On the semiclassical three-point function in AdS_{3}

P. Bozhilov, P. Furlan, V. B. Petkova, and M. Stanishkov

Phys. Rev. D **86**, 066005 — Published 12 September 2012

DOI: 10.1103/PhysRevD.86.066005

On the semiclassical 3-point function in AdS_3

P. Bozhilov[a], P. Furlan[b,c], V.B. Petkova[a] and M. Stanishkov[a]

[a)] *Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Sofia, Bulgaria*

[b)] *Dipartimento di Fisica dell'Università di Trieste, Italy,*
[c)] *Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Trieste, Italy,*

We reconsider the problem of determining the semiclassical 3-point function in the Euclidean $\mathrm{Ad}S_3$ model. Exploiting the affine symmetry of the model we use solutions of the classical Knizhnik-Zamolodchikov (KZ) equation to compute the saddle point of the action in the presence of three vertex operators. This alternative derivation reproduces the "heavy charge" classical limit of the quantum 3-point correlator. It is different from the recently proposed expression obtained by generalised Pohlmeyer reduction in AdS_2.

---

plbozhilov@inrne.bas.bg, furlan@ts.infn.it, petkova@inrne.bas.bg, marian@inrne.bas.bg

1. Introduction

The AdS/CFT conjecture [1],[2],[3] implies that the correlation functions in the dual (boundary) quantum field theory can be computed alternatively in string theory, i.e., essentially by the methods of a two - dimensional theory. The first computations however were mostly performed in the supergravity approximation, representing the correlators in terms of integrals over the target (bulk) coordinates [4], [5].

On the string side one may start with a semiclassical approach, when the string path integral for the correlation functions is evaluated in the saddle-point approximation with large 't Hooft coupling $\lambda >> 1$. In this calculation one has to identify the correct vertex operators [6], [7] and to find the corresponding classical solutions, which provide the appropriate saddle-point approximation. Some preliminary results for the three point function of three heavy operators are already available [8], see also [9], [10], [11]. Here we propose to use our knowledge of the quantum Liouville and WZW theories to such semiclassical computations.

Semiclassical considerations of the (euclidean) AdS_3 string theory have been initiated e.g., in [12], where classical solutions of the equations of motion in the absence of sources, or with one vertex insertion have been constructed. On the other hand one can compute straightforwardly the classical limit of the known quantum 3-point OPE coefficients, i.e., for $b^2 \sim \frac{1}{\sqrt{\lambda}} \to 0$, and heavy charges $\triangle_i = -2j_i = \frac{2\eta_i}{b^2}$, s.t., η_i are finite; here $\triangle$ stands for the scaling dimension in the dual CFT. Such configurations dominate the saddle point of the action in the presence of sources, i.e., classical vertex operators. The resulting expression in this limit is similar to the semiclassical Liouville result [13], due to the fact, that the quantum (euclidean) AdS_3 3-point function is expressed by a formula [14], [15] closely related to the Liouville one.

This semiclassical limit of the quantum AdS_3 3-point function yields an expression which differs from the recently proposed one [8]. The latter is interpreted as the AdS_2 part of the correlator of three heavy strings propagating in $\mathrm{AdS}_2 \times S^5$ model, and is assumed to be universal for heavy (scalar) AdS_{d+1} operators. This motivated us to compute directly the semiclassical constant, generalising the method in [13], which was proposed originally as a semiclassical check of the quantum Liouville 3-point constant. While the quantum AdS_3 theory and its applications to the superstring on $AdS_3 \times S^3 \times M^4$ in the NS-NS background ([16], [17] and references therein) is well studied, and, thus, this is no more than a toy model in the semiclassical context under consideration, the elaboration of 2d

CFT techniques is important for the analogous unsolved problems in more realistic and less known cases.

Our derivation is based on the affine algebra symmetry of the model generated by a current and (in the euclidean version) its complex conjugate. This leads to a chiral equation, a classical version of the KZ equation [18]. The equation determines the classical fundamental vertex V of isospin $j = 1/2$ as a function of the coordinates of the three vertex sources. The solution is then used, analogously to [13], to evaluate the contribution of the sources to the saddle point of the action and thus to compute the semiclassical 3-point function, confirming the direct classical limit of the quantum correlator.

2. Summary of the AdS$_3$ data

In this mostly preliminary section we summarise some basic data on the non-compact $\hat{sl}(2)$ WZW model [19], [14].

- The Euclidean AdS$_3$ is the coset $\simeq SL(2,\mathbb{C})/SU(2)$

$$-X_{-1}^2 + \sum_{i=1}^{3} X_i^2 = -1 \tag{2.1}$$

parametrised in $SL(2,\mathbb{C})$ as

$$g(X) = X_{-1}\mathbf{1}_2 - X_i\sigma_i = \begin{pmatrix} e^{-\phi} + |\gamma|^2 e^{\phi} & e^{\phi}\gamma \\ e^{\phi}\bar{\gamma} & e^{\phi} \end{pmatrix} = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\phi} & 0 \\ 0 & e^{\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{\gamma} & 1 \end{pmatrix}, \tag{2.2}$$

$$ds^2 = \frac{1}{2}\mathrm{tr}(g^{-1}dg)^2 = d\phi^2 + d\gamma\, d\bar{\gamma} e^{2\phi}\,.$$

In the WZW classical action the coordinates $\phi(z,\bar{z})\,, \gamma(z,\bar{z})$ are 2d fields

$$S_{AdS} = \frac{k}{\pi}\int d^2z(\partial_z\phi\,\partial_{\bar{z}}\phi + \partial_z\bar{\gamma}\,\partial_{\bar{z}}\gamma e^{2\phi})\,. \tag{2.3}$$

The classical equations of motion

$$\begin{aligned} &\partial_{\bar{z}}\partial_z\phi = e^{2\phi}\partial_z\bar{\gamma}\,\partial_{\bar{z}}\gamma\,, \\ &\partial_z\big((\partial_{\bar{z}}\gamma)e^{2\phi}\big) = 0\,, \quad \partial_{\bar{z}}\big((\partial_z\bar{\gamma})e^{2\phi}\big) = 0 \end{aligned} \tag{2.4}$$

imply that the currents

$$J(z) = k\,\partial_z g g^{-1} = k \begin{pmatrix} -\partial_z\phi + \gamma\partial_z\bar{\gamma}\, e^{2\phi} & 2\gamma\partial_z\phi - \gamma^2\partial_z\bar{\gamma}e^{2\phi} + \partial_z\gamma \\ \partial_z\bar{\gamma}e^{2\phi} & \partial_z\phi - \gamma\partial_z\bar{\gamma}e^{2\phi} \end{pmatrix} = J^a t^a \tag{2.5}$$

and $\bar{J}(\bar{z}) = k\, g^{-1}\bar{\partial}_z g$ are conserved (chiral), $\partial_{\bar{z}} J(z) = 0\,,\ \partial_z \bar{J}(\bar{z}) = 0$ and vice versa. In other words $g(z,\bar{z})$ satisfies two chiral first order equations

$$k\,\partial_z g = J(z) g = J^a(z) t^a g\,,\quad k\,\partial_{\bar{z}} g = g \bar{J}^a(\bar{z}) t^a\,. \tag{2.6}$$

The translations in the diagonal action of $SL(2,\mathbb{C})$ on the coset shift $\gamma \to \gamma - x$. On the projected group element

$$V = V_{j=\frac{1}{2}}(z,\bar{z};x,\bar{x}) = e^{-\phi} + |\gamma - x|^2 e^{\phi} = (1,-x) g(z,\bar{z}) \begin{pmatrix} 1 \\ -\bar{x} \end{pmatrix} \tag{2.7}$$

the generators t^a of $sl(2)$ are realised by standard differential operators with respect to the isospin variable x. They are determined from

$$k\,\partial_z V = (1,-x) t^a g(z) \begin{pmatrix} 1 \\ -\bar{x} \end{pmatrix} J^a(z) =: J^a(z) D^a(x) V\,,$$

and analogously $\bar{D}^a(\bar{x})$ are determined from the right action of t^a. General vertex operators are given by

$$V_j(z,\bar{z},x,\bar{x}) = (e^{-\phi(z,\bar{z})} + |\gamma(z,\bar{z}) - x|^2 e^{\phi(z,\bar{z})})^{2j}\,. \tag{2.8}$$

• In the quantum theory $k \to k-2 = 1/b^2$ and furthermore a curvature term is added. In the AdS_3/CFT_2 correspondence (2.8) is the kernel of the integral boundary-bulk operator with boundary conformal dimension $\triangle = -2j$. Its "world sheet" (Sugawara) scaling dimension is

$$\delta^{Su}(j) = -b^2\, j(j+1) = \alpha(b-\alpha) = \delta^L(\alpha) - \alpha/b\,. \tag{2.9}$$

We have used the notation $\alpha_i = -j_i b$ for the vertex charges to compare with the Virasoro theory of central charge $c > 25$ (Liouville theory). The two Virasoro theories with generic $c = 13 - 6(b^2 + \frac{1}{b^2}) < 1$ and $c = 13 + 6(b^2 + \frac{1}{b^2}) > 25$ can be realised via quantum Hamiltonian reduction of the $\hat{sl}(2)$ (respectively, compact and non-compact) WZW models. Accordingly the 3-point WZW OPE constants are closely related to the Virasoro ones. In particular, in the non-compact case the 3-point constant is given [14] by the DOZZ Liouville expression up to a simple $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ factor

$$\begin{aligned} C(j_1,j_2,j_3) &= (\nu(b))^{1+j_{123}}\, \frac{\Upsilon_b(b)}{\Upsilon_b(\alpha_{123}-b)} \prod_{i=1}^{3} \frac{\Upsilon_b(2\alpha_i)}{\Upsilon_b(\alpha_{123}-2\alpha_i)} \\ &\sim \gamma((\alpha_{123}-Q)\frac{1}{b}) C_L(\alpha_1,\alpha_2,\alpha_3) \end{aligned} \tag{2.10}$$

where $\Upsilon_b(x) = \Upsilon_b(Q-x)$ is expressed by Barnes double Gamma functions, $\nu(b)$ is an arbitrary constant and $\alpha_{123} = \alpha_1+\alpha_2+\alpha_3$.[1]

• With k replaced by the shifted $k-2 = 1/b^2$ in the action (2.3) the semiclassical limit corresponds to $b^2 \to 0$ and "heavy" charges j

$$j = -\frac{\alpha}{b} = -\frac{\eta}{b^2}\,, \eta - \text{finite}\,, \quad b^2 \delta^{Su}(j) \to -\eta^2\,. \tag{2.11}$$

In this limit, described for the Liouville theory in [13], the function Υ_b goes to

$$\log \Upsilon_b(\frac{\eta}{b}) \to \frac{1}{b^2} F(\eta) := \frac{1}{b^2} \int_{1/2}^{\eta} dx \log \gamma(x) \tag{2.12}$$

so that for the 3-point constant (2.10) one obtains

$$\begin{aligned} -b^2 \log C(-\frac{\eta_1}{b^2}, -\frac{\eta_2}{b^2}, -\frac{\eta_3}{b^2}) &\to - b^2 \log C^{(cl)}(\eta_1, \eta_2, \eta_3) \\ &= (\eta_{123}) \log \nu(b) + F(\eta_{123}) - F(0) + \sum_i (F(\eta_{123} - 2\eta_i) - F(2\eta_i))\,. \end{aligned} \tag{2.13}$$

The heavy charge classical limit describes the semiclassical 3-point function of vertex operators which is dominated by the saddle point of the classical action, i.e., on some solution of the classical equations of motion (2.4) which depend on the charges and coordinates of the sources - the vertex operators. Following and extending the approach in [13] in the Liouville theory we shall reproduce in the next section formula (2.13) by first describing explicitly these classical solutions and then directly computing the semiclassical 3-point function.

• Just for comparison recall the "light charge" classical limit of (2.10): for $b \to 0$ consider $2\alpha_i = \triangle_i b$ with fixed $\triangle_i = -2j_i$. In this limit [22]

$$\frac{\Upsilon_b(b\,\sigma)}{\Upsilon_b(b)} \to \frac{b^{1-\sigma}}{\Gamma(\sigma)} \tag{2.14}$$

and thus from (2.10) one reproduces, up to trivial field renormalisation, the expression for the AdS_3 3-point constant computed in the supergravity approximation [4].[2]

[1] We restrict here to three spectrally unflowed representations, cf. [20], [21] for the full spectrum of the model.

[2] The formulae in the semiclassical considerations in [9],[11] correspond to "light charge" classical limit, in which all $\triangle_i$ are furthermore taken big, exploiting the Stirling formula for the asymptotics of the Gamma functions. In this limit of the supergravity AdS_{2d+1} constants, as well as of their S^{2d+1} analogs, the dependence on d is erased, which in particular trivialises the cancellations for BPS type operators for $d=2$, in contrast with the full consideration in [5].

3. Alternative derivation of the quasiclassical OPE constant

The derivation follows and generalises the approach of [13] in the Liouville theory, so let us sketch the main steps. The solutions of the classical equations of motion for the Liouville field

$$\partial\bar{\partial}\varphi = \pi\mu b^2 e^{2\varphi}$$

can be recovered from the solutions of the second order chiral equation (see, e.g. [23] and earlier references therein)

$$(\partial_z^2 + b^2 T_L(z))e^{-\varphi(z,\bar{z})} = 0\,, \quad \partial_{\bar{z}} T_L = 0 \tag{3.1}$$

and its $\bar{T}_L(\bar{z})$ counterpart. The Liouville equation of motion ensures the conservation of the energy momentum tensor and vice versa. In the presence of (three) sources the classical tensor is determined through the limit $b^2 \to 0$ of its normalised 4-point correlator with the three vertex operators $e^{\frac{2\alpha_i\varphi}{b}}$ of "heavy" charges $\alpha_i = \eta_i/b$, or

$$\begin{aligned}\hat{T}_L(z; z_a) &= \lim_{b\to 0} b^2 T(z; z_a) = \sum_i \Big(\frac{h_i}{(z-z_i)^2} + \frac{\partial_i}{z-z_i}\log\frac{1}{(z_{12})^{h_{12}^3}(z_{23})^{h_{23}^1}(z_{13})^{h_{13}^2}}\Big)\\ &= \Big(\frac{z_{12}z_{23}}{(z-z_2)^2 z_{13}}\Big)^2\Big(\frac{h_1}{w^2} + \frac{h_3}{(1-w)^2} + \frac{h_{13}^2}{w(1-w)}\Big) =: (\partial_z w)^2\,\hat{T}_L(w)\,,\end{aligned} \tag{3.2}$$

where $h_{ij}^k = h_i + h_j - h_k$ and

$$h_i = \eta_i(1-\eta_i) = \lim_{b\to 0} b^2\alpha_i(Q-\alpha_i)\,, \quad w = \frac{(z-z_1)z_{23}}{(z-z_2)z_{13}}\,. \tag{3.3}$$

Then the solution for $e^{-\varphi(z,\bar{z})}$ as a function of the coordinates $z_i, \bar{z}_i$ of the sources is given, up to a prefactor (determined by its classical dimension $-1/2 = \lim_{b\to 0}\delta^L(-b/2)$), by a monodromy invariant diagonal combination of two solutions of

$$(\partial_w^2 + \hat{T}_L(w))G^{\pm}(w) = 0\,. \tag{3.4}$$

Equation (3.1) is the classical version of the BPZ equation, resulting from the decoupling of a level 2 singular vector. The solution of (3.1) is identified up to an overall constant with the classical limit of the 4-point function of the fundamental quantum vertex operator $e^{-b\phi}$ ($b\phi = \varphi$) and the three arbitrary vertex operators, normalised by the 3-point function of these operators. Finally, the solution for $\varphi(z; z_i)$ is used to compute the saddle point action with sources which determines the semiclassical 3-point function [13].

• In the related to a WZW model AdS_3 case, the BPZ equation is replaced by the KZ equation [18]. The solution for the field $V_{j=1/2}(z,\bar{z};x,\bar{x})$ in (2.7) as a function of the coordinates $\{z_i,\bar{z}_i,x_i,\bar{x}_i\}$ of the three sources is the classical limit of the corresponding 4-point function with one such vertex operator [24],[14] and three vertex operators (2.8) of isospins $j_i=-\eta_i/b^2$, normalised by the corresponding 3-point function. For completeness let us sketch the derivation of the KZ equation directly in the classical limit. In the presence of three sources the chiral equation (2.6) for (2.7) becomes

$$(\partial_z - \hat{J}^a(z;z_i;x_i)t^a)V(z,x)=0\,, \tag{3.5}$$

with the current defined through the classical limit of its 4-point function normalised with the 3-point function $< V_{j_1}V_{j_2}V_{j_3} >$

$$\begin{aligned}\hat{J}^a(z;z_i,x_i) :&= \lim_{b\to 0}\frac{b^2}{< V_{j_1}V_{j_2}V_{j_3} >}\sum_i \frac{t_i^a}{z-z_i} < V_{j_1}V_{j_2}V_{j_3} > \\ &= x_{12}^{\eta_{12}^3}x_{23}^{\eta_{23}^1}x_{13}^{\eta_{13}^2}\sum_i\Big(\frac{D_i^a(-\eta_i)}{z-z_i}-\frac{D_i^a(-\eta_i)}{z-z_2}\Big)\,x_{12}^{-\eta_{12}^3}x_{23}^{-\eta_{23}^1}x_{13}^{-\eta_{13}^2} \\ &=: (\partial_z w)\hat{J}^a(w;x_i)\end{aligned} \tag{3.6}$$

The $sl(2)$ generators are represented by the standard differential operators $D_i^a(j_i)$ in x_i. In the last two lines we have used the Ward identities (projective invariance) with w defined in (3.3). The current $\hat{J}(w,x;x_i)=\hat{J}^a(w;x_i)t^a$ in the equation (3.5) becomes a differential operator with respect to the isospin projective invariant y when acting on the function $\hat{V}(w,y)$ in

$$V(z,x;z_i,x_i)=\frac{|(x-x_2)^2x_{13}|}{|x_{12}x_{23}|}\hat{V}(w,y)\,,\qquad y=\frac{(x-x_1)x_{23}}{(x-x_2)x_{13}}\,. \tag{3.7}$$

Here the dependence of V and $\hat{V}$ on the complex conjugated variables is suppressed . Since the Sugawara dimension of $V=V_{j=1/2}$ vanishes in the limit $b\to 0$, there is no z-dependent prefactor in (3.7).

One obtains the equation (written for the chiral constituents of V)

$$\begin{aligned}\partial_w G(w,y) = &-\Big(\eta_{13}^2\big(\frac{(y-w)^2}{w(w-1)}-1\big)+2\eta_1\big(\frac{y-w}{w}+1\big)+2\eta_3\big(\frac{y-w}{w-1}+1\big)\Big)\partial_y G(w,y) \\ &+\Big(\frac{\eta_{13}^2(y-w)}{w(w-1)}+\frac{\eta_1}{w}+\frac{\eta_3}{w-1}\Big)G(w,y)\,.\end{aligned} \tag{3.8}$$

• The equation (3.8) is equivalent to a pair of differential equations for the components of

$$G(w,y) = G_0(w) + (y-w)G_2(w)\,. \tag{3.9}$$

In matrix form (3.8) reads for the vector $G = (G_0, G_2)^t$

$$\frac{d}{dw}\,G(w) - \begin{pmatrix} \frac{\eta_1}{w} + \frac{\eta_3}{w-1} & 1-\eta_{123} \\ \frac{\eta_{13}^2}{w(w-1)} & -(\frac{\eta_1}{w} + \frac{\eta_3}{w-1}) \end{pmatrix} G(w) = 0\,. \tag{3.10}$$

The component $G_0(w) = G(w, y=w)$ satisfies the second order equation (3.4), with the Liouville classical energy-momentum tensor $\hat{T}_L(w)$.[3] Thus the problem is reduced to that in the Liouville case. This is an illustration of the Drinfeld-Sokolov reduction: gauge transformation by a lower triangular group element which preserves $G_0 = G_0^{\rm gau}$ and brings the matrix equation (3.10) to the form

$$\frac{d}{dw}G^{\rm gau}(w) - \begin{pmatrix} 0 & 1-\eta_{123} \\ -\frac{\hat{T}_L(w)}{1-\eta_{123}} & 0 \end{pmatrix} G^{\rm gau}(w) = 0\,. \tag{3.11}$$

Combining the left and right solutions one obtains for V

$$V(z,x;z_i,x_i) = \frac{|x-x_2|^2|x_{13}|}{|x_{23}||x_{12}|N_{WZW}}(|G^{(+)}(w,y)|^2 + N_{WZW}^2|G^{(-)}(w,y)|^2)\,, \tag{3.12}$$

where

$$G^{(\pm)}(w,y) = G_0^{(\pm)}(w) + (y-w)G_2^{(\pm)}(w) = G_1^{(\pm)}(w) + y\,G_2^{(\pm)}(w) \tag{3.13}$$

and the two solutions are given explicitly as

$$\begin{aligned} &G^{(+)}(w,y) = w^{\eta_1}(1-w)^{\eta_3}\times \\ &\big({}_2F_1(\eta_{13}^2, -1+\eta_{123}, 2\eta_1; w) - (y-w)\frac{\eta_{13}^2}{2\eta_1}{}_2F_1(1+\eta_{13}^2, \eta_{123}, 1+2\eta_1; w)\big) \\ &= \frac{w^{\eta_1}(1-w)^{\eta_3}}{B(\eta_{12}^3, \eta_{13}^2)}\int_1^\infty du\, u^{\eta_{23}^1-1}(u-1)^{\eta_{12}^3-1}(u-w)^{1-\eta_{123}}\frac{u-y}{u-w} \\ &G^{(-)}(w,y) = w^{1-\eta_1}(1-w)^{\eta_3}\times \\ &\big(\frac{1-\eta_{123}}{1-2\eta_1}\,{}_2F_1(1-\eta_{12}^3, \eta_{23}^1, 2-2\eta_1; w) + \frac{y-w}{w}\,{}_2F_1(1-\eta_{12}^3, \eta_{23}^1, 1-2\eta_1; w)\big) \\ &= \frac{w^{\eta_1}(1-w)^{\eta_3}}{B(1-\eta_{123}, \eta_{23}^1)}\int_0^w du\, u^{\eta_{23}^1-1}(1-u)^{\eta_{12}^3-1}(w-u)^{1-\eta_{123}}\frac{y-u}{w-u} \end{aligned} \tag{3.14}$$

[3] I.e., with the "classical" scaling dimensions given as in (3.3) by $h_i = \eta_i(1-\eta_i)$, while in the WZW classical stress tensor, which we do not exploit here, they are given by $(-\eta_i^2)$.

with constants expressed by beta functions $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. The last lines in (3.14) correspond to the expansions in (3.13) in powers of y.

The relative constant in (3.12)

$$N^2_{WZW} = \frac{\gamma(\eta^1_{23})\gamma(2\eta_1)^2}{\gamma(\eta_{123})\gamma(\eta^3_{12})\gamma(\eta^2_{13})} = -\frac{(1-2\eta_1)^2}{(\eta_{123}-1)^2}\, N^2_L \tag{3.15}$$

is determined from the requirement of permutation invariance of the solution (crossing symmetry, or locality of the corresponding quantum 4-point correlator, see [24] for the compact WZW model, and [14] for the non-compact analog); N_{WZW} is expressed by ratio of products of fusing matrices and N_L in the r.h.s. of (3.15) is the corresponding Liouville constant. Note that the basis of contour integrals in (3.14) transforms (moving simultaneously the pairs of space-time and isospin coordinates (z_a, x_a)) with the same braiding (fusing) matrices as in the Liouville case: the shift from N_L to N_{WZW} in (3.15) is due to different coefficient in front of the second contour integral in (3.14) when compared with the corresponding Liouville combination.[4] The Liouville solution itself is reproduced identifying in (3.12) the isospin variables with space time coordinates $x \to z\,, x_i \to z_i$. The overall constant in (3.12) is fixed by the equations of motion, see below.

• Given the solution for V we can extract the expressions for the analogs of the matrix elements in the classical formulae (2.2), (2.7), i.e, ϕ and $\gamma, \bar\gamma$ as functions of $\{z_i, \bar z_i, x_i, \bar x_i\,, i = 1,2,3\}$ and check the equations of motion. There is a certain arbitrariness in it since only V, not its ingredients, are monodromy invariant. We shall expand (3.12) in powers of $(x - x_1)$ in the vicinity of $x \sim x_1$. More precisely, introduce normalised chiral $u^{(\pm)}_i(w, x; x_i)$ as

$$V = |\psi^{(+)}_1 + \psi^{(+)}_2|^2 + |\psi^{(-)}_1 + \psi^{(-)}_2|^2 \tag{3.16}$$

$$\begin{aligned}
\psi^{(\pm)}_1 &= u^{(\pm)}_1(w, x; x_i) = \frac{x-x_2}{x_{12}}\Big(\frac{x_{13}x_{12}}{x_{23}}\Big)^{1/2}\frac{1}{(N_{WZW})^{\pm\frac12}}\big(G^{(\pm)}_0(w) - wG^{(\pm)}_2(w)\big) \\
\psi^{(\pm)}_2 &= (x-x_1)\frac{x-x_2}{x_{12}}u^{(\pm)}_2(w, x; x_i) \\
&\Rightarrow u^{(\pm)}_2(w, x; x_i) = \frac{x_{12}}{x-x_2}\Big(\frac{x_{13}x_{12}}{x_{23}}\Big)^{-1/2}\frac{1}{(N_{WZW})^{\pm\frac12}}G^{(\pm)}_2(w)
\end{aligned} \tag{3.17}$$

[4] The contour integrals have simple transformations described by linear combination of phases, which organise in sin - functions [25]. On the other hand these bases are not normalised to 1 when approaching one of the three sources. When accounting for the additional constants the fusing matrix elements are expressed by Γ - functions, leading to the γ - functions in (3.15). The related consideration of [8] seems to us not sufficiently clear at this point. Recall that the gauge freedom in the braiding matrices is correlated with the normalisation of the chiral vertex operators ${}^{j_3}V^{j_1}_{j_2}$.

where $G_0^{(\pm)}(w)\,, G_2^{(\pm)}(w)$ can be read from (3.14), (3.9). We then identify, taking $u_i^{(\pm)} := u_i^{(\pm)}(w, x = x_1; x_i)$

$$X_{-1} + X_3 = e^{\phi} = \sum_{\pm} |u_2^{(\pm)}|^2\,,$$
$$X = -X_1 + iX_2 = \gamma e^{\phi} = -\sum_{\pm} u_1^{(\pm)} \bar{u}_2^{(\pm)}\,,\ \bar{X} = \bar{\gamma} e^{\phi} = -\sum_{\pm} \bar{u}_1^{(\pm)} u_2^{(\pm)} \tag{3.18}$$
$$X_{-1} - X_3 = e^{-\phi} + |\gamma|^2 e^{\phi} = \sum_{\pm} |u_1^{(\pm)}|^2$$

where $\bar{u}_i$ is the complex conjugate of u_i. Or, in a matrix form for the group element (2.2) we have (chiral factorisation)

$$g(X) = hh^+\,, h = \begin{pmatrix} u_1^{(+)} & u_1^{(-)} \\ u_2^{(+)} & u_2^{(-)} \end{pmatrix}\,. \tag{3.19}$$

The last equality in (3.18) (or, equivalently, the validity of (2.1)) requires that

$$|u_1^{(+)} u_2^{(-)} - u_1^{(-)} u_2^{(+)}|^2 = |\det h|^2 = 1\,. \tag{3.20}$$

This is checked to hold true and more precisely

$$u_1^{(+)} u_2^{(-)} - u_1^{(-)} u_2^{(+)} = 1\,. \tag{3.21}$$

To prove (3.21) one has to use the KZ equation to express $G_2^{(\pm)}$ in terms of $G_0^{(\pm)}$ and their derivatives. The evaluation of the above difference is then reduced to the computation of the Wronskian of the two normalised to 1 solutions in the Liouville case, which is a constant due to (3.4).

Using once again (3.20) one checks that the classical equations of motion (2.4) are indeed satisfied by (3.18); this in particular fixes the overall constant in (3.12).

- The check of the equations of motion is done for $z, \bar{z}$ far from the locations of the sources. Let us now look at the behaviour of the solutions near one of the sources when $z \to z_1\,, x \to x_1$. For $b \to 0$ one has $\delta^{Su}(j) + \delta^{Su}(1/2) - \delta^{Su}(j \pm 1/2) \to \mp\eta\,.$

The leading contribution in the fusion $j \to j + 1/2$ is given by the first vector component of the first solution, namely (taking here only the chiral factors)

$$\psi_1^{(+)} \sim u_1^{(+)} \sim \frac{1}{\sqrt{N_{WZW}}} \frac{1}{(z - z_1)^{-\eta_1}} (\frac{z_{23}}{z_{13} z_{12}})^{\eta_1} (\frac{x_{13} x_{12}}{x_{23}})^{\frac{1}{2}}\,, \tag{3.22}$$

while the second vector component describes a descendant with respect of the finite subalgebra

$$\psi_2^{(+)} \sim (x-x_1)u_2^{(+)} \sim -\frac{\eta_{13}^2}{2\eta_1}\frac{1}{\sqrt{N_{WZW}}}\frac{x-x_1}{(z-z_1)^{-\eta_1}}(\frac{z_{23}}{z_{13}z_{12}})^{\eta_1}(\frac{x_{13}x_{12}}{x_{23}})^{-\frac{1}{2}}\,. \tag{3.23}$$

For the contribution of $j \to j-1/2$ the leading term is given by the second vector component of the second solution

$$\psi_2^{(-)} \sim (x-x_1)u_2^{(-)} \sim \sqrt{N_{WZW}}\frac{(x-x_1)}{(z-z_1)^{\eta_1}}(\frac{z_{23}}{z_{13}z_{12}})^{-\eta_1}(\frac{x_{13}x_{12}}{x_{23}})^{-\frac{1}{2}}\,, \tag{3.24}$$

while the first vector component reads

$$\psi_1^{(-)} \sim u_1^{(-)} \sim -\frac{\eta_{23}^1}{1-2\eta_1}\sqrt{N_{WZW}}\frac{1}{(z-z_1)^{\eta_1-1}}(\frac{z_{23}}{z_{13}z_{12}})^{1-\eta_1}(\frac{x_{13}x_{12}}{x_{23}})^{\frac{1}{2}}\,. \tag{3.25}$$

The behaviour of the solutions for $z \sim z_2$, or $z \sim z_3$ is described permuting $(1,2,3) \to (2,3,1)$, or $(1,2,3) \to (3,1,2)$.

From (3.22) and (3.24) one has that near $z \sim z_1\,, x \sim x_1$

$$\log|\psi_1^{(+)}|^2 \sim -(\log|\psi_2^{(-)}|^2 - \log|x-x_1|^2) \sim -\log|u_2^{(-)}|^2\,. \tag{3.26}$$

The solution in the r.h.s of (3.26) gives for $\eta_1 > 0$ the leading contribution to ϕ, defined by the first equality in (3.18), i.e., near the source

$$2\phi \sim 2\phi^{(-)} = 2\log|\psi_2^{(-)}|^2 - 2\log|x-x_1|^2 \sim -2\eta_1\log|z-z_1|^2 + X_1\,, \tag{3.27}$$

where

$$X_1 = 2\eta_1\log|\frac{z_{13}z_{12}}{z_{23}}|^2 - \log|\frac{x_{13}x_{12}}{x_{23}}|^2 - \log\frac{\gamma(\eta_{123})\gamma(\eta_{12}^3)\gamma(\eta_{13}^2)}{\gamma(\eta_{23}^1)\gamma(2\eta_1)^2}\,. \tag{3.28}$$

Due to the symmetry of the classical solution analogous formulae hold in the vicinity of all three singular points. We now add to the classical action $S^{(cl)} = b^2 S'_{AdS}$ terms which account for the three vertex insertions

$$\begin{aligned}\hat{S}^{(cl)} &= S^{(cl)} + S^{(src)},\\ S^{(src)} &= -\int d^2z\sum_i \delta^2(z-z_i,\bar{z}-\bar{z}_i)(2\eta_i\phi^{(-)}(z,\bar{z}) + 2\eta_i^2\log|z-z_i|^2)\\ &= -\sum_i \eta_i X_i\,.\end{aligned} \tag{3.29}$$

Here it is assumed that the integration in the first term $S^{(cl)}$ is on the Riemann sphere with the points of insertion of the three sources excluded. The action (3.29) is regularised, the logarithmic singularities compensate those in the classical solution.

At the saddle point of $S^{(cl)} = S^{(cl)}[\phi, \gamma, \bar{\gamma}]$, i.e., on a solution of the classical equations, only the second term in (3.29) contributes to the derivative of the full action with respect to any of the charges η_i. Thus one obtains

$$\frac{\partial}{\partial \eta_i} \hat{S}^{(cl)} = -X_i \, , i = 1, 2, 3. \tag{3.30}$$

This set of equations integrates to

$$\begin{aligned} \frac{\hat{S}^{(cl)}}{b^2} = & \sum_{i \neq j \neq k \neq i} (\delta_{ij}^k \log |z_{ij}|^2 - j_{ij}^k \log |x_{ij}|^2) \\ & + \frac{1}{b^2} (F(\eta_{123}) + \sum_i F(\eta_{123} - 2\eta_i) - \sum_i F(2\eta_i) - F(0)) \, , \end{aligned} \tag{3.31}$$

$$\text{where} \;\; \delta_{ik}^l = \delta_i + \delta_k - \delta_l \, , \quad \delta_i = -\frac{\eta_i^2}{b^2} \, , \; j_{ik}^l = j_i + j_k - j_l \, , \quad j_i = -\frac{\eta_i}{b^2} \, . \tag{3.32}$$

From the r.h.s. of (3.31) one reproduces the 3-point function with coefficient as in (2.10) up to the dependence on the normalisation factor $\nu(b)$

$$e^{-\frac{\hat{S}^{(cl)}}{b^2}} = C^{(cl)}(\eta_1, \eta_2, \eta_3) \frac{|x_{13}|^{2j_{13}^2} |x_{12}|^{2j_{12}^3} |x_{23}|^{2j_{23}^1}}{|z_{13}|^{2\delta_{13}^2} |z_{12}|^{2\delta_{12}^3} |z_{23}|^{2\delta_{23}^1}} \, . \tag{3.33}$$

Since $F(x) = F(1 - x)$ the choice $F(0) = F(1)$ for the arbitrary integration constant ensures that $C^{(cl)} = 1$ for $\sum_i \eta_i = 1$, a charge conservation condition, corresponding to absence of screening charges in the quantum case.

• The semiclassical 3-point contribution for S^3 is computed in precisely the same way, exploiting the solutions of the classical KZ equation in the compact WZW model ($b^2 \to -b^2$); see, e.g., [26] for the analogous consideration for the related $c < 1$ Virasoro theory. In the full superstring theory the interrelation of the AdS_3 and S^3 contributions to the quantum 3-point correlator of BPS type fields has been described in [16], [17].

4. Discussion

The difference with the AdS_2 result is reduced essentially to the function $\alpha\tilde{h}(\alpha)$ in (9.4) of [8] vs $F(\alpha)$ in (2.12). Though formally AdS_2 is a special case of AdS_3 it does not possess the affine symmetry of the WZW model. If at all legitimate to compare the semiclassics of these two different models, the disagreement might be also related to a different normalisation of the 3-point vertex itself. In both cases one analyses the solutions of a 2×2 matrix differential equation with the same qualitative asymptotics around the singular points. The braiding matrix elements computed in [8] coincide for a trivial value of the spectral parameter with those in the WZW model, but in a gauge corresponding to the particular bases of contour integrals, see the comment in footnote 4 above.

Irrespectively of the possible applications, the determination of the OPE coefficients in higher rank theories is an important difficult problem in CFT by itself, which might be easier to attack in the semiclassical region; see [27] for partial results for Toda theories. The approach followed here can be extended in principle to the computation of the WZW AdS_5, respectively S^5, contributions to the 3-point OPE constant of three scalar operators. Note that extrapolation from the formulae computed in the supergravity approximation [4], which can be interpreted as "light charge" limits of the (unknown) quantum expressions (cf. (2.14)), suggests that the quantum formula for the AdS_{2n+1} 3-point scalar constant for $n = 2$ might be very close to the $n = 1$ expression in (2.10) - with the shift $-b$ in the first factor in the denominator replaced by $-nb = -2b$. If so, it would not affect the heavy charge limit of the constant, up to field renormalisations.

The WZW AdS_3 semiclassical OPE constants can hardly be expected to provide an approximation of the corresponding scalar semiclassical OPEs in the (non-conformal) AdS_5 sigma model. On the other hand the contribution of the fermionic WZ term in the action [28] ensures the conformal invariance of the supersymmetric $AdS_5 \times S^5$ model in RR background, confirmed in perturbation theory (see also the discussion in [29] and [30]). This justifies and requires further elaboration of CFT techniques.

Acknowledgments

We thank Romuald Janik, Ivan Kostov and Tristan McLoughlin for useful comments. VBP acknowledges the hospitality of the Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Trieste, Italy, where this work was started. This research is partially supported by the Bulgarian NSF grant *DO 02-257*.

References

[1] J. M. Maldacena, The large N limit of superconformal field theories and supergravity", *Adv. Theor. Math. Phys.* **2** (1998) 231, hep-th/9711200.

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Gauge theory correlators from non-critical string theory, *Phys. Lett.* **B 428** (1998) 105, hep-th/9802109.

[3] E. Witten, Anti-de Sitter space and holography, *Adv. Theor. Math. Phys.* **2** (1998) 253, arXiv:hep-th/9802150.

[4] D. Freedman, S. Mathur, A. Matusis and L. Rastelli, Correlation functions in the $\mathrm{CFT}_d/\mathrm{AdS}_{d+1}$ correspondence, *Nucl. Phys.* **B 546** (1999) 96, hep-th/9804058.

[5] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, Three point functions of chiral operators in $D = 4, \mathcal{N} = 4$ SYM at large N, *Adv. Theor. Math. Phys.* **2** (1998) 697 hep-th/9806074.

[6] A.M. Polyakov, Gauge Fields and Space-Time, *Int. J. Mod. Phys.* **A17** S1 (2002) 119, hep-th/0110196.

[7] A. A. Tseytlin, On semiclassical approximation and spinning string vertex operators in $AdS_5 \times S^5$, *Nucl. Phys.* **B 664** (2003) 247, hep-th/0304139.

[8] R. Janik and A. Wereszczynski, Correlation functions of three heavy operators - the AdS_5 contribution, *JHEP* **1112** (2011) 095, arXiv:1109.6262.

[9] T. Klose and T. McLoughlin, A light-cone approach to three-point functions in $\mathrm{AdS}_5 \times S_5$, arXiv:1106.0495.

[10] Y. Kazama, S. Komatsu, On holographic three point functions for GKP strings from integrability, *JHEP* **1201** (2012), 110, arXiv:hep-th/1110.3949.

[11] E. I. Buchbinder and A. A. Tseytlin, Semiclassical correlators of three states with large S^5 charges in string theory in $AdS_5 \times S^5$, *Phys. Rev.* **D85** 026001 (2012) arXiv:hep-th/1110.5621.

[12] J. de Boer, H. Ooguri, H. Robins, J. Tannenhauser, String Theory on AdS_3, *JHEP* **9812** (1998) 026, arXiv:hep-th/9812046.

[13] A. Zamolodchikov and Al. Zamolodchikov, Structure constants and conformal bootstrap in Liouville field theory, *Nucl. Phys.* **B 477** (1996) 577, hep-th/9506136.

[14] J. Teschner, On structure constants and fusion rules in the $SL(2,\mathbb{C})/SU(2)$ WZNW model *Nucl. Phys.* **B 546** (1999) 390, hep-th/9712256.

[15] J. Teschner, Operator product expansion and factorization in the H_3^+-WZNW model, *Nucl. Phys.* **B 546** (1999) 390, hep-th/9906215.

[16] M.R. Gaberdiel and I. Kirsch, Worldsheet correlators in AdS(3)/CFT(2), *JHEP* **0704** (2007), 050, hep-th/0703001.

[17] A. Dabholkar and A. Pakman, Exact chiral ring of $\mathrm{AdS}_3/\mathrm{CFT}_2$, *Adv. Theor. Math. Phys.* **13** (2009) 409, arXiv:hep-th/0703022.

[18] V.G. Knizhnik and A.B. Zamolodchikov, Current algebra Wess-Zumino model in two dimensions, *Nucl. Phys.* **B 247** (1984) 83.

[19] K. Gawędzki, Non-compact WZW conformal field theories, in: Proceedings of NATO ASI Cargese 1991, "New symmetry principles in quantum field theory", eds. , J. Froehlich, G. 'T Hooft, A. Jaffe, G. Mack, P.K.Mitter, R. Stora, Plenum Press (1992) p. 247, hep-th/9110076.

[20] J. Maldacena and H. Ooguri, Strings in AdS_3 and the SL(2,R) WZW Model. Part 1: The spectrum, *J. Math. Phys.* **42** (2001) 2929, hep-th/0001053.

[21] J. Maldacena, H. Ooguri and J Son, Strings in AdS_3 and the SL(2,R) WZW Model. Part 2: Euclidean Black Hole, hep-th/0005183.

[22] C. Thorn, Liouville perturbation theory, *Phys. Rev.* **D 66**, 027702 (2002), hep-th/0204142.

[23] P. Ginsparg and G. Moore, Lectures on 2D gravity and 2D string theory, hep-th/9304011.

[24] V. A. Fateev and A. B. Zamolodchikov, Operator algebra and correlation functions of the two-dimensional Wess-Zumino $SU(2) \times SU(2)$ chiral model, *Sov. J. Nucl. Phys.* **43** (1986) 657.

[25] Vl. S. Dotsenko and V.A. Fateev, Conformal algebra and multipoint correlation functions in 2D statistical models, *Nucl. Phys.* **B 240** (1984) 312.

[26] D. Harlow, J. Maltz and E. Witten, Analytic Continuation of Liouville Theory, arXiv:1108.4417.

[27] V.A. Fateev and A.V. Litvinov, Correlation functions in conformal Toda field theory I, JHEP 11 (2007) 002, arXiv.0709.3806; V.A. Fateev and A.V. Litvinov, Correlation functions in conformal Toda field theory II, JHEP 01 (2009) 033, arXiv:0810.3020.

[28] R.R. Metsaev and A.A. Tseytlin, Type IIB superstring action in $AdS_5 \times S^5$ background, *Nucl. Phys.* **B 533** (1998) 109, hep-th/9805028.

[29] R. Kallosh and A.A. Tseytlin, Simplifying superstring action on $AdS_5 \times S^5$, JHEP **10** (1998) 016, hep-th/9808088.

[30] P. Wiegmann, Extrinsic geometry of superstrings, *Nucl. Phys.* **B 323** (1982) 330.