard formulation of Witten diagrams

Witten diagrams are a standard phrasing of the AdS/CFT correspondence for CFT operator correlators. Here we show that our bulk-boundary propagator is indeed (proportional to) the standard one in the literature, and that all relative factors associated to k-point correlators, for different k, automatically agree with the standard prescription as finally understood in the literature. The overall normalization of any local operator is of course a convention, and we have chosen ours to keep the source term in Eq. (6.9) with unit coefficient.

First, let us study the delicate $w \to 0$ limit of the bulk-boundary propagator, K. As long as $x \neq x'$, we can straightforwardly replace the Bessel function in Eq. (6.12) by its $w \to 0$ asymptotics,

$$K(x-x',w) \underset{w\to 0}{\longrightarrow} \frac{w^{\Delta}}{(2^{\Delta-2}\Gamma(\Delta-1))^2}\int dm\; m^{2\Delta-3}G_m(x-x'), \quad x \neq x' \tag{6.14}$$

because the oscillatory behavior in $G_m(x-x')$ is enough to ensure the convergence of the m-integral. However, as x' approaches x, this oscillatory behavior is lost and we must be more careful since the large argument behavior of the Bessel function is now needed for m-integral convergence. The situation is most straightforwardly understood by first going to 4D momentum space,

$$\begin{aligned}
K(p,w) &= \frac{w^2}{(2^{\Delta-2}\Gamma(\Delta-1))^2}\int dm\; m^{\Delta-1}J_{\Delta-2}(mw)\frac{i}{p^2-m^2+i\epsilon} \\
&\underset{\xi\equiv mw}{=} \frac{w^{4-\Delta}}{(2^{\Delta-2}\Gamma(\Delta-1))^2}\int d\xi\; \xi^{\Delta-1}J_{\Delta-2}(\xi)\; \frac{i}{w^2p^2-\xi^2+i\epsilon} \\
&\underset{w\to 0}{\longrightarrow} -i\frac{w^{4-\Delta}}{(2^{\Delta-2}\Gamma(\Delta-1))^2}\int d\xi\; \xi^{\Delta-3}J_{\Delta-2}(\xi).
\end{aligned} \tag{6.15}$$

The naive $w \to 0$ limit is justified in going to the last line from the second because the oscillatory Bessel asymptotics guarantee ξ-integral convergence for large ξ. The last line is just $w^{4-\Delta}$ multiplied by a constant. Returning to position space, we conclude that

$$K(x-x',w) \underset{w\to 0}{\longrightarrow} \text{constant } w^{4-\Delta}\delta^4(x-x'). \tag{6.16}$$

The final property of the bulk-boundary propagator to note is that it satisfies the AdS Klein-Gordon equation away from the boundary, $w > 0$,

$$\begin{aligned}
[\Box_5 + \Delta(\Delta-4)]K(x-x',w) &\equiv \underset{w'\to 0}{\text{limit}}[\Box_5 + \Delta(\Delta-4)]\frac{\langle 0|T\phi(x,w)\phi(x',w')|0\rangle}{w'^{\Delta}} \\
&= \underset{w'\to 0}{\text{limit}} \; -i\delta^4(x-x')\delta(w-w')\frac{w^5}{w'^{\Delta}} \\
&= 0.
\end{aligned} \tag{6.17}$$

The second line follows from Eq. (4.16). The last line follows for finite w. If we want to probe small w we still take the limit $w' \to 0$ defining K *first*, before letting w approach the boundary.

It is these properties, Eqs.(6.16, 6.17), that are essentially those used to specify K in the literature [7]. There are two differences, however. It is standard to take the constant in Eq. (6.16) to be unity. Since in interacting Witten diagrams there is precisely one factor of K for each local operator $\mathcal{O}$ in the CFT correlator being computed, the choice of unity as the constant in Eq. (6.16) appears to be a normalization convention for local operators. The constant displayed in the last line of Eq. (6.15) has been absorbed into $\mathcal{O}$. There is however an important exception to this rule, namely the non-interacting ($N = \infty$) two-point diagrams of

Eq. (6.13), where a single K connects *two* local CFT operators. Therefore if one absorbs the constant of Eqs. (6.15, 6.16) into the normalization of $\mathcal{O}$, the two-point correlator diagram of Eq. (6.13) must be modified as follows:

$$\operatorname*{limit}_{w\to 0}\frac{K(x-x',w)}{w^{\Delta}} \longrightarrow \frac{1}{\text{constant}}\operatorname*{limit}_{w\to 0}\frac{K(x-x',w)}{w^{\Delta}} \tag{6.18}$$

This modification was missed in the original discussion of Ref. [7], but was caught in Ref. [40]. Here, we have understood it in straightforward terms, but in our convention the constant is retained in Eq. (6.16), and the two-point modification is unnecessary.

The planar limit of the CFT was originally cast as being dual to a *classical* AdS theory, which then has a tree-level perturbative expansion. Here, we have directly derived the tree-level expansion with the modifications of Ref. [40]. To return to a classical AdS prescription, see Ref. [41], which identifies new classical AdS-boundary conditions needed to correctly obtain these modifictions of Ref. [40] (that is, to agree with the derivation of this paper).

The literature often works in Euclidean signature CFT and AdS. The passage to that signature is straightforward in K and the AdS propagator, which are both written as superpositions of 4D propagators, $G_m(x)$. The Euclidean formulas then follow by straightforwardly replacing $G_m(x)$ by its Euclidean equivalent,

$$G_m(x) \to G_m^E(x) = \int \frac{d^4p_E}{(2\pi)^4}\frac{e^{ip.x}}{p_E^2+m^2}, \tag{6.19}$$

and using the Euclidean version of the interaction vertices in standard fashion.

The Euclidean formulation is useful in computing Witten diagrams with AdS effective field theory. Because of the subtlety of large AdS red-shifts this is not entirely straightforward in Lorentzian signature. We now turn to this.

6.4 Obstruction to AdS effective theory for Lorentzian correlators

If very heavy AdS particle lines, $m_5^{heavy} > \Delta_{gap} \gg 1$, were far off-shell in tree-level Witten diagrams for CFT correlators, then we could effectively shrink such lines to points. That is, we could imagine having integrated out heavy AdS particles at tree level, and could simply work with the AdS effective theory with finitely many light fields. This would obviously be of great advantage. But this is *not* the case, no matter how soft the momentum flowing through the CFT operators (source momenta), as we show in this subsection. The root of the problem is that Witten diagrams, with external lines on the boundary and vertices in the bulk, necessarily traverse an infinite number of AdS radii, whereas our effective field theory intuition is based on Minkowski spacetime, valid only well inside a single AdS radius. Also see the discussion of Ref. [42].

Let us take the local CFT operator sources, $j_n(x)$, to be smoothly varying packets, separated to avoid coincident points, with typical momenta, p, in their Fourier transform. Such momenta are injected into external lines of Witten diagrams in AdS. Very naively, if $|p_0| \ll m_5^{heavy}$, we would not have the energy in a diagram to put a heavy AdS particle on-shell. But of course, from the CFT-viewpoint we know there cannot be such an intrinsic energy scale, defining "high" and "low" energy. The AdS/CFT compatibility is enforced by the non-trivial AdS metric. From the AdS side, a heavy particle can be localized inside an AdS radius, say with w: $w_0 < w < w_0 + 1$ for some $w_0 \gg 1$. In this vicinity, the AdS metric is approximated by 5D Minkowski spacetime,

$$ds^2 \approx \frac{\eta_{MN} dX^M dX^N}{w_0^2}, \tag{6.20}$$

where η_{MN} is the standard 5D Minkowski metric, and $X^M \equiv x^\mu, w$. But there is an overall redshift factor of w_0 between the AdS coordinates we are using, which follow naturally from the CFT side, and standard 5D Minkowski spacetime coordinates. Therefore, our CFT-coordinate energy p_0, needed to produce such a heavy state is not the naive $\sim m_5^{heavy}$, but the much smaller $\sim m_5^{heavy}/w_0$. Thus no matter how small the typical p of CFT correlators, Witten diagrams with external legs attached to the boundary, $w = 0$, can have internal lines stretching across to $w \sim w_0$, with redshift ("warp") factor w_0 large enough that $p_0 > m_5^{heavy}/w_0$, so that subsequent heavy lines in the vicinity of w_0 can go on-shell. This means we cannot integrate out heavy AdS particles, no matter how small our p_{CFT}.

A somewhat similar situation occurs in QCD predictions for hadronic processes, especially in large-N QCD. For example, consider a two-point correlator of QCD gauge-invariant local operators, which is already non-trivial without conformal invariance. The spectral decomposition takes the form

$$-i \int d^4x e^{ip.x} \langle 0|T\mathcal{O}(x)\mathcal{O}(0)|0\rangle(p) = \int dm^2 \frac{|\langle m|\mathcal{O}(0)|0\rangle|^2}{p^2 - m^2 + i\epsilon}, \tag{6.21}$$

where the numerator is the non-trivial spectral weight, or probability density for $\mathcal{O}$ to create a hadronic state of invariant-mass m. Even at $N = \infty$ this is a superposition of hadron poles, with non-perturbatively determined masses and residues. Knowing this correlator is equivalent to knowing $|\langle m|\mathcal{O}(0)|0\rangle|^2$, as is clear by taking the imaginary parts of both sides. Naively, far above the confinement scale, $p^2 \gg \Lambda_{QCD}^2$, the correlator should be perturbatively computable in terms of quark-gluon Feynman diagrams, but at large or infinite N this is not true. Perturbation theory is badly behaved due to IR divergences and the correlator is dominated by non-perturbatively determined poles for arbitrarily large timelike p. If we want to know every detail of the location and strength of these poles, perturbative QCD cannot tell us.

But perturbative QCD can reliably predict a suitably "smeared" [43] version of the non-perturbative structure, smoothly aggregating many poles. One of the simplest versions of such a smeared quantity is the correlator for *spacelike* p, or equivalently the Euclidean field theory correlator,

$$\langle \mathcal{O}\mathcal{O}\rangle(p_E) = \int dm^2 \frac{|\langle m|\mathcal{O}(0)|0\rangle|^2}{p_E^2 + m^2}, \tag{6.22}$$

where the matrix element in the numerator is the same as in Minkowski spacetime but the denominator has been continued to Euclidean space. As can be seen this is a smooth p_E^2-dependent integral over the hadronic spectrum. One cannot take an imaginary part to reconstruct exclusive information about an individual pole. Furthermore, quark-gluon perturbation theory is well-behaved in Euclidean space, so a perturbative calculation of this Euclidean correlator is to be trusted for $p_E^2 \gg \Lambda_{QCD}^2$.

6.5 Resolution in Euclidean space

In our AdS/CFT theory, again the general CFT correlators in Minkowski spacetime probe very exclusive information in the sense of being sensitive to the entire AdS spectrum and interactions, as explained above. And again, the cure is to appropriately "smear" the questions we are asking to a more inclusive form, most familiarly by going to Euclidean CFT correlators.

Let us understand how Euclidean CFT correlators escape the fate of their Minkowski counterparts. As discussed earlier, the Witten diagrams are straightforwardly continued to Euclidean signature. Even though we can no longer literally put an intermediate line on-shell in this signature, it is still true that an internal line with large starting and ending values of w can only be integrated out (approximated as a point rather than a line) if $|p_E| < m_5^{heavy}/w$, due to the redshift effect. No matter how large m_5^{heavy} for a heavy particle, and how small p_E, there is a large enough w to prevent us integrating out the heavy particle. This seems to threaten AdS effective field theory in Euclidean CFT correlators, as much as in Minkowski correlators. But any dangerous Witten diagram, with external lines attached to the AdS boundary, $w = 0$, must have at least one propagator traversing from modest w' to large w,

$$w \sim m_5^{heavy}/|p_E|, \tag{6.23}$$

which then in turn connects to the heavy particle line discussed above. Unlike Minkowski signature however, in Euclidean signature this traversing propagator is highly suppressed, regardless of its AdS mass, as we now show.

Since we are necessarily considering non-coincident 5D points, $w \neq w'$, the AdS propagator must obey the free-field Euclidean-AdS Klein-Gordon equation (the inhomogeneous δ-function term vanishing),

$$(w^2 p_E^2 - w^5 \partial_w \frac{1}{w} \partial_w + m_5^2) \langle \phi(p_E, w) \phi(-p_E, w') \rangle = 0, \tag{6.24}$$

where we Fourier-transformed to 4D Euclidean momentum space. For large w we can drop the mass term relative to the p_E term, and find the possible large-w asymptotics,

$$\langle \phi(p_E, w) \phi(-p_E, w') \rangle \propto w^{3/2} e^{\pm |p_E| w}. \tag{6.25}$$

Let us now determine which sign to choose. In more detail, the Euclidean-signature AdS propagator is given by replacing the 4D Minkowski space propagator, $G_m(x)$, by its Euclidean equivalent, in Eq. (4.13) (and Fourier-transforming),

$$\langle\phi(p_E,w)\phi(-p_E,w')\rangle = (ww')^2\int dmm J_{\sqrt{4+m_5^2}}(mw)J_{\sqrt{4+m_5^2}}(mw')\frac{1}{p_E^2+m^2}. \quad (6.26)$$

For large w, the Bessel asymptotics, Eq. (4.15), implies a rapidly oscillating phase and suppression of the m-integral, except at small $m < 1/w$. Since there is no singular behavior at small m in the rest of the integrand, we can minimally conclude that the ϕ propagator at least does not grow exponentially for large w. Together with Eq. (6.25), we can conclude that

$$\langle\phi(p_E,w)\phi(-p_E,w')\rangle \sim w^{3/2}e^{-|p_E|w}, \quad (6.27)$$

for large w. Eq. (6.23) gives us the minimal w needed to get redshifts large enough to stop us being able to integrate out a heavy particle of mass m_5^{heavy}. This corresponds to the propagator traversing from modest w' to this w behaving as

$$\langle\phi(p_E,w)\phi(-p_E,w')\rangle \sim w^{3/2}e^{-m_5^{heavy}}. \quad (6.28)$$

It should be stressed that this traversing propagator may well correspond to a light field, not to m_5^{heavy} itself. (Units are balanced in the exponent by restoring $R_{AdS}\equiv 1$.)

We have introduced a scaling dimension gap parameter, Δ_{gap}, to separate heavy and light AdS particles. Thus the above suppression is $< e^{-\Delta_{gap}}$. Given that AdS effective field theory is essentially an expansion in $1/\Delta_{gap}$ as discussed in subsections 5.4 and 5.5, we see that the naively dangerous diagrams are in fact parametrically smaller than any order in that expansion.

Therefore in Euclidean signature, we can indeed integrate out heavy particles and use AdS effective field theory. The Witten diagrams which make heavy particle exchanges appear non-pointlike require large-w redshifts, but in Euclidean signature propagation out to such large w is suppressed beyond all orders in effective field theory.

6.6 Restricting sources to avoid $w \to 0$ divergences

Our derivation is based on the assumption we made that source terms take the same form as at $N = \infty$, Eq. (6.2), and that we can take the limit $w' \to 0$ straightforwardly on external lines to write them in terms of K. These assumptions are related, and they do not always hold. Let us see why.

The limit $w' \to 0$ we took to get the boundary-bulk propagator in Eq. (6.12) is only straightforward on the external lines of Witten diagrams if the other end of such lines is dominated away from $w = 0$, justifying taking $w' \to 0$ with w fixed, as we implicitly did. To see how this can fail, consider the simplest AdS-invariant coupling,

$$\int d^4x\int dw\sqrt{G}\phi_1\phi_2\phi_3 \;\propto\; \int d^4x\int dw\frac{\phi_1\phi_2\phi_3}{w^5}. \quad (6.29)$$

Naively, Eq. (4.9) and Eq. (4.6) imply that each ϕ behaves like w^{Δ} as $w \to 0$, and therefore this region is unimportant in the w integrals for fields satisfying the Breitenlohner-Freedman bound, $\Delta > 2$. But this scaling for small w can fail for a ϕ that connects to an external line, because of the concentrated support at $w \to 0$ of the source term in Eq. (6.3), as we see in Eq. (6.16). Away from a source the naive scaling holds as we see in Eq. (6.14). Our interaction vertex can at most approach one such source on the boundary since we are restricting to CFT correlators at non-coincident points. Therefore, at most one of the ϕ's in our interaction vertex, say ϕ_1, can scale as $w^{4-\Delta_1}$ for small w by connecting to an external line extending to this source. The other lines from ϕ_2, ϕ_3 are either internal or extend to other sources away from the interaction vertex and therefore continue to have the naive near-boundary scaling $w^{\Delta_2}, w^{\Delta_3}$. Consequently, the w integral of the interaction vertex behaves most singularly as $\int dw w^{\Delta_2+\Delta_3-\Delta_1-1}$ for small w. This is well-defined if

$$\Delta_1 < \Delta_2 + \Delta_3, \tag{6.30}$$

but not otherwise.

Notice that adding more fields to the interaction vertex only improves the convergence of the w integral of the vertex, since they scale as positive powers of w. (At most one field in the vertex can behave as $w^{4-\Delta}$ as argued above, and we have already assumed this is "ϕ_1".) Adding ∂_w derivatives reduces the power of w being integrated for small w, but this is off-set by the powers of inverse metric, $G^{MN} \propto w^2$, needed for AdS-invariance of the vertex. Adding x-derivatives obviously does not change w-scaling.

We conclude that the Witten diagrams are well-defined if we restrict ourselves to turning on sources only for $\mathcal{O}_n$ with Δ_n smaller than the sum of any two (or more) other $\Delta_{n'}$.

6.7 Derivation of source matching in Eq. (6.9)

We will see that with this restriction on source terms in place, we can justify our assumption that sources match between CFT and AdS as assumed earlier in writing Eq. (6.9). Further, we will understand better what is behind the restriction on sources.

Let us first consider a source for an unrestricted $\mathcal{O}_n$. For familiarity's sake, let us start in the confining deformation of the CFT, where the source term for a local CFT operator is equivalent to a source term for one or more glueball fields,

$$j_n(x)\mathcal{O}_n(x) \equiv j_n(x)\ell_n(\chi_i(x), \partial_\mu), \tag{6.31}$$

where ℓ_n is a local operator made from glueball fields, χ_i, and x-derivatives. As the deformation is removed, $\sigma \to 0$, we get

$$j_n(x)\mathcal{O}_n(x) \equiv j_n(x)\ell_n(\chi_m(x), \partial_\mu), \tag{6.32}$$

an x-local operator made from the continuum of $\chi_m(x)$ fields and derivatives. Now, the planar limit is only sensitive to tree diagrams made from χ_m, so the two-point correlator of $\mathcal{O}_n$ is determined in this approximation entirely by the term in ℓ_n *linear* in χ_m. But since we have defined $\mathcal{O}_n^{(N<\infty)}$ to be a primary operator even for finite, large N, its two-point correlator is entirely determined by conformal invariance and Δ_n. The fact that Δ_n is unchanged in planar approximation from its $N = \infty$ value, means that the planar two-point correlator of $\mathcal{O}_n$ is uncorrected from $N = \infty$. Hence the linear term in ℓ_n must be precisely the same coupling to j_n as at $N = \infty$, so as to ensure this same correlator.

What remains is to show that there are no *non-linear* corrections in χ appearing in ℓ_n. Suppose there were such non-linear corrections, say an order χ^2 term for simplicity. This would imply that in planar approximation, $\mathcal{O}_n(0)|0\rangle$ has a $1/N$-suppressed overlap with a two-χ Fock state, $|\chi, \chi'\rangle$, corresponding to two free χ 4D particles. Note that the operator is evaluated at time $t = 0$, so that there is no time evolution where interaction Hamiltonian terms in $1/N$-perturbation theory can appear that might cancel the two creation operators in the order χ^2 term in ℓ_n. Of course, since χ states are just 4D modes of ϕ, it follows that $|\chi, \chi'\rangle \equiv c_{n'n''}|\phi_{n'}, \phi_{n''}\rangle$ is some free two-ϕ state in AdS. As discussed in subsection 4.5, such two-particle (or multi-particle) AdS states can be decomposed into eigenstates of dilatations, with eigenvalues of the form, $\sqrt{4 + m_5^2} + \sqrt{4 + m_5'^2} + 4 + k$, where k is a non-negative integer. Equivalently, in terms of the primary scaling dimensions, Δ, Δ', dual to the 5D masses, the possible dilatation eigenvalues of the two-ϕ state take the form $\Delta + \Delta' + k$, $k \geq 0$ integer. But on the other hand, $\mathcal{O}_n(0)|0\rangle$ obviously is a dilatation eigenstate with eigenvalue Δ_n. Therefore, $\mathcal{O}_n(0)|0\rangle$ can only overlap the two-ϕ state if

$$\Delta_n \geq \Delta + \Delta'. \tag{6.33}$$

This is essentially a kinematic constraint in AdS.

It thereby follows that if we make the restriction at the end of the last subsection, that we only give source terms to n: Δ_n is smaller than the sum of any other two $\Delta_{n'}$, then there can be no non-linear corrections in ℓ_n, and Eq. (6.9) indeed holds.

This restriction is similar to the situation in a standard perturbative S-matrix construction in Minkowski spacetime. At zeroth order in perturbation theory all fields correspond to free, and therefore stable, particles. But many of the heavier fields can decay once perturbations are turned on, if their zeroth order mass exceeds the sum of two zeroth order masses of lighter fields. Such unstable particles should not appear as asymptotic states in the S-matrix construction. A particle whose mass is smaller than the sum of any two others is however stable by kinematics alone, and does represent an asymptotic state.

Of course, there can be other reasons, not purely kinematic in origin (other quantum numbers), that can ensure the stability of even a very heavy particle in Minkowski spacetime, for example a proton in the real world relative to electrons and positrons, so that it does appear as an asymptotic state. Similarly, in AdS/CFT there can be special situations/symmetries for which sources for operators with large Δ_n can be included without difficulty. In other words, while our restriction on sources is sufficient, it may not always be necessary.

Of course, in the CFT there is no restriction on correlators of any local operators, but it is only the *simplicity* of their translation into AdS, via Eq. (6.9), that is at stake. For discussion of a (special) situation in $\mathcal{N} = 4$ supersymmetric Yang-Mills in which Eq. (6.33) (just) fails to hold but a subtler AdS prescription can nevertheless be given, see Refs. [44].

7 Vector Primaries, Conserved Currents, and AdS Gauge Theory

Let us finally move beyond Lorentz-scalar primary operators to the next simplest case, Lorentz-vector primaries. (We will not treat spinor primaries in this paper, but the methodology in the vector case should guide the reader.)

7.1 General non-conserved vector primaries

We begin with a general vector primary, $\mathcal{O}_\mu(x)$, which is *not* a conserved current. Generalizing the approach of Section 3, we try to realize this operator (acting on the vacuum for the reasons of subsection 3.5, 3.6), as a free AdS_5 vector field, $A_M(x, w)$, which contains a 4D Lorentz-vector, A_μ. More precisely, we try to identify the irreducible representation of conformal symmetry given by $\mathcal{O}_\mu(x)$ with an irreducible representation of the isomorphic AdS spacetime symmetry, labelled by a particular AdS_5 mass and spin, realized in terms of $A_M(x, w)$ and a suitable AdS free-field wave equation.

As in subsections 3.1–3.4, the simplest "geometrization" of dilatations provides suitable near-boundary asymptotics,

$$A_\mu(x, w) \underset{w \to 0}{\longrightarrow} w^{\Delta - 1} \mathcal{O}_\mu(x). \tag{7.1}$$

Note that the power of w in Eq. (7.1) required by dilatations is different from the scalar case. This is because the requirement of being a scalar field under a spacetime symmetry transformation,

$$\phi'(x', w') = \phi(x, w), \tag{7.2}$$

is replaced by

$$A'_M(x', w') dX'^M = A_M(x, w) dX^M. \tag{7.3}$$

For dilatations, $x' = x/\lambda,\ w' = w/\lambda$, this implies

$$\phi'(x, w) = \phi(\lambda x, \lambda w), \tag{7.4}$$

compared with

$$A'_M(x, w) = \lambda A_M(\lambda x, \lambda w). \tag{7.5}$$

On the CFT side the vector index makes no difference to the dilatation transformation, which is determined entirely by the scaling dimension,

$$\mathcal{O}'_\mu(x) = \lambda^\Delta \mathcal{O}_\mu(\lambda x). \tag{7.6}$$

The vector/scalar difference in power of w in Eq. (7.1) follows. As in Section 3, one can directly check using Eq. (7.1) that for very small w, the AdS and CFT versions of K_μ match up. The 4D Poincare transformations trivially match up too. Our next job is to extend this near-boundary matching to finite w, with a suitably AdS-covariant wave equation.

The free wave equation for A_M that projects the field onto an irreducible representation of spacetime symmetry is given by the AdS_5 generalization of the Proca equation for massive spin-1, following from the invariant 5D action,

$$S_{5D} = \int d^4x dw \sqrt{G}\{-\frac{1}{4}G^{MN}G^{KL}F_{MK}F_{NL} + \frac{1}{2}m_5^2 G^{KL}A_K A_L\}, \tag{7.7}$$

where

$$F_{MN} \equiv \partial_M A_N - \partial_M A_N. \tag{7.8}$$

The equation of motion is then

$$\partial_M(\sqrt{G}G^{MN}G^{KL}F_{NL}) = m_5^2\sqrt{G}G^{KL}A_L. \tag{7.9}$$

It can be broken down as

$$\begin{aligned} w^2\partial^\nu F_{\mu\nu} + w^3\partial_w(\frac{1}{w}(\partial_w A_\mu - \partial_\mu A_w)) &= m_5^2 A_\mu \\ w^2\partial^\mu(\partial_w A_\mu - \partial_\mu A_w) &= m_5^2 A_w. \end{aligned} \tag{7.10}$$

The number of independent components of A_M and (the non-conserved) $\mathcal{O}_\mu$ is seen to match by taking ∂_K of Eq. (7.9),

$$m_5^2\partial_K(\sqrt{G}G^{KL}A_L) = 0. \tag{7.11}$$

That is,

$$\partial_w(\frac{A_w}{w^3}) = \frac{\partial_\mu A^\mu}{w^3}, \tag{7.12}$$

so that A_w is not an independent dynamical field, but rather is given by

$$A_w(x,w) = w^3\int_0^w dw' \frac{\partial_\mu A^\mu(x,w')}{w'^3}. \tag{7.13}$$

The near-boundary condition on A_μ, Eq. (7.1), then implies the near-boundary condition on A_w,

$$A_w(x,w) \underset{w\to 0}{\longrightarrow} \frac{w^\Delta}{\Delta - 3}\,\partial^\mu\mathcal{O}_\mu(x). \tag{7.14}$$

Thus, all components of A_M have boundary conditions, and there is a unique solution to the equation of motion. By the same logic as in Section 3, this realizes conformal transformations on $\mathcal{O}_\mu$ as AdS isometry transformations of A_M.

As we did for scalars, we can match m_5^2 with Δ by focusing on solutions to the equations of motion near the AdS boundary,

$$m_5^2 = (\Delta - 3)(\Delta - 1). \tag{7.15}$$

One can proceed for such vector primaries very much as for scalar primaries, in discussing the large-N expansion, focussing on single-trace primaries, the planar/tree duality, source terms and Witten diagrams, and so on. But something qualitatively new happens in the special case of CFT conserved currents.

7.2 (Improved) Conserved Noether Currents

Let us suppose that the CFT has a global symmetry, with an associated Noether current operator which is conserved,

$$\partial^\mu \mathcal{O}_\mu = 0. \tag{7.16}$$

As is standard in quantum field theory, such a current is not renormalized (vanishing anomalous dimensions) and therefore has a scaling dimension equal to its naive dimension of 3.

Let us first ask if $\mathcal{O}_\mu$ is a primary operator, or merely a scaling operator. If it were not a primary operator, it would have the form

$$\mathcal{O}_\mu = c\ \hat{\mathcal{O}}_\mu(x) + \partial\tilde{\mathcal{O}}(x), \tag{7.17}$$

where we expand in a variety of scaling operators of dimension 3. $\hat{\mathcal{O}}_\mu$ is a possible primary operator of dimension 3, and $\partial\tilde{\mathcal{O}}$ is a linear combination of descendent operators, each of which is necessarily a derivative of other operators. In order for $\int d^3\vec{x}\mathcal{O}_0(x)$ to be a total charge, conserved in time, $\mathcal{O}_0$ must not vanish at zero-momentum. Since (the Fourier transform of) derivative terms vanish at zero momentum, there must be a primary $\hat{\mathcal{O}}_\mu$ with non-zero coefficient c.

If $\hat{\mathcal{O}}_\mu$ were also conserved, then in it would be an "improved" symmetry current, in that it is also a primary operator with $\Delta = 3$, which would specify its conformal transformations. We now prove this is the case. The proof follows by studying the Jacobi identity:

$$\begin{aligned} 0 &= [P^\nu, [K_\mu, \hat{\mathcal{O}}_\nu(0)]] - [[P^\nu, K_\mu], \hat{\mathcal{O}}_\nu(0)] + [[P^\nu, \hat{\mathcal{O}}_\nu(0)], K_\mu] \\ &= 2i[J^\nu{}_\mu - \delta^\nu{}_\mu S, \hat{\mathcal{O}}_\nu(0)] + [K_\mu, \partial.\hat{\mathcal{O}}(0)] \\ &= [K_\mu, \partial.\hat{\mathcal{O}}(0)], \end{aligned} \tag{7.18}$$

where the first term on the right-hand side of the first line vanishes since $\hat{\mathcal{O}}_\nu$ is primary, and the first term on the second line vanishes by a cancellation between the Lorentz and scale transformations at precisely $\Delta_{\hat{\mathcal{O}}} = 3$. We thereby deduce that $\partial.\hat{\mathcal{O}}(0)$ is either a primary operator as well, or it vanishes. But $\partial.\hat{\mathcal{O}}$ is manifestly a decendent of $\hat{\mathcal{O}}_\mu$, so it cannot be primary. We conclude that $\partial.\hat{\mathcal{O}}(0) = 0$, and so by translation invariance, $\partial.\hat{\mathcal{O}}(x) = 0$.

We thereby conclude that each CFT global symmetry is associated to a conserved current, $\hat{\mathcal{O}}_\mu$, which is a *vector primary* of scale dimension 3.

7.3 AdS gauge invariance from conserved CFT current

The case of a vector primary with $\Delta = 3$ corresponds to the massless limit of Eq. (7.15). The resulting gauge invariance of the AdS equations of motion means that A_w becomes indeterminate, and that the near boundary asymptotics do not yield a unique solution in the "bulk" of AdS. Any solution to the AdS Maxwell equations with the boundary behavior of Eq. (7.1) can be gauge-transformed with a gauge transformation that vanishes near the boundary, to yield a new Maxwell solution also satisfying Eq. (7.1). Clearly, in this $\Delta = 3$ case, the "geometrization" of $\mathcal{O}_\mu$ should map it to an AdS *gauge connection*, that is the whole gauge *equivalence class* of AdS vector fields. This is the global/gauge (symmetry) aspect of CFT/AdS duality.

We can phrase the entire CFT/AdS mapping in gauge-invariant terms, by expressing the near-boundary behavior in terms of the gauge field strength,

$$F_{\mu w}(x, w) \underset{w \to 0}{\longrightarrow} w \hat{\mathcal{O}}_\mu(x), \tag{7.19}$$

and using it to solve the AdS Maxwell equation.

It is also useful to view this map in "axial gauge", $A_w(x, w) = 0$. This condition still leaves a residual gauge invariance, which can be fixed by the auxiliary gauge condition, $\partial^\mu A_\mu(x, w) \underset{w \to 0}{\to} 0$. Together with Eq. (7.1), this provides a full set of boundary conditions for $A_\mu(x, w)$ in order to solve the gauge-fixed Maxwell equations,

$$\begin{aligned} \partial^\nu F_{\mu\nu} + w\partial_w \frac{1}{w}\partial_w A_\mu &= 0 \\ \partial_w \partial^\mu A_\mu &= 0. \end{aligned} \tag{7.20}$$

Given the auxiliary gauge-fixing condition, the second of these equations implies

$$\partial^\mu A_\mu(x, w) = 0, \tag{7.21}$$

so that the first equation then reads simply,

$$-\Box_4 A_\mu + w\partial_w \frac{1}{w}\partial_w A_\mu = 0. \tag{7.22}$$

It is then straightforward (but tedious) to parallel our scalar discussion in deriving the free AdS propagator, matching the CFT-correlator source terms with AdS, and deriving the boundary-bulk propagator. One point to note is that the boundary-bulk propagator, $K_{M\mu}(x -$

$x', w)$, where $K_{w\mu}(x-x', w) = 0$ in axial gauge but not other gauges, satisfies the naive near-boundary $\sim w^2$ scaling (see Eq. (7.1)) for $x \neq x'$, but with the dominant behavior arising at coincidence,

$$K_{\mu\nu}(x-x', w) \underset{w\to 0}{\propto} \delta_{\mu\nu}\delta^4(x-x'), \tag{7.23}$$

scaling without a power of w (zeroth power) near the boundary. This means that $K_{M\mu}(x, w)$ is literally the Green function that allows one to solve for a 5D gauge field whose boundary value *is* the CFT source, rather than in terms of the subtler limiting behavior of massive scalar fields.

To complete the Witten diagrammatic rules we turn to the issue of interactions.

7.4 AdS effective gauge theories

Consider a large-N CFT with a global $U(1)$ symmetry and associated primary conserved current, $\hat{\mathcal{O}}_\mu$, which is a single-trace primary, at least at $N = \infty$. In addition, imagine that at $N = \infty$ there is a complex scalar single-trace primary, $\mathcal{O}$, charged under the global $U(1)$, with low dimension Δ. Imagine other single-trace primaries have very high dimension. All this translates into AdS having a massless $U(1)$ gauge field and charged scalar field with mass $m_5^2 = \Delta(\Delta - 4)$, with all other AdS fields being very heavy. Integrating out the heavy fields, the general AdS scalar-QED effective field theory has the leading gauge-invariant form,

$$\begin{aligned} S_{eff} = \int d^4x dw \sqrt{G}\{ -\frac{1}{4} G^{MN} G^{KL} F_{MK} F_{NL} \\ + G^{MN}(\partial_M + igA_M)\phi^*(\partial^M - igA^M)\phi - m_5^2|\phi|^2 - \lambda|\phi|^4\}. \end{aligned} \tag{7.24}$$

Again, tree-level in this effective theory corresponds to the planar limit of the CFT.

In addition to these gauge-invariant terms, the effective theory may also contain the *almost* gauge-invariant Chern-Simons action,

$$S_{CS} = 4\kappa \int d^4x dw \; \epsilon^{JKLMN} A_J \partial_K A_L \partial_M A_N. \tag{7.25}$$

It transforms under a 5D $U(1)$ gauge transformation, $\delta A_M(X) = \partial_M \Lambda(X)$, as

$$\begin{aligned} \delta S_{CS} &= 4\kappa \int d^4x \int_0^\infty dw \; \epsilon^{JKLMN} \partial_J \Lambda \partial_K A_L \partial_M A_N \\ &= \kappa \int d^4x \; \Lambda(x, w=0) \epsilon^{\kappa\lambda\mu\nu} F_{\kappa\lambda}(x, w=0) F_{\mu\nu}(x, w=0), \end{aligned} \tag{7.26}$$

where the last line is the boundary term that follows by integration by parts on the first line, with the non-boundary term in the integration by parts vanishing using the anti-symmetry of the ϵ-tensor and the symmetry of successive derivatives. While gauge invariance is central to the effective field theory description of the massless spin-1 particle, the gauge-variation above does indeed manifestly vanish for the propagating field satisfying the boundary condition, Eq. (7.19).

As pointed out at the end of the last subsection, by Eq. (7.23) it is the CFT *source* for the conserved current that corresponds to the well-defined non-vanishing $w \to 0$ limit component of the A_μ field. The gauge non-invariance at the boundary of the Chern-Simons action is therefore a statement about these sources and the associated current correlators. Let us examine this from the CFT perspective. If we add source terms for the conserved current in the CFT,

$$S_{CFT} \longrightarrow S_{CFT} + \int d^4x A_\mu(x)\hat{\mathcal{O}}^\mu(x), \tag{7.27}$$

and compute the generating functional $W[A_\mu]$ for correlators of currents at *non-coincident points*, then current conservation simply reads,

$$\partial_\mu \frac{\delta W}{\delta A_\mu} = 0, \tag{7.28}$$

which one can think of as a gauge invariance for the source,

$$W[A_\mu + \partial_\mu \Lambda] = W[A_\mu], \tag{7.29}$$

where $\Lambda(x)$ is a 4D gauge transformation. But at coincident points for correlators both these equations can be corrected. A $U(1)^3$ triangle anomaly in the global symmetry current of the CFT corresponds to just such a correction,

$$W[A_\mu + \partial_\mu \Lambda] = W[A_\mu] + \kappa \int d^4x\ \Lambda(x)\epsilon^{\kappa\lambda\mu\nu}F_{\kappa\lambda}(x)F_{\mu\nu}(x). \tag{7.30}$$

This is a perfect match to Eq. (7.26).

The Chern-Simons action is the unique 5D action that breaks gauge-invariance on the boundary, so as to match in AdS a possible CFT anomaly in the global currents [7], while remaining gauge-invariant in the AdS "bulk" as required for the effective field theory description of massless spin-1. In principle, we restricted ourselves in this paper to correlators of local operators at non-coincident points, while the above subtleties occur at coincidence. Nevertheless, one can think of the non-coincident correlators as giving a point-splitting regularization of the anomaly, and one can carefully take the limit as the regularization is removed to find the anomaly at coincidence. See Ref. [40].

As a second example of AdS effective field theory, consider the case where the global symmetry is non-abelian, say $SU(2)$, and all single-trace operators except the associated conserved currents are very high dimension. The only light fields are the dual gauge fields, $A_M^{a=1,2,3}$. One can decompose these under a $U(1)$ subgroup, say that corresponding to gauge field $A_M \equiv A_M^3$, with $U(1)$-charged vector massless "matter" field $W_M^\pm \equiv A_M^1 \pm iA_M^2$. The unique effective theory that is gauge-invariant under all such $U(1)$ subgroups is non-abelian gauge-invariant,

$$S_{eff} = -\frac{1}{4}\int d^4x dw \sqrt{G} G^{MN} G^{KL} \mathcal{F}^a_{MK}\mathcal{F}^a_{NL}, \tag{7.31}$$

where

$$\mathcal{F}^a_{MN} \equiv \partial_M A^a_N - \partial_N A^a_M - g\epsilon^{abc} A^b_M A^c_N \tag{7.32}$$

is the non-abelian field strength.

Again, one can also have (more complicated) non-abelian Chern-Simons terms to match CFT anomalies in the non-abelian currents.

8 Tensor Primaries, the Energy-Momentum Tensor, and AdS Gravity

The study of tensor primaries parallels many of the steps we took in the last section for vector primaries. We move briskly through those aspects which are most similar, and give more care to those that are new.

8.1 General non-conserved tensor primaries

Since CFT primaries come in irreducible representations of 4D Lorentz symmetry, a 2-tensor primary, $\mathcal{O}_{\mu\nu}$ must either be symmetric and traceless or be anti-symmetric and traceless. Let us focus on the symmetric case, so that spin-2 states are among the states $\mathcal{O}_{\mu\nu}$ interpolates on the CFT vacuum. In general, $\mathcal{O}_{\mu\nu}$ is not conserved. We will realize this operator in terms of a free AdS_5 symmetric tensor field, $h_{MN}(x, w)$, satisfying a free AdS wave equation that picks out a particular irreducible representation of spacetime symmetry. The near-boundary asymptotics are given by now-familiar considerations,

$$h_{\mu\nu}(x, w) \underset{w \to 0}{\longrightarrow} w^{\Delta - 2} \mathcal{O}_{\mu\nu}(x). \tag{8.1}$$

Let us review the construction of the massive spin-2 AdS wave equation, by starting with the analogous equation in 5D Minkowski spacetime,

$$\begin{aligned} -\partial_S \partial^S h_{MN} + \partial_N \partial^S h_{MS} + \partial_M \partial^S h_{NS} - \partial_M \partial_N h^S{}_S - \eta_{MN} \partial^S \partial^T h_{ST} + \eta_{MN} \partial^S \partial_S h^T{}_T \\ = -\eta_{MN} m_5^2 h^T{}_T + m_5^2 h_{MN}. \end{aligned} \tag{8.2}$$

The choice of tensor structure can be understood as follows. Taking ∂^M of this equation implies

$$\partial^M h_{MN} = \partial_N h^S{}_S. \tag{8.3}$$

Using ∂^N of this in the trace of the equation of motion then implies

$$h^S{}_S = 0, \tag{8.4}$$

which in turn reduces Eq. (8.3) to

$$\partial^M h_{MN} = 0. \tag{8.5}$$

The equation of motion then reduces to

$$(\partial_S \partial^S + m_5^2) h_{MN} = 0. \tag{8.6}$$

Thus the action has been chosen to give the Klein-Gordon equation, but projecting out all but the transverse and traceless parts of h_{MN}. One can check that it is the unique local action with this property.

Now, Eq. (8.2) has the form of the *linearized* 5D Einstein Equation, with the addition of a "Pauli-Fierz" mass term on the right-hand side, if one thinks of $\mathcal{G}_{MN} \equiv \eta_{MN} + h_{MN}$ as a dynamical spacetime metric. The corresponding classical action,

$$S_{Mink} = \int d^5X \frac{1}{2}(\partial_S h_{MN})^2 - (\partial_N h^{MN})^2 + \partial_N h^{MN} \partial_N h^S{}_S - \frac{1}{2}(\partial_M h^S{}_S)^2 - \frac{m_5^2}{2} h^{MN} h_{MN} + \frac{m_5^2}{2}(h^S{}_S)^2, \tag{8.7}$$

consists of precisely the *quadratic fluctations* about Minkowski space of the Einstein-Hilbert 5D action, with Pauli-Fierz mass terms added,

$$S = M_5^3 \int d^5X \{\sqrt{\mathcal{G}}\mathcal{R} - \frac{m_5^2}{2} h^{MN} h_{MN} + \frac{m_5^2}{2}(h^S{}_S)^2\}. \tag{8.8}$$

The generalization to AdS is given by covariantizing derivatives in Eq. (8.7) with respect to the AdS metric G_{MN}. The result of doing this is summarized by quadratic fluctions about AdS of the gravitational action with Pauli-Fierz mass terms and a *negative cosmological constant*,

$$\begin{aligned} S &= M_5^3 \int d^5X \{\sqrt{\mathcal{G}}[\mathcal{R} + 12] - \frac{m_5^2}{2} G_{AdS}^{KM} G_{AdS}^{LN} h_{KL} h_{MN} + \frac{m_5^2}{2}(G_{AdS}^{MN} h_{MN})^2\}, \\ \mathcal{G}_{MN} &\equiv G_{MN}^{AdS} + h_{MN}. \end{aligned} \tag{8.9}$$

$\mathcal{R}$ denotes the Ricci scalar curvature constructed from $\mathcal{G}_{MN}$. The cosmological constant (in our $R_{AdS} \equiv 1$ units) is such that $h_{MN} = 0$, $\mathcal{G}_{MN} = G_{MN}^{AdS}$, is an extremum of the action, so that the expansion for small fluctuations, h_{MN}, makes sense.

We can parallel the remaining steps of subsection 7.1, and straightforwardly (if tediously) see that the equations of motion determine h_{Mw} and $\eta^{\mu\nu} h_{\mu\nu}$ in terms of the non-4D-trace parts of $h_{\mu\nu}$, so that the number of independent components agrees with $\mathcal{O}_{\mu\nu}$. The matching between m_5^2 and Δ follows from matching near-boundary behavior of solutions,

$$m_5^2 = \Delta(\Delta - 4). \tag{8.10}$$

8.2 (Improved) Conserved Energy-Momentum Tensor

4D Poincare invariance of a general CFT implies the existence of a Noether current, the conserved energy-momentum tensor:

$$\partial^\mu T_{\mu\nu} = 0. \tag{8.11}$$

In standard fashion it is not renormalized (vanishing anomalous dimensions) and therefore has true (= naive) scaling dimension, $\Delta = 4$. As for conserved currents of internal global symmetries, we can "improve" the energy-momentum tensor [45] for our purposes.

If $T_{\mu\nu}$ is not itself primary, it can be expanded in a variety of scaling operators,

$$T_{\mu\nu}(x) = c_n \mathcal{O}^n_{\mu\nu}(x) + \partial\tilde{\mathcal{O}}(x), \tag{8.12}$$

where $\mathcal{O}^n_{\mu\nu}$ are dimension-4 primary operators, c_n are constants, and the last term consists of descendent operators of various types (with total scale dimension 4). Analogously to the conserved current case of the last section, in order for $\int d^3\vec{x}\ T_{\mu 0}$ to be the total conserved 4-momentum, P_μ, some c_n must be non-zero.

If $c_n \mathcal{O}^n_{\mu\nu}(x)$ were also conserved, then it would be a "partially improved" energy-momentum tensor, in that it is a sum of primary operators, and therefore "primary" in the sense that

$$[K_\sigma, c_n \mathcal{O}^n_{\mu\nu}(0)] = 0. \tag{8.13}$$

We now prove that it is indeed conserved by studying the Jacobi identity,

$$\begin{aligned} 0 &= [P^\nu, [K_\sigma, c_n \mathcal{O}^n_{\mu\nu}(0)]] - [[P^\nu, K_\sigma], c_n \mathcal{O}^n_{\mu\nu}(0)] + [[P^\nu, c_n \mathcal{O}^n_{\mu\nu}(0)], K_\sigma] \\ &= 2i[J^\nu{}_\mu - \delta^\nu{}_\mu S, c_n \mathcal{O}^n_{\mu\nu}(0)] + [K_\sigma, \partial^\mu c_n\ \mathcal{O}^n_{\mu\nu}(0)] \\ &= [K_\sigma, \partial^\mu c_n \mathcal{O}^n_{\mu\nu}(0)], \end{aligned} \tag{8.14}$$

where the first term on the second line vanishes by a cancellation between between the Lorentz and dilatation transformations at precisely $\Delta_{\mathcal{O}} = 4$. If $\partial^\mu c_n \mathcal{O}^n_{\mu\nu}(0)$ does not vanish, then the equation above shows that it is primary, while also being a superposition of derivatives of primaries. This would be a contradiction, so we must conclude that it vanishes, $\partial^\mu c_n \mathcal{O}^n_{\mu\nu}(0) = 0$. By translation invariance, $c_n \mathcal{O}^n_{\mu\nu}(x)$ is then conserved for all x,

$$c_n\ \partial^\mu \mathcal{O}^n_{\mu\nu}(x) = 0. \tag{8.15}$$

If more than one c_n were non-zero, Eq. (8.15) would imply an operator relationship between descendents of different primaries. This is inconsistent with such primaries labelling different conformal representations. Therefore precisely one such primary has non-vanishing c, which we will denote $\hat{\mathcal{O}}_{\mu\nu}$. Since primaries come in irreducible representations of 4D Lorentz symmetry, $\hat{\mathcal{O}}_{\mu\nu}$ is one of (i) symmetric, traceless tensor, (ii) anti-symmetric tensor, (iii) scalar $\times \eta_{\mu\nu}$. However, anti-symmetry is inconsistent with having an energy operator, $\int d^3\vec{x}\ T_{00}$, while a scalar is inconsistent with having a momentum operator, $\int d^3\vec{x}\ T_{0i}$. We conclude that every CFT has an "improved" conserved, symmetric, traceless energy-momentum tensor, $\hat{\mathcal{O}}_{\mu\nu}$, which is a primary operator of scale dimension 4.

8.3 Linearized general covariance from CFT energy-momentum tensor

The improved CFT energy-momentum tensor, $\mathcal{O}_{\mu\nu}$, with $\Delta = 4$ corresponds to the massless limit of Eq. (8.9), namely the expansion in h_{MN} at quadratic order of 5D General Relativity with negative cosmological constant. In this limit, the linearized Einstein equation that follows is invariant under *linearized* general coordinate transformations,

$$h_{MN} \to h_{MN} + G^{AdS}_{KN} D^{AdS}_{M} \xi^{K} + G^{AdS}_{ML} D^{AdS}_{N} \xi^{L}. \tag{8.16}$$

Consequently, the near-boundary asymptotics and equations of motion no longer uniquely determine $h_{MN}(x, w)$ in the AdS "bulk", since for any such solution there is another given by an infinitesimal coordinate transformation $X^M \to X^M + \xi^M(X)$ where ξ^M vanishes in the vicinity of the AdS boundary. Instead, $\mathcal{O}_{\mu\nu}$ determines a unique (infinitesimal) coordinate transformation *equivalence class* of h_{MN}'s. That is, $\mathcal{O}_{\mu\nu}$ determines a unique dynamical geometry, represented by the metric field $\mathcal{G}_{MN} \equiv G^{AdS}_{MN} + h_{MN}$.

As in the last section, one can again work in "axial" gauge, $h_{Mw} \equiv 0$, with auxiliary gauge-fixing given by $\partial^\mu h_{\mu\nu} \underset{w\to 0}{\to} 0$, so that (with analogous analysis to the last section) both near-boundary asymptotics and the bulk equation of motion are given in terms of just the transverse and traceless $h_{\mu\nu}(x, w)$. This has the same number of components as $\mathcal{O}_{\mu\nu}(x)$. The linearized Einstein equation (about AdS) reads in this gauge,

$$(w^2 \Box_4 - w^2 \partial_w^2 - w\partial_w + 4) h_{\mu\nu}(x, w) = 0. \tag{8.17}$$

Again, we can study the boundary-bulk propagator behavior near the boundary. At non-coincident points it is the naive $\sim w^2$ of Eq. (8.1), but at coincidence it behaves as $h \sim \delta^4(x - x')/w^2$. That is, Witten diagrams perturbatively determine the dynamical metric $\mathcal{G}_{MN} = G^{AdS}_{MN} + h_{MN}$ about the AdS metric, such that the near-boundary behavior remains $\propto 1/w^2$ as in G^{AdS}_{MN}, but with x-dependence given by the CFT source term for $\mathcal{O}_{\mu\nu}$. This matches the defining features of the ansatz in the literature for the AdS/CFT correspondence for CFT energy-momentum correlators [7].

8.4 AdS effective General Relativity

The improved energy-momentum tensor in a large-N CFT is a single-trace primary with dimension 4. Let us first suppose that all other single-trace primaries have very large dimension. At $N = \infty$ the energy-momentum tensor is dual to a massless spin-2 free field, while all other AdS fields are very heavy. Therefore, in the planar large-N limit of the CFT, the AdS effective field theory, valid to distances much smaller than the AdS radius, contains only the massless spin-2 particle, with self-interactions. Such self-interactions must minimally account for the fact that the spin-2 AdS state itself must be dual to a CFT state which carries energy and momentum.

In Minkowski spacetime, approximately valid at distances smaller than the AdS radius, the only such self-interacting theory is fully non-linear (5D) General Relativity [46] [47]. This requires the linearized general coordinate invariance of the last subsection to be extended to *full* general coordinate invariance. At larger distances, the only more-relevant coordinate-invariant interaction is a cosmological constant term. Therefore, the AdS effective theory describing the planar limit is 5D General Relativity with negative cosmological constant. It is just given by Eq. (8.9) with vanishing mass and eliminating the restriction keeping terms only quadratic in h. That is,

$$\begin{aligned} S &= M_5^3 \int d^5X \sqrt{\mathcal{G}} \{\mathcal{R} + 12\}, \\ \mathcal{G}_{MN} &\equiv G_{MN}^{AdS} + h_{MN}. \end{aligned} \tag{8.18}$$

As a second example, suppose the CFT also has a $U(1)$ global symmetry and associated conserved single-trace current with dimension 3, and a complex scalar single-trace primary with order-one dimension Δ, which is charged under the $U(1)$. Let all other single-trace primaries have very large dimension. General coordinate invariance and $U(1)$ gauge invariance powerfully restrict the structure of the AdS effective field theory,

$$\begin{aligned} S &= \int d^5X \sqrt{\mathcal{G}} \{M_5^3 \mathcal{R} + 12M_5^3 - \frac{1}{4}\mathcal{G}^{MN}\mathcal{G}^{KL}F_{MK}F_{NL} \\ &+ \mathcal{G}^{MN}(\partial_M + igA_M)\phi^*(\partial^M - igA^M)\phi - m_5^2|\phi|^2 - \lambda|\phi|^4\}. \end{aligned} \tag{8.19}$$

The simplicity of these leading terms in the AdS effective field theory, founded on only broadly stated features of the CFT, illustrates the power of the AdS/CFT correspondence.

9 Emergent Relativity

Let us ask whether the pre-requisite CFT can itself emerge from something even more basic and less symmetric. It is common for *equilibrium* condensed matter systems, which may be discrete lattice theories at short distances, to approach conformal field theories in the IR, at second order phase transitions. Because time is out of the picture at equilibrium, these are *Euclidean* CFTs. However, in real-time systems even the approach to emergent special relativity, let alone Lorentzian conformal invariance, is subtler.

9.1 Weak-coupling examples

Since, emergent rotational and translational symmetry and locality of couplings is both common and familiar, let us start by assuming we have a continuum quantum field theory with these properties, but without insisting on Lorentz invariance. We begin with a simple example in which Lorentz invariance does emerge robustly. Let ϕ be a weakly coupled scalar field, whose dynamics has $\phi \to -\phi$ symmetry and $\phi(t, \vec{x}) \to \phi(-t, \vec{x})$ symmetry. Just imposing translational and rotational invariance, the effective Lagrangian must have the form,

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial_t \phi)^2 - \frac{c^2}{2}(\partial_i \phi)^2 - \frac{m^2}{2}\phi^2 - \lambda\phi^4 + \text{non} - \text{renormalizable couplings}, \tag{9.1}$$

where c^2 is an arbitrary constant. The renormalizable terms are accidentally Lorentz invariant, if we identify c with the speed of light in the Lorentz algebra. The non-renormalizable terms can violate this symmetry while still being translation and rotational invariants, such as a term $\partial_i\phi\partial_i\phi\partial_j\phi\partial_j\phi$ without accompanying time-derivative terms, but at low enough energies/momenta (and for small enough m^2), these terms would be irrelevant.

But now consider a theory of two such scalars coupled to each other, without any symmetry *between* them (but each with their own symmetries as in the example above). The general rotationally symmetric effective Lagrangian is

$$\begin{aligned} \mathcal{L}_{eff} \;=\; & \frac{1}{2}(\partial_t \phi)^2 - \frac{c_\phi^2}{2}(\partial_i \phi)^2 - \frac{m_\phi^2}{2}\phi^2 - \lambda_\phi^2\phi^4 \\ & + \frac{1}{2}(\partial_t \chi)^2 - \frac{c_\chi^2}{2}(\partial_i \chi)^2 - \frac{m_\chi^2}{2}\chi^2 - \lambda_\chi^2\chi^4 - \lambda_{mix}\phi^2\chi^2 \\ & + \text{non} - \text{renormalizable couplings}, \end{aligned} \tag{9.2}$$

where now c_ϕ and c_χ are two independent (separately renormalized) constants. In general, infrared Lorentz invariance is badly broken by these different maximal speeds.

9.2 Strong-coupling robustness of emergent relativity

This problem is quite general in weakly coupled field theories for multiple particle species, but at strong coupling the flow to Lorentz invariance can be robust. To assess the robustness of emergent Lorentz invariance, let us start with a Lorentz-invariant "target" field theory, say specified by a quantum Hamiltonian, $H_{relativistic}$, and ask whether a Lorentz-violating but rotationally-symmetric and local deformation would robustly flow towards the target in the infrared. Such a deformation can be written as

$$H_{deformed} = H_{relativistic} + \int d^3\vec{x} g \mathcal{O}(\vec{x}), \tag{9.3}$$

where $\mathcal{O}$ is a rotational scalar local operator, and g is the deformation strength at some renormalization scale. If all such operators are IR-irrelevant for small g, then there is a robust flow towards IR Lorentz invariance, $g \underset{IR}{\to} 0$. But there is one such operator which we know on general grounds is marginal, not irrelevant, namely the energy-momentum tensor, $T_{\mu\nu}$. The translational invariance of the theory implies its conservation as a Noether current, and that it is not renormalized, so that it has scale dimension exactly 4 without anomalous dimension corrections. T_{00} and T_{ii} are each Lorentz-violating rotational scalars, but since their difference is a Lorentz-scalar, $\eta^{\mu\nu}T_{\mu\nu}$, there is only one independent Lorentz-violating operator, which we take to be T_{00} without loss of generality. In a strongly coupled field theory, this is the *only* marginal Lorentz-violating rotational scalar that *must* be present. Other Lorentz-violating operators might very well be significantly irrelevant and flow rapidly to zero in the IR. But this marginal operator is nothing but energy-density, so plugging it into Eq. (9.3) yields,

$$\begin{aligned} H_{deformed} &= H_{relativistic} + \int d^3\vec{x} g T_{00}(\vec{x}) \\ &= (1+g) H_{relativistic}. \end{aligned} \tag{9.4}$$

We have merely recovered a rescaled version of our relativistic theory! If we rescale time $t \to (1+g)t$, the conjugate Hamiltonian becomes

$$H_{deformed} = H_{relativistic}. \tag{9.5}$$

Rescaling time, but not space, changes the undeformed speed of light, $c \to c/(1+g)$. But since this an overall rescaling of c for all particle species, the theory remains relativistic.

9.3 Instability of relativity due to weak inter-sector couplings

Given this optimistic conclusion above, one might ask what fails in the weakly coupled case. More generally, consider two sectors, A and B, which are weakly coupled to each other, although there may be either strong or weak couplings within each sector. In the limit in which the two sectors are completely decoupled, there are two, separately conserved energy-momentum tensors, $T^A_{\mu\nu}, T^B_{\mu\nu}$, each with scale dimension exactly 4. In the presence of weak $A-B$ couplings however, each of these operators receives perturbative corrections to their anomalous dimension (matrix), in such a way that these corrections cancel in the *total* energy-momentum tensor, $T^A_{\mu\nu} + T^B_{\mu\nu} + T^{AB\ interaction}_{\mu\nu}$. Thus, for example, $\mathcal{O} = T^A_{00} - T^B_{00}$ is an almost marginal deformation which will not flow rapidly away in the infrared (though it might flow away logarithmically slowly [48] [49]). Clearly, the effect of this deformation is (minimally) to give sectors A and B different speeds of light and spoil Lorentz invariance.

At strong coupling, $\mathcal{O}$ will generically get a substantial anomalous dimension, which can be positive, making it order one irrelevant, and there is a robust flow to Lorentz invariance. At some point into the IR the strong coupling might transition to weak coupling, but the very precise Lorentz invariance is now imprinted on the effective theory by matching at this threshold to the UV strong coupling theory. Alternatively, the Lorentz invariant quantum field theory might flow to a strongly coupled CFT with an AdS low-curvature dual, with emergent higher-dimensional general relativity. An irrelevant Lorentz-violating operator like $\mathcal{O} = T^A_{\mu\nu} - T^B_{\mu\nu}$ would then be dual to a *massive* tensor field, by the analysis of subsection 8.1. See Ref. [19] for the proposal to use AdS/CFT duality to infer the IR-irrelevance of Lorentz violation in strong-coupling $\mathcal{N} = 4$ supersymmetric Yang-Mills.

9.4 Lorentz violation by conserved currents: chemical potentials

There is an interesting generalization of the plot of emergent Lorentz invariance from strong coupling, which takes place if the dynamics has a global internal symmetry, say $U(1)$. In that case, there is a conserved Noether current, $\mathcal{O}_\mu$, with non-renormalized scale dimension 3. Therefore, we now have a possible *relevant* deformation of the relativistic target theory,

$$H_{deformed} = H_{relativistic} + \int d^3\vec{x} g \mathcal{O}_0(\vec{x}). \tag{9.6}$$

Such a deformation grows rapidly in important in the IR and therefore it appears that Lorentz invariance will not robustly emerge. However, this need not be the case if we insist on charge conjugation invariance, under which the $\mathcal{O}_\mu$ current is odd. It is therefore technically natural (radiatively stable) for the dimensionful conjugation-violating coupling, g, to be so small that the theory first flows very close to Lorentz invariance, before this specific deformation becomes important. In that case, g would be nothing but a chemical potential for a (very nearly) Lorentz-invariant theory. This still represents Lorentz-violation, but of a familiar kind.

In the case in which the emergent special relativitic dynamics is a CFT with $U(1)$ global symmetry and which enjoys a low-curvature AdS dual, let us work out the dual of turning on the small chemical potential. Thinking of g as a constant CFT source, we see that the generalization of Eq. (6.9) to a conserved vector primary and energy-momentum tensor primary is given on the AdS side by

$$\begin{aligned} S &= \int d^5X\sqrt{\mathcal{G}}\{M_5^3\mathcal{R} + 12M_5^3 - \frac{1}{4}\mathcal{G}^{MN}\mathcal{G}^{KL}F_{MK}F_{NL}\} \\ &\quad + \underset{w'\to 0}{\text{limit}} \int d^4x \frac{gA_0(x,w')}{w'^2}. \end{aligned} \tag{9.7}$$

The leading effect of the chemical potential, g, on the ground state is given by solving the classical equations of motion of this effective theory. We will try the static and $\vec{x}$-translation independent ansatz that all fields are x-independent, that only A_0 is non-vanishing within A_M, and that the metric takes the diagonal form,

$$\mathcal{G}_{00} = \frac{f(w)}{w^2},\ \mathcal{G}_{ij} = -\frac{\delta_{ij}}{w^2},\ \mathcal{G}_{ww} = -\frac{1}{f(w)w^2}. \tag{9.8}$$

The Maxwell equation then reads

$$\partial_w \frac{1}{w}\partial_w A_0(w) = -\underset{w'\to 0}{\text{limit}} \frac{g\delta(w-w')}{w'^2}, \tag{9.9}$$

with solution,

$$\begin{aligned} A_0(w) &= aw^2 + \underset{w'\to 0}{\text{limit}}\{\frac{gw^2}{2w'^2}\theta(w'-w) + \frac{g}{2}\theta(w-w')\} \\ &= aw^2 + \frac{g}{2}, \end{aligned} \tag{9.10}$$

where a is a constant. With this last line providing a gravitational source, in addition to the cosmological constant term, the solution to Einstein's equations is given by

$$f(w) = 1 - 3(\frac{g^2w^2}{24M_5^3})^2 + 2(\frac{g^2w^2}{24M_5^3})^3. \tag{9.11}$$

This function vanishes at $w = \sqrt{24M_5^3}/g$, signally the horizon of a charged "black 3-brane" solution, a generalization of the Reissner-Nordstrom charged black hole. Regularity of the gauge field at this horizon requires it to vanish there [50], which then determines the constant, $a = -g^3/(48M_5^3)$.

This illustrates just one instance of how strongly-coupled many-body physics on the CFT side can be connected to black-hole physics on the AdS-side. See the reviews in Ref. [20] for greater elaboration.

In a similar way, emergent supersymmetry/supergravity have been discussed in Refs. [19] [51] [52] [53] [54] [55], with a general treatment, paralleling the one above for Lorentz invariance, given in Ref. [56].

10 Concluding Remarks

We have seen how a strongly-coupled CFT (or even its discrete progenitors) can robustly lead, "holographically", to emergent General Relativity and gauge theory in the AdS description. We first saw how a general CFT is dual to some AdS theory, but then proceeded to a more detailed understanding in the planar large-N limit of the CFT, which we saw was dual to the tree-level expansion of the AdS theory. Of course, the CFT at some finite N is fully interacting and quantum mechanical, so this must also be true on the AdS side. In particular, this requires AdS loop diagrams to unitarize the trees, corresponding to $1/N$ subleading corrections to the planar limit of the CFT. At the level of AdS effective field theory, UV loop divergences will arise, which must then be treated in the usual manner of low-energy non-renormalizable effective field theory, adding new counter-terms and input couplings at each new loop order in precision. This is still predictive when one works to a fixed loop order. But the *full* AdS theory with an infinite tower of particles must give UV-finite loop results, since it is exactly equivalent to the already-renormalized CFT in the $1/N$ expansion. It thereby UV-completes AdS effective field theory. Of course, this is why some type of string theory, with its famously good UV behavior, is such a good bet for the AdS dual. In any case, the CFT is dual to a fully unitary and well-defined quantum gravity on AdS.

There may be corrections, say scaling as e^{-N}, which are smaller than any order in the $1/N$-expansion. By the general form of the CFT/AdS mapping, these effects must be present on the AdS side, but they must be parametrically smaller than any order in the AdS loop expansion. In quantum gravity, we are in general poorly equipped to understand these effects, either in semi-classical General Relativity or in string theory. On the CFT side, we are faced with strong coupling at the quantitative level. Nevertheless, we at least have a *qualitative* understanding of quantum field theory for finite N, which translates into some qualitative understanding of AdS quantum gravity. The challenge is to mine this observation for new precisely-stated insights.

In a sense, *every* CFT has an energy-momentum tensor which is dual to some "graviton" with an associated quantum gravity theory on AdS. But that alone does not guarantee that the AdS description has a recognizable semi-classical General Relativity regime. That requires a finite number of light particle species (one being the 5D graviton) and a (approximate) Minkowski regime, conditions which are dual to having a large scaling-dimension gap, which in turn requires very strongly coupled CFTs. While a large dimension gap is certainly a non-trivial requirement, it seems a small price to pay for a full-blown theory of quantum gravity!

The set of such strongly coupled CFTs, supersymmetric and non-supersymmetric, are, not surprisingly, still far from fully explored. It may well be that the "landscape" of such UV-complete AdS/CFT theories is very rich, similar to the richness of the 4D string "landscape" that has emerged in recent years as completions of numerous 4D effective field theories containing gravity and gauge theory (reviewed in Ref. [57]). If this is true, in phenomenological modeling of strong coupling physics (CFT-side), one should develop an AdS effective theory, guided by IR self-consistency as well as experimental considerations, relatively confident that a UV completion, or equivalently a well-defined CFT, exists. This is how a great deal of particle physics model-building is being done, in the context of warped 5D compactifications. See Ref. [58] for a review, and Refs. [59] for AdS/CFT interpretation. Once promising phenomenological models have been developed, one can search for AdS UV completions. Ref. [55] is a particularly explicit and careful, but not fully realistic, example of this type in string theory, based on the earlier prototype of Ref. [19]. In this regard, it would be very helpful to continue to develop tools for engineering AdS string theories with specified properties.

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