Determinants for the lightcone worldsheet

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Determinants for the Lightcone Worldsheet [1]

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Abstract

The evaluation of the determinant of the Laplacian defined on two dimensional regions of various shapes is an essential ingredient in calculating the scattering amplitudes of strings. In lightcone parameterization the regions are rectangular in shape with several slits of different length and location cut parallel to the τ axis of the rectangle. This paper offers a compendium of applications of the methods of Kac and McKean and Singer to the calculation of such worldsheet determinants. Particular attention is paid to the effect of corners on the determinants. The effect of corners joining edges with like boundary conditions is implicit in Kac's results. We discuss the generalization to a corner joining a Dirichlet edge to a Neumann edge, and apply it to a scattering amplitude involving D-branes.

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1 Introduction

The lightcone quantization of string [1, 2] was employed by Mandelstam [3, 4] to describe interacting string theory via the sum over path histories in which interactions between strings are interpreted simply as breaking and joining processes as depicted in Fig. 1. The lightcone

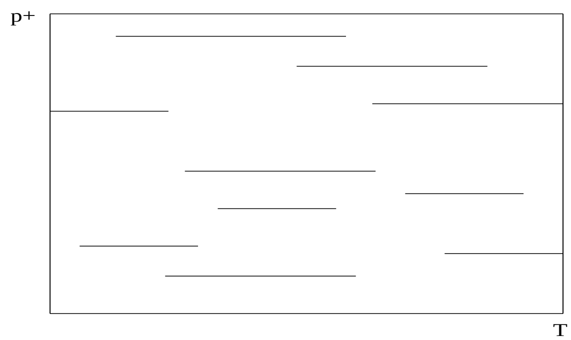

Figure 1: Mandelstam Interacting String Diagram

worldsheet is parameterized by taking the evolution parameter τ to be ix^+, where the i reflects a Wick rotation to imaginary x^+; and by labelling points on the string by a parameter σ defined so that density of P^+ momentum is unity. Then the dimensions of the world sheet are

$$T = ix^+ = i(t + z)/\sqrt{2}, \qquad P^+ = (p^0 + p^z)/\sqrt{2}, \tag{1}$$

where x^μ and p^μ are the spacetime coordinates and total four momentum of the string. The diagram in Fig. 1 describes the time evolution of a system of open strings, breaking and rejoining as shown by the horizontal lines.

For the critical open bosonic string (i.e. the spacetime dimension $D = 26$), the worldsheet path history integrates over the transverse coordinates $\boldsymbol{x}(\sigma, \tau)$ and uses the lightcone action for the free open string:

$$S_{l.c.} = \frac{1}{2}\int_0^T d\tau \int_0^{P^+} d\sigma \left[\left(\frac{\partial \boldsymbol{x}}{\partial \tau}\right)^2 + T_0^2 \left(\frac{\partial \boldsymbol{x}}{\partial \sigma}\right)^2\right] \tag{2}$$

The transverse coordinates, defined on the domain of the lightcone diagram, are discontinuous across horizontal lines. For each beginning and end of a horizontal line there is a factor of string coupling g. Then the sum over all planar open string loops is simply the sum over the number, lengths and locations of those horizontal lines.

It is a remarkable fact that the normalization of diagrams implied by this simple prescription, defined concretely by introducing a rectangular grid in σ, τ, correctly reproduces all of the multistring tree amplitudes of the dual resonance model. This means in particular that the continuum limit of the worldsheet lattice, introduced by Giles and Thorn [5] (GT), is Lorentz covariant (in the critical dimension. The simplest process which reflects this is the

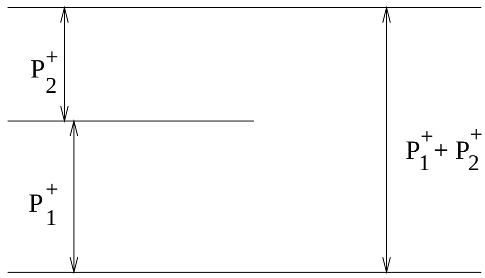

Figure 2: The lightcone diagram for the three string vertex.

three string vertex described by Fig. 2. Since lightcone diagrams are properly normalized probability amplitudes, Lorentz covariance dictates the P^+ dependence

$$\text{Vertex} \sim \frac{1}{\sqrt{P_1^+ P_2^+ (P_1^+ + P_2^+)}} \tag{3}$$

for the vertex involving spin 0 states. This factor must come from the determinant factor arising from the Gaussian integral over $\boldsymbol{x}$. So under $P_i^+ \to \lambda P_i^+$, which is just a scaling of the size of the diagram, the above diagram should scale as $\lambda^{-3/2}$. A lightcone worldsheet lattice calculation gives $\lambda^{-(D-2)/16}$, which makes clear the need for the critical dimension $D = 26$ to obtain Lorentz covariance [5].

More generally the complete evaluation of a lightcone interacting string diagram proceeds in two steps. First the dependence on the initial and final data is extracted by shifting the $\boldsymbol{x}(\sigma, \tau)$ by a solution of the classical equations of motion. The Gaussian integral that remains is then expressed in terms of the determinant of the Laplacian defined on the lightcone worldsheet. In this article we bring together in one place results on worldsheet determinants scattered throughout the literature with the addition of some new results and applications[3].

Central to our discussion will be an insightful paper by Mark Kac studying what he called "Hearing the Shape of a Drum" [6]. His idea was to connect the distribution of allowed normal mode frequencies λ_k, which are the eigenvalues of the Laplacian $-\nabla^2/2$, to the shape of a two dimensional membrane. Technically, he considered a general polygonal shape and demonstrated the small t behavior of

$$\text{Tr } e^{t\nabla^2/2} \;=\; \sum_k e^{-\lambda_k t} \sim \frac{\text{Area}}{2\pi t} \mp \frac{\text{Perimeter}}{4\sqrt{2\pi t}} + \sum_{\text{corners}} \frac{1}{24}\left(\frac{\pi}{\theta_i} - \frac{\theta_i}{\pi}\right) + o(1) \tag{4}$$

where the minus sign is valid for Dirichlet and the plus sign for Neumann boundary conditions on all edges. Kac derived this formula for Dirichlet boundary conditions on all polygon edges, but it also applies to Neumann boundary conditions on all edges. If some edges have

[3]For example, some of the Dirichlet boundary condition choices for the lattice determinants and their duality properties calculated in Section 2, have not been previously discussed in this context. As far as I know, the derivation in section 3 of the previously known result for ND corners is new, as are the scattering applications discussed in Section 4 and much of the work described in the appendices.

Dirichlet and others Neumann boundary conditions, the perimeter is replaced by $L_D - L_N$. The contribution of each DD and NN corner is as above but DN and ND corners are different. The suitable generalization is given, for example in [7]:

$$\mathrm{Tr}\; e^{t\nabla^2/2} \sim \frac{\mathrm{Area}}{2\pi t} - \frac{L_D - L_N}{4\sqrt{2\pi t}} + \sum_{\substack{\text{NN,DD}\\ \text{corners}}} \frac{1}{24}\left(\frac{\pi}{\theta_i} - \frac{\theta_i}{\pi}\right) - \sum_{\substack{\text{DN,ND}\\ \text{corners}}} \frac{1}{48}\left(\frac{\pi}{\theta_i} + \frac{2\theta_i}{\pi}\right) + o(1). \qquad (5)$$

In appendix A we confirm the ND contribution in an elementary way for the special cases $\theta = \pi/2M$, $M = 1, 2, 3 \ldots$, for which the method of images can be successfully applied to the solutions of the diffusion equation. Later on we show that the general θ case is also a consequence of the 90° case (which can be inferred from the exact calculations for a rectangle in the next section and the conformal transformation formula inferred from [8] and discussed in Section 3).

For the case of 90° corners a DD, NN corner contributes $+1/16$ whereas a DN corner contributes $-1/16$. The lightcone open string vertex is a 360° NN corner. Putting $\theta = 2\pi$,

$$\frac{1}{24}\left(\frac{\pi}{\theta} - \frac{\theta}{\pi}\right) \rightarrow -\frac{1}{16}. \qquad (6)$$

For 24 transverse dimensions the scaling power is thus $24/16 = 3/2$, explaining the scaling law required by Lorentz covariance. By taking an n sided polygon with angles $\theta_i = \pi - \epsilon_i$ and the limit $n \rightarrow \infty$ with $\sum_i \epsilon_i = 2\pi$, Kac showed that a smooth closed curve will contribute a term to the above expression of $1/6$. In particular a semi-circular arc subtending an angle θ would contribute a term $\theta/12\pi$.

In Section 2 we quote or derive expressions for the Gaussian path integrals defined on a rectangular lattice. Most of these results are known: see for example [5]. We obtain the results with all possible choices of Dirichlet (D) and Neumann (N) boundary conditions. Each determinant can be expressed as a single infinite product corresponding to diagonalizing the transfer matrix in either the horizontal or vertical directions. The equality of the two representations is a lattice analog of the Jacobi imaginary transformation in the theory of elliptic functions.

In Section 3 we discuss some applications of the McKean-Singer result for the relation of the determinants of the Laplacian on two regions related by a conformal transformation. In Section 4 we review Mandelstam's evaluation of the determinant for bosonic tree diagrams and then discuss some possible interpretations for the case of subcritical dimensions $D < 26$, when some aspect of Lorentz invariance fails. Also in Section 4 we discuss two applications for string scattering. Technical details are relegated to two appendices.

2 Rectangles: Lattice Results

Recall that by virtue of lightcone parametrization (1), the lightcone worldsheet is a rectangle of dimensions $P^+ \times T$, and only the transverse coordinates participate in the worldsheet path integral. In the following we shall impose Dirichlet and Neumann boundary conditions on the transverse coordinates, in all possible permutations and combinations, at the edges of the rectangle. Dirichlet boundary conditions are appropriate when open strings end on Dp-branes. Notice that with lightcone parametrization the $x^+ = \tau$ and $x^{-\prime} = \boldsymbol{x}' \cdot \dot{\boldsymbol{x}}$ coordinates of the string automatically satisfy Neumann boundary conditions when $\boldsymbol{x}$ satisfies either Dirichlet or Neumann conditions, so that the Dp-branes allowed for here always have $p \geq 1$. An application involving D1-branes will be discussed in Section 4.

A brute force way to calculate lightcone worldsheet determinants is to explicitly evaluate Gaussian path integrals on a lattice [5]. So take an $M \times N$ finite rectangular lattice, with a (transverse) coordinate x at each point on the lattice. Then we have

$$\det{}^{-1/2}(-\nabla^2) \to \int dx_{kl} \exp\left\{-\frac{1}{2}\sum_{kl}\left[(x_{k,l+1} - x_{k,l})^2 + (x_{k+1,l} - x_{k,l})^2\right]\right\} \tag{7}$$

In each case N, M, K are the number of integration variables in a row of column of the lattice.

Points on the boundary of the lattice can be fixed (Dirichlet) or freely integrated (Neumann). The bilinear forms can be diagonalized by expanding in normal modes, for which the eigenvalues are:

$$\alpha_n \equiv 4\sin^2\frac{n\pi}{2(N+1)}, \qquad n = 1, 2, \ldots, N \tag{8}$$

$$\beta_n \equiv 4\sin^2\frac{m\pi}{2M}, \qquad m = 0, 1, \ldots, M-1 \tag{9}$$

$$\gamma_k \equiv 4\sin^2\frac{(k+1/2)\pi}{2K+1}, \qquad k = 0, 1, \ldots, K-1 \tag{10}$$

The α's are appropriate to a bilinear form with fixed ends (DD), the β's to a form with free ends (NN), and the γ's to a form with one fixed and one free end. Then we are interested in the following determinants:

$$\det{}^{-1/2}_{\rm DDDD} = \prod_{n=1}^{N}\prod_{m=1}^{M}(\alpha_n + \alpha_m)^{-1/2}, \qquad \det{}^{-1/2}_{\rm DNDN} = \prod_{n=1}^{N}\prod_{m=0}^{M-1}(\alpha_n + \beta_m)^{-1/2} \tag{11}$$

$$\det{}^{-1/2}_{\rm DDDN} = \prod_{n=1}^{N}\prod_{k=0}^{K-1}(\alpha_n + \gamma_k)^{-1/2}, \qquad \det{}^{-1/2}_{\rm DNNN} = \prod_{m=0}^{M-1}\prod_{k=0}^{K-1}(\beta_m + \gamma_k)^{-1/2} \tag{12}$$

$$\det{}^{-1/2}_{\rm DDNN} = \prod_{j=0}^{J-1}\prod_{k=0}^{K-1}(\gamma_j + \gamma_k)^{-1/2} \tag{13}$$

In each of these formulas one of the products can be evaluated exactly on the lattice. The following product identities can be easily derived:

$$\prod_{n=1}^{N}(\alpha_n - z) = \frac{\sin(N+1)\kappa}{\sin\kappa}, \qquad \prod_{k=0}^{K-1}(\gamma_k - z) = \frac{\cos[(2K+1)\kappa/2]}{\cos[\kappa/2]} \tag{14}$$

where z and κ are related by $z = 4\sin^2[\kappa/2]$. Applying these identities at $z = 0, \kappa = 0$ shows immediately that $D_{\rm DNDN} = D_{\rm DDDD}/\sqrt{N+1}$ and $D_{\rm DNNN} = D_{\rm DDDN}$.

2.1 DDDD

We then find

$$\begin{aligned} \det^{-1/2}_{\rm DDDD} &= \prod_{m=1}^{M}\left[\frac{\sinh(2(N+1)\sinh^{-1}(\sin(m\pi/2(M+1))))}{\sinh(2\sinh^{-1}(\sin(m\pi/2(M+1))))}\right]^{-1/2} \\ &= (M+1)^{1/4}\left(\frac{\sinh[2(M+1)\sinh^{-1}1]}{2\sqrt{2}}\right)^{1/4} \\ &\quad e^{-(N+1)\sum_{m=1}^{M}\sinh^{-1}\sin m\pi/2(M+1)}\prod_{m=1}^{M}\left\{1-e^{-4(N+1)\sinh^{-1}\sin m\pi/2(M+1)}\right\}^{-1/2} \end{aligned} \tag{15}$$

where we used

$$\prod_{m=1}^{M}\left(2\sin\frac{m\pi}{2(M+1)}\right) = \sqrt{M+1} \tag{16}$$

$$\prod_{m=1}^{M}\sqrt{4+4\sin^2\frac{m\pi}{2(M+1)}} = \left(\frac{\sinh(2(M+1)\sinh^{-1}1)}{\sinh(2\sinh^{-1}1)}\right)^{1/2} \tag{17}$$

and $\sinh(2\sinh^{-1}1) = 2\sqrt{2}$.

The continuum limit is $M, N \to \infty$ with $L = (M+1)a$, and $T = (N+1)a$ fixed. For this we need

$$\sum_{m=1}^{M}\sinh^{-1}\sin\frac{m\pi}{2(M+1)} \sim \frac{2(M+1)G}{\pi} - \frac{1}{2}\sinh^{-1}1 - \frac{\pi}{24(M+1)} \tag{18}$$

where $G = \sum_{k=0}^{\infty}(-)^k/(2k+1)^2$ is Catalan's constant. Then

$$\det^{-1/2}_{\rm DDDD} \sim \left(\frac{L}{2a\sqrt{2}}\right)^{1/4} e^{-\alpha LT+\beta(T+L)+\pi T/24L}\prod_{m=1}^{\infty}\left\{1-e^{-2m\pi T/L}\right\}^{-1/2} \tag{19}$$

with $\alpha \equiv 2G/\pi a^2$ and $\beta = (2a)^{-1}\sinh^{-1}1$.

We see that, apart from the coefficient $a^{-1/4}$, the divergences associated with the continuum limit reside in the terms in the exponent proportional to the area or perimeter of the rectangle. These terms are inconsequential and can be dropped in order to define a finite continuum determinant

$$\det^{-1/2}_{DDDD,C} \equiv L^{1/4}e^{\pi T/24L}\prod_{m=1}^{\infty}\left\{1-e^{-2m\pi T/L}\right\}^{-1/2} \tag{20}$$

The factor of $L^{1/4}$ accounts for the corner contribution in the Kac formula, in this case 4 90° corners or $4\times(1/16)$. The remaining factors depend on the shape T/L of the rectangle. The symmetry $T \leftrightarrow L$ of the rectangle and boundary conditions implies the equality

$$L^{1/4}e^{\pi T/24L}\prod_{m=1}^{\infty}\left\{1-e^{-2m\pi T/L}\right\}^{-1/2} = T^{1/4}e^{\pi L/24T}\prod_{m=1}^{\infty}\left\{1-e^{-2m\pi L/T}\right\}^{-1/2} \tag{21}$$

which is simply the Jacobi transform in the theory of elliptic functions.[4]

2.2 DNDN

From the identity $\det^{-1/2}_{\rm DNDN} = \det^{-1/2}_{\rm DDDD}/\sqrt{N+1}$ we can immediately write down

$$\det{}^{-1/2}_{DNDN,C} \quad\equiv\quad L^{-1/4}\sqrt{L/T}e^{\pi T/24L}\prod_{m=1}^{\infty}\left\{1-e^{-2m\pi T/L}\right\}^{-1/2} \tag{23}$$

The scaling power is now $-1/4$ corresponding to 4 90° ND corners in the Kac formula. Because the boundary conditions break the symmetry $T \leftrightarrow L$ the determinant doesn't have the symmetry:

$$L^{-1/4}\sqrt{L/T}e^{\pi T/24L}\prod_{m=1}^{\infty}\left\{1-e^{-2m\pi T/L}\right\}^{-1/2} = T^{-1/4}e^{\pi L/24T}\prod_{m=1}^{\infty}\left\{1-e^{-2m\pi L/T}\right\}^{-1/2} \tag{24}$$

The factor $\sqrt{L/T}$ on the left reflects the propagation in T of the zero mode of an NN string. The right shows the propagation in L which is that of a DD string with no zero mode.

2.3 DDDN and DNNN

Next we turn to the $DDDN$ determinant. Doing the product over n, we find

$$\begin{aligned}\det{}^{-1/2}_{\rm DDDN} &= \prod_{k=0}^{K-1}\left[\frac{\sinh(2(N+1)\sinh^{-1}(\sin((k+1/2)\pi/(2K+1))))}{\sinh(2\sinh^{-1}(\sin((k+1/2)\pi/(2K+1))))}\right]^{-1/2} \\ &= \left(\frac{\cosh[(2K+1)\sinh^{-1}1]}{\sqrt{2}}\right)^{1/4} e^{-(N+1)\sum_{k=0}^{K-1}\sinh^{-1}\sin(k+1/2)\pi/(2K+1)} \\ &\quad \prod_{k=0}^{K-1}\left\{1-e^{-4(N+1)\sinh^{-1}\sin(k+1/2)\pi/(2K+1)}\right\}^{-1/2}\end{aligned} \tag{25}$$

where we used

$$\prod_{k=0}^{K-1}\left(2\sin\frac{(k+1/2)\pi}{2K+1}\right) \quad=\quad 1 \tag{26}$$

$$\prod_{k=0}^{K-1}\sqrt{4+4\sin^2\frac{(k+1/2)\pi}{2K+1}} \quad=\quad \left(\frac{\cosh((2K+1)\sinh^{-1}1}{\cosh(\sinh^{-1}1)}\right)^{1/2} \tag{27}$$

[4] In standard notation with $q \equiv e^{i\pi\tau} = e^{-2\pi T/L}$ and $\dot{q} = e^{i\pi\dot{\tau}}$ this identity reads

$$q^{-1/48}\prod_{n=1}^{\infty}(1-q^n)^{-1/2} \quad=\quad \left(\frac{-i\tau}{2\pi}\right)^{1/4}\dot{q}^{-1/48}\prod_{n=1}^{\infty}(1-\dot{q}^n)^{-1/2}. \tag{22}$$

and $\cosh(\sinh^{-1} 1) = \sqrt{2}$. For the continuum limit we need

$$\sum_{k=0}^{K-1} \sinh^{-1} \sin \frac{(k+1/2)\pi}{2K+1} \sim \frac{(2K+1)G}{\pi} - \frac{1}{2}\sinh^{-1} 1 + \frac{\pi}{48(K+1/2)} \tag{28}$$

With the understanding that the "length" of an ND string is $L = a(K+1/2)$ we see the bulk and boundary terms are identical to the DDDD and DNDN cases. So the continuum limit is

$$\det{}_{\rm DDDN}^{-1/2} \;=\; 2^{-3/8} e^{-\alpha LT+\beta(L+T)-\pi T/48L} \prod_{k=0}^{\infty} \left\{1 - e^{-2(k+1/2)\pi T/L}\right\}^{-1/2} \tag{29}$$

with $\alpha = 2G/\pi a^2$ and $\beta = (2a)^{-1} \sinh^{-1} 1$ as before. The corresponding continuum determinant can be taken to be

$$\det{}_{\rm DDDN,C}^{-1/2} \;=\; e^{-\pi T/48L} \prod_{k=0}^{\infty} \left\{1 - e^{-2(k+1/2)\pi T/L}\right\}^{-1/2} \tag{30}$$

This determinant is scale invariant in accord with the fact that this rectangle has two ND 90° corners and 2 DD 90° corners which contribute with cancelling signs. In this form the determinant displays the propagation of a DN string in T. A Jacobi transform displays the propagation of a DD string in L[5]:

$$e^{-\pi T/48L} \prod_{k=0}^{\infty} \left\{1 - e^{-2(k+1/2)\pi T/L}\right\}^{-1/2} \;=\; 2^{-1/4} e^{\pi L/24T} \prod_{n=1}^{\infty} (1 + e^{-2n\pi L/T})^{-1/2} \tag{32}$$

We have already noted that the determinant for the NNND case is identical to the DDDN case we just discussed.

2.4 DDNN

Finally for completeness we analyze the DDNN rectangle, which reflects a DN string propagating in both T and L. Like the DDDD case the result should possess the symmetry $T \leftrightarrow L$. Doing the product over j gives

$$\det{}_{\rm DDNN}^{-1/2} \;=\; \prod_{k=0}^{K-1} \left[\frac{\cosh((2J+1)\sinh^{-1}(\sin((k+1/2)\pi/(2K+1))))}{\cosh(\sinh^{-1}(\sin((k+1/2)\pi/(2K+1))))}\right]^{-1/2}$$

[5] In terms of $q, \dot{q}$ defined in the previous footnote this identity reads

$$q^{1/96} \prod_{k=0}^{\infty} (1 - q^{k+1/2})^{-1/2} \;=\; 2^{-1/4} \dot{q}^{-1/48} \prod_{n=1}^{\infty} (1 + \dot{q}^n)^{-1/2}. \tag{31}$$

$$
\begin{aligned}
&= \left(\frac{\cosh[(2K+1)\sinh^{-1}1]}{\sqrt{2}}\right)^{1/4} e^{-(J+1/2)\sum_{k=0}^{K-1}\sinh^{-1}\sin(k+1/2)\pi/(2K+1)} \\
&\quad \prod_{k=0}^{K-1}\left\{1+e^{-2(2J+1)\sinh^{-1}\sin(k+1/2)\pi/(2K+1)}\right\}^{-1/2} \qquad (33)\\
&\sim 2^{-3/8}e^{-\alpha LT+\beta(L+T)-\pi T/48L}\prod_{k=0}^{\infty}\left\{1+e^{-2(k+1/2)\pi T/L}\right\}^{-1/2} \qquad (34)
\end{aligned}
$$

where the last line is the continuum limit with the identifications $L = K + 1/2$ and $T = J + 1/2$. Just as in the previous cases the continuum determinant can then be chosen as

$$
\det{}^{-1/2}_{DDNN,C} = e^{-\pi T/48L}\prod_{k=0}^{\infty}\left\{1+e^{-2(k+1/2)\pi T/L}\right\}^{-1/2} \qquad (35)
$$

And the symmetry under $T \leftrightarrow L$ is yet another Jacobi relation

$$
e^{-\pi T/48L}\prod_{k=0}^{\infty}\left\{1+e^{-2(k+1/2)\pi T/L}\right\}^{-1/2} = e^{-\pi L/48T}\prod_{k=0}^{\infty}\left\{1+e^{-2(k+1/2)\pi L/T}\right\}^{-1/2} \qquad (36)
$$

A noteworthy feature of the lattice definition of the various determinants is that the bulk and boundary terms are identical in all cases: $-\alpha LT + \beta(L+T)$ regardless of how Dirichlet and Neumann conditions are assigned. This contrasts with the diffusion equation method of Kac and McKean-Singer. Of course this statement entails a varying identification between the continuum lengths and the number of degrees of freedom: $L/a = M, N+1, K+1/2$ for NN, DD, DN conditions respectively, and similarly for T. That is, there is an intrinsic ambiguity in identifying a unique "continuum" length. If we express these three lengths in terms of the ND one $L_0 = a(K+1/2)$, they are $L_0 - a/2$, L_0, and $L_0 + a/2$. This is a variation that mirrors the results of the diffusion equation continuum method.

2.5 NNNN

We end this section with a brief aside on the NNNN case, which requires special handling because of the zero mode. This zero mode is due to the translational invariance of the Gaussian integral (7) that we have used to define determinants. To interpret it add a source term $i\sum_{kl} x_{kl}J_{kl}$ to the exponent. Then insert (a la Fadeev-Popov)

$$
1 = \int da\delta\left(a - \frac{1}{MN}\sum_{kl} x_{kl}\right) \qquad (37)
$$

in the x integrand. A change of variables $x_{kl} \to x_{kl} + a$ transfers the a dependence to the exponent and then integration over a produces a delta function factor

$$
\int da e^{ia\sum_{kl} J_{kl}} = 2\pi\delta(\sum_{kl} J_{kl}). \qquad (38)
$$

Since J is conjugate to x, we see that this factor simply enforces momentum conservation. This interprets the infinite factor due to translation invariance as $2\pi\delta(0) = \infty$. The coefficient of the delta function has a finite zero source limit $J_{kl} \to 0$. We define $\det_{\rm NNNN}^{-1/2}$ as this coefficient at zero source:

$$\det{}_{\rm NNNN}^{-1/2} \equiv \int dx_{kl}\delta\left(\frac{1}{MN}\sum_{kl} x_{kl}\right)\exp\left\{-\frac{1}{2}\sum_{kl}\left[(x_{k,l+1}-x_{k,l})^2+(x_{k+1,l}-x_{k,l})^2\right]\right\} \quad (39)$$

When we change variables to normal modes q_{mn}, normalized so that the Jacobian is unity, we find that $q_{00}\sqrt{MN} = \sum_{kl} x_{kl}$. Hence the effect of the delta function is to multiply by $\sqrt{MN}$ and delete the contribution of the zero mode:

$$\begin{aligned} \det{}_{\rm NNNN}^{-1/2} &= \sqrt{MN}\prod_{(m,n)\neq(0,0)}(\beta_m+\beta_n)^{-1/2} \\ &= \sqrt{MN}\prod_{m=1}^{M-1}\beta_m^{-1/2}\prod_{n=1}^{N-1}\beta_n^{-1/2}\det{}_{\rm DDDD}^{-1/2} = \det{}_{\rm DDDD}^{-1/2} \end{aligned} \quad (40)$$

This formula confirms that NN 90° corners have identical effect to DD 90° corners.

3 Conformal Transformation

More generally, under a conformal scaling $g_{ab} \to e^{2\Sigma} g_{ab}$, the change in the determinant of the Laplacian is given by [8]

$$\begin{aligned} -\frac{1}{2}\delta(\ln(-\nabla^2)) &= \frac{1}{24\pi}\int dA g^{ab}\frac{d\Sigma}{dz_a}\frac{d\Sigma}{dz_b} + \frac{1}{12\pi}\int d\ell k\Sigma + \frac{1}{24\pi}\int dAR\Sigma \\ &\quad + \frac{1}{24}\sum_{\substack{\rm DD,NN \\ \rm corners}}\left(\frac{\pi}{\theta_i}-\frac{\theta_i}{\pi}\right)\Sigma(z_i) - \frac{1}{48}\sum_{\substack{\rm DN \\ \rm corners}}\left(\frac{\pi}{\theta_i}+\frac{2\theta_i}{\pi}\right)\Sigma(z_i) \end{aligned} \quad (41)$$

Here it is understood that the two determinants have the bulk and boundary terms dropped. Actually this formula does not explicitly appear in [8]. Rather the first three terms in the asymptotic behavior as $t \to 0$ of ${\rm Tr}e^{t\nabla^2}$ are explicitly calculated in terms of the geometry of an arbitrary smooth manifold endowed with a metric g_{ab}. The change formula then follows after a straightforward evaluation of the difference of their results for two manifolds related by a conformal transformation (see, for example [9]). When the boundary is only piecewise smooth, the corner terms that appear can be inferred from Kac's results, and their generalization to DN corners.

3.1 DD Corners from Conformal Transform of a Rectangle

We can use (41) to obtain the measure for the region on the right of Fig. 3 from the measure for the figure on the left, or, more fundamentally, from the measure for a rectangle, which

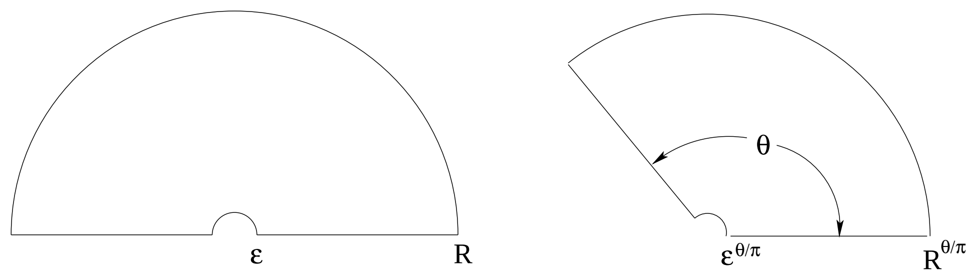

Figure 3: Two geometries related by the conformal transformation $y = z^{\theta/\pi}$.

we have explicitly evaluated in Section 2 by introducing a rectangular lattice. The shapes in Fig. 3 are related by the transformation

$$y \;=\; \frac{\pi}{\theta} z^{\theta/\pi}, \qquad \Sigma = \ln\left|\frac{dy}{dz}\right| = \left(\frac{\theta}{\pi} - 1\right)\ln|z| \tag{42}$$

However, we begin by recalling a formula for the determinant of an $M \times N$ rectangular grid [5].

$$\begin{aligned} \det{}^{-1/2}(-\nabla^2) \;&=\; (M+1)^{1/4}\left(\frac{\sinh[2(M+1)\sinh^{-1}1]}{\sinh[2\sinh^{-1}1]}\right)^{1/4} e^{-(N+1)\sum_{m=1}^{M}\sinh^{-1}\sin m\pi/2(M+1)} \\ &\quad\times \prod_{m=1}^{M}\left\{1 - e^{-4(N+1)\sinh^{-1}\sin m\pi/2(M+1)}\right\}^{-1/2} \end{aligned} \tag{43}$$

In the continuum limit $M, N \to \infty$, with $T/L \equiv (N+1)/(M+1)$ fixed, this reduces to:

$$\det{}^{-1/2}(-\nabla^2) \;\sim\; K e^{\alpha LT + \beta(L+T)} L^{1/4} e^{\pi T/24L} \prod_{m=1}^{\infty}\left\{1 - e^{-2m\pi T/L}\right\}^{-1/2} \tag{44}$$

The factor of $L^{1/4}$ reflects the scaling predicted by Kac for the four 90° corners of the rectangle. We may therefore choose a standard rectangle setting $L = \pi$ and dropping the divergent area and perimeter terms in the exponential prefactor,

$$\det{}^{-1/2}_{\mathrm{DDDD\ rect}}(-\nabla^2) \;\sim\; e^{T/24}\prod_{m=1}^{\infty}\left\{1 - e^{-2mT}\right\}^{-1/2} \tag{45}$$

It is convenient to coordinatize the rectangle by the complex variable $\rho = \tau + i\sigma$, with $0 < \sigma < \pi$ and $T_1 < \tau < T_2$. Then the conformal transformation, $z = e^{\frac{\theta\rho}{\pi}}$ maps the rectangle onto a wedge of an annulus of angle θ inner radius $\epsilon = e^{\theta T_1/\pi}$ and outer radius $R = e^{\theta T_2/\pi}$. To get the determinant for this new region, we first compute

$$\frac{dz}{d\rho} \;=\; \frac{\theta}{\pi}e^{\frac{\theta\rho}{\pi}}, \qquad \Sigma = \ln\frac{\theta}{\pi} + \frac{\theta}{\pi}\mathrm{Re}\,\rho, \qquad \partial_n\Sigma = \begin{cases} \theta/\pi & \tau = T_2 \\ -\theta/\pi & \tau = T_1 \\ 0 & \sigma = 0, \pi \end{cases} \tag{46}$$

$$\frac{1}{24\pi}\oint d\rho\Sigma\partial_n\Sigma \;=\; \frac{\theta^2}{24\pi^2}(T_2 - T_1) = \frac{\theta}{24\pi}\ln\frac{R}{\epsilon} \tag{47}$$

From the 90° corners we need

$$\frac{1}{16}\sum_i \Sigma_i = \frac{1}{8}\left(\frac{\theta}{\pi}(T_1+T_2) + 2\ln\frac{\theta}{\pi}\right) = \frac{1}{8}\left(\ln R + \ln\epsilon + 2\ln\frac{\theta}{\pi}\right) \tag{48}$$

Thus we have

$$\begin{aligned}
-\frac{1}{2}\ln\det{}_{\rm DD\ annular\ wedge} &= -\frac{1}{2}\ln\det{}_{\rm DDDD\ rect} + \frac{\theta}{24\pi}\ln\frac{R}{\epsilon} + \frac{1}{8}\left(\ln R + \ln\epsilon + 2\ln\frac{\theta}{\pi}\right)\\
&= \frac{T_2-T_1}{24} - \frac{1}{2}\sum_{m=1}^{\infty}\ln(1-e^{-2m(T_2-T_1)}) + \frac{\theta}{24\pi}\ln\frac{R}{\epsilon} + \frac{1}{8}\left(\ln\frac{\theta R}{\pi} + \ln\frac{\theta\epsilon}{\pi}\right)\\
&= -\frac{1}{2}\sum_{m=1}^{\infty}\ln(1-(\epsilon/R)^{2m\pi/\theta}) + \frac{1}{24}\left(\frac{\pi}{\theta}+\frac{\theta}{\pi}\right)\ln\frac{R}{\epsilon} + \frac{1}{8}\left(\ln\frac{\theta R}{\pi} + \ln\frac{\theta\epsilon}{\pi}\right)\\
&\sim \frac{1}{24}\left(\frac{\pi}{\theta}+\frac{\theta}{\pi}\right)\ln\frac{R}{\epsilon} + \frac{1}{8}\left(\ln\frac{\theta R}{\pi} + \ln\frac{\theta\epsilon}{\pi}\right)
\end{aligned} \tag{49}$$

where the last line is valid for $\epsilon \ll R$. If we drop the ϵ terms in this limit, the R terms that remain should give the determinant for the wedge with the annular hole removed

$$\begin{aligned}
-\frac{1}{2}\ln\det{}_{\rm DD\ wedge} &\sim \frac{1}{24}\left(\frac{\pi}{\theta}+\frac{\theta}{\pi}\right)\ln R + \frac{1}{8}\ln R\\
&= \frac{1}{24}\left(\frac{\pi}{\theta}-\frac{\theta}{\pi}+3\right)\ln R + \frac{\theta}{12\pi}\ln R
\end{aligned} \tag{50}$$

where we also dropped the scale independent term $(1/4)\ln(\theta/\pi)$. The first term agrees with Kac's formula for corners: one corner of angle θ and two corners of angle $\pi/2$. The last term is the contribution from the circular arc, which we have seen follows from the Kac formula for a limiting polygon with corner angles $\sim \pi$. In this way we see that the conformal transformation formula embodies Kac's result as well as it's McKean-Singer generalization.

3.2 DN Corners from Conformal Transform of a Rectangle.

If we replace the DDDD rectangle used in the previous subsection with a DDDN rectangle we learn about DN corners. In that case the corner contributions to the conformal transformation formula cancel and we have

$$\begin{aligned}
&-\frac{1}{2}\ln\det{}_{\rm DN\ annular\ wedge}\\
&\quad= -\frac{1}{2}\ln\det{}_{\rm DDDN\ rect} + \frac{\theta}{24\pi}\ln\frac{R}{\epsilon}\\
&\quad= -\frac{T_2-T_1}{48} - \frac{1}{2}\sum_{k=0}^{\infty}\ln(1-e^{-(2k+1)(T_2-T_1)}) + \frac{\theta}{24\pi}\ln\frac{R}{\epsilon}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{k=0}^{\infty}\ln(1-(\epsilon/R)^{(2k+1)\pi/\theta}) + \frac{1}{24}\left(-\frac{\pi}{2\theta}+\frac{\theta}{\pi}\right)\ln\frac{R}{\epsilon} \\
&\sim \frac{1}{24}\left(-\frac{\pi}{2\theta}+\frac{\theta}{\pi}\right)\ln\frac{R}{\epsilon}, \qquad \frac{\epsilon}{R}\to 0. \qquad (51)
\end{aligned}
$$

Dropping the ϵ terms produces the determinant for a wedge of angle θ

$$
-\frac{1}{2}\ln\det_{\text{DN wedge}} = \frac{1}{24}\left(-\frac{\pi}{2\theta}+\frac{\theta}{\pi}\right)\ln R = -\frac{1}{24}\left(\frac{\pi}{2\theta}+\frac{\theta}{\pi}\right)\ln R + \frac{\theta}{12\pi}\ln R. \qquad (52)
$$

Again the last term accounts for the contribution from the circular arc that closes the wedge, whence the first term must be associated with the DN angle itself. (The corners at the end of the arc contribute opposite signs and cancel.) It is seen to agree with our generalized Kac formula.

4 Lightcone Bosonic Tree

From Mandelstam's work [10], the measure factor for an N point tree is

$$
\left|\frac{\partial T}{\partial Z}\right|\det{}^{-(D-2)/2}(-\nabla^2) = Z_{N-1}\prod_{k=1}^{N}\frac{1}{|\alpha_k|^{(D-2)/48}}\left[|\alpha_N|^{N-3}\frac{\prod_{r<t}|x_t-x_r|}{\prod_{m<l}|Z_l-Z_m|}\right]^{(26-D)/24} \qquad (53)
$$

where $\alpha_r = 2p_r^+$. The rest of the Koba-Nielsen integrand is just the usual $\prod_{i<j}|Z_j-Z_i|^{2k_i\cdot k_j}$ (in units where $\alpha'=1$) and for which $k_j^2 = (D-2)/24$.

The quantities Z_k, with $k = 1\cdots(N-1)$, and x_r, with $r = 1\cdots(N-2)$ are determined from the map from the upper-half Koba-Nielsen plane (z) to the lightcone world sheet ($\rho = \tau + i\sigma$):

$$
\rho = \sum_{k=1}^{N-1}\alpha_k\ln(z-Z_k), \qquad \left.\frac{d\rho}{dz}\right|_{z=x_r} = 0 \qquad (54)
$$

$$
\frac{d\rho}{dz} = \sum_{k=1}^{N-1}\frac{\alpha_k}{z-Z_k} = \frac{\sum_{k=1}^{N-1}\alpha_k\prod_{l\neq k}(z-Z_l)}{\prod_k(z-Z_k)} = -\alpha_N\frac{\prod_r(z-x_r)}{\prod_k(z-Z_k)} \qquad (55)
$$

so that the asymptotic strings at $\tau = \pm\infty$ are mapped from the Z_k. In this notation $Z_N = \infty$, $Z_1 = 0$. A useful identity follows by setting $z = Z_m$ in the identity

$$
-\alpha_N\prod_r(z-x_r) = \sum_{k=1}^{N-1}\alpha_k\prod_{l\neq k}(z-Z_l) \qquad (56)
$$

$$
-\alpha_N\prod_r(Z_m-x_r) = \sum_{k=1}^{N-1}\alpha_k\prod_{l\neq k}(Z_m-Z_l) = \alpha_m\prod_{l\neq m}(Z_m-Z_l) \qquad (57)
$$

$$
|\alpha_N|^N\prod_{m,r}|Z_m-x_r| = \prod_{m=1}^{N}|\alpha_m|\prod_{l\neq k}|Z_k-Z_l| \qquad (58)
$$

Then the measure can be put in the more suggestive form

$$\begin{aligned}
&\left|\frac{\partial T}{\partial Z}\right| \det{}^{-(D-2)/2}(-\nabla^2) \\
&= Z_{N-1}\prod_{k=1}^{N}\frac{1}{|\alpha_k|^{(D-2)/48}}\left[|\alpha_N|^{N-3}\frac{\prod_{r<t}|x_t-x_r|\prod_{m<l}|Z_l-Z_m|}{\prod_{m\neq l}|Z_l-Z_m|}\right]^{(26-D)/24} \\
&= Z_{N-1}\prod_{k=1}^{N}\frac{1}{|\alpha_k|^{(D-2)/48}}\left[\frac{\prod_k|\alpha_k|}{|\alpha_N|^3}\frac{\prod_{r<t}|x_t-x_r|\prod_{m<l}|Z_l-Z_m|}{\prod_{l,r}|Z_l-x_r|}\right]^{(26-D)/24} \\
&= Z_{N-1}\prod_{k=1}^{N}\frac{1}{\sqrt{|\alpha_k|}}\left[\frac{\prod_k|\alpha_k|^{3/2}}{|\alpha_N|^3}\frac{\prod_{r<t}|x_t-x_r|\prod_{m<l}|Z_l-Z_m|}{\prod_{l,r}|Z_l-x_r|}\right]^{(26-D)/24} \quad (59)\\
&= Z_{N-1}\prod_{k=1}^{N}\frac{1}{\sqrt{|\alpha_k|}}\left[\frac{\prod_{k<N}|\alpha_k|}{|\alpha_N|}\right]^{(26-D)/16}\left[\frac{\prod_{r<t}|x_t-x_r|\prod_{m<l}|Z_l-Z_m|}{\prod_{l,r}|Z_l-x_r|}\right]^{(26-D)/24} \quad (60)
\end{aligned}$$

4.1 Interpretation of $D < 26$

The factors in square brackets spoil Lorentz covariance for $D < 26$. However the x_r, Z_k dependence of these factors is in a form that can be cancelled by inserting an operator of the form $e^{\pm i\gamma\phi(\rho)}$ at each x_r, Z_k. Here $\phi(\rho)$ is one of the transverse string coordinates. Take the D indices of x^μ to be $0, 1, 2, \cdots, (D-1)$. Then $x^\pm = (x^0 \pm x^1)/\sqrt{2}$ and the transverse components are $2, \cdots (D-1)$. We choose $\phi(\rho) = x^{D-1}$. Clearly inserting such operators sacrifices the full $SO(D-1,1)$ Lorentz invariance. But if the Lorentz violating measure can indeed be cancelled in this way, the scattering amplitudes will be invariant under $SO(D-2,1)$ Lorentz invariance. For example, to get a subcritical string theory that respects 3+1 Lorentz invariance, we should start with $5 = 4+1$ dimensional space-time.

The contribution of the field ϕ to the Boltzmann factor of the worldsheet path integral will be

$$B(\phi) = \exp\left\{-\frac{1}{4\pi}\int d^2\rho(\nabla\phi)^2 + i\gamma\sum_r\phi(\rho(x_r)) + i\sum_k\frac{p_k}{2\pi p_k^+}\int d\sigma_k\phi(\sigma_k,\tau_k)\right\} \quad (61)$$

The last term converts the initial and final state description from coordinate space to momentum space, and we specialize to a constant momentum density on each string at initial and final times. Here, to conform with Mandelstam's (and also GT's) conventions, we have taken $\alpha' = 1$ and scale the worldsheet spatial coordinate $\sigma_{old} = T_0\sigma_{new} = \sigma_{new}/2\pi$ so that on a given string $0 < \sigma_{new} < 2\pi p_k^+ \equiv \pi\alpha_k$. Thus $\rho = \tau + i\sigma_{new}$, and henceforth $\sigma = \sigma_{new}$.

We can extract the γ dependence of the path integral by completing the square in the usual way. We shift $\phi \to \phi + c$ and choose c to cancel the linear terms:

$$-\nabla^2 c = 2\pi i\gamma\sum_r\delta(\rho-\rho(x_r)), \qquad \dot{c}|_{\tau_f} = i\frac{p_k}{p_k^+}, \quad \dot{c}|_{\tau_i} = -i\frac{p_k}{p_k^+}, \qquad c'|_\partial = 0 \quad (62)$$

$$\ln \frac{B(\phi+c)}{B(\phi)|_0} = +\frac{i\gamma}{2}\sum_s c(x_s) + \frac{i}{2}\sum_k \frac{p_k}{2\pi p_k^+}\int d\sigma_k c(\sigma_k,\tau_k) \tag{63}$$

The answer can be expressed in terms of the Neumann function

$$-\nabla^2 N(\rho,\rho') = -2\pi\delta(\rho-\rho'), \qquad \partial_n N(\rho,\rho')|_{\rho\in\partial} = f(\rho) \tag{64}$$

Then applying Green's theorem we have

$$\begin{aligned} c(\rho') &= -i\gamma\sum_r N(\rho(x_r),\rho') - i\sum_{k\in f}\frac{p_k}{2\pi p_k^+}\int d\sigma_k N(\rho,\rho') + i\sum_{k\in i}\frac{p_k}{2\pi p_k^+}\int d\sigma_k N(\rho,\rho') \\ &\quad + \frac{1}{2\pi}\int d\sigma(cf)\Big|_{\tau_i}^{\tau_f} \end{aligned} \tag{65}$$

The last term, independent of ρ' drops out of $\ln B/B_0$:

$$\begin{aligned} \ln\frac{B(\phi+c)}{B(\phi)|_0} &= \frac{\gamma^2}{2}\sum_{r,s} N(\rho(x_r),\rho(x_s)) + \gamma\sum_{r,k}\frac{p_k}{2\pi p_k^+}\int d\sigma_k N(\rho(x_r),\rho_k) \\ &\quad + \frac{1}{2}\sum_{kl}\int d\sigma_k d\sigma_l \frac{p_k p_l}{4\pi^2 p_k^+ p_l^+} N(\rho_k,\rho_l) \end{aligned} \tag{66}$$

The Neumann function on the upper half plane is

$$N(z,z') = \ln|z-z'| + \ln|z-z'^*| \to 2\ln|z-z'| \tag{67}$$

when one or both z's are on the real axis. Then, with $z(\rho)$ the conformal map from the string diagram to the upper half plane we find

$$\begin{aligned} \ln\frac{B(\phi+c)}{B(\phi)|_0} &= \gamma^2\sum_{r\neq s}\ln|x_r-x_s| + 2\gamma\sum_{r,k} p_k\ln|x_r-Z_k| + \sum_{k\neq l} p_k p_l \ln|Z_k-Z_l| \\ &\quad + \gamma^2\sum_r \ln|x_r-x_r| + \frac{1}{2}\sum_k\int d\sigma_k d\sigma_k' \frac{p_k^2}{4\pi^2 p_k^{+2}} N(\rho_k,\rho_k') \end{aligned} \tag{68}$$

Note that Z_N, which we have set to ∞, appears on the right side in the combination

$$2p_N(\gamma + \sum_{k=1}^{N-1} p_k)\ln Z_N = -2p_N^2 \ln Z_N \tag{69}$$

so the terms involving Z_N for this special dimension will combine just as with the other dimensions into the terms that lead to the mass shell condition on the Nth leg. We therefore can drop them. The self-contractions on the last line need further discussion. Those in the last term are of the same form for all transverse dimensions and combined give the mass shell condition. Let us denote the first $D-3$ transverse components as a vector $\boldsymbol{p}_k$ in bold face type, retaining roman type for the last one. Then the lightcone mass shell condition reads

$$\boldsymbol{p}_k^2 - 2p_k^+ p_k^- = \frac{D-2}{24} - p_k^2 \tag{70}$$

The left side of this equation is Lorentz invariant $p_{k\mu}p_k^\mu$ which should be $+1$ to describe the subcritical Veneziano model. This requires that $p_k^2 = (D-26)/24$, or $p_k = \pm i\sqrt{(26-D)/24}$. In this case the requirement

$$\sum_k p_k = -(N-2)\gamma \tag{71}$$

can be met if $p_k = -i\sqrt{(26-D)/24} = -\gamma$ for $N-1$ values of k and the Nth momentum is $+i\sqrt{(26-D)/24} = +\gamma$.

Finally we need an interpretation of the self contractions at the interaction points. Infinities in these contractions can be absorbed into the coupling constant, provided they are independent of the geometry of the worldsheet. Since the lightcone worldsheet is the fundamental starting point, we should set any regulator cutoffs in the ρ coordinate. Let us examine $\rho(z)$ near $z = x_r$, where $d\rho/dz = 0$:

$$\rho(z) \approx \rho(x_r) + \frac{1}{2}\frac{d^2\rho}{dz^2}\Big|_{z=x_r}(z-x_r)^2 \tag{72}$$

$$\frac{d^2\rho}{dz^2}\Big|_{z=x_r} = -\alpha_N \frac{\prod_{s\neq r}(x_r - x_s)}{\prod_k (x_r - z_k)} \tag{73}$$

$$z - x_r \approx \sqrt{\rho - \rho(x_r)}\left[\frac{\prod_k (x_r - z_k)}{-\alpha_N \prod_{s\neq r}(x_r - x_s)}\right]^{1/2} \tag{74}$$

$$|z(\rho) - z(\rho')| \approx |\sqrt{\rho-\rho(x_r)} - \sqrt{\rho'-\rho(x_r)}| \frac{\prod_k \sqrt{|x_r - z_k|}}{\sqrt{|\alpha_N|}\prod_{s\neq r}\sqrt{|x_r - x_s|}} \tag{75}$$

$$\gamma^2 \sum_e \ln|x_r - x_r| \to \frac{\gamma^2}{2}\left[(N-2)\ln\epsilon + \ln\frac{\prod_{r,k}|x_r - Z_k|}{|\alpha_N|^{N-2}\prod_{s\neq r}|x_r - x_s|}\right] \tag{76}$$

$$\to \frac{\gamma^2}{2}\left[(N-2)\ln\epsilon + \ln\frac{|\alpha_N|}{\prod_{k=1}^{N-1}|\alpha_k|} + \ln\frac{\prod_{r,k}|x_r - Z_k|^2}{\prod_{s\neq r}|x_r - x_s|\prod_{k\neq l}|Z_k - Z_l|}\right] \tag{77}$$

The last line is our interpretation of the self contractions at the interaction points where we have let ϵ be a measure of the cutoff regularization on the lightcone world sheet.

Having taken care of the terms involving Z_N and the self contractions at the external states, and setting $p_k = -\gamma$, for $k < N$ and $p_N = +\gamma$, what is left of the contribution from the insertion operator is the correction factor

$$C = \epsilon^{(N-2)\gamma^2/2}\left[\frac{\prod_{r\neq s}|x_r - x_s|\prod_{k\neq l<N}|Z_k - Z_l|}{\prod_{r,k<N}|x_r - Z_k|^2}\right]^{\gamma^2/2}\left[\frac{|\alpha_N|}{\prod_{k=1}^{N-1}|\alpha_k|}\right]^{\gamma^2/2}$$

$$= \epsilon^{(N-2)\gamma^2/2}\left[\frac{\prod_{r<s}|x_s-x_r|\prod_{l<k<N}|Z_k-Z_l|}{\prod_{r,k<N}|x_r-Z_k|}\right]^{\gamma^2}\left[\frac{\prod_{k=1}^{N-1}|\alpha_k|}{|\alpha_N|}\right]^{-\gamma^2/2} \tag{78}$$

More generally we can choose another momentum $p_n=+\gamma$, with $p_k=-\gamma$ for $k\neq n$. In that case the terms in $\ln C$ *linear* in p_n change sign, that is

$$\begin{aligned}
-2\gamma^2\sum_r \ln|x_r-Z_n| &+ 2\gamma^2\sum_{k\neq n,N}\ln|Z_k-Z_n| \\
&\to +2\gamma^2\sum_r \ln|x_r-Z_n| - 2\gamma^2\sum_{k\neq n,N}\ln|Z_k-Z_n| \\
&= -2\gamma^2\sum_r \ln|x_r-Z_n| + 2\gamma^2\sum_{k\neq n,N}\ln|Z_k-Z_n| + 4\gamma^2\ln\frac{|\alpha_n|}{|\alpha_N|}
\end{aligned} \tag{79}$$

where we have used the identity $-\alpha_N\prod_r(Z_n-x_r)=\alpha_n\prod_{k\neq n,N}(Z_n-Z_k)$, which we have proven earlier. Thus in more generality the correction factor becomes

$$C = \epsilon^{(N-2)\gamma^2/2}\left[\frac{\prod_{r<s}|x_s-x_r|\prod_{l<k<N}|Z_k-Z_l|}{\prod_{r,k<N}|x_r-Z_k|}\right]^{\gamma^2}\left[\frac{\prod_{k=1}^{N-1}|\alpha_k|}{|\alpha_N|}\right]^{-\gamma^2/2}\left[\frac{|\alpha_n|}{|\alpha_N|}\right]^{4\gamma^2} \tag{80}$$

Note that this formula embraces the previously obtained special case $n=N$.

Finally we combine this correction factor with the measure, setting $Z_{N-1}=1$:

$$\begin{aligned}
C\left|\frac{\partial T}{\partial Z}\right|\det{}^{-(D-2)/2}(-\nabla^2) &= \epsilon^{(N-2)\gamma^2/2}\prod_{k=1}^N\frac{1}{\sqrt{|\alpha_k|}} \\
&\left[\frac{|\alpha_n|}{|\alpha_N|}\right]^{4\gamma^2}\left[\frac{\prod_{k<N}|\alpha_k|}{|\alpha_N|}\right]^{(26-D)/16-\gamma^2/2}\left[\frac{\prod_{r<t}|x_t-x_r|\prod_{m<l<N}|Z_l-Z_m|}{\prod_{r,\ l<N}|Z_l-x_r|}\right]^{(26-D)/24+\gamma^2} \\
&= \epsilon^{(N-2)\gamma^2/2}\prod_{k=1}^N\frac{1}{\sqrt{|\alpha_k|}} \\
&\left[\frac{\prod_{k\neq n}|\alpha_k|}{|\alpha_n|}\right]^{(26-D)/16-\gamma^2/2}\left[\frac{|\alpha_n|^3}{|\alpha_N|^3}\frac{\prod_{r<t}|x_t-x_r|\prod_{m<l<N}|Z_l-Z_m|}{\prod_{r,\ l<N}|Z_l-x_r|}\right]^{(26-D)/24+\gamma^2}
\end{aligned} \tag{81}$$

We recall that the N point tree amplitude is obtained by multiplying this measure factor by the factor

$$dZ_2\cdots dZ_{N-2}\prod_{m<l<N}|Z_l-Z_m|^{2p_l\cdot p_m}, \qquad p_k^2=\boldsymbol{p}^2-2p^+p^-=\frac{D-2}{24}-\gamma^2 \tag{82}$$

and integrating the Z's over the range $Z_1=0<Z_2<Z_3<\cdots<Z_{N-2}<Z_{N-1}=1$, where $Z_1=0, Z_{N-1}=1, Z_N=\infty$ are held fixed.

It is of interest to write the formula for the scattering amplitude in a general projective frame where $Z_1 < Z_{N-1} < Z_N$ are fixed to general values. This is done by making a change of variables by a projective transformation $Z_k \to Y_k = (aZ_k + b)/(cZ_k + d)$ under which $Z_1 = 0 \to Y_1 = b/d$, $Z_{N-1} = 1 \to Y_{N-1} = (a+b)/c+d)$, and $Z_N = \infty \to Y_N = a/c$. In this case the map from the z-plane to the lightcone diagram includes all N terms:

$$\rho \;=\; \sum_{k=1}^{N} \alpha_k \ln(z - Y_k), \qquad \frac{d\rho}{dz} = \frac{\sum_k \alpha_k \prod_{l\neq k}(z - Y_l)}{\prod_k (z - Y_k)} \tag{83}$$

The numerator of $d\rho/dz$ is a polynomial of degree $N-2$ because $\sum_k \alpha_k = 0$. Let its roots be ξ_r which are the images of x_r under the projective transformation $\xi_r = (ax_r + b)/(cx_r + d)$.

$$\begin{aligned} \sum_k \alpha_k \prod_{l\neq k}(z - Y_l) &= \left(\sum_l \alpha_l Y_l\right) \prod_{r=1}^{N-2}(z - \xi_r) \\ \alpha_m \prod_{l\neq m}(Y_m - Y_l) &= \left(\sum_l \alpha_l Y_l\right) \prod_{r=1}^{N-2}(Y_m - \xi_r) \end{aligned} \tag{84}$$

The scattering amplitude then becomes

$$A_N \;=\; (Y_N - Y_{N-1})(Y_N - Y_1)(Y_{N-1} - Y_1) \int dY_2 \cdots dY_{N-2} \prod_{m<l} |Y_l - Y_m|^{2p_l\cdot p_m} \mathcal{M} \tag{85}$$

$$\begin{aligned} \mathcal{M} \;=\; & \epsilon^{(N-2)\gamma^2/2} \prod_{k=1}^{N} \frac{1}{\sqrt{|\alpha_k|}} \left[\frac{\prod_{k\neq n}|\alpha_k|}{|\alpha_n|}\right]^{(26-D)/16-\gamma^2/2} \\ & \left[\frac{|\alpha_n|^3}{|\sum_l \alpha_l Y_l|^3} \frac{\prod_{r<t}|\xi_t - \xi_r| \prod_{m<l}|Y_l - Y_m|}{\prod_{r,\, l}|Y_l - \xi_r|}\right]^{(26-D)/24+\gamma^2} \end{aligned} \tag{86}$$

where now factors involving Y_N are included in the various products.

We see that since there are two noncovariant factors raised to different powers, the only Lorentz covariant choice is $\gamma = 0, D = 26$. For $D < 26$ the best one can do is either remove the noncovariant ξ_r dependence by setting $\gamma^2 = (D-26)/24$ or remove the other factor with only α dependence by setting $\gamma^2 = (26 - D)/8$. We have already seen that in the first case the external states have $\boldsymbol{p}^2 - 2p^+p^- = 1$. Then the amplitude is just the generalized N-point Veneziano amplitude times the noncovariant function of the α

$$\left[\frac{\prod_{k\neq n}|\alpha_k|}{|\alpha_n|}\right]^{(26-D)/16-\gamma^2/2} \quad \to \quad \left[\frac{\prod_{k\neq n}|\alpha_k|}{|\alpha_n|}\right]^{(26-D)/12} \tag{87}$$

This noncovariant factor distinguishes the particle n which is assigned $+\gamma$ from the $N-1$ others assigned $-\gamma$, and it is the only feature that does so. It is interesting that this factor satisfies tree factorization by itself. This means that removing it by hand leaves a covariant amplitude that distinguishes none of the particles and that factorizes as unitarity demands. This ad hoc procedure would however destroy a local lightcone worldsheet description,

The second choice $\gamma^2 = (26 - D)/8$ maintains the scaling behavior demanded by Lorentz invariance, but sacrifices Lorentz invariance in the behavior of excited states. In this case the external state momenta satisfy

$$\boldsymbol{p}^2 - 2p^+p^- = \frac{D-2}{24} - \frac{26-D}{8} = \frac{D-20}{6} \tag{88}$$

The N-point scattering amplitude is then proportional to

$$\begin{aligned}
A_N &= \int dZ_2 \cdots dZ_{N-2} \prod_{k<l<N} |Z_l - Z_k|^{2p_k\cdot p_l} \left[\frac{|\alpha_n|^3}{|\alpha_N|^3} \frac{\prod_{r<t}|x_t - x_r| \prod_{m<l<N}|Z_l - Z_m|}{\prod_{l,r}|Z_l - x_r|}\right]^{(26-D)/6} \\
&= \int dZ_2 \cdots dZ_{N-2} \prod_{k<l<N} |Z_l - Z_k|^{2p_k\cdot p_l} \left[\frac{|\alpha_N|^{N-3}|\alpha_n|^3}{\prod_k |\alpha_k|} \frac{\prod_{r<t}|x_t - x_r|}{\prod_{k<l<N}|Z_l - Z_k|}\right]^{(26-D)/6} \\
&= \int dZ_2 \cdots dZ_{N-2} \prod_{k<l<N} |Z_l - Z_k|^{2p_k\cdot p_l - (26-D)/6} \left[\frac{|\alpha_N|^{N-3}|\alpha_n|^3}{\prod_k |\alpha_k|} \prod_{r<t}|x_t - x_r|\right]^{(26-D)/6}
\end{aligned}$$

Note that the role played by the field ϕ in this discussion is similar to that of the Liouville field in Polyakov's treatment of the subcritical string [11–14].

4.2 Four Point Examples

We have seen that some aspect of Lorentz invariance is lost when $D < 26$. To illustrate this we work out the 4-point amplitude in various cases. We first look at the unmodified lightcone 4-point amplitude at general D (taking $Z_1 = 0, Z_2 = Z, Z_3 = 1, Z_4 = \infty$):

$$\begin{aligned}
A_4 &= \int dZ \prod_{k=1}^{4} \frac{1}{|\alpha_k|^{(D-2)/48}} \left[|\alpha_4| \frac{|x_2 - x_1|}{Z(1-Z)}\right]^{(26-D)/24} Z^{2p_1\cdot p_2}(1-Z)^{2p_2\cdot p_3} \\
&= \int dZ \prod_{k=1}^{4} \frac{1}{|\alpha_k|^{(D-2)/48}} \left[|\alpha_4||x_2 - x_1|\right]^{(26-D)/24} Z^{-\alpha(s)-1}(1-Z)^{-\alpha(t)-1}
\end{aligned} \tag{89}$$

Let us define

$$\begin{aligned}
2p_1 \cdot p_2 - (26-D)/24 &= (p_1 + p_2)^2 - 2(D-2)/24 - (26-D)/24 \\
&\equiv -s - (D-2)/24 - 1 \equiv -\alpha(s) - 1 && (90) \\
2p_2 \cdot p_3 - (26-D)/24 &\equiv -\alpha(t) - 1 && (91) \\
\alpha_{ij} &\equiv \alpha_i + \alpha_j. && (92)
\end{aligned}$$

Then

$$\begin{aligned}
|\alpha_4|^2|x_2 - x_1|^2 &= (\alpha_1 + \alpha_2 + Z(\alpha_1 + \alpha_3))^2 + 4Z\alpha_1\alpha_4 \\
&= \alpha_{12}^2(1-Z)^2 + \alpha_{23}^2 Z^2 + (\alpha_{12}^2 + \alpha_{23}^2 - \alpha_{13}^2)Z(1-Z)
\end{aligned} \tag{93}$$

The last form shows that, in spite of the lack of Lorentz invariance, A_4 is crossing symmetric, which immediately follows from the change of variables $Z \to 1 - Z$. To check factorization, note that the poles in s arise from $Z \sim 0$, for which

$$|\alpha_4|^2 |x_2 - x_1|^2 \quad \to \quad (\alpha_1 + \alpha_2)^2 \tag{94}$$

so the contribution of this factor to the residue is $|\alpha_{12}|^{1-(D-2)/24}$. Thus we have

$$\begin{aligned} A_4 \quad &\sim \quad \prod_{k=1}^{4} \frac{1}{|\alpha_k|^{(D-2)/48}} \frac{|\alpha_{12}|^{1-(D-2)/24}}{\boldsymbol{p}^2 - (D-2)/24 - |\alpha_{12}| p^-} \\ &\sim \quad |\alpha_1 \alpha_2 \alpha_{12}|^{-(D-2)/48} \frac{1}{[\boldsymbol{p}^2 - (D-2)/24] / |\alpha_{12}| - p^-} |\alpha_3 \alpha_4 \alpha_{34}|^{-(D-2)/48} \end{aligned} \tag{95}$$

which is precisely the desired factorization property. In this way we see that the unmodified scattering amplitudes for $D < 26$ are crossing symmetric (cyclic), and unitary (factorizing poles), but lack Lorentz invariance because of the p_k^+ dependence.

4.3 Branion Branion Scattering

In a four dimensional theory the transverse space is two dimensional. To describe this situation with critical 26 dimensional open strings we make a 2+22 split of the 24 transverse coordinates $(x^1, x^2; y^1, \cdots, y^{22})$ and impose Dirichlet conditions $y^a = 0$ at both ends of each open string [15].

The concept of branions was introduced in the context of quantum field theory [16] to get a handle on the force between external sources in lightcone quantization, where it is beneficial to maintain p^+ conservation: the sources are fixed in transverse space but free to move in the longitudinal direction. In addition to bulk gauge fields, we introduced dynamical source fields that lived at a point in transverse space, but were free to move in the lightcone longitudinal direction. In other words the source fields lived on 1-branes. We called excitations of these source fields "branions". Of course, the branions interact with the bulk gauge fields.

To translate this situation to string theory, we associate the bulk gauge fields with open strings ending on a stack of D3-branes. They have two NN dimensions and 22 DD dimensions. We associate the branion of [16] with an open string with one end on a 1-brane within the D3-brane, and the other end free to move in the bulk of the D3 brane. Thus, the physical situation of branion-branion scattering in string theory is an open string, with one end free to move in the two-dimensional $\boldsymbol{x}$ space and the other end fixed at say $\boldsymbol{x} = 0$, scattering from another open string, also with one free end and the other end fixed to a different point, say $\boldsymbol{x} = \boldsymbol{R}$. Only the free ends participate in the interactions (see Fig. 4). Examples of such string amplitudes have been obtained long ago in [17] in the context of building dual resonance amplitudes with Regge trajectories with intercepts less than 1.

It is just as easy to analyze a $D = d + 2$ dimensional theory using a $(d, 24 - d)$ slit of transverse space. The ground state mass of each branion string is then given by

$$\alpha' m_{\text{branion}}^2 = -\frac{24 - d}{24} + \frac{d}{48} = \frac{d - 16}{16}. \tag{96}$$

Figure 4: Worldsheet for branion scattering

Because the scattering kinematics is 1+1 dimensional, $p^\pm$ conservation implies either forward or backward scattering. The figure shows backward scattering $p_4^+ = -p_2^+$ and $p_3^+ = -p_1^+$, with corresponding relations among the α_k's. In the mapping to the upper half plane we choose $Z_1 = 1$, $Z_2 = U$, $Z_3 = 0$, $Z_4 = \infty$. Then the scattering amplitude is

$$\begin{aligned}
\mathcal{M} &= \frac{g^2}{4p_1^+p_2^+}\int_0^1 dU\mathcal{D}(U)^d q^{T_0R^2/4\pi}(1-U)^{-(d-16)(p_1^+/p_2^++p_2^+/p_1^+)/16}U^{(d-16)(p_1^+/p_2^++p_2^+/p_1^+)/16} \\
&= \frac{g^2}{4p_1^+p_2^+}\int_0^1 dU\mathcal{D}(U)^d q^{T_0R^2/4\pi}(1-U)^{2p_1\cdot p_2}U^{-2p_1\cdot p_2} \\
&= \frac{g^2}{4p_1^+p_2^+}\int_0^1 \frac{4(1-k)dk}{(1+k)^3} q^{T_0R^2/4\pi}\left(\frac{4k}{(1-k)^2}\right)^{2p_1\cdot p_2}\mathcal{D}(k)^d \qquad (97)
\end{aligned}$$

here $q = e^{-\pi K'/K}$ is the modulus associated with the map of the rectangle to the upper half plane (see Fig. 5). The R dependence arises after shifting the string coordinates by the classical solution, $\boldsymbol{x}_c = \boldsymbol{R}(K-x)/2K$ in the $z = x+iy$ plane, that sets both Dirichlet boundary conditions to $\boldsymbol{x} = 0$. In the discussion of that figure, we established that $U = (1-k)^2/(1+k)^2$, which was used to obtain the last line.

Consulting (183) in Appendix B, we have

$$\mathcal{D}^{24} = \det{}^{-12}_{\rm DNDN}\frac{(1+k)^2}{4k^2(1-k)^4} = (2K)^{-6}q^{-1/2}\prod_m(1-q^m)^{-12}\frac{(1+k)^2}{4k^2(1-k)^4} \qquad (98)$$

$$\mathcal{M} = \frac{g^2}{p_1^+p_2^+}\int_0^1 \frac{(1+k)^{(d-16)/4}dk}{(2k)^{d/12}(1-k^2)^{(d-6)/6}}\left(\frac{4k}{(1-k)^2}\right)^{2\alpha' p_1\cdot p_2}\frac{q^{T_0R^2/4\pi-d/48}}{(2K)^{d/4}\prod_m(1-q^m)^{d/2}} \qquad (99)$$

We recall the relations between k, K and q:

$$k^2 = \frac{\theta_2(0)^4}{\theta_3(0)^4} = 16q\prod_{n=1}^\infty\frac{(1+q^{2n})^8}{(1+q^{2n-1})^8} \qquad (100)$$

$$1-k^2 = \frac{\theta_4(0)^4}{\theta_3(0)^4} = \prod_{n=1}^{\infty}\frac{(1-q^{2n-1})^8}{(1+q^{2n-1})^8} \tag{101}$$

$$(2K)^2 = \pi^2\theta_3(0)^4 = \pi^2\prod_{n=1}^{\infty}(1+q^{2n})^8(1-q^{2n})^4 \tag{102}$$

From these relations we see that $q \sim k^2/16$ for $k \to 0$. In this limit the integrand then behaves as

$$\frac{dk}{(2k)^{d/12}}(4k)^{2\alpha' p_1\cdot p_2}\frac{k^{T_0R^2/2\pi - d/24}}{\pi^{d/4}} = \frac{4^{2\alpha' p_1\cdot p_2}}{(2\pi^3)^{d/12}}dk k^{T_0R^2/2\pi+2\alpha' p_1\cdot p_2 - d/8} \tag{103}$$

We note that the invariant (mass)^2 in the 12 channel $-M^2 = (p_1+p_2)^2 = 2p_1\cdot p_2 - (d-16)/8\alpha'$ since $\alpha' p_k^2 = -(d-16)/16$. Integration near $k=0$ then generates a pole at

$$\alpha' M^2 = \frac{T_0R^2}{2\pi} - 1 \qquad \text{or} \qquad M^2 = T_0^2R^2 - \frac{1}{\alpha'} \tag{104}$$

since $\alpha' = 1/2\pi T_0$. This is in accord with the presence of a stretched string of mass $M \sim T_0R$ between $\boldsymbol{x}=0$ and $\boldsymbol{x}=\boldsymbol{R}$ in the 12 channel. The zero point energy squared $-1/\alpha'$ is also in accord with that of a DD ground string.

Singularities in the 23 channel arise from integrating near $k=1$. To analyze them it is best to do a Jacobi transformation on the various infinite products. Define $\bar{q}$ via the relation $\ln q \ln\bar{q} = \pi^2$, so $q\to 1$ implies $\bar{q}\to 0$. Then

$$\left(\frac{-\ln\bar{q}}{\pi}\right)^{1/2}\bar{q}^{1/12}\prod_{k=1}^{\infty}(1-\bar{q}^{2k}) = q^{1/12}\prod_{k=1}^{\infty}(1-q^{2k}) \tag{105}$$

$$\bar{q}^{-1/24}\prod_{k=1}^{\infty}(1+\bar{q}^{2k-1}) = q^{-1/24}\prod_{k=1}^{\infty}(1+q^{2k-1}) \tag{106}$$

$$\bar{q}^{-1/24}\prod_{k=1}^{\infty}(1-\bar{q}^{2k-1}) = 2^{1/2}q^{1/12}\prod_{k=1}^{\infty}(1+q^{2k}) \tag{107}$$

$$2^{1/2}\bar{q}^{1/12}\prod_{k=1}^{\infty}(1+\bar{q}^{2k}) = q^{-1/24}\prod_{k=1}^{\infty}(1-q^{2k-1}) \tag{108}$$

From these identities we infer

$$k^2 = \prod_{n=1}^{\infty}\frac{(1-\bar{q}^{2n-1})^8}{(1+\bar{q}^{2n-1})^8}, \qquad 1-k^2 = 16\bar{q}\prod_{n=1}^{\infty}\frac{(1+\bar{q}^{2n})^8}{(1+\bar{q}^{2n-1})^8} \tag{109}$$

$$(2K)^2 = \pi^2\theta_3(0)^4 = \pi^2\left(\frac{-\ln\bar{q}}{\pi}\right)^2\prod_{n=1}^{\infty}(1+\bar{q}^{2n})^8(1-\bar{q}^{2n})^4 \tag{110}$$

$$\prod_m(1-q^m) = \prod_k(1-q^{2k})(1-q^{2k-1})$$

$$= \bar{q}^{1/6} q^{-1/24} \left(\frac{-\ln\bar{q}}{\pi}\right)^{1/2} \sqrt{2} \prod_k (1-\bar{q}^{4k}) \tag{111}$$

$$(2K)^{d/4} \prod_m (1-q^m)^{d/2} = (2\pi)^{d/4} \left(\frac{-\ln\bar{q}}{\pi}\right)^{d/2} \prod_{n=1}^{\infty} (1+\bar{q}^{2n})^{d/2} (1-\bar{q}^{4n})^d \bar{q}^{d/12} q^{-d/48} \tag{112}$$

Thus we see that $\bar{q} \to 0$ implies that $k \to 1$. To analyze integration near $k = 1$, we substitute these relations in the integrand, dropping terms that vanish like a power of $(1-k)$ or a power of $\bar{q}$:

$$\begin{aligned} \mathcal{I} &\sim \frac{dk}{8(1-k)^{(d-6)/6}} \left(\frac{4}{(1-k)^2}\right)^{2\alpha' p_1\cdot p_2} \frac{q^{T_0 R^2/4\pi}}{(2\pi)^{d/4}\bar{q}^{d/12}} \left(\frac{-\ln\bar{q}}{\pi}\right)^{-d/2} \\ &\sim \frac{d\bar{q}}{2(2\pi)^{d/4}\bar{q}} \left(\frac{1}{16\bar{q}^2}\right)^{2\alpha' p_1\cdot p_2 + (d-8)/8} \left(\frac{-\pi}{\ln\bar{q}}\right)^{d/2} e^{\pi T_0 R^2/4\ln\bar{q}} \end{aligned} \tag{113}$$

Integration near $\bar{q} = 0$ generates a branch point (because of the powers of $\ln\bar{q}$) in the variable $(p_2+p_3)^2 = (p_1-p_2)^2 = -2p_1\cdot p_2 - (d-16)/8\alpha'$ at $1/\alpha'$. This precisely reflects the propagation of the open string tachyon (with $(\text{mass})^2 = -1/\alpha'$) between two points in transverse space.

The small $\bar{q}$ region of integration controls the large R behavior of the scattering amplitude. To clarify this point, it is helpful to change variables to $T = -\ln\bar{q}$ which is large for small $\bar{q}$. Then we apply a saddle point approximation to the integral

$$\begin{aligned} I &= \int_\Lambda^\infty dT \left(\frac{\pi}{T}\right)^{d/2} \exp\left\{-2(\alpha' p_{23}^2 - 1)T - \frac{\pi T_0 R^2}{4T}\right\} \\ &\approx \frac{\pi^{3/4}(T_0R^2)^{1/4}}{2^{5/4}(\alpha' p_{23}^2-1)^{3/4}} \left(\frac{8\pi(\alpha' p_{23}^2-1)}{T_0R^2}\right)^{d/4} \exp\left\{-R\sqrt{p_{23}^2 - 1/\alpha'}\right\} \end{aligned} \tag{114}$$

where the saddle point is at $T = \sqrt{\pi T_0 R^2/8(\alpha' p_{23}^2 - 1)}$ which is large for large R. Notice that for 4 dimensional spacetime ($d = 2$), this large R behavior is precisely that of the Kelvin Bessel function $K_0(R\sqrt{p_{23}^2 - 1/\alpha'})$, which is the R dependence of the corresponding scattering amplitude in quantum field theory.

To compare our results to those of Siegel [17], we make use of some identities from the theory of elliptic functions [18] to rewrite our expression for $\mathcal{D}$. First note that

$$2^{1/6} q^{1/24} \prod_m (1-q^m) = \theta_4(0)^{2/3}\theta_2(0)^{1/6}\theta_3(0)^{1/6} = \sqrt{\frac{2K}{\pi}}(1-k^2)^{1/6} k^{1/12} \tag{115}$$

Then

$$\begin{aligned} \mathcal{D}^{24} &= \frac{\pi^6}{16}(2K)^{-12}\frac{1}{4k^3(1-k)^6} = \frac{\pi^6}{16}[2(1+k)K(k)]^{-12}\frac{(1+k)^{12}}{4k^3(1-k)^6} \\ &= \pi^6\left[2K(\sqrt{1-U})\right]^{-12} U^{-3}(1-U)^{-3} \end{aligned} \tag{116}$$

Then (97) can be written

$$\mathcal{M} \;=\; \frac{g^2\pi^{d/4}}{4p_1^+p_2^+}\int_0^1 dU\left[2K(\sqrt{1-U})\right]^{-d/2} q^{T_0R^2/4\pi}(1-U)^{2p_1\cdot p_2-d/8}U^{-2p_1\cdot p_2-d/8} \tag{117}$$

The amplitudes calculated in [17] would have $R = 0$ and would not necessarily be for backward scattering. We obtain agreement if we set $R = 0$ in the above formula and $2p_2 \cdot p_3 = -2p_1 \cdot p_2$ in [17].

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A Method of Images

The empty space solution of the diffusion equation is simply

$$P(\boldsymbol{\rho}-\boldsymbol{\rho}',t) = \frac{1}{2\pi t}e^{-(\boldsymbol{\rho}-\boldsymbol{\rho}')^2/2t} \tag{118}$$

For a wedge of angle $\theta = \pi/2M$ D or N boundary conditions can be arranged by placing sources at angles $\pm\alpha + n\pi/M$, $n = 0, 1 \ldots, 2M-1$ where the source in the wedge is at angle α. To impose N conditions on both edges of the angle choose the same sign for all sources. To impose Dirichlet conditions on both edges, the image charges alternate in sign. Finally to arrange D conditions on the abscissa and N conditions on the ray $\theta = \pi/2M$, choose the sign pattern $++--++--++\cdots++--$ counterclockwise around the circle. Here the first $+$ is the sign of the original source.

For all cases, as one goes counterclockwise around the circle, $(\boldsymbol{\rho}-\boldsymbol{\rho}_{n\mp})^2$ assumes the values $2\rho^2(1-\cos(n\pi/M-2\alpha)$, $2\rho^2(1-\cos(n\pi/M)$. Then

$$\begin{aligned}\mathrm{Tr}e^{t\nabla^2/2} \;=\; \int_{\rm wedge} d^2\rho\frac{1}{2\pi t}\Bigg[1 + \sum_{n=1}^{2M}\begin{pmatrix} - \\ + \\ (-)^{n-1}\end{pmatrix} e^{-\rho^2[1-\cos(n\pi/M-2\alpha)]/t} \\ + \sum_{n=1}^{2M-1}\begin{pmatrix} + \\ + \\ (-)^{n}\end{pmatrix} e^{-\rho^2[1-\cos(n\pi/M)]/t}\Bigg]\end{aligned} \tag{119}$$

where the signs in the sums are for boundary conditions DD, NN, DN respectively.

Except for the 1 term and the $n = 1, 2M$ terms of the first sum, the integral of ρ over the whole infinite wedge is finite. The integral of the 1 term is simply $A/2\pi t$ where A is the area of the wedge. The integral of the $n = 2M$ term of the first sum, restricting $0 < \rho < R(\alpha)$,

$$\frac{1}{2\pi t}\int_0^{\pi/2M} d\alpha \int_0^R \rho d\rho e^{-2(\rho^2/t)\sin^2\alpha} = \frac{1}{4\pi}\int_0^{\pi/2M} d\alpha\frac{1}{2\sin^2\alpha}(1-e^{-2(R^2/t)\sin^2\alpha})$$

$$
\begin{aligned}
&= \frac{1}{8\pi}\int_0^{\pi/2M} d\alpha(1-e^{-2(R^2/t)\sin^2\alpha})\frac{d}{d\alpha}(-\cot\alpha) \\
&= -\frac{1}{8\pi}\cot\frac{\pi}{2M}(1-e^{-2(R^2/t)\sin^2\pi/2M}) + \frac{1}{4\pi t}\int_0^{\pi/2M} d\alpha e^{-2(R^2/t)\sin^2\alpha}\cot\alpha\frac{d}{d\alpha}(R^2\sin^2\alpha) \\
&\sim -\frac{1}{8\pi}\cot\frac{\pi}{2M} + \frac{1}{4\pi t}\int_0^\infty \frac{d\alpha}{\alpha}e^{-2(\alpha^2R^2/t)}2(RR'\alpha^2+R^2\alpha) \\
&\sim -\frac{1}{8\pi}\cot\frac{\pi}{2M} + \frac{1}{2\pi}\int_0^\infty d\eta e^{-\eta^2}\left(\frac{R'}{2R}\eta + \frac{R}{\sqrt{2t}}\right) = -\frac{1}{8\pi}\cot\frac{\pi}{2M} + \frac{R(0)}{4\sqrt{2\pi}} + O\left(\frac{R'}{R}\right) \\
&\sim -\frac{1}{8\pi}\cot\frac{\pi}{2M} + \frac{R(0)}{4\sqrt{2\pi}} \qquad (120)
\end{aligned}
$$

The integral in the $n=1$ term of the first sum gives the same result with $R(\pi/2M)$ in place of $R(0)$. The remainder of the first sum gives

$$
\begin{aligned}
&\frac{1}{2\pi t}\sum_{n=2}^{2M-1}\begin{pmatrix} - \\ + \\ (-)^{n-1}\end{pmatrix}\int_0^{\pi/2M} d\alpha\int_0^\infty \rho d\rho e^{-\rho^2[1-\cos(n\pi/M-2\alpha)]/t} \\
&\qquad = \frac{1}{8\pi}\sum_{n=2}^{2M-1}\begin{pmatrix} - \\ + \\ (-)^{n-1}\end{pmatrix}\left[\cot\frac{(n-1)\pi}{2M} - \cot\frac{n\pi}{2M}\right] \qquad (121)
\end{aligned}
$$

In the first two cases (DD, NN) the inner terms in the sum cancel in pairs leaving the first term for $n=2$ and the second term for $n=2M-1$:

$$
\begin{aligned}
\frac{1}{8\pi}\sum_{n=2}^{2M-1}\left[\cot\frac{(n-1)\pi}{2M} - \cot\frac{n\pi}{2M}\right] &= \frac{1}{8\pi}\left[\cot\frac{\pi}{2M} - \cot\frac{(2M-1)\pi}{2M}\right] \\
&= \frac{1}{4\pi}\cot\frac{\pi}{2M} \qquad (122)
\end{aligned}
$$

In the last case a complete cancellation occurs "outside-in":

$$
\begin{aligned}
\sum_{n=2}^{2M-1}(-)^{n-1}\left[\cot\frac{(n-1)\pi}{2M} - \cot\frac{n\pi}{2M}\right] &= \sum_{n=1}^{2M-2}(-)^n\cot\frac{n\pi}{2M} + \sum_{n=2}^{2M-1}(-)^n\cot\frac{n\pi}{2M} \\
&= \sum_{n=1}^{2M-2}(-)^n\left[\cot\frac{n\pi}{2M} + \cot\frac{(2M-n)\pi}{2M}\right] \\
&= 0 \qquad (123)
\end{aligned}
$$

The 1 term together with all the contributions to the first sum in square brackets contribute simply

$$
\frac{A}{2\pi t}\begin{pmatrix} - \\ + \\ -\end{pmatrix}\frac{R(0)}{4\sqrt{2\pi t}}\begin{pmatrix} - \\ + \\ +\end{pmatrix}\frac{R(\pi/2M)}{4\sqrt{2\pi t}} = \frac{A}{2\pi t} - \frac{L_D - L_N}{4\sqrt{2\pi t}} \qquad (124)
$$

where L_D, L_N are the total lengths of the Dirichlet and Neumann boundaries respectively.

Finally we turn to the second sum in square brackets, which will be responsible for the corner contributions. The ρ integration is finite and elementary:

$$\mathrm{Tr}\{e^{t\nabla^2/2}\}_{\rm corner} = \frac{1}{16M}\sum_{n=1}^{2M-1}\begin{pmatrix}+\\+\\(-)^n\end{pmatrix}\frac{1}{\sin^2(n\pi/2M)} = -\frac{1}{4M}\sum_{n=1}^{2M-1}\begin{pmatrix}+\\+\\(-)^n\end{pmatrix}\frac{e^{-in\pi/M}}{(1-e^{-in\pi/M})^2} \quad (125)$$

This sum can be represented as a contour integral because the quantities $z_n \equiv e^{-in\pi/M}$ are all the non unit $2M$th roots of unity: $z_n^{2M} - 1 = 0$ and $z_n^M = (-)^n$. We have

$$-\frac{(1, z^M)}{2(z^{2M}-1)(z-1)^2} \sim \frac{(1,(-)^n)}{16M\sin^2(n\pi/2M)}\frac{1}{z-z_n}, \qquad \text{as} \quad z \to z_n \quad (126)$$

Thus

$$\mathrm{Tr}\{e^{t\nabla^2/2}\}_{\rm corner} \;=\; -\oint_C \frac{dz}{2\pi i}\begin{pmatrix}1\\1\\z^M\end{pmatrix}\frac{1}{2(z^{2M}-1)(z-1)^2} \quad (127)$$

where C is a counterclockwise contour encircling all the z_n, for $n = 1, 2, \ldots, 2M-1$. This contour can be deformed to a clockwise contour encircling the (triple) pole at $z = 1$. Then the integral is just $(-)$ times the residue of that triple pole. In terms of the functions

$$f(z) \equiv \frac{z-1}{2(z^{2M}-1)}, \qquad f_1(z) \equiv \frac{z^M(z-1)}{2(z^{2M}-1)} = \frac{(z-1)}{2(z^M - z^{-M})} \quad (128)$$

these residues are just $f''(1)/2$, $f''(1)/2$, and $f_1''(1)/2$ respectively. An efficient way to evaluate these derivatives it to put $g(t) = f(e^t)$, so that $\dot{(g)} = e^t f'$ and $\ddot{(g)} = e^t f' + e^{2t} f''$.Then $f''(1) = \ddot{g}(0) - \dot{g}(0)$. So we expand g to order t^2:

$$\begin{aligned} g(t) &= \frac{1}{4M}\frac{1+t/2+t^2/6}{1+Mt+2M^2t^2/3} = \frac{1}{4M} + \left(\frac{1}{8M}-\frac{1}{4}\right)t + \left(\frac{1}{24M}+\frac{M}{12}-\frac{1}{8}\right)t^2 + O(t^3) \\ \frac{1}{2}f''(1) &= \frac{1}{24M}+\frac{M}{12}-\frac{1}{16M} = \frac{M}{12}-\frac{1}{48M} = \frac{1}{24}\left(2M - \frac{1}{2M}\right) \end{aligned} \quad (129)$$

which confirms the formula for a DD or NN corner of angle $\theta = \pi/2M$.

To handle the ND case we expand

$$\begin{aligned} g_1(t) &= \frac{1}{4M}\frac{1+t/2+t^2/6}{1+M^2t^2/6} = \frac{1}{4M} + \frac{t}{8M} - \frac{M^2-1}{24M}t^2 + O(t^3) \\ \frac{1}{2}f''(1) &= -\frac{M}{24}+\frac{1}{24M}-\frac{1}{16M} = -\frac{M}{24}-\frac{1}{48M} = -\frac{1}{48}\left(2M+\frac{2}{2M}\right) \end{aligned} \quad (130)$$

which confirms the formula for a DN corner of angle $\theta = \pi/2M$.

B Determinant for the Lightcone Worldsheet Tree

The quantities Z_k, with $k = 1 \cdots (N-1)$, and x_r, with $r = 1 \cdots (N-2)$ are determined from the map from the upper-half Koba-Nielsen plane (z) to the lightcone world sheet $(\rho = \tau + i\sigma)$:

$$\rho = \sum_{k=1}^{N-1} \alpha_k \ln(z - Z_k), \qquad \left.\frac{d\rho}{dz}\right|_{z=x_r} = 0 \tag{131}$$

$$\frac{d\rho}{dz} = \sum_{k=1}^{N-1} \frac{\alpha_k}{z - Z_k} = \frac{\sum_{k=1}^{N-1} \alpha_k \prod_{l\neq k}(z - Z_l)}{\prod_k (z - Z_k)} = -\alpha_N \frac{\prod_r (z - x_r)}{\prod_k (z - Z_k)} \tag{132}$$

$$\left.\frac{d^2\rho}{dz^2}\right|_{z=x_s} = \sum_{k=1}^{N-1} \frac{\alpha_k}{z - Z_k} = -\alpha_N \frac{\prod_{r\neq s}(x_s - x_r)}{\prod_k (x_s - Z_k)} \tag{133}$$

where the last line is true because the factor $(z - x_s)$ in the numerator must be killed by the derivative to get a nonzero contribution. The asymptotic strings at $\tau = \pm\infty$ are mapped from the Z_k. In this notation $Z_N = \infty, Z_1 = 0$. A useful identity follows by setting $z = Z_m$ in the identity

$$-\alpha_N \prod_r (z - x_r) = \sum_{k=1}^{N-1} \alpha_k \prod_{l\neq k} (z - Z_l) \tag{134}$$

$$-\alpha_N \prod_r (Z_m - x_r) = \sum_{k=1}^{N-1} \alpha_k \prod_{l\neq k} (Z_m - Z_l) = \alpha_m \prod_{l\neq m} (Z_m - Z_l) \tag{135}$$

$$|\alpha_N|^N \prod_{m,r} |Z_m - x_r| = \prod_{m=1}^{N} |\alpha_m| \prod_{l\neq k} |Z_k - Z_l| \tag{136}$$

We next consider the transformation of the determinant.

$$\Sigma = \ln|\alpha_N| - \sum_{k=1}^{N-1} \ln|z - Z_k| + \sum_{r=1}^{N-2} \ln|z - x_r| \tag{137}$$

Clearly $\partial_y \Sigma = 0$ on the real axis. Since the points $z = Z_k, x_s$ are singular, we deform the boundary near those points into small semicircles, in the upper half plane, of radii ϵ_k, ϵ_r respectively. The radius ϵ_k near Z_k can be interpreted in terms of a large time T_k for the asymptotic string k. From the mapping function we find

$$\epsilon_k = e^{T_k/\alpha_k} \prod_{l\neq k} |Z_l - Z_k|^{-\alpha_l/\alpha_k} \tag{138}$$

The string N is asymptotic at large z. If R is the radius of a large semi-circle, we have from the mapping function

$$T_N \sim -\alpha_N \ln R, \qquad R \sim e^{-T_N/\alpha_N}. \tag{139}$$

On the other hand the radius ϵ_s near x_s is a temporary regulator, which maps onto a circular deformation of the boundary near the corresponding interaction point on the lightcone worldsheet. From the mapping function we see that the radius of this regulating circle on the worldsheet is given by

$$\delta_s = \frac{1}{2}\epsilon_s^2\left|\frac{d^2\rho}{dz^2}\right|_{z=x_s} = \frac{1}{2}\epsilon_s^2|\alpha_N|\frac{\prod_{r\neq s}|x_s-x_r|}{\prod_k|x_s-Z_k|} \tag{140}$$

$$\epsilon_s = \sqrt{\frac{2\delta_s}{|\alpha_N|}}\frac{\prod_k|x_s-Z_k|^{1/2}}{\prod_{r\neq s}|x_s-x_r|^{1/2}} \tag{141}$$

$$\prod_s\epsilon_s = |\alpha_N|^{-N+3/2}\prod_k|\alpha_k|^{1/2}\prod_s\sqrt{2\delta_s}\frac{\prod_{l\neq k}|Z_l-Z_k|^{1/2}}{\prod_{r\neq s}|x_s-x_r|^{1/2}} \tag{142}$$

To calculate the determinant for the lightcone worldsheet, we start with the determinant for the region in the upper-half z-plane bounded by the real axis, the large radius R semi-circle, and the small radius ϵ_k, ϵ_r semi-circles. Then we apply the generalized McKean-Singer formula to transform to the determinant for the worldsheet.

B.1 Unmixed Boundary Conditions

In this case, the boundary conditions are either Dirichlet everywhere or Neumann everywhere. Then in the limit of large R and small ϵ, factorization implies that the z-plane figure determinant has the behavior

$$-\frac{1}{2}\mathrm{Tr}\ln(-\nabla^2)_z \sim \frac{5}{24}\ln R + \frac{1}{24}\sum_k\ln\epsilon_k + \frac{1}{24}\sum_r\ln\epsilon_r + \mathrm{const} \tag{143}$$

where the constant term, representing the determinant for the upper half plane with the same boundary conditions everywhere, has nothing to depend on! We treat mixed boundary conditions in the next subsection, where the corresponding term can depend on the relative locations of the points that separate Dirichlet from Neumann boundary conditions.

Next we develop the transformation of the determinant from this z-plane figure to the lightcone worldsheet:

$$\Sigma = \ln|\alpha_N| - \sum_{k=1}^{N-1}\ln|z-Z_k| + \sum_{r=1}^{N-2}\ln|z-x_r| \tag{144}$$

Clearly $\partial_y\Sigma = -\partial_n\Sigma = 0$ on the real axis. Thus the change formula receives contributions from the corners and semi-circles only. For z near Z_k put $z = Z_k + re^{i\varphi}$ and approximate

$$\Sigma \approx \ln|\alpha_N| - \ln r - \sum_{l\neq k}^{N-1}\ln|Z_l-Z_k| + \sum_{r=1}^{N-2}\ln|Z_k-x_r|, \qquad \partial_n\Sigma \approx \frac{1}{r} \tag{145}$$

Then

$$\Delta_{\epsilon_k} \quad = \quad \left[\frac{1}{24} - \frac{1}{12} + \frac{1}{8}\right]\Sigma = \frac{1}{12}\ln\left(\frac{|\alpha_N|}{\epsilon_k}\frac{\prod_r |Z_k - x_r|}{\prod_{l\neq k}|Z_k - Z_l|}\right) = \ln\left(\frac{|\alpha_k|}{\epsilon_k}\right)^{1/12} \tag{146}$$

The three terms in square brackets are the $\int dl\Sigma\partial_n\sigma$ term the extrinsic curvature term (negative here) and the two corners at this semi-circle respectively.

For $z = x_s + re^{i\varphi}$, on the other hand we have

$$\Sigma \approx \ln|\alpha_N| + \ln r - \sum_{l}^{N-1}\ln|Z_l - x_s| + \sum_{r\neq s}\ln|x_s - x_r|, \qquad \partial_n\Sigma \approx -\frac{1}{r} \tag{147}$$

Then

$$\Delta_{\epsilon_s} = \left[-\frac{1}{24} - \frac{1}{12} + \frac{1}{8}\right]\Sigma = 0 \tag{148}$$

Finally for the large semi-circle, $\Sigma \approx -\ln(r/|\alpha_N|)$, $\partial_n\Sigma \approx -1/r$, and

$$\Delta_R = \left[-\frac{1}{24} + \frac{1}{12} + \frac{1}{8}\right]\Sigma = -\frac{1}{6}\ln\frac{R}{|\alpha_N|} \tag{149}$$

Combining all the contributions, we have

$$\begin{aligned}
\det{}^{-1/2}(-\nabla^2)_\rho &= \det{}^{-1/2}(-\nabla^2)_z\left(\frac{|\alpha_N|}{R}\right)^{1/6}\prod_k\left(\frac{|\alpha_k|}{\epsilon_k}\right)^{1/12} \\
&= C|\alpha_N|^{1/6}R^{1/24}\prod_k\epsilon_k^{-1/24}\prod_r\epsilon_r^{1/24}\prod_k|\alpha_k|^{1/12} \\
&= C|\alpha_N|^{1/6}\exp\left\{-\sum_{k=1}^{N}\frac{T_k}{24\alpha_k}\right\}\prod_{k\neq l}|Z_k - Z_l|^{\alpha_k/24\alpha_l}\prod_k|\alpha_k|^{1/12} \\
&\quad \left[\frac{\prod_r(2\delta_r)}{|\alpha_N|^{N-2}}\frac{\prod_k\prod_r|x_r - Z_k|}{\prod_{r\neq s}|x_s - x_r|}\right]^{1/48} \\
&= C|\alpha_N|^{1/6}\exp\left\{-\sum_{k=1}^{N}\frac{T_k}{24\alpha_k}\right\}\prod_{k\neq l}|Z_k - Z_l|^{\alpha_k/24\alpha_l}\prod_k|\alpha_k|^{1/12} \\
&\quad \left[\frac{\prod_r(2\delta_r)\prod_k|\alpha_k|}{|\alpha_N|^{2N-3}}\frac{\prod_{k\neq l}|Z_l - Z_k|}{\prod_{r\neq s}|x_s - x_r|}\right]^{1/48} \\
&= C\prod_{k=1}^{N}\frac{|\alpha_k|^{1/8}}{|\alpha_k|^{1/48}}\prod_{k\neq l}|Z_k - Z_l|^{\alpha_k/24\alpha_l} \\
&\quad \left[\frac{\prod_r\sqrt{2\delta_r}}{|\alpha_N|^{N-3}}\frac{\prod_{k<l}|Z_l - Z_k|}{\prod_{r<s}|x_s - x_r|}\right]^{1/24}\exp\left\{-\sum_{k=1}^{N}\frac{T_k}{24\alpha_k}\right\}
\end{aligned} \tag{150}$$

If there are $d = D-2$ transverse dimensions this entire factor should be raised to the power d.

The worldsheet path integral is this determinant factor times a factor e^{iW_c} which arises from removing boundary data in the path integral by shifting the $\boldsymbol{x}$ by the classical solution that satisfies those boundary data. Among other things e^{iW_c} includes factors $R^{-\boldsymbol{p}^2}\prod_k \epsilon_k^{\boldsymbol{p}^2}$ in the limit that the $-T_k/\alpha_k$ get large. If $W_c = \sum_{kl} p_k N(\rho_k,\rho_l)p_l$ is expressed in terms of a Neumann function, these factors arise from the diagonal $l=k$ terms. The rest of these diagonal terms, combined with the factors $|\alpha_k|^{1/8}$, provide a factor of the ground string wave function for each external string. The N ground string scattering amplitude is obtained by amputating these ground state wave functions together with the factors $e^{\sum_k(\boldsymbol{p}_k^2-d/24)T_k/\alpha_k} = e^{\sum_k T_k P_k^-}$ from the path integral and integrating over the interaction times $\int d\tau_1\cdots d\tau_{N-2}$ where $\rho_r = \tau_r + i\sigma_r$ are the locations of the $N-2$ interaction points on the worldsheet. By translational invariance in x^+ the integrand after amputation will acquire a factor $e^{a\sum_k P_k^-}$ if all the τ_r are translated by a. This means that integrating over one of the τ_r simply produces a P^- conserving delta function. The coefficient of this delta function is just the integral over only $N-3$ of the τ_r. Note that $\sum_k \alpha_k = 0$ by the lightcone worldsheet construction and $\sum_k P_k = 0$ when Neumann conditions are chosen for the $\boldsymbol{x}$ integrals as explained in Section 3 (see (38)).

$$\mathcal{M} = \int d\tau_2\cdots d\tau_{N-2}\left[\det{}^{-d/2}(-\nabla^2)_\rho e^{iW_c}\right]_{\text{amputated}} \tag{151}$$

where we have set $\tau_1 = 0$ and understand that $\sum_k P_k^- = 0$.

The final result for $[e^{iW_c}]_{\text{amputated}}$ includes the off diagonal terms in its Neumann function representation, together with the parts of ϵ_k that remain after amputating $e^{\sum_k T_k P_k^-}$:

$$\begin{aligned}
\left[e^{iW_c}\right]_{\text{amputated}} &= \prod_{k<l}|Z_l-Z_k|^{2\boldsymbol{p}_k\cdot\boldsymbol{p}_l}\left(\prod_{k\neq l}|Z_k-Z_l|\right)^{-\alpha_l\boldsymbol{p}_k^2/\alpha_k}\\
\left[\det{}^{-d/2}(-\nabla^2)_\rho\right]_{\text{amputated}} &= C\prod_{k=1}^N\frac{1}{|\alpha_k|^{d/48}}\prod_{k\neq l}|Z_k-Z_l|^{d\alpha_k/24\alpha_l}\\
&\qquad\left[\frac{\prod_r\sqrt{2\delta_r}}{|\alpha_N|^{N-3}}\frac{\prod_{k<l}|Z_l-Z_k|}{\prod_{r<s}|x_s-x_r|}\right]^{d/24}\\
\left[\det{}^{-d/2}(-\nabla^2)_\rho e^{iW_c}\right]_{\text{amputated}} &= C\prod_{k=1}^N\frac{1}{|\alpha_k|^{d/48}}\prod_{k<l}|Z_k-Z_l|^{2p_k\cdot p_l}\\
&\qquad\left[\frac{\prod_r\sqrt{2\delta_r}}{|\alpha_N|^{N-3}}\frac{\prod_{k<l}|Z_l-Z_k|}{\prod_{r<s}|x_s-x_r|}\right]^{d/24}
\end{aligned} \tag{152}$$

where we have used $p_k\cdot p_l = \boldsymbol{p}_k\cdot\boldsymbol{p}_l - p_k^+p_l^- - p_k^-p_l^+ = \boldsymbol{p}_k\cdot\boldsymbol{p}_l - \alpha_k(\boldsymbol{p}_l^2-d/24)/2\alpha_l - \alpha_l(\boldsymbol{p}_k^2-d/24)/2\alpha_k$ It is convenient to change integration variables from the τ's to the Z's.

Mandelstam's result for the Jacobian is (taking $Z_1, Z_{N-1}, Z_N = 0, 1, \infty$ respectively)

$$\frac{\partial(\tau_2, \cdots, \tau_{N-2})}{\partial(Z_2, \cdots, Z_{N-2})} = \left[\frac{1}{|\alpha_N|^{N-3}} \frac{\prod_{k<l} |Z_l - Z_k|}{\prod_{r<s} |x_s - x_r|}\right]^{-1}, \tag{153}$$

so that the scattering amplitude becomes

$$\begin{aligned} \mathcal{M} &= C \prod_r (2\delta_r)^{d/48} \prod_{k=1}^{N} \frac{1}{|\alpha_k|^{d/48}} \int dZ_2 \cdots dZ_{N-2} \prod_{k<l} |Z_k - Z_l|^{2p_k \cdot p_l} \\ &\quad \left[\frac{1}{|\alpha_N|^{N-3}} \frac{\prod_{k<l} |Z_l - Z_k|}{\prod_{r<s} |x_s - x_r|}\right]^{(D-26)/24} \end{aligned} \tag{154}$$

The factor raised to the power $D-26$ depends on the Lorentz frames so the critical dimension $D = 26$ is necessary for Lorentz invariance [2], in which case $\mathcal{M}$ is proportional to the N particle dual resonance amplitude. Of course factorization implies that $C = g^{N-2}$ and $\delta_r = \delta$, independent of r. Then $\prod_r (2\delta_r) = (2\delta)^{N-2}$ so δ can be absorbed in the coupling constant.

B.2 Mixed Boundary Conditions

Here we consider the cases where the boundary consists of several segments with either Dirichlet or Neumann boundary. Call the points that separate different boundary conditions P_a. The asymptotic strings on the world sheet can now have two free ends (NN), one free end (ND), or no free ends (DD). It will be convenient to choose to close the asymptotic world sheet with N, D, and D boundary conditions respectively, in order to minimize the number of ND corners.

The contributions associated with the NN and DD asymptotic strings will therefore be exactly as in the previous subsection, since they involve no ND corners. Also the contributions associated with the interaction points will be the same. Only the contributions from the ND strings need modification. Since we have to have an even number of ND strings, in this section we might as well assume there are at least 2 and take one of them to map to the $z = \infty$. Then by factorization the z-plane determinant has the behavior for large R and small ϵ

$$-\frac{1}{2}\mathrm{Tr}\ln(-\nabla^2)_z \sim \frac{1}{48}\ln R - \frac{1}{48}\sum_{k\in\mathrm{DN}} \ln\epsilon_k + \frac{1}{24}\sum_{k\in\mathrm{NN}} \ln\epsilon_k + \frac{1}{24}\sum_r \ln\epsilon_r + \ln\mathcal{D} \tag{155}$$

where in the last term $\mathcal{D}(P_a)$, representing the determinant for the z-plane stripped of the semi-circles, can now depend on the locations of the Dirichlet-Neumann transitions points P_a.

The transform to the worldsheet involves the same Σ (144), the same change factors associated with x_r (148) and Z_k for $k \in \mathrm{NN}$ (146) as in the previous subsection. Modifications occur in the change factor associated with R

$$\Delta_R^{\mathrm{DN}} = \left[-\frac{1}{24} + \frac{1}{12} + 0\right]\Sigma = -\frac{1}{24}\ln\frac{R}{|\alpha_N|} \tag{156}$$

and in the change factor associated with Z_k with $k \in \mathrm{DN}$.

$$\Delta^{\mathrm{DN}}_{\epsilon_k} \;=\; \left[\frac{1}{24}-\frac{1}{12}+0\right]\Sigma = -\frac{1}{24}\ln\left(\frac{|\alpha_k|}{\epsilon_k}\right) \tag{157}$$

Combining all the contributions, we have for the determinant on the worldsheet:

$$\begin{aligned}
\det{}^{-1/2}(-\nabla^2)_\rho &= \det{}^{-1/2}(-\nabla^2)_z \left(\frac{|\alpha_N|}{R}\right)^{1/24} \prod_{k\in\mathrm{NN}}\left(\frac{|\alpha_k|}{\epsilon_k}\right)^{1/12} \prod_{k\in\mathrm{DN}}\left(\frac{|\alpha_k|}{\epsilon_k}\right)^{-1/24} \\
&= \mathcal{D}|\alpha_N|^{1/24}R^{-1/48}\prod_{k\in\mathrm{NN}}\epsilon_k^{-1/24}\prod_{k\in\mathrm{DN}}\epsilon_k^{1/48}\prod_{k\in\mathrm{NN}}|\alpha_k|^{1/12}\prod_{k\in\mathrm{DN}}|\alpha_k|^{-1/24}\prod_r \epsilon_r^{1/24} \\
&= \mathcal{D}|\alpha_N|^{1/24}R^{-1/48}\prod_{k\in\mathrm{NN}}\epsilon_k^{-1/24}\prod_{k\in\mathrm{DN}}\epsilon_k^{1/48}\prod_{k\in\mathrm{NN}}|\alpha_k|^{1/12}\prod_{k\in\mathrm{DN}}|\alpha_k|^{-1/24} \\
&\quad \left[\frac{\prod_r(2\delta_r)\prod_k|\alpha_k|}{|\alpha_N|^{2N-3}}\frac{\prod_{k\neq l}|Z_l-Z_k|}{\prod_{r\neq s}|x_s-x_r|}\right]^{1/48} \\
&= \mathcal{D}R^{-1/48}\prod_{k\in\mathrm{NN}}\epsilon_k^{-1/24}\prod_{k\in\mathrm{DN}}\epsilon_k^{1/48}\prod_{k\in\mathrm{NN}}|\alpha_k|^{1/8}\prod_{k=1}^{N}|\alpha_k|^{-1/48} \\
&\quad \left[\frac{\prod_r\sqrt{2\delta_r}}{|\alpha_N|^{N-3}}\frac{\prod_{k<l}|Z_l-Z_k|}{\prod_{r<s}|x_s-x_r|}\right]^{1/24}
\end{aligned} \tag{158}$$

Remembering (138) we see that the different powers of ϵ_k for the NN and DN cases simply reflect the different ground state masses for the open string in those cases

$$\alpha' M_G^2 = -\frac{d_{\mathrm{NN}}}{24}+\frac{d_{\mathrm{DN}}}{48} \tag{159}$$

where $d_{\mathrm{NN(DN)}}$ is the dimension of NN(DN) string coordinates. Each NN external string can carry a momentum, so we we collect them as the components of a d_{NN} dimensional vector $\boldsymbol{p}$. Then the p^- of the kth string is $p_k^- = (\boldsymbol{p}^2+M_G^2)/2p_k^+ = (\boldsymbol{p}^2+M_G^2)/\alpha_k$. Then

$$\begin{aligned}
\epsilon_k^{\boldsymbol{p}_k^2-d_{\mathrm{NN}}/24+d_{\mathrm{DN}}/48} &= e^{T_k p_k^-}\prod_{l\neq k}|Z_k-Z_l|^{-2\alpha' p_l^+ p_k^-} \\
&= e^{T_k p_k^-}\prod_{l<k}|Z_k-Z_l|^{-2\alpha' p_l^+ p_k^- - 2\alpha' p_k^+ p_l^-}
\end{aligned} \tag{160}$$

The extra factor of $|\alpha_k|^{d_{\mathrm{NN}}/8}$ for $k\in\mathrm{NN}$ simply reflects the normalization of the NN ground state compared to the DN ground state.

B.3 Determining $\mathcal{D}$

We now consider the dependence of $\mathcal{D}$ on the DN transition points. We will content ourselves with working out that dependence for no more than 2 Dirichlet boundaries (i.e. no more than 4 DN transition points. For the case of only one Dirichlet boundary, the two transition points can be taken to be two of the fixed Koba-Nielsen variables, and $\mathcal{D}$ will therefore not depend on any of the integration variables. For the case of two Dirichlet boundaries there are four transition points, three of which can be taken fixed, but $\mathcal{D}$ can depend on the fourth, which will be an integration variable.

To calculate $\mathcal{D}$ for the case of two Dirichlet boundaries we consider the conformal map of a DNDN rectangle to the upper half plane (see Fig. (5). If the lightcone worldsheet is mapped to that figure, the asymptotic strings would be mapped to the centers of the circular arcs on the vertical boundaries. For the purpose of calculating $\mathcal{D}$ we only need to

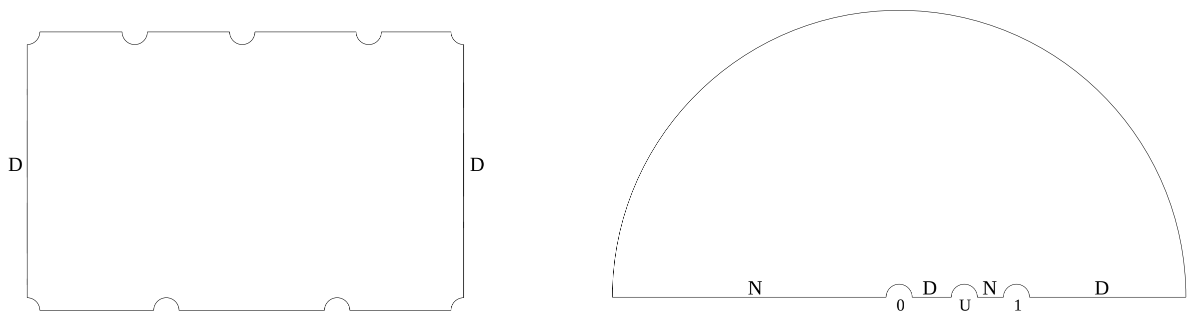

Figure 5: Rectangle mapped to the upper half plane by an elliptic function. The horizontal boundaries can be taken Neumann, and the vertical ones can be taken Dirichlet. The small circular arcs in both figures are all meant to be infinitesimal and are centered on potentially singular points of the mapping. The large semicircle on the right is meant to be infinite.

keep the quarter circles at the corners. We map their centers to the points $0, U, 1, \infty$, labelled counterclockwise starting at the upper left corner. Situate the rectangle in the upper half z-plane with the bottom side on the real axis, $-K < x < +K$, with the upper boundary on the line $z = x + iK'$. Let u be the complex variable of the target upper half plane. Then

$$u = \frac{(k\ \mathrm{sn}(z,q)+1)(k-1)}{(k\ \mathrm{sn}(z,q)-1)(k+1)}, \qquad \frac{du}{dz} = \frac{-2k(k-1)\mathrm{sn}'(z,q)}{(k+1)(k\ \mathrm{sn}(z,q)-1))^2} \tag{161}$$

where sn is one of the Jacobian elliptic functions of modulus k and $q = e^{-\pi K'/K}$. With this notation, $U = (k-1)^2/(k+1)^2$.

The determinant for the u-plane figure is in the limit $\bar{\epsilon}_{1,2,3} \to 0$ and $R \to \infty$

$$-\frac{1}{2}\ln\det{}_u \sim \frac{1}{48}\ln R - \frac{1}{48}\ln\bar{\epsilon}_1\bar{\epsilon}_2\bar{\epsilon}_3 + \ln\mathcal{D} \tag{162}$$

where R is the radius of the large semicircle and $\bar{\epsilon}_{1,2,3}$ are the radii of the small semicircles. This is related by a conformal transformation to the determinant for the z-plane figure given by

$$-\frac{1}{2}\ln\det{}_z \sim \frac{1}{48}\ln\epsilon_1\epsilon_2\epsilon_3\epsilon_4 - \frac{1}{2}\ln\det{}_{\mathrm{DNDN}} \tag{163}$$

Since $\partial_n \Sigma = 0$ on all of the straight line segments of the boundary of the rectangle, we only get a contribution from the change formula near each of the corners. So we approximate Σ for each corner in turn. Starting with the upper left corner, put $z = -K + iK' + re^{i\varphi}$ with r small. then

$$\begin{aligned}
\mathrm{sn}(-K+iK') &= -\frac{1}{k}, \quad \mathrm{sn}'(-K+iK') = 0, \quad \mathrm{sn}''(-K+iK') = -\frac{1-k^2}{k} && (164)\\
\mathrm{sn}(z) &\approx -\frac{1}{k} - \frac{1-k^2}{2k} r^2 e^{2i\varphi}, \quad u \approx -\frac{(k-1)^2}{4} r^2 e^{2i\varphi}, \quad \bar{\epsilon}_1 = \frac{(k-1)^2\epsilon_1^2}{4} && (165)\\
\Sigma &\approx \ln \frac{(k-1)^2 r}{2}, \qquad \partial_n \Sigma = -\frac{1}{r} && (166)\\
\Delta_1 &= \left(-\frac{1}{48} - \frac{1}{24}\right)\Sigma = -\frac{1}{16} \ln \frac{(k-1)^2\epsilon_1}{2} && (167)
\end{aligned}$$

For the lower left corner $z = -K + re^{i\varphi}$

$$\begin{aligned}
\mathrm{sn}(-K) &= -1, \qquad \mathrm{sn}'(-K) = 0, \qquad \mathrm{sn}''(-K) = 1 - k^2 && (168)\\
\mathrm{sn}(z) &\approx -1 + \frac{1-k^2}{2} r^2 e^{2i\varphi}, \quad u \approx \frac{(k-1)^2}{(k+1)^2}[1 + kr^2 e^{2i\varphi}], \quad \bar{\epsilon}_2 = \frac{k(k-1)^2\epsilon_2^2}{(k+1)^2} && (169)\\
\Sigma &\approx \ln \frac{2k(k-1)^2 r}{(k+1)^2}, \qquad \partial_n \Sigma = -\frac{1}{r} && (170)\\
\Delta_2 &= \left(-\frac{1}{48} - \frac{1}{24}\right)\Sigma = -\frac{1}{16} \ln \frac{2k(k-1)^2\epsilon_2}{(k+1)^2} && (171)
\end{aligned}$$

For the lower right corner $z = K + re^{i\varphi}$,

$$\begin{aligned}
\mathrm{sn}(K) &= 1, \qquad \mathrm{sn}'(K) = 0, \qquad \mathrm{sn}''(K) = -(1-k^2) && (172)\\
\mathrm{sn}(z) &\approx 1 - \frac{1-k^2}{2} r^2 e^{2i\varphi}, \quad u \approx 1 - kr^2 e^{2i\varphi}, \quad \bar{\epsilon}_3 = k\epsilon_3^2 && (173)\\
\Sigma &\approx \ln 2kr, \qquad \partial_n \Sigma = -\frac{1}{r} && (174)\\
\Delta_3 &= \left(-\frac{1}{48} - \frac{1}{24}\right)\Sigma = -\frac{1}{16} \ln 2k\epsilon_3 && (175)
\end{aligned}$$

For the final (upper right) corner, $z = K + iK' + re^{i\varphi}$,

$$\begin{aligned}
\mathrm{sn}(K+iK') &= \frac{1}{k}, \quad \mathrm{sn}'(-K+iK') = 0, \quad \mathrm{sn}''(K+iK') = \frac{1-k^2}{k} && (176)\\
\mathrm{sn}(z) &\approx \frac{1}{k} + \frac{1-k^2}{2k} r^2 e^{2i\varphi}, \quad u \approx -\frac{4}{(k+1)^2 r^2 e^{2i\varphi}}, \quad R = \frac{4}{(k+1)^2\epsilon_4^2} && (177)\\
\Sigma &\approx \ln \frac{8}{(1+k)^2 r^3}, \qquad \partial_n \Sigma = \frac{3}{r} && (178)\\
\Delta_4 &= \left(\frac{3}{48} - \frac{1}{24}\right)\Sigma = \frac{1}{48} \ln \frac{8}{(1+k)^2\epsilon_4^3} && (179)
\end{aligned}$$

Then we have

$$
\begin{aligned}
-\frac{1}{2}\ln\det{}_u &= -\frac{1}{2}\ln\det{}_z + \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 && (180)\\
&\sim -\frac{1}{24}\ln\epsilon_1\epsilon_2\epsilon_3\epsilon_4 - \frac{1}{2}\ln\det{}_{\mathrm{DNDN}} - \frac{1}{16}\ln\frac{2k^2(k-1)^4}{(k+1)^2} + \frac{1}{48}\ln\frac{8}{(1+k)^2}\\
&\sim -\frac{1}{24}\ln\epsilon_1\epsilon_2\epsilon_3\epsilon_4 - \frac{1}{2}\ln\det{}_{\mathrm{DNDN}} - \frac{1}{16}\ln k^2(k-1)^4 + \frac{1}{24}\ln(1+k)^2 && (181)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\frac{1}{48}\ln R - \frac{1}{48}\ln\bar\epsilon_1\bar\epsilon_2\bar\epsilon_3 &\sim \frac{1}{48}\ln\frac{4}{(k+1)^2\epsilon_4^2} - \frac{1}{48}\ln\frac{k^2(k-1)^4\epsilon_1^2\epsilon_2^2\epsilon_3^2}{4(k+1)^2}\\
&\sim -\frac{1}{24}\ln\frac{k(k-1)^2\epsilon_1\epsilon_2\epsilon_3\epsilon_4}{4} && (182)
\end{aligned}
$$

So comparing we deduce

$$
\begin{aligned}
\ln\mathcal{D} &= \frac{1}{24}\ln\frac{k(k-1)^2}{4} - \frac{1}{2}\ln\det{}_{\mathrm{DNDN}} - \frac{1}{8}\ln k(k-1)^2 + \frac{1}{24}\ln(1+k)^2\\
&= -\frac{1}{2}\ln\det{}_{\mathrm{DNDN}} - \frac{1}{12}\ln 2k(k-1)^2 + \frac{1}{24}\ln(1+k)^2\\
&= -\frac{1}{2}\ln\det{}_{\mathrm{DNDN}} - \frac{1}{24}\ln\frac{4k^2(k-1)^4}{(1+k)^2} && (183)
\end{aligned}
$$

It is important to bear in mind that this formula applies only when the corners of the rectangle are mapped to $0, U = (1-k)^2/(1+k)^2, 1, \infty$, which mark the DN transition points. The formula to use when the transition points are at general locations, can be obtained by executing a projective conformal transformation

$$
w = \frac{au+b}{cu+d}, \qquad ad - bc = 1. \tag{184}
$$

Carefully transforming the corresponding determinants, regulated by suitable circular arcs to avoid singular points, leads to the result

$$
\ln\mathcal{D}_w = \ln\mathcal{D} + \frac{1}{8}\ln cd(cU+d)(c+d) \tag{185}
$$

For example, a symmetrical and canonical choice is to map the corners of the rectangle to $-1/k, -1, +1, +1/k$ respectively. For this case, $ad(c+d)(cU+d) = k^2/(1+k)^2$, and The corresponding determinant $\mathcal{D}_0$ is given by

$$
\begin{aligned}
\ln\mathcal{D}_0 &= -\frac{1}{2}\ln\det{}_{\mathrm{DNDN}} - \frac{1}{24}\ln\frac{4k^2(k-1)^4}{(1+k)^2} + \frac{1}{8}\ln\frac{k^2}{(1+k)^2}\\
&= -\frac{1}{2}\ln\det{}_{\mathrm{DNDN}} + \frac{1}{12}\ln\frac{k^2}{2(1-k^2)^2} && (186)
\end{aligned}
$$

B.3.1 D for Unmixed Boundary conditions

As we have noted, for unmixed boundary conditions the analog of $\mathcal{D}$ had nothing to depend on, and so had to be a constant. It is instructive to see this using the methods of the present subsection. The determinant for the u-plane figure changes, in the unmixed case, to

$$\begin{aligned} -\frac{1}{2}\ln\det{}_u &\sim \frac{5}{24}\ln R + \frac{1}{24}\ln\bar{\epsilon}_1\bar{\epsilon}_2\bar{\epsilon}_3 + \ln\mathcal{D}^{\mathrm{N}} \\ &\sim \frac{1}{12}\ln\frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_4^5} + \frac{1}{12}\ln\frac{k(k-1)^2}{(k+1)^6} + \ln\mathcal{D}^{\mathrm{N}}. \end{aligned} \tag{187}$$

And the determinant for the z-plane figure becomes

$$-\frac{1}{2}\ln\det{}_z \sim \frac{1}{48}\ln\epsilon_1\epsilon_2\epsilon_3\epsilon_4 - \frac{1}{2}\ln\det{}_{\mathrm{DDDD}} \tag{188}$$

To relate these we need to adapt the Δ_i to the unmixed case. The only difference is that the corner contributions for each quarter circle add instead of cancel:

$$\begin{aligned} \Delta_1 &= \left(-\frac{1}{48} - \frac{1}{24} + \frac{1}{8}\right)\Sigma = \frac{1}{16}\ln\frac{(k-1)^2\epsilon_1}{2} \\ \Delta_2 &= \frac{1}{16}\ln\frac{2k(k-1)^2\epsilon_2}{(k+1)^2}, \qquad \Delta_3 = \frac{1}{16}\ln 2k\epsilon_3 \\ \Delta_4 &= \left(\frac{3}{48} - \frac{1}{24} + \frac{1}{8}\right)\Sigma = \frac{7}{48}\ln\frac{8}{(1+k)^2\epsilon_4^3} \\ \Delta &= \sum_i \Delta_i = \frac{1}{16}\ln\frac{2k^2(k-1)^4\epsilon_1\epsilon_2\epsilon_3}{(k+1)^2} + \frac{7}{48}\ln\frac{8}{(1+k)^2\epsilon_4^3} \end{aligned} \tag{189}$$

Then

$$\Delta - \frac{1}{2}\ln\det{}_z \sim \frac{1}{12}\ln\frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_4^5} - \frac{1}{2}\ln\det{}_{\mathrm{DDDD}} + \frac{1}{16}\ln\frac{2k^2(k-1)^4}{(k+1)^2} + \frac{7}{48}\ln\frac{8}{(1+k)^2} \tag{190}$$

Since this quantity should be the determinant in the u-plane, we must have

$$\begin{aligned} \ln\mathcal{D}^{\mathrm{N}} &= -\frac{1}{2}\ln\det{}_{\mathrm{DDDD}} + \frac{1}{16}\ln\frac{2k^2(k-1)^4}{(k+1)^2} + \frac{7}{48}\ln\frac{8}{(1+k)^2} - \frac{1}{12}\ln\frac{k(k-1)^2}{(k+1)^6} \\ &= -\frac{1}{2}\ln\det{}_{\mathrm{DDDD}} + \frac{1}{48}\ln k^2(k^2-1)^4 + \frac{1}{2}\ln 2 \end{aligned} \tag{191}$$

To see that the right side is a constant we use

$$\begin{aligned} k^{1/24}(1-k^2)^{1/12} &= \frac{\theta_2(0)^{1/12}\theta_4(0)^{1/3}}{\theta_3(0)^{5/12}} = \frac{(\theta_2(0)\theta_3(0)\theta_4(0))^{1/12}\theta_4(0)^{1/4}}{\theta_3(0)^{1/2}} \\ &= 2^{1/12}q^{1/48}\frac{\prod(1-q^{2n-1})^{1/2}}{\prod(1+q^{2n-1})} = 2^{1/12}q^{1/48}\frac{\prod(1-q^n)^{1/2}}{\sqrt{\theta_3(0)}} \\ &= 2^{1/12}q^{1/48}\frac{\prod(1-q^n)^{1/2}}{(2K/\pi)^{1/4}} = \pi^{1/4}2^{1/12}\det{}_{\mathrm{DDDD}}^{+1/2} \\ \ln\mathcal{D}^{\mathrm{N}} &= \frac{1}{12}\ln(2\pi^3) + \frac{1}{2}\ln 2 = \frac{1}{12}\ln 2^7\pi^3 \end{aligned} \tag{192}$$

In a similar vein, executing a projective conformal transformation shows that $\mathcal{D}^N$ is a projective invariant.

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