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Cosmic string interactions induced by gauge and scalar fields

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We study the interaction between two parallel cosmic strings induced by gauge fields and by scalar fields with non-minimal couplings to curvature. For small deficit angles the gauge field behaves like a collection of non-minimal scalars with a specific value for the non-minimal coupling. We check this equivalence by computing the interaction energy between strings at first order in the deficit angles. This result provides another physical context for the “contact terms” which play an important role in the renormalization of black hole entropy due to a spin-1 field.

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1 Introduction

For a single cosmic string in four Euclidean dimensions the metric is [1, 2]

$$ds^2 = dr^2 + r^2 d\psi^2 + d\tau^2 + dz^2 \quad (1)$$

The string tension produces a deficit angle, $\psi \approx \psi + \beta$ where

$$\beta = 2\pi - 8\pi\lambda \quad (2)$$

Here $\lambda = G\mu$ where G is Newton's constant and μ is the mass per unit length of the string.

We will be interested in the interaction between two parallel cosmic strings. At the classical level there is no force between strings,¹ but (as in the Casimir effect) an interaction potential can be generated at one loop by a quantum field propagating on this background. For simplicity we will take a perturbative approach, and calculate the interaction energy at first order in the product of the two deficit angles. We consider two types of fields – scalar fields with a non-minimal coupling to curvature, and abelian gauge fields – as the main point of this paper is to highlight a relation between these two cases. Vacuum polarization in the presence of a single cosmic string has been studied before; see for example [5, 6, 7] for scalar fields and [8, 9] for gauge fields. For related calculations in the presence of multiple cosmic strings see [10, 11].

We begin by recalling the argument that, to first order in the background curvature, there should be a relation between gauge fields and scalar fields with specific non-minimal couplings to curvature. To our knowledge this relation was first stated in [12], although the essence of the following argument is taken from [13]. Consider a spacetime which is a product $\mathcal{M}_n \times \mathbb{R}^{d-n}$ of a weakly-curved n -dimensional Einstein manifold \mathcal{M}_n with flat space \mathbb{R}^{d-n} . The metric takes the form

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + \delta_{ij} dx^i dx^j \quad (3)$$

where x^α are coordinates on \mathcal{M}_n and x^i are coordinates on \mathbb{R}^{d-n} . The Einstein manifold has Ricci curvature $R_{\alpha\beta} = \frac{1}{n} g_{\alpha\beta} R$.² Choose a vielbein $g_{\alpha\beta} = e_\alpha^a e_\beta^b \delta_{ab}$ and denote the corresponding spin connection ω_α .

To establish the relation between gauge and scalar fields we compare their equations of motion. For a gauge field, the equations of motion in Feynman gauge are

$$-\nabla_\nu \nabla^\nu A_\mu + R_{\mu\nu} A^\nu = 0 \quad (4)$$

¹In classical gravity there is, however, a non-trivial scattering amplitude which results from the conical boundary conditions [3, 4].

²By Einstein manifold we mean a manifold with Ricci curvature locally proportional to the metric, $R_{\alpha\beta}(x) = f(x)g_{\alpha\beta}(x)$. In two dimensions all manifolds are Einstein. In higher dimensions the contracted Bianchi identity $\nabla^\mu (R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0$ requires that f be a constant. In either case it follows from the definition that $R_{\alpha\beta} = \frac{1}{n}g_{\alpha\beta}R$.

where $x^\mu = (x^\alpha, x^i)$. There are ghosts associated with this choice of gauge which behave like a pair of minimally-coupled scalar fields [14]. The components of the gauge field tangent to \mathbb{R}^{d-n} obey

$$-\nabla_\beta \nabla^\beta A_i - \partial_j \partial^j A_i = 0 \quad (5)$$

where the covariant derivative ∇_α treats A_i as a singlet of $SO(n)$. That is, the components A_i behave like minimally-coupled scalar fields. The components of the gauge field tangent to \mathcal{M}_n , on the other hand, obey

$$-\nabla_\beta \nabla^\beta A_a - \partial_j \partial^j A_a + \frac{1}{n} R A_a = 0 \quad (6)$$

Here ∇_α acts on $A_a = e_a^\alpha A_\alpha$ in the fundamental representation of $SO(n)$, and we've made use of the fact that $R_{\alpha\beta} = \frac{1}{n} g_{\alpha\beta} R$. So the components A_a are in the fundamental representation of $SO(n)$ and have an explicit non-minimal coupling to curvature.

Physical quantities can be computed perturbatively, as an expansion in powers of the background curvature. As a concrete example imagine computing the effective action for the background which results from integrating out A_μ . The spin connection can appear in the effective action, but only through its field strength $F = d\omega + \omega^2$. In fact the field strength can first appear in the effective action in terms such as $F_{\alpha\beta} F^{\alpha\beta}$ that are quadratic in the curvature. So to first order in the background curvature we can forget about the spin connection and treat A_a as a collection of n scalar fields with a non-minimal coupling to curvature. The equation of motion for a non-minimal scalar is

$$-\nabla_\beta \nabla^\beta \phi - \partial_j \partial^j \phi + \xi R \phi = 0, \quad (7)$$

and comparing to (6) we identify the effective non-minimal coupling parameter $\xi = 1/n$. Thus to first order in the background curvature a gauge field is equivalent to n scalar fields with $\xi = 1/n$, plus $d - n$ minimally-coupled scalars.

This discussion is relevant to parallel cosmic strings because in two dimensions every manifold is an Einstein manifold. The argument suggests that, to first order in the product of the deficit angles, the interaction between two cosmic strings induced by a gauge field should be the same as the interaction induced by an appropriate collection of non-minimal scalars.

In the remainder this paper we verify this claim, by computing the interaction energy between cosmic strings perturbatively. In section 2 we compute the interaction energy for a scalar field, and in section 3 we carry out the corresponding computation for a gauge field. We conclude in section 4, where we comment on our results and point out the relation to studies of black hole entropy.

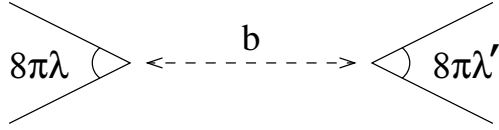


Figure 1: Two parallel cosmic strings, separated by a distance b .

2 Non-minimal scalar energy

The Euclidean action is

$$S = \int d^4x \sqrt{g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} \xi R \phi^2 \right)$$

For the conical geometry (1) the scalar curvature is³

$$R = 16\pi\lambda\delta^2(x)/\sqrt{g} \quad (8)$$

The action on a cone can be split into three pieces,

$$S_{\text{cone}} = S_0 + S_{\text{int}}, \quad S_{\text{int}} = S_{\text{wedge}} + S_{\text{tip}} \quad (9)$$

where

$$S_0 = \int d^4x \frac{1}{2} \delta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \quad (10)$$

is the action in flat space,

$$S_{\text{wedge}} = - \int d\tau dz \int_0^\infty r dr \int_{-4\pi\lambda}^{4\pi\lambda} d\psi \frac{1}{2} \delta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \quad (11)$$

cancels the flat-space action in the region corresponding to the deficit angle, and

$$S_{\text{tip}} = \int d\tau dz 8\pi\lambda\xi\phi^2 \quad (12)$$

arises from the non-minimal coupling to curvature. It's straightforward to extend this to a pair of cosmic strings, just by putting the deficit angles in opposite directions as shown in Fig. 1.

³The easiest way to see this is to note that a truncated cone, i.e. a disc with a conical singularity at the center, has Euler characteristic $\chi = \frac{1}{4\pi} \int d^2x \sqrt{g} R + \frac{1}{2\pi} \beta = 1$.

We will treat S_{int} as a perturbation.⁴ To find the interaction energy per unit length along the strings \mathcal{H}_{int} we use [15]

$$\int d\tau dz \mathcal{H}_{\text{int}} = \langle 1 - e^{-S_{\text{int}}} \rangle_{C,0} \quad (13)$$

where the subscript $C,0$ denotes a connected correlation function computed in the unperturbed theory (10). Expanding in powers of S_{int} , the leading $\mathcal{O}(\lambda\lambda')$ interaction between the strings comes from

$$\int d\tau dz \mathcal{H}_{\text{int}} \approx -\langle S_{\text{int}}^{(1)} S_{\text{int}}^{(2)} \rangle_{C,0} \quad (14)$$

where the superscripts (1), (2) refer to the first and second cosmic string, respectively. Some useful unperturbed correlators are

$$\langle \phi(x) \phi(x') \rangle = \frac{1}{4\pi^2} \frac{1}{(x - x')^2} \quad (15)$$

and

$$\begin{aligned} \langle (\partial\phi)^2(x) (\partial\phi)^2(x') \rangle &= \frac{6}{\pi^4} \frac{1}{(x - x')^8} \\ \langle (\partial\phi)^2(x) \phi^2(x') \rangle &= \frac{1}{2\pi^4} \frac{1}{(x - x')^6} \\ \langle \phi^2(x) \phi^2(x') \rangle &= \frac{1}{8\pi^4} \frac{1}{(x - x')^4} \end{aligned}$$

There are three types of interactions. For generality we can imagine that the two strings have different non-minimal couplings ξ, ξ' .

wedge – wedge

To first order in λ and λ' the wedges can be treated as very narrow, so that

$$\begin{aligned} \mathcal{H}_{\text{int}} &= -16\pi^2 \lambda \lambda' \int d\tau dz \int_0^\infty x dx \int_0^\infty x' dx' \frac{6}{\pi^4} \frac{1}{(\tau^2 + z^2 + (x + x' + b)^2)^4} \\ &= -\frac{4\lambda\lambda'}{15\pi b^2} \end{aligned}$$

⁴This is somewhat subtle, since it's not manifest that perturbation theory in S_{int} will enforce the proper conical boundary condition $\phi(r, \psi) = \phi(r, \psi + \beta)$. Fortunately the boundary conditions are controlled by the spin connection on the cone, which as we argued in the introduction can only enter at second order in the deficit angle.

wedge – tip

For wedge 1 with tip 2 we have

$$\begin{aligned}\mathcal{H}_{\text{int}} &= 32\pi^2\lambda\lambda'\xi'\int d\tau dz \int_0^\infty x dx \frac{1}{2\pi^4} \frac{1}{(\tau^2 + z^2 + (x+b)^2)^3} \\ &= \frac{4\lambda\lambda'\xi'}{3\pi b^2}\end{aligned}$$

tip – tip

The interaction between the two tips is

$$\begin{aligned}\mathcal{H}_{\text{int}} &= -64\pi^2\lambda\lambda'\xi\xi'\int d\tau dz \frac{1}{8\pi^4} \frac{1}{(\tau^2 + z^2 + b^2)^2} \\ &= -\frac{8\lambda\lambda'\xi\xi'}{\pi b^2}\end{aligned}$$

Assembling these results, to first order in λ and λ' the interaction energy per unit length due to a non-minimally coupled scalar field is

$$\mathcal{H}_{\text{int}} = \frac{\lambda\lambda'}{\pi b^2} \left(-\frac{4}{15} + \frac{4}{3}(\xi + \xi') - 8\xi\xi' \right) \quad (16)$$

To check the validity of our perturbative approach consider computing $\langle\phi^2\rangle$ for a minimally-coupled scalar field in the presence of a single cosmic string. At first order in perturbation theory, after subtracting the divergence which is present in flat space, we have

$$\langle\phi^2\rangle = -\langle\phi^2 S_{\text{wedge}}\rangle_{C,0} = \frac{\lambda}{6\pi^2 r^2} \quad (17)$$

where r is the distance from the tip of the cone. On the other hand $\langle\phi^2\rangle$ can be computed exactly,

$$\langle\phi^2\rangle = \int_0^\infty ds K(s, x, x) \quad (18)$$

where the scalar heat kernel on a cone is⁵

$$K(s, x, x) = -\frac{1}{2\beta} \frac{1}{(4\pi s)^2} \int_{-\infty}^\infty dy e^{-\frac{r^2}{s} \cosh^2(y/2)} \left(\cot \frac{\pi}{\beta}(\pi + iy) + \cot \frac{\pi}{\beta}(\pi - iy) \right) \quad (19)$$

Expanding the heat kernel to first order in the deficit angle and integrating over s reproduces (17).

⁵See for example [16]. We dropped the term in the heat kernel $1/(4\pi s)^2$ which is responsible for the divergence in flat space.

3 Gauge field energy

We start from the Euclidean action

$$\begin{aligned} S &= S_{\text{Maxwell}} + S_{\text{gauge fixing}} \\ &= \int d^d x \sqrt{g} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\nabla_\mu A^\mu)^2 \right) \end{aligned}$$

There are ghosts associated with this choice of gauge that behave like a pair of minimally-coupled scalars.

If we smooth out the conical singularities, so that we can freely integrate by parts, the action becomes

$$\begin{aligned} S &= \int d^d x \sqrt{g} \left(\frac{1}{2} \nabla_\mu A_\nu \nabla^\mu A^\nu - \frac{1}{2} A^\mu (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) A^\nu \right) \\ &= \int d^d x \sqrt{g} \left(\frac{1}{2} \nabla_\mu A_\nu \nabla^\mu A^\nu + \frac{1}{2} R_{\mu\nu} A^\mu A^\nu \right) \end{aligned}$$

In the second line we used $[\nabla_\mu, \nabla_\nu] A^\nu = -R_{\mu\nu} A^\nu$. We work on a space which is a product of a two-dimensional cone with coordinates x^α and a $(d-2)$ -dimensional flat space with coordinates x^i .

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + \delta_{ij} dx^i dx^j$$

In two dimensions the Ricci tensor is proportional to the metric, so from (8)

$$R_{\alpha\beta} = 8\pi\lambda g_{\alpha\beta} \delta^2(x) / \sqrt{g} \quad (20)$$

where $8\pi\lambda$ is the deficit angle. Thus the action for a gauge field on a cone can be decomposed into

$$S_{\text{cone}} = S_0 + S_{\text{int}}, \quad S_{\text{int}} = S_{\text{wedge}} + S_{\text{tip}} \quad (21)$$

For example in four dimensions

$$S_0 = \int d^4 x \frac{1}{2} (\partial_\mu A_\nu)^2 \quad (22)$$

is the Feynman gauge action in flat space,

$$S_{\text{wedge}} = - \int d\tau dz \int_0^\infty r dr \int_{-4\pi\lambda}^{4\pi\lambda} d\psi \frac{1}{2} (\partial_\mu A_\nu)^2 \quad (23)$$

cancels the flat-space action in the region corresponding to the deficit angle, and

$$S_{\text{tip}} = 4\pi\lambda \int d\tau dz g_{\alpha\beta} A^\alpha A^\beta \quad (24)$$

arises from the explicit coupling to curvature. Aside from the sums over photon polarizations, this is identical to the decomposition of the non-minimal scalar action (9).

The interaction between two cosmic strings can be calculated perturbatively, just as for a non-minimal scalar field.⁶ In fact the two calculations are identical. There are $d-2$ polarizations transverse to the cone which behave in perturbation theory just like minimally-coupled scalars. Two of these polarizations are canceled by the ghosts, leaving no contribution in four dimensions. The two polarizations tangent to the cone behave like non-minimal scalars with $\xi = 1/2$. So the overall interaction energy coming from a gauge field in four dimensions is simply twice the scalar result (16) evaluated at $\xi = 1/2$. That is, for a gauge field in four dimensions

$$\mathcal{H}_{\text{int}} = \frac{2\lambda\lambda'}{\pi b^2} \left(-\frac{14}{15} \right) \quad (25)$$

To check the validity of our perturbative approach consider computing $\langle A_\mu A^\mu \rangle$ around a single cosmic string. In perturbation theory, after subtracting the divergence present in flat space, we have

$$\langle A_\mu A^\mu \rangle = \langle A_\mu A^\mu (-S_{\text{wedge}} - S_{\text{tip}}) \rangle_{C,0} = \frac{4\lambda}{6\pi^2 r^2} - \frac{\lambda}{\pi^2 r^2} \quad (26)$$

The first term comes from S_{wedge} and is four times the scalar field result (17). The second term comes from S_{tip} and reflects the non-minimal coupling to curvature. The same quantity can be computed exactly,

$$\langle A_\mu A^\mu \rangle = \int_0^\infty ds g_{\mu\nu} K_{\text{vector}}^{\mu\nu}(s, x, x) \quad (27)$$

where the vector heat kernel is [16]

$$g_{\mu\nu} K_{\text{vector}}^{\mu\nu} = 4K_{\text{scalar}}(s, x, x) + \frac{2}{r} \partial_r s K_{\text{scalar}}(s, x, x) \quad (28)$$

Expanding to first order in the deficit angle and integrating over s reproduces (26).⁷

4 Conclusions

In this paper we considered a cosmic string spacetime and argued that to first order in the deficit angle there is an equivalence between a gauge field and a collection scalar fields with

⁶Again it's not manifest that perturbation theory in S_{int} enforces the proper conical boundary conditions on A_α , but this effect is controlled by the spin connection which can only enter at second order in the deficit angle.

⁷Note that the last term in (28), which in the black hole context captures the contact interaction of a gauge field with the horizon, corresponds at first order in perturbation theory to effects associated with S_{tip} .

specific non-minimal couplings to curvature. More generally the equivalence holds on the product of any weakly-curved Einstein manifold with flat space. We tested the equivalence by computing the interaction energy between two cosmic strings to first order in perturbation theory, showing that it indeed matched for the appropriate value of the non-minimal coupling parameter.

Throughout this paper we worked in Feynman gauge, which is adequate for studying gauge-invariant quantities. However it would be interesting to study the relation between gauge and scalar fields in other choices of gauge. Also it would be interesting to study the interaction between strings at higher orders in perturbation theory. Beyond leading order there is no reason to expect an equivalence between gauge and scalar fields, since the spin connection distinguishes between the two types of fields and can appear in the interaction energy at second order in the deficit angle.

Besides their direct application to cosmic strings, our results also have relevance to the thermodynamics of black holes. In a Euclidean formalism the entropy of a black hole measures the response of the partition function to an infinitesimal conical deficit angle inserted at the horizon [17, 18]. This has been used to study the renormalization of black hole entropy due to matter fields, with the somewhat surprising conclusion that a gauge field can make a negative contribution to the entropy. In [16] it was shown that this is due to a contact term in the partition function for a gauge field, associated with particle paths that begin and end on the horizon. Here we've shown that, to first order in the deficit angle, a gauge field is equivalent to a collection of non-minimal scalars. So the contact interaction of [16] is visible at the level of the equations of motion, as the explicit non-minimal coupling to curvature seen in (6). This makes the negative renormalization of black hole entropy less mysterious, since it maps a gauge field to the well-studied problem of a non-minimally coupled scalar field in a black hole background [19]. Our results also show the physical relevance of these contact interactions: besides contributing to black hole entropy, they make a (finite, observable, gauge invariant) contribution to the force between two cosmic strings.

We conclude with some additional evidence in support of the relation between gauge and scalar fields at first order in the background curvature. The partition function for a gauge field on a cone was evaluated in [16]. Including the ghosts, the result is

$$\beta F_{\text{gauge}} = (d-2)\beta F_{\text{scalar}}^{\text{minimal}} + A_{\perp}(2\pi - \beta) \int_{\epsilon^2}^{\infty} \frac{ds}{(4\pi s)^{d/2}} \quad (29)$$

Here d is the total number of spacetime dimensions, A_{\perp} is the area of the $d-2$ transverse dimensions corresponding to the horizon, s is a Schwinger parameter, and ϵ is a UV cutoff. The partition function for a non-minimal scalar was evaluated to first order in the deficit angle in [19], with the result

$$\beta F_{\text{scalar}}^{\xi} = \beta F_{\text{scalar}}^{\text{minimal}} + \xi A_{\perp}(2\pi - \beta) \int_{\epsilon^2}^{\infty} \frac{ds}{(4\pi s)^{d/2}} \quad (30)$$

Comparing the partition functions again shows that a gauge field corresponds to two non-minimal scalars with $\xi = 1/2$, together with $d - 2$ minimal scalars (two of which are canceled by the ghosts). The same relation can be seen in the one-loop renormalization of Newton's constant,

$$\frac{1}{4G_{N,\text{ren}}} = \frac{1}{4G_N} + \frac{c_1}{(4\pi)^{\frac{d-2}{2}}(d-2)\epsilon^{d-2}} \quad (31)$$

where the Seeley – de Witt coefficients are [20]

$$c_1 = \begin{cases} \frac{1}{6} - \xi & \text{non-minimal scalar} \\ \frac{d-2}{6} - 1 & \text{gauge field including ghosts} \end{cases} \quad (32)$$

On a d -dimensional Einstein manifold the gauge field result corresponds to d non-minimal scalars with $\xi = 1/d$, plus two minimally-coupled scalar ghosts.

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