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ABSTRACT

Using Exact Renormalization Group Equation approach and background field method, we investigate the one-loop problem in a six-dimensional conformal gravity theory whose Lagrangian takes the same form as holographic Weyl anomaly of multiple coincident M5-branes. We choose the backgrounds to be the symmetric Einstein spaces including S^6, CP^3, $S^2 \times S^4$, $S^2 \times CP^2$, $S^3 \times S^3$ and $S^2 \times S^2 \times S^2$. Evaluating the functional sums gives power-law and logarithmic divergences. We extract from the specific values of logarithmic divergence on above backgrounds, the coefficient in front of Euler density and two linear equations constraining the coefficients in front of three type-B conformal invariants. As a test of the effectiveness of Exact Renormalization Group Equation approach to quantum conformal gravity, we reexamine the one-loop problem in four-dimensional conformal gravity and confirm the logarithmic divergence derived from generalized Schwinger-DeWitt method.

1 Introduction

Conformal gravity is a class of gravity theory full of interests. It is constructed from Polynomials and conformally covariant derivatives of Weyl tensor. A Weyl transformation of the metric, $g_{\mu\nu} \to \Omega^2(x)g_{\mu\nu}$, is an exact symmetry of conformal gravity. Conformal gravity has appeared periodically in the literature for various reasons. In four dimensions, conformal gravity was considered as a possible UV completion of gravity in [1, 2, 3]. It also appeared as counter term in $\text{AdS}_5/\text{CFT}_4$ computations [4]. Asymptotically AdS and Lifshitz solutions in four-dimensional conformal gravity have been studied in [5, 6, 7, 8]. The recent work [8] reveals that these solutions can be related to AdS (dS)-Schwarzschild black holes in Einstein gravity by conformal transformations which also introduce new hair for asymptotically AdS black holes [8]. The principal reason that conformal gravity has not received general acceptance is because, since it is a higher-derivative theory, the propagating degrees of freedom contain ghost-like particles. Interestingly, it is suggested by [9] that in four-dimensional conformal gravity, the ghosts can be removed from the physical spectrum by imposing Neumann boundary condition which also selects Eintein solutions out of more numerous solutions of conformal graviy. Compared with Einstein gravity, conformal gravity possesses an advantage that the renormalized on-shell action of AdS-Schwarzschild black holes in Einstein gravity can be computed from the action of conformal gravity without referring to any regularization scheme. In [10], this scenario has been generalized to a six-dimensional conformal gravity theory whose Lagrangian takes the same form as the holographic Weyl anomaly of multiple coincident M5-branes [4].

The one-loop problem of four-dimensional conformal gravity has been tackled first in [2], subsequently in [11] and recently revisited by [12] which adopted generalized Schwinger-DeWitt method. One crucial point in the application of generalized Schwinger-DeWitt method is to express the 4th order operator appearing in the gauge fixed quadratic action as the minimal form

$$\mathbf{\Delta}^{(4)} = (\nabla^2)^2 \mathbf{1} + \mathbf{U}^{\mu\nu}\nabla_\mu\nabla_\nu + \mathbf{V}^\mu\nabla_\mu + \mathbf{W}, \tag{1.1}$$

by selecting a convenient gauge. With the aid of dimensional regularization, the one-loop effective action can be read from the $\mathbf{b}_4$ (or $\mathbf{a}_2$) coefficient in the heat kernel expansion of the 4th order minimal operator [13].

We recall that in the study of quantum gravity, Exact Renormalization Group Equation (ERGE) is also a powerful tool which has been applied to explore the asymptotic safety in Einstein gravity [14], extended gravity with R^2 term [15, 16] and topologically massive gravity [17]. For a nice review on the application of ERGE in quantum gravity, the reader is

referred to [18]. In this work, we will investigate the one-loop problem of the six-dimensional conformal gravity theory proposed in [10] by using Exact Renormalization Group Equation. In the calculation of one-loop divergences, we use background field method, therefore these divergences should be composed by terms invariant under general coordinate transformations. The backgrounds are chosen to be symmetric Einstein spaces including S^6, CP^3, $S^2 \times S^4$, $S^2 \times CP^2$, $S^3 \times S^3$ and $S^2 \times S^2 \times S^2$. Since the spectra of differential operators appearing in the gauge fixed quadratic action can be solved exactly for above backgrounds, the calculation of ERGE is converted to the sums of eigenvalues restricted by a "built-in" cutoff. By computing these sums, we obtain the power-law and logarithmic divergences. Generically, the one-loop logarithmic divergence in a classically conformally invariant theory is also conformally invariant. Here we assume such property is retained in six-dimensional quantum conformal gravity. Therefore, the logarithmic divergence consists of six-dimensional Euler density E_6, three type-B conformal invariants $I_i(i = 1, 2, 3)$ and total derivatives [19]. From the specific values of logarithmic divergence on above backgrounds, we extract the coefficient in front Euler density and two linear equations constraining the coefficients in front of three type-B conformal invariants. Before calculating the divergences in six-dimensional conformal gravity, we carry out a test of compatibility between ERGE approach and conformality of logarithmic divergence in quantum conformal gravity by applying ERGE approach to the one-loop problem in four-dimensional conformal gravity. It is shown that the results in [2, 11, 12] can be reproduced from ERGE approach. The power-law divergences corresponding to quantum generations of cosmological constant, Einstein-Hilbert term and quadratic curvature terms are presented. In the framework of ERGE approach, these power-law divergences will depend on the explicit form of cutoff, however they appear to be qualitatively stable [18]. The appearance of non-conformally invariant terms seems to imply the breaking of conformality in conformal gravity at quantum level and the renormalizability would render the final theory to be a general cubic curvature theories. Therefore from the point of view of perturbatively renormalizability, a more accurate treatment will take into account the contribution of non-conformally invariant terms to ERGE. However, in the same spirit of [20] we will focus on the pure conformal gravity truncation in order to see the divergences solely from the quantum conformal gravity effects.

The rest of this paper is organized as follows. In section 2, we briefly review the essence of Exact Renormalization Group Equation. In section 3, we calculate the one-loop divergences in four-dimensional conformal gravity by evaluating the ERGE. In section 4, we apply the ERGE approach to a six-dimensional conformal gravity theory and obtain the one-loop divergences. Especially, we elaborate on the derivation of logarithmic divergence. Conclusion are made in

section 5.

2 Preliminary of Exact Renormalization Equation

The effective action describing physical phenomena at a typical momentum scale k can be thought of as the result of integrating out all fluctuations of field with momenta larger than k. In this sense, k can be regarded as the IR cutoff of the functional integration. The dependence of the effective action on k is the Wilsonian RG flow. There are several ways of realizing this idea for practical use. One of the widely used approaches was proposed in [21]. According to [21], a suppression term $\Delta S_k[\phi]$

$$\Delta S_k[\phi] = \frac{1}{2} \int dx \phi(x) R_k(\mathcal{O}) \phi(x) \tag{2.1}$$

is added to the bare action $S[\phi]$ for the sake of suppressing the contribution from fluctuations with momentum lower than k. The cutoff function $R_k(z)$ is required to be a monotonically decreasing function in both z and k. Due to the properties of $R_k(z)$, k effectively plays the role of UV cutoff in the evaluation of the beta functions. One can define the k-dependent generating functional of connected Green functions by

$$e^{-W_k[J]} = \int D\phi \exp\left\{-S[\phi] - \Delta S_k[\phi] - \int dx J\phi\right\} \tag{2.2}$$

and a modified k-dependent Legendre transformation

$$\Gamma_k[\phi] = W_k[J] - \int dx J\phi - \Delta S_k[\phi]. \tag{2.3}$$

The functional Γ_k is called the effective average action and satisfies the following "Exact Renormalization Group Equation"

$$k\frac{d\Gamma_k}{dk} = \frac{1}{2}\mathrm{Tr}[\Gamma_k^{(2)} + R_k]^{-1} k\frac{dR_k}{dk}, \tag{2.4}$$

where $\Gamma_k^{(2)}$ is understood to be the exact connected two point function defined by

$$\Gamma_k^{(2)} = \frac{\delta^2 \Gamma_k}{\delta\phi(x)\delta\phi(y)}. \tag{2.5}$$

Given a cutoff function $R_k(z)$, Eq.(2.4) describes the evolution of effective action along momentum scale k. The ERGE can also be applied to the theory which does not possess renormalizability and have infinitely many couplings as long as the system permits a useful truncation.

The ERGE can also been seen as a RG-improved evolution equation for one-loop effective action. To see this, recall that given a bare action S, the one-loop effective action $\Gamma^{(1)}$ is

$$\Gamma^{(1)} = S + \frac{1}{2}\mathrm{Tr}\log\left[\frac{\delta^2 S}{\delta\phi\delta\phi}\right]. \tag{2.6}$$

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If we add to S the cutoff term Eq.(2.1), the functional becomes

$$\Gamma_k^{(1)} = S + \frac{1}{2}\text{Tr}\log\left[\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k\right].$$

(2.7)

It satisfies the equation

$$k\frac{d\Gamma_k^{(1)}}{dk} = \frac{1}{2}\text{Tr}\left[\frac{\delta^2 S}{\delta\phi\delta\phi} + R_k\right]^{-1}k\frac{dR_k}{dk},$$

(2.8)

which takes a similar form as Eq.(2.4) except that in the r.h.s, the renormalized coupling constants are replaced by the bare ones. Therefore, the ERGE can be perceived as RG-improved effective action since the running of coupling constants is retained in the exact propagator. However, in one-loop calculation, usually it is good enough to consider a weakly improved ERGE by neglecting the derivatives of the coupling constants occurring in the r.h.s of ERGE [18]. This weakly improved ERGE will be adopted in later calculations.

3 One-Loop Divergences in Four-Dimensional Conformal Gravity: A Test

3.1 Quadratic Gauge Fixed Action of Four-Dimensional Conformal Gravity

In the following, we work throughout with a Euclidean metric of positive signature. Four-dimensional conformal gravity is described by

$$S = \int d^4x\sqrt{g}\left(\alpha C_{\mu\nu\sigma\rho}^2 + \beta E_4\right),$$

(3.1)

where $C_{\mu\nu\sigma\rho}$ is the Weyl tensor and E_4 is the four-dimensional Euler density. Expanding the action around background metric $\bar{g}_{\mu\nu}$ as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, In terms of powers of $h_{\mu\nu}$ we get

$$S[g_{\mu\nu}] = \sum_n S^{(n)}[\bar{g}_{\mu\nu}, h_{\mu\nu}].$$

(3.2)

Invariance of the original action under the conformal transformation

$$g_{\mu\nu} \rightarrow e^{2\omega}g_{\mu\nu},$$

(3.3)

implies each $S^{(n)}$ should be invariant under the following transformations

$$\bar{g}_{\mu\nu} \rightarrow e^{2\omega}\bar{g}_{\mu\nu}, \quad h_{\mu\nu} \rightarrow e^{2\omega}h_{\mu\nu}.$$

(3.4)

Equivalently, to the leading order of ω we have

$$\bar{g}_{\mu\nu}\frac{\delta S^{(n)}[\bar{g}_{\mu\nu}, h_{\mu\nu}]}{\delta\bar{g}_{\mu\nu}} + h_{\mu\nu}\frac{\delta S^{(n)}[\bar{g}_{\mu\nu}, h_{\mu\nu}]}{\delta h_{\mu\nu}} = 0.$$

(3.5)

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On the other hand, the linearized conformal scaling symmetry

$$\bar{g}_{\mu\nu} \to \bar{g}_{\mu\nu}, \quad h_{\mu\nu} \to h_{\mu\nu} + 2\omega\bar{g}_{\mu\nu}, \tag{3.6}$$

states that $S^{(2)}$ is independent of $h \equiv g^{\mu\nu}h_{\mu\nu}$. Next, to fix the conformal invariance, we choose the following gauge fixing conditions for the local conformal symmetry

$$h \equiv g^{\mu\nu}h_{\mu\nu} = 0. \tag{3.7}$$

The gauge-fixing action for the diffeomorphism invariance is given by

$$S_{GF} = \int d^4x \sqrt{g}\chi_\mu Y^{\mu\nu}\chi_\nu, \tag{3.8}$$

where

$$\chi_\mu = \nabla^\mu(h_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}h), \quad Y^{\mu\nu} = -\alpha(\Box g^{\mu\nu} - \tfrac{1}{3}\nabla^\mu\nabla^\nu). \tag{3.9}$$

Here we make a simplest choice for $Y^{\mu\nu}$, so that in $S^{(2)} + S_{GF}$ the 4th order operator sandwiched by $h_{\mu\nu}$ can be expressed as standard minimal form. Since h does not enter χ^μ, it can be checked that the ghost fields associated with fixing the local conformal invariance will not contribute to the one-loop divergences. The Faddeev-Popov ghost action corresponding to χ_μ is determined by

$$S_{FP} = -\int d^4x \sqrt{g}\bar{C}_\mu \left(\Box g^{\mu\nu} + \tfrac{1}{2}\nabla^\mu\nabla^\nu + R^{\mu\nu} \right) C_\nu. \tag{3.10}$$

Since $Y^{\mu\nu}$ contains differential operators, it will contribute $-\tfrac{1}{2}\ln\det Y^\mu{}_\nu$ to the one-loop effective action. Exponentiating this logarithmic determinant, we obtain the action for Nielsen-Kallosh ghosts as follows

$$S_{NK} = \int d^4x \sqrt{g}\bar{\omega}_\mu Y^{\mu\nu}\omega_\nu + \int d^4x \sqrt{g}f_\mu Y^{\mu\nu}f_\nu. \tag{3.11}$$

In above expressions, $\bar{\omega}_\mu$ and ω_μ are complex anticommuting vectors, f_μ is a real commuting vector.

In order to achieve diagonalization of the inverse propagator we decompose the fluctuation $h_{\mu\nu}$ into irreducible parts

$$h_{\mu\nu} = h_{\mu\nu}^{TT} + \bar{\nabla}_\mu\xi_\nu^T + \bar{\nabla}_\nu\xi_\mu^T + \bar{\nabla}_\mu\bar{\nabla}_\nu\sigma - \tfrac{1}{4}\bar{g}_{\mu\nu}\Box\sigma + \tfrac{1}{4}\bar{g}_{\mu\nu}h, \tag{3.12}$$

where ξ_λ^T satisfies $\bar{\nabla}^\lambda\xi_\lambda^T = 0$ and $h_{\mu\nu}^{TT}$ satisfies $\bar{g}^{\mu\nu}h_{\mu\nu}^{TT} = 0$ and $\bar{\nabla}^\mu h_{\mu\nu}^{TT} = 0$. To achieve the maximal factorization of higher order operators, we choose the backgrounds to be symmetric Einstein spaces, for which

$$\bar{\nabla}_\lambda\bar{R}_{\mu\nu\sigma\rho} = 0; \quad \bar{R}_{\mu\nu} = \tfrac{\bar{R}}{4}\bar{g}_{\mu\nu}. \tag{3.13}$$

It is noted that in passing from a path integral over field $h_{\mu\nu}$ to one over the fields $(h_{\mu\nu}^{TT}, \xi_\nu^T, \sigma, h)$, one has to take into account the appropriate Jacobian factors. These Jacobian will be exactly canceled by other Jacobians arising from suitable redefinitions of ξ_ν^T and σ (see below). In the following, we drop the "bar" for notational simplicity. Inserting Eq.(3.12) into $S^{(2)} + S_{GF}$, we find

$$S^{(2)} + S_{GF} = -\frac{\alpha}{2} \int d^4x \sqrt{g} \left(h^{TT\mu\nu} (\Delta_L - \tfrac{R}{3})(\Delta_L - \tfrac{R}{2}) h_{\mu\nu}^{TT} \right)$$
$$-\alpha \int d^4x \sqrt{g} \left(\xi_\mu^T \Box(\Box + \tfrac{R}{4})^2 \xi^{T\mu} - \tfrac{3}{8}\sigma\Box(\Box + \tfrac{R}{3})^2(\Box + \tfrac{3R}{8})\sigma \right), \quad (3.14)$$

where Δ_L is the Lichnerowicz operator acting on $h_{\mu\nu}$. On a d-dimensional Einstein manifold with $R_{\mu\nu} = \frac{R}{d}g_{\mu\nu}$, the Lichnerowicz operator is defined by

$$\Delta_L h_{\mu\nu} = -\Box h_{\mu\nu} - 2R_{\mu\sigma\nu\rho}h^{\sigma\rho} + \tfrac{2R}{d}h_{\mu\nu}. \quad (3.15)$$

Because any Einstein metric solves the equaitons of motion of four-dimensional conformal gravity, the one-loop calculation performed in this paper is "on-shell". Similarly, we decompose the ghost fields as

$$C_\mu = C_\mu^T + \partial_\mu C; \quad \omega_\mu = \omega_\mu^T + \partial_\mu \omega; \quad f_\mu = f_\mu^T + \partial_\mu f, \quad (3.16)$$

and similarly for $\bar{C}_\mu$ and $\bar{\omega}_\mu$. This leads to

$$S_{FP} = \int d^4x \sqrt{g} \left(\bar{C}^{T\mu}(-\Box - \tfrac{R}{4})C_\mu^T + \tfrac{3}{2}\bar{C}\Box(\Box + \tfrac{R}{3})C \right),$$
$$S_{NK} = -\alpha \int d^4x \left(\bar{\omega}^{T\mu}\Box\omega_\mu^T - \tfrac{2}{3}\bar{\omega}\Box(\Box + \tfrac{3R}{8})\omega \right)$$
$$-\alpha \int d^4x \left(f^{T\mu}\Box f_\mu^T - \tfrac{2}{3}f\Box(\Box + \tfrac{3R}{8})f \right). \quad (3.17)$$

3.2 The Cutoff

In this section, we define the cutoff and express the ERGE as functional traces. We make the field redefinitions

$$\hat{\xi}_\mu^T = \sqrt{(-\Box - \tfrac{R}{4})}\xi_\mu^T; \quad \hat{\sigma} = \sqrt{-\Box(-\Box - \tfrac{R}{3})}\sigma; \quad (3.18)$$
$$\hat{C} = \sqrt{-\Box}C; \quad \hat{\omega} = \sqrt{-\Box}\omega; \quad \hat{f} = \sqrt{-\Box}f, \quad (3.19)$$

whose Jacobian factors cancel those coming from Eq.(3.12) and Eq.(3.16) [22, 23]. Then the action Eq.(3.14) becomes

$$S^{(2)} + S_{GF} = -\frac{\alpha}{2} \int d^4x \sqrt{g} \left[h^{TT\mu\nu} \Delta_{h\mu\nu}{}^{\rho\sigma} h_{\rho\sigma}^{TT} + q_1 \hat{\xi}^{T\mu} \Delta_{\xi\mu}{}^\nu \hat{\xi}_\nu^T + q_2 \hat{\sigma} \Delta_\sigma \hat{\sigma} \right], \quad (3.20)$$

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where we have defined the operators

$$\Delta_{h\,\mu\nu}^{\ \ \rho\sigma} = \Delta_{h,1\mu\nu}^{\ \ \ \lambda\delta}\Delta_{h,2\lambda\delta}^{\ \ \ \rho\sigma}, \quad \Delta_{h,1\mu\nu}^{\ \ \ \rho\sigma} = (\Delta_L - \tfrac{R}{3})\delta^{(\rho}_{(\mu}\delta^{\ \sigma)}_{\nu)}, \quad \Delta_{h,2\mu\nu}^{\ \ \ \rho\sigma} = (\Delta_L - \tfrac{R}{2})\delta^{(\rho}_{(\mu}\delta^{\ \sigma)}_{\nu)},$$

$$\Delta_{\xi\,\mu}^{\ \ \nu} = \Delta_{\xi,1\mu}^{\ \ \ \lambda}\Delta_{\xi,2\lambda}^{\ \ \ \nu}, \quad \Delta_{\xi,1\mu}^{\ \ \ \nu} = \Box\delta_\mu^\nu, \quad \Delta_{\xi,2\mu}^{\ \ \ \nu} = (-\Box - \tfrac{R}{4})\delta_\mu^\nu,$$

$$\Delta_\sigma = \Delta_{\sigma1}\Delta_{\sigma2}, \quad \Delta_{\sigma1} = (-\Box - \tfrac{R}{3}), \quad \Delta_{\sigma2} = (-\Box - \tfrac{3R}{8}), \tag{3.21}$$

and coefficients

$$q_1 = 2; \quad q_2 = -\frac{3}{4}. \tag{3.22}$$

Similarly, the ghost actions Eq.(3.23) become

$$S_{FP} = \int d^4x\sqrt{g}\Big(\bar{C}^{T\mu}\Delta_{C^T\mu}^{\ \ \ \nu}C_{T\nu} + q_3\hat{\bar{C}}\Delta_C\hat{C}\Big),$$

$$S_{NK} = \alpha\int d^4x\Big(\bar{\omega}^{T\mu}\Delta_{\omega^T\mu}^{\ \ \ \nu}\omega_\nu^T + q_4\hat{\bar{\omega}}\Delta_C\hat{\omega} + f^{T\mu}\Delta_{f^T\mu}^{\ \ \ \nu}f_\nu^T + q_5\hat{f}\Delta_f\hat{f}\Big) \tag{3.23}$$

where we defined operators

$$\Delta_{C^T\mu}^{\ \ \ \nu} = (-\Box - \tfrac{R}{4})\delta_\mu^\nu, \quad \Delta_C = (-\Box - \tfrac{R}{3})$$

$$\Delta_{\omega^T\mu}^{\ \ \ \nu} = -\Box\delta_\mu^\nu, \quad \Delta_\omega = (-\Box - \tfrac{3R}{8}),$$

$$\Delta_{f^T\mu}^{\ \ \ \nu} = -\Box\delta_\mu^\nu, \quad \Delta_f = (-\Box - \tfrac{3R}{8}) \tag{3.24}$$

and coefficients

$$q_3 = \tfrac{3}{2}, \quad q_4 = \tfrac{2}{3}, \quad q_5 = \tfrac{2}{3}. \tag{3.25}$$

We define the gauge fixed inverse propagator

$$\mathcal{O} = -\frac{\alpha}{2}\begin{pmatrix} \Delta_h & & \\ & \Delta_\xi & \\ & & \Delta_\sigma \end{pmatrix}. \tag{3.26}$$

For each spin component of $h_{\mu\nu}$, we choose the cutoff to have the following forms

$$\mathcal{R}_k = -\frac{\alpha}{2}\begin{pmatrix} R_k(\Delta_h) & & \\ & R_k(\Delta_\xi) & \\ & & R_k(\Delta_\sigma) \end{pmatrix}, \tag{3.27}$$

where $R_k = (k^4 - z)\theta(k^4 - z)$. Discussions of cutoffs for ghost fields is similar, however since the corresponding operators are second order, $R_k = (k^2 - z)\theta(k^2 - z)$. With above preparations, employing the ERGE, we can express the evolution equation of effective action with respect

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to the renormalization group "time $t \equiv \ln k$" as

$$\partial_t \Gamma_k = 2 \sum_n d_n^{h^{TT}} \theta(k^4 - \lambda_n^{h^{TT}}) + 2 \sum_n{}' d_n^{\xi^T} \theta(k^4 - \lambda_n^{\xi^T}) + 2 \sum_n{}'' d_n^{\sigma} \theta(k^4 - \lambda_n^{\sigma})$$

$$-2 \sum_n d_n^{C^T} \theta(k^2 - \lambda_n^{C^T}) - 2 \sum_n{}' d_n^{C} \theta(k^2 - \lambda_n^{C}) - 2 \sum_n d_n^{\omega^T} \theta(k^2 - \lambda_n^{\omega^T}),$$

$$-2 \sum_n d_n^{\omega} \theta(k^2 - \lambda_n^{\omega}) + \sum_n d_n^{f^T} \theta(k^2 - \lambda_n^{f^T}) + \sum_n{}' d_n^{f} \theta(k^2 - \lambda_n^{f}), \tag{3.28}$$

where λ_n and d_n stand for the level-n eigenvalue and corresponding degeneracy respectively. Note that single "prime" for ξ states that the vector modes corresponding to the Killing vectors are excluded. Single "prime" for scalars denotes that the constant scalar modes should not be taken into account. Double "prime" for σ means both the constant modes and the modes corresponding to conformal Killing vectors are not taken into account.

3.3   Evaluation of Functional Traces

By dimensional analysis, the evolution equation should have the following form

$$\partial_t \Gamma_k = \left(\mathcal{A}_0(k) R^{-2} + \mathcal{A}_2(k) R^{-1} + \mathcal{A}_4 + \mathcal{O}(R) \right), \tag{3.29}$$

where R is the scalar curvature of background metric. Utilizing the spectra of Hodge-de Rahm and Lichnerowicz operators on S^4, CP^2 (see Appendix) and $S^2 \times S^2$[1], We obtain all the divergences in the evolution equation of one-loop effective action. These results are exhibited in Table 1. In the calculation, it is assumed that $k^2 \gg R$. Upon integrating $\partial_t \Gamma_k$ to certain

Manifold	$\mathcal{A}_0(k)$	$\mathcal{A}_2(k)$	$\mathcal{A}_4$
S^4	$72k^4$	$-136k^2$	$-\frac{87}{5}$
CP^2	$54k^4$	$-102k^2$	$-\frac{31}{5}$
$S^2 \times S^2$	$48k^4$	$-\frac{272k^2}{3}$	$\frac{26}{45}$

Table 1: In this table, we present the divergences appearing in $\partial_t \Gamma_k$.

UV cutoff Λ, one can obtain the power-law and logarithmic divergences in one-loop effective action. Among these divergences, the most interesting one is the logarithmic divergence which can be parameterized as

$$\Gamma_{\log\Lambda} = \frac{\log \Lambda}{(4\pi)^2} \int d^4x \sqrt{g} \left(aE_4 + cC_{\mu\nu\sigma\rho}^2 + d\nabla_\mu J^\mu \right), \tag{3.30}$$

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[1]Eigenfunctions of Hodge-de Rahm and Lichnerowicz on $S^2 \times S^2$ can be constructed from the bilinear of Harmonics on S^2. For the recent application of harmonics on S^2, the reader is referred to [24].

Manifold	Vol/$(4\pi)^2$	$C^2_{\mu\nu\sigma\rho}$	E_4
S^4	$\frac{24}{R^2}$	0	$\frac{R^2}{6}$
CP^2	$\frac{18}{R^2}$	$\frac{R^2}{6}$	$\frac{R^2}{3}$
$S^2 \times S^2$	$\frac{16}{R^2}$	$\frac{R^2}{3}$	$\frac{R^2}{2}$

Table 2: In this table, we present the values of Euler density and Weyl squared on S^4, CP^2 and $S^2 \times S^2$.

When evaluated on S^4, CP^2 and $S^2 \times S^2$, Eq.(3.30) should reproduce the corresponding $\mathcal{A}_4$ coefficients. Since $\nabla_\mu J^\mu$ vanishes on Einstein spaces, using results of Table 2, we find

$$a = -\frac{87}{20}, \quad c = \frac{199}{30}. \tag{3.31}$$

These two coefficients agree with those obtained in [2, 11, 12] after a Wick rotation from Euclidean signature to Lorentzian.

Before proceeding to six dimensions, we would like to discuss another way of calculating the one-loop divergences in four-dimensional conformal gravity based on heat kernel expansion. By a straightforward calculation, we notice that on S^4, CP^2 and $S^2 \times S^2$ the functional sums in Eq.(3.28) is equivalent to the following expression up to $\mathcal{O}(k^{-2})$

$$
\begin{aligned}
\partial_t \Gamma_k = {}& \sum_n d_n^{h^{TT}} \theta(k^2 - \lambda_{n,1}^{h^{TT}}) + {\sum_n}' d_n^{\xi^T} \theta(k^2 - \lambda_{n,1}^{\xi^T}) + {\sum_n}'' d_n^\sigma \theta(k^2 - \lambda_{n,1}^\sigma) \\
& + \sum_n d_n^{h^{TT}} \theta(k^2 - \lambda_{n,2}^{h^{TT}}) + {\sum_n}' d_n^{\xi^T} \theta(k^2 - \lambda_{n,2}^{\xi^T}) + {\sum_n}'' d_n^\sigma \theta(k^2 - \lambda_{n,2}^\sigma) \\
& - 2\sum_n d_n^{C^T} \theta(k^2 - \lambda_n^{C^T}) - 2{\sum_n}' d_n^C \theta(k^2 - \lambda_n^C) - 2\sum_n d_n^{\omega^T} \theta(k^2 - \lambda_n^{\omega^T}), \\
& - 2{\sum_n}' d_n^\omega \theta(k^2 - \lambda_n^\omega) + \sum_n d_n^{f^T} \theta(k^2 - \lambda_n^{f^T}) + {\sum_n}' d_n^f \theta(k^2 - \lambda_n^f) + \mathcal{O}(k^{-2}),
\end{aligned} \tag{3.32}
$$

where each sum over eigenvalues of a 4th order operator has been splitted into two sums about eigenvalues of two 2nd order operators which factor the 4th order operator (see Eq.(3.21)).

Generically, for a d-dimensional 2nd order operator Δ

$$\mathrm{Tr}W(\Delta) = \left(Q_{\frac{d}{2}}(W)\mathbf{B}_0(\Delta) + Q_{\frac{d}{2}-1}(W)\mathbf{B}_2(\Delta) + \ldots + Q_0(W)\mathbf{B}_d(\Delta) + \ldots\right), \tag{3.33}$$

where $W(\Delta)$ is a function of Δ, $\mathbf{B}_n$ is related to the DeWitt-Seeley coefficient $\mathbf{b}_n$ by

$$\mathbf{B}_n = \frac{1}{(4\pi)^{\frac{d}{2}}} \int d^d x \sqrt{g} \, \mathbf{b}_n, \tag{3.34}$$

and based on $\tilde{W}$, the Laplace anti-tranform of W, Q_n is defined by

$$Q_n(W) = \int_0^\infty ds s^{-n} \tilde{W}(s). \tag{3.35}$$

Applying Eq.(3.33) to Eq.(3.32), we otain

$$
\begin{aligned}
\partial_t \Gamma_k &= \mathrm{Tr} W \left(\Sigma_i \gamma_i \Delta_i \right) \\
&= \left(Q_{\frac{d}{2}}(W) \mathbf{B}_0^t + Q_{\frac{d}{2}-1}(W) \mathbf{B}_2^t + \ldots + Q_0(W) \mathbf{B}_d^t + \ldots \right), \quad \mathbf{B}_n^t = \Sigma_i \gamma_i \mathbf{B}_n(\Delta_i), \tag{3.36}
\end{aligned}
$$

where $W(z) = \theta(k^2 - z)$, Δ_is are 2nd order operators whose eigenvalues appearing in the sums Eq.(3.32) and coefficients γ_is can be read from Eq.(3.32) directly. On symmetric Einstein spaces, we find

$$
\begin{aligned}
\mathcal{A}_0 &= \frac{1}{(4\pi)^2} \int d^d x \sqrt{g} \mathbf{b}_0^t Q_2, \quad \mathbf{b}_0^t = 6, \quad Q_2 = \frac{1}{2} k^4, \\
\mathcal{A}_2 &= \frac{1}{(4\pi)^2} \int d^d x \sqrt{g} \mathbf{b}_2^t Q_1, \quad \mathbf{b}_2^t = -\frac{17R}{3}, \quad Q_1 = k^2; \\
\mathcal{A}_4 &= \frac{1}{(4\pi)^2} \int d^d x \sqrt{g} \mathbf{b}_4^t Q_0, \quad \mathbf{b}_4^t = \frac{137}{60} R_{\mu\nu\sigma\rho}^2 - \frac{1}{30} R_{\mu\nu}^2 - \frac{79}{72} R^2, \quad Q_0 = 1. \tag{3.37}
\end{aligned}
$$

In the derivation of above formulae, the property of symmetric Einstein spaces has been used. Inserting the specific metrics on S^4, CP^2 and $S^2 \times S^2$ to above formulae, one can immediately reproduce the results listed in Table 1. Since $\mathbf{b}_0^t$ is equivalent to the number of degrees of freedom in a theory, thus we see that four-dimensional conformal gravity contains 6 degrees of freedom.

4 One-Loop Divergences in Six-Dimensional Conformal Gravity

4.1 Quadratic Gauge Fixed Action of Six-Dimensional Conformal Gravity

In six dimensions, there are three type-B conformal invariants consisting of[2] polynomials and covariant derivatives of Weyl tensor

$$
\begin{aligned}
I_1 &= C_{\mu\rho\sigma\nu} C^{\mu\alpha\beta\nu} C_\alpha{}^{\rho\sigma}{}_\beta, \\
I_2 &= C_{\mu\nu\rho\sigma} C^{\rho\sigma\alpha\beta} C_{\alpha\beta}{}^{\mu\nu}, \\
I_3 &= C_{\mu\rho\sigma\lambda} \left(\delta_\nu^\mu \Box + 4R^\mu{}_\nu - \frac{6}{5} R \delta_\nu^\mu \right) C^{\nu\rho\sigma\lambda} + \nabla_\mu J^\mu, \tag{4.1}
\end{aligned}
$$

where $\nabla_\mu J^\mu$ vanishes on symmetric Einstein spaces. In general, a Lagrangian constructed from general linear combinations of $I_i (i = 1, 2, 3)$ will give equations of motion that are not satisfied

---

[2]Here we following the notations of [25]

by arbitrary Einstein metrics. However, for a specific choice of the combination coefficients, the equations of motion will be satisfied by any Einstein metric. This same linear combination has the feature that, modulo total derivatives, all terms of cubic and quadratic order in the Riemann tensor are absent [10]. Then we achieve the following Lagrangian taking the same form as holographic Weyl anomaly of multiple coincident M5-branes.

$$e^{-1}\mathcal{L}_{\mathrm{conf}}^{6D} = \alpha(4I_1 + I_2 - \tfrac{1}{3}I_3 - \tfrac{1}{24}E_6). \tag{4.2}$$

Note that E_6 is the Euler density defined by

$$E_6 = \epsilon_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\epsilon^{\rho_1\sigma_1\rho_2\sigma_2\rho_3\sigma_3}R^{\mu_1\nu_1}{}_{\rho_1\sigma_1}R^{\mu_2\nu_2}{}_{\rho_2\sigma_2}R^{\mu_3\nu_3}{}_{\rho_3\sigma_3}. \tag{4.3}$$

After some algebraic manipulations the above Lagrangian can be recast into

$$e^{-1}\mathcal{L}_{\mathrm{conf}}^{6D} = \alpha\left(RR^{\mu\nu}R_{\mu\nu} - \tfrac{3}{25}R^3 - 2R^{\mu\nu}R^{\rho\sigma}R_{\mu\rho\nu\sigma} - R^{\mu\nu}\Box R_{\mu\nu} + \tfrac{3}{10}R\Box R\right) + t.d. \tag{4.4}$$

Expanding above action around background metric $\bar{g}_{\mu\nu}$ as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and using the six-dimensional field decomposition

$$h_{\mu\nu} = h_{\mu\nu}^{TT} + \bar{\nabla}_\mu\xi_\nu^T + \bar{\nabla}_\nu\xi_\mu^T + \bar{\nabla}_\mu\bar{\nabla}_\nu\sigma - \tfrac{1}{6}\bar{g}_{\mu\nu}\bar{\Box}\sigma + \tfrac{1}{6}\bar{g}_{\mu\nu}h, \tag{4.5}$$

we find that to the quadratic order in $h_{\mu\nu}$ the action of six-dimensional conformal gravity is given by

$$S^{(2)} = -\frac{\alpha}{2}\int d^6x\sqrt{g}\left(h^{TT\mu\nu}(\Delta_L - \tfrac{2\bar{R}}{15})(\Delta_L - \tfrac{\bar{R}}{5})(\Delta_L - \tfrac{\bar{R}}{3})h_{\mu\nu}^{TT}\right), \tag{4.6}$$

where ξ_ν^T, σ, h disappear, because they are gauge degrees of freedom in a conformally invariant gravity theory and $S^{(2)}$ is gauge independent. The background is understood to be a symmetric Einstein space with

$$\bar{\nabla}_\lambda\bar{R}_{\mu\nu\sigma\rho} = 0; \quad \bar{R}_{\mu\nu} = \tfrac{\bar{R}}{6}\bar{g}_{\mu\nu}. \tag{4.7}$$

In the following, "bar" is removed from the notation of background quantities for simplicity. Similar to the four-dimensional case, Jacobian factors coming from replacing $h_{\mu\nu}$ by $(h_{\mu\nu}^{TT}, \xi_\nu^T, \sigma, h)$ will be exactly canceled by other Jacobians arising from suitable redefinitions of ξ_ν^T and σ (see below). To fix the conformal invariance, we choose the following gauge fixing conditions for the local conformal symmetry

$$h \equiv g^{\mu\nu}h_{\mu\nu} = 0. \tag{4.8}$$

In order that the 6th order operator sandwiched by $h_{\mu\nu}$ takes the standard minimal form

$$\boldsymbol{\Delta}^{(6)} = (\nabla^2)^3\mathbf{1} + \mathbf{U}^{\mu\nu\rho\sigma}\nabla_\mu\nabla_\nu\nabla_\rho\nabla_\sigma + \mathbf{V}^{\mu\nu\rho}\nabla_\mu\nabla_\nu\nabla_\rho + \mathbf{W}^{\mu\nu}\nabla_\mu\nabla_\nu + \mathbf{X}^\mu\nabla_\mu + \mathbf{Z}, \tag{4.9}$$

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the gauge-fixing action for the diffeomorphism invariance is chosen to be

$$S_{GF} = \int d^4x \sqrt{g} \chi_\mu Y^{\mu\nu} \chi_\nu, \tag{4.10}$$

where

$$\chi_\mu = \nabla^\mu (h_{\mu\nu} - \tfrac{1}{6} g_{\mu\nu} h), \quad Y^{\mu\nu} = \tfrac{\alpha}{2} (\Box^2 g^{\mu\nu} - \tfrac{2}{5} \nabla^\mu \Box \nabla^\nu). \tag{4.11}$$

Inserting the decomposition Eq.(4.5) into the gauge-fixing action leads to

$$S_{GF} = \frac{\alpha}{2} \int d^6x \sqrt{g} \left(\xi^{T\mu} \Box^2 (\Box + \tfrac{R}{6})^2 \xi_\mu^T - \tfrac{5}{12} \sigma \Box (\Box + \tfrac{R}{5})^2 (\Box^2 + \tfrac{5R}{9}\Box + \tfrac{5R^2}{108}) \sigma \right). \tag{4.12}$$

Since h does not enter χ^μ, it can be checked that the conformal ghost fields associated with fixing the local conformal invariance will not contribute to the one-loop divergences. The Faddeev-Popov ghost action corresponding to χ_μ is determined by

$$S_{FP} = -\int d^6x \sqrt{g} \bar{C}_\mu \left(\Box g^{\mu\nu} + \tfrac{2}{3} \nabla^\mu \nabla^\nu + R^{\mu\nu} \right) C_\nu. \tag{4.13}$$

Because $Y^{\mu\nu}$ contains differential operators, it will contribute $-\tfrac{1}{2} \ln \det Y^\mu_{\;\nu}$ to the one-loop effective action. Exponentiating this logarithmic determinant, we obtain the action for Nielsen-Kallosh ghosts as follows

$$S_{NK} = \int d^6x \sqrt{g} \bar{\omega}_\mu Y^{\mu\nu} \omega_\nu + \int d^6x \sqrt{g} f_\mu Y^{\mu\nu} f_\nu. \tag{4.14}$$

In above expressions, $\bar{\omega}_\mu$ and ω_μ are complex anticommuting vectors, f_μ is a real commuting vector. We decompose the ghost fields as

$$C_\mu = C_\mu^T + \partial_\mu C; \quad \omega_\mu = \omega_\mu^T + \partial_\mu \omega; \quad f_\mu = f_\mu^T + \partial_\mu f, \tag{4.15}$$

and similarly for $\bar{C}_\mu$ and $\bar{\omega}_\mu$. This leads to

$$S_{FP} = \int d^6x \sqrt{g} \left(\bar{C}^{T\mu} (-\Box - \tfrac{R}{6}) C_\mu^T + \tfrac{5}{3} \bar{C} \Box (\Box + \tfrac{R}{5}) C \right),$$
$$S_{NK} = \frac{\alpha}{2} \int d^6x \left(\bar{\omega}^{T\mu} \Box^2 \omega_\mu^T - \tfrac{3}{5} \bar{\omega} \Box (\Box^2 + \tfrac{5R}{9}\Box + \tfrac{5R^2}{108}) \omega \right)$$
$$+ \frac{\alpha}{2} \int d^6x \left(f^{T\mu} \Box^2 f_\mu^T - \tfrac{3}{5} f \Box (\Box^2 + \tfrac{5R}{9}\Box + \tfrac{5R^2}{108}) f \right). \tag{4.16}$$

4.2 The Cutoff

In this section, we define the cutoff and express the ERGE as functional traces. We employ the field redefinitions

$$\hat{\xi}_\mu^T = \sqrt{(-\Box - \tfrac{R}{6})} \xi_\mu^T; \quad \hat{\sigma} = \sqrt{-\Box(-\Box - \tfrac{R}{5})} \sigma; \tag{4.17}$$

$$\hat{C} = \sqrt{-\square}C; \quad \hat{\omega} = \sqrt{-\square}\omega; \quad \hat{f} = \sqrt{-\square}f, \tag{4.18}$$

whose Jacobian factors cancel those coming from Eq.(4.5) and Eq.(4.15). Then the actions Eq.(4.6) and Eq.(4.12) become

$$S^{(2)} + S_{GF} = -\frac{\alpha}{2}\int d^6x\sqrt{g}\left[h^{TT\mu\nu}\Delta_{h\mu\nu}{}^{\rho\sigma}h^{TT}_{\rho\sigma} + p_1\hat{\xi}^{T\mu}\Delta_{\xi\mu}{}^{\nu}\hat{\xi}^T_\nu + p_2\hat{\sigma}\Delta_\sigma\hat{\sigma}\right], \tag{4.19}$$

where we have defined the operators

$$\Delta_{h\mu\nu}{}^{\rho\sigma} = \Delta_{h,1\mu\nu}{}^{\lambda\delta}\Delta_{h,2\lambda\delta}{}^{\alpha\beta}\Delta_{h,3\alpha\beta}{}^{\rho\sigma}$$

$$\Delta_{h,1\mu\nu}{}^{\rho\sigma} = (\Delta_L - \tfrac{2R}{15})\delta^{(\rho}_{(\mu}\delta^{\sigma)}_{\nu)}, \quad \Delta_{h,2\mu\nu}{}^{\rho\sigma} = (\Delta_L - \tfrac{R}{5})\delta^{(\rho}_{(\mu}\delta^{\sigma)}_{\nu)}, \quad \Delta_{h,3\mu\nu}{}^{\rho\sigma} = (\Delta_L - \tfrac{R}{3})\delta^{(\rho}_{(\mu}\delta^{\sigma)}_{\nu)},$$

$$\Delta_{\xi\mu}{}^{\nu} = \Delta_{\xi,1\mu}{}^{\lambda}\Delta_{\xi,2\lambda}{}^{\sigma}\Delta_{\xi,3\sigma}{}^{\nu},$$

$$\Delta_{\xi,1\mu}{}^{\nu} = \Delta_{\xi,2\mu}{}^{\nu} = -\square\delta^\nu_\mu, \quad \Delta_{\xi,3\mu}{}^{\nu} = (-\square - \tfrac{R}{6})\delta^\nu_\mu,$$

$$\Delta_\sigma = \Delta_{\sigma1}\Delta_{\sigma2},$$

$$\Delta_{\sigma1} = (-\square - \tfrac{R}{5}), \quad \Delta_{\sigma2} = (\square^2 + \tfrac{5R}{9}\square + \tfrac{5R^2}{108}), \tag{4.20}$$

and coefficients

$$p_1 = -1; \quad p_2 = -\tfrac{5}{12}. \tag{4.21}$$

Similarly, the ghost actions Eq.(4.16) become

$$S_{FP} = \int d^6x\sqrt{g}\left(\bar{C}^{T\mu}\Delta_{C^T\mu}{}^\nu C^T_\nu + p_3\hat{\bar{C}}\Delta_C\hat{C}\right),$$

$$S_{NK} = \frac{\alpha}{2}\int d^6x\left(\bar{\omega}^{T\mu}\Delta_{\omega^T\mu}{}^\nu\omega^T_\nu + p_4\hat{\bar{\omega}}\Delta_\omega\hat{\omega}\right)$$

$$+\frac{\alpha}{2}\int d^6x\left(f^{T\mu}\Delta_{f^T\mu}{}^\nu f^T_\nu + p_5\hat{f}\Delta_f\hat{f}\right), \tag{4.22}$$

where we defined operators

$$\Delta_{C^T\mu}{}^\nu = (-\square - \tfrac{R}{6})\delta_\mu{}^\nu, \quad \Delta_C = (-\square - \tfrac{R}{5})$$

$$\Delta_{\omega^T\mu}{}^\nu = \square^2\delta_\mu{}^\nu, \quad \Delta_\omega = (\square^2 + \tfrac{5R}{9}\square + \tfrac{5R^2}{108}),$$

$$\Delta_{f^T\mu}{}^\nu = \square^2\delta_\mu{}^\nu, \quad \Delta_f = (\square^2 + \tfrac{5R}{9}\square + \tfrac{5R^2}{108}) \tag{4.23}$$

and coefficients

$$p_3 = \tfrac{5}{3}, \quad p_4 = \tfrac{3}{5}, \quad p_5 = \tfrac{3}{5}. \tag{4.24}$$

We define the gauge fixed inverse propagator

$$\mathcal{O} = -\frac{\alpha}{2}\begin{pmatrix} \Delta_h & & \\ & \Delta_\xi & \\ & & \Delta_\sigma \end{pmatrix}. \tag{4.25}$$

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For each spin component of $h_{\mu\nu}$, we choose the cutoff to have the following forms

$$\mathcal{R}_k = -\frac{\alpha}{2}\begin{pmatrix} R_k(\Delta_h) & & \\ & R_k(\Delta_\xi) & \\ & & R_k(\Delta_\sigma) \end{pmatrix},\tag{4.26}$$

where $R_k = (k^6 - z)\theta(k^6 - z)$. Discussions of cutoffs for ghost fields is similar, however we choose $R_k = (k^2 - z)\theta(k^2 - z)$ for Faddeev-Popov ghost and $R_k = (k^4 - z)\theta(k^4 - z)$ for Nielsen-Kallosh ghosts, since the former is related to 2nd order operators while the latter is associated with 4th order operators. The evolution equation of effective action with respect to the renormalization group "time $t \equiv \ln k$" can be expressed as

$$\partial_t \Gamma_k = 3\sum_n d_n^{h^{TT}}\theta(k^6 - \lambda_n^{h^{TT}}) + 3{\sum_n}'d_n^{\xi^T}\theta(k^6 - \lambda_n^{\xi^T}) + 3{\sum_n}''d_n^\sigma\theta(k^6 - \lambda_n^\sigma)$$

$$-2\sum_n d_n^{C^T}\theta(k^2 - \lambda_n^{C^T}) - 2{\sum_n}'d_n^C\theta(k^2 - \lambda_n^C) - 4\sum_n d_n^{\omega^T}\theta(k^4 - \lambda_n^{\omega^T}),$$

$$-4{\sum_n}'d_n^\omega\theta(k^4 - \lambda_n^\omega) + 2\sum_n d_n^{f^T}\theta(k^4 - \lambda_n^{f^T}) + 2{\sum_n}'d_n^f\theta(k^4 - \lambda_n^f),\tag{4.27}$$

where single "prime" for ξ states that the vector modes corresponding to the Killing vectors are excluded. Single "prime" for scalars denotes that the constant scalar modes should not be taken into account. Double "prime" for σ means both the constant modes and the modes corresponding to conformal Killing vectors are not taken into account.

4.3 Evaluation of Functional Traces

In six dimensions, $\partial_t\Gamma_k$ should take the following general form

$$\partial_t\Gamma_k = \left(\mathcal{A}_0(k)R^{-3} + \mathcal{A}_2(k)R^{-2} + \mathcal{A}_4(k)R^{-1} + \mathcal{A}_6 + \mathcal{O}(R)\right).\tag{4.28}$$

By using the spectra of harmonics on S^6 (See Appendix), we obtained

$$\partial_t\Gamma_k = \frac{2700k^6}{R^3} - \frac{1470k^4}{R^2} - \frac{19415k^2}{54R} - \frac{601}{21} + \mathcal{O}(k^{-2}).\tag{4.29}$$

Here we emphasize that splitting a sum over eigenvalues of a 6th order operator into sums over eigenvalues of three 2nd order operators which factor the 6th order operator will give result differing from the original one by terms with both negative and positive powers of k. For instance, on S^6

$$3\sum_n d_n^{h^{TT}}\theta(k^6 - \lambda_n^{h^{TT}}) = \sum_{i=1}^3\sum_n d_n^{h^{TT}}\theta(k^2 - \lambda_{n,i}^{h^{TT}}) - \frac{98k^2}{3R} + \mathcal{O}(k^{-2}).\tag{4.30}$$

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However, one can straightforwardly prove that $\mathcal{A}_0(k)$ is intact when splitting a sum over eigenvalues of a 6th order operator into three sums over eigenvalues of three 2nd order operators which factor the 6th order operator. Meanwhile, if we denote a $2p$th order operator by Δ_1, a $2q$th order operator by Δ_2 and corresponding eigenvalues by λ_1, λ_2, we find by using the property of generalized zeta function, that the k independent terms in $(p+q)\sum_n \theta(k^{2(p+q)} - \lambda_{n,1}\lambda_{n,2})$ and $p\sum_n \theta(k^{2p} - \lambda_{n,1}) + q\sum_n \theta(k^{2q} - \lambda_{n,2})$ are equal. With this observation, we find that the k independent term in Eq.(4.27) is equal to that in a simpler expression only consisting of sums over eigenvalues of 2nd order operators. The sums over eigenvalues of a 2nd order operator can be computed from DeWitt-Seeley coefficients. By this easier way, we obtain the $\mathcal{A}_6$ coefficients in Eq.(4.27) on symmetric Einstein spaces including S^6, CP^3, $S^2 \times S^4$, $S^2 \times CP^2$, $S^3 \times S^3$ and $S^2 \times S^2 \times S^2$. These results can be found in table 3. One can also derive them by using the spectra of Hodge-de Rahm and Lichnerowicz operators on above backgrounds (see Appendix[3]).

Manifold	$\mathrm{Vol}/(4\pi)^3$	I_1	I_2	I_3	E_6	$\mathcal{A}_6$
S^6	$\frac{450}{R^3}$	0	0	0	$\frac{16}{75}R^3$	$-\frac{601}{21}$
CP^3	$\frac{576}{R^3}$	$-\frac{29}{7200}R^3$	$\frac{31}{1800}R^3$	$-\frac{4}{75}R^3$	$\frac{2}{3}R^3$	$-\frac{58526}{2625}$
$S^2 \times S^4$	$\frac{324}{R^3}$	$-\frac{34}{6075}R^3$	$\frac{104}{6075}R^3$	$-\frac{128}{2075}R^3$	$\frac{16}{27}R^3$	$-\frac{79462}{7875}$
$S^2 \times CP^2$	$\frac{243}{R^3}$	$-\frac{44}{6075}R^3$	$\frac{214}{6075}R^3$	$-\frac{208}{2025}R^3$	$\frac{32}{27}R^3$	$-\frac{11168}{2625}$
$S^3 \times S^3$	$\frac{108\pi}{R^3}$	$-\frac{3}{400}R^3$	$\frac{1}{300}R^3$	$-\frac{4}{75}R^3$	0	$-\frac{1417\pi}{500}$
$S^2 \times S^2 \times S^2$	$\frac{216}{R^3}$	$-\frac{2}{225}R^3$	$\frac{4}{75}R^3$	$-\frac{32}{225}R^3$	$\frac{16}{9}R^3$	$-\frac{732}{875}$

Table 3: In this table, we present the values of conformal invariants and Euler density on S^6, CP^3, $S^2 \times S^4$, $S^2 \times CP^2$, $S^3 \times S^3$ and $S^2 \times S^2 \times S^2$. The values of $\mathcal{A}_6$ coefficients on above backgrounds are also given.

Upon integrating $\partial_t \Gamma_k$ to certain UV cutoff Λ, we can obtain the power-law and logarithmic divergences in one-loop effective action. The logarithmic divergence can be parameterized as

$$\Gamma_{\log\Lambda} = \frac{\log\Lambda}{(4\pi)^3} \int d^6x \sqrt{g}\left(aE_6 + c_1 I_1 + c_2 I_2 + c_3 I_3 + d\nabla_\mu J^\mu\right). \qquad (4.31)$$

When evaluated on S^6, CP^3, $S^2 \times S^4$, $S^2 \times CP^2$, $S^3 \times S^3$ and $S^2 \times S^2 \times S^2$, Eq.(4.31) should reproduce the corresponding $\mathcal{A}_6$ coefficients. Because $\nabla_\mu J^\mu$ vanishes on symmetric Einstein spaces, using results of Table 3, we find

$$a = -\frac{601}{2016}, \quad c_1 = \frac{5633}{105} - 4c_2, \quad c_3 = -\frac{35543}{5040} + \frac{5c_2}{8}, \qquad (4.32)$$

---

[3]The spectra of Hodge-de Rahm and Lichnerowicz operators on S^3 can be found in appendix of [17]

That $c_i(i = 1, 2, 3)$ can not be completely fixed is because of the following equality satisfied on symmetric Einstein spaces.

$$R^{\mu\nu}{}_{\sigma\rho}R^{\sigma\rho}{}_{\lambda\delta}R^{\lambda\delta}{}_{\mu\nu} = -4R^{\mu}{}_{\sigma}{}^{\nu}{}_{\rho}R^{\sigma}{}_{\lambda}{}^{\rho}{}_{\delta}R^{\lambda}{}_{\mu}{}^{\delta}{}_{\nu} + \tfrac{1}{3}RR^{\mu\nu}{}_{\sigma\rho}R^{\sigma\rho}{}_{\mu\nu}. \qquad (4.33)$$

Taking into account above equality, we find that in $\sum_i c_i I_i$, the coefficients of $R^{\mu}{}_{\sigma}{}^{\nu}{}_{\rho}R^{\sigma}{}_{\lambda}{}^{\rho}{}_{\delta}R^{\lambda}{}_{\mu}{}^{\delta}{}_{\nu}$ and $RR^{\mu\nu}{}_{\sigma\rho}R^{\sigma\rho}{}_{\mu\nu}$ take definite values. They are given by $-\frac{5633}{105}$ and $\frac{20389}{18900}$ respectively. To obtain these two numerical values, we have expanded Weyl tensor in terms of Rieman tensor, Ricci tensor and Ricci scalar as well as replacing $R_{\mu\nu}$ by $\frac{R}{6}g_{\mu\nu}$.

5 Conclusion

Employing the Exact Renormalization Group Equation and background field method, we explore the one-loop problem in a six-dimensional conformal gravity theory whose Lagrangian takes the same form as holographic Weyl anomaly of multiple coincident M5-branes. The backgrounds are chosen to be the symmetric Einstein spaces including S^6, CP^3, $S^2 \times S^4$, $S^2 \times CP^2$, $S^3 \times S^3$ and $S^2 \times S^2 \times S^2$. By evaluating the functional sums, we obtain power-law and logarithmic divergences. The power-law divergences do not appear in [2, 12], because they are annihilated by dimensional regularization. Since any Einstein metric solves the equations of motion of six-dimensional conformal gravity considered in this paper, the one-loop calculation is on-shell and therefore according to [31] the logarithmic divergence should not depend on the explicit form of gauge fixing action. Because an equality satisfied on symmetric Einstein spaces relates different cubic curvature structures to each other, there are only three independent cubic curvature structures R^3, $R^{\mu}{}_{\sigma}{}^{\nu}{}_{\rho}R^{\sigma}{}_{\lambda}{}^{\rho}{}_{\delta}R^{\lambda}{}_{\mu}{}^{\delta}{}_{\nu}$ and $RR^{\mu\nu}{}_{\sigma\rho}R^{\sigma\rho}{}_{\mu\nu}$ on symmetric Einstein spaces. As a consequence, we cannot fix the coefficients in front of four non-trivial conformal invariants in the one-loop logarithmic divergence. However, if the backgrounds are chosen to be the more general Einstein spaces, we will not achieve the maximal factorization in the gauge fixed quadratic action, and the calculation becomes infeasible. From the explicit values of logarithmic divergence on above backgrounds, we extract the coefficient in front of Euler density and two linear equations constraining the coefficients in front of three type-B conformal invariants. It should be interesting and challenging to conceive a new way by which the three coefficients in front of type-B conformal invariants can be entirely fixed. Direct application of generalized Schwinger-DeWitt method to six dimensions would be extremely tedious.

Acknowledgement

A   Spectrum of Laplacian Operator on S^4

Spin	Operator	Eigenvalue	Degeneracy	Lowest value of n
0	$-\Box$	$\frac{R}{12}(n^2+3n)$	$\frac{1}{6}(n+1)(n+2)(2n+3)$	0
1	$-\Box$	$\frac{R}{12}(n^2+3n-1)$	$\frac{3}{6}n(n+3)(2n+3)$	1
2	$-\Box$	$\frac{R}{12}(n^2+3n-2)$	$\frac{5}{6}(n-1)(n+4)(2n+3)$	2

Table 4: In this table, we present the spectra of scalar, vector and tensor harmonics on S^4.

B   Spectra of Hodge-de Rahm and Lichnerowicz operators on CP^2

Spin	Operator	Eigenvalue	Degeneracy	Lowest value of n
0	$-\Box$	$\frac{R}{6}(n^2+2n)$	$(n+1)^3$	0
1	$-\Box+\frac{R}{4}$	$\frac{R}{6}(n^2+2n)$	$(n+1)^3$	1
1	$-\Box+\frac{R}{4}$	$\frac{R}{6}(n^2+5n+6)$	$(n+1)(n+4)(2n+5)$	0
2	Δ_L	$\frac{R}{6}(n^2+2n)$	$(n+1)^3$	2
2	Δ_L	$\frac{R}{6}(n^2+5n+6)$	$(n+1)(n+4)(2n+5)$	1
2	Δ_L	$\frac{R}{6}(n^2+8n+18)$	$(n+1)(n+7)(2n+8)$	0

Table 5: In this table, the spectra of Hodge-de Rahm and Lichnerowicz operators related to scalar, vector and tensor on CP^2 are given. An explicit construction of these harmonics can be found in [27, 28] and a group theoretical way of obtaining the spectra has been discussed in [29]

Spin	Operator	Eigenvalue	Degeneracy	Lowest value of n
0	$-\Box$	$\frac{R}{30}(n^2+5n)$	$\frac{1}{120}(n+1)(n+2)(n+3)(n+4)(2n+5)$	0
1	$-\Box$	$\frac{R}{30}(n^2+5n-1)$	$\frac{5}{120}n(n+2)(n+3)(n+5)(2n+5)$	1
2	$-\Box$	$\frac{R}{30}(n^2+5n-2)$	$\frac{14}{120}(n-1)(n+2)(n+3)(n+6)(2n+5)$	2

Table 6: In this table, we exhibit the spectra of scalar, vector and tensor harmonics on S^6.

C Spectrum of Laplacian Operator on S^6

D Spectra of Hodge-de Rahm and Lichnerowicz operators on CP^3

Since $CP^3 \simeq SU(4)/(SU(3) \times U(1))$ is a coset space, one can use the method presented in [30] to calculate the spectrum.

Spin	Operator	Eigenvalue	Degeneracy	Lowest value of n
0	$-\Box$	$\frac{R}{12}(n^2+3n)$	$\frac{1}{12}(1+n)^2(2+n)^2(3+2n)$	0
1	$-\Box+\frac{R}{6}$	$\frac{R}{12}(n^2+3n)$	$\frac{1}{12}(1+n)^2(2+n)^2(3+2n)$	1
1	$-\Box+\frac{R}{6}$	$\frac{R}{12}(n^2+6n+8)$	$\frac{8}{12}(1+n)(3+n)^3(5+n)$	0
2	Δ_L	$\frac{R}{12}(n^2+3n)$	$\frac{1}{12}(1+n)^2(2+n)^2(3+2n)$	2
2	Δ_L	$\frac{R}{12}(n^2+6n+8)$	$\frac{8}{12}(1+n)(3+n)^3(5+n)$	1
2	Δ_L	$\frac{R}{12}(22+9n+n^2)$	$\frac{6}{12}(1+n)(4+n)(5+n)(8+n)(9+2n)$	0
2	Δ_L	$\frac{R}{12}(n^2+5n+6)$	$\frac{3}{12}(1+n)^2(4+n)^2(5+2n)$	0

Table 7: In this table, we present the spectra of Hodge-de Rahm and Lichnerowicz operators on CP^3.

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