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Strange Metals in One Spatial Dimension

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We consider $1 + 1$ dimensional $SU(N)$ gauge theory coupled to a multiplet of massive Dirac fermions transforming in the adjoint representation of the gauge group. The only global symmetry of this theory is a $U(1)$ associated with the conserved Dirac fermion number, and we study the theory at variable, non-zero densities. The high density limit is characterized by a deconfined Fermi surface state with Fermi wavevector equal to that of free gauge-charged fermions. Its low energy fluctuations are described by a coset conformal field theory with central charge $c = (N^2 - 1)/3$ and an emergent $\mathcal{N} = (2, 2)$ supersymmetry: the $U(1)$ fermion number symmetry becomes an R -symmetry. We determine the exact scaling dimensions of the operators associated with Friedel oscillations and pairing correlations. For $N > 2$, we find that the symmetries allow relevant perturbations to this state. We discuss aspects of the $N \rightarrow \infty$ limit, and its possible dual description in AdS_3 involving string theory or higher-spin gauge theory. We also discuss the low density limit of the theory by computing the low lying bound state spectrum of the large N gauge theory numerically at zero density, using discretized light cone quantization.

I. INTRODUCTION

An important aim of many applications of the AdS/CFT correspondence to condensed matter physics is the description of quantum matter at variable, non-zero densities. Here ‘density’ refers to the conserved charge, Q , of a global $U(1)$ symmetry of the underlying quantum field theory in d spatial dimensions. Our interest here will be restricted to zero temperature states which do not break translational symmetry or the global $U(1)$ symmetry. Thus, we will not consider ‘solids,’ ‘charge density waves’ or ‘superfluids.’ In the traditional phases of condensed matter physics, the only remaining possibilities for non-zero density states are the Landau Fermi liquid in dimension $d \geq 2$, and the Luttinger liquid in dimension $d = 1$. Both these states are characterized by a Fermi momentum, k_F , whose value obeys the Luttinger relation: the volume enclosed by the $(d - 1)$ -dimensional surface in momentum space at k_F is proportional to the density, Q/L^d , with the same proportionality constant as that for free fermions (L is the spatial size which we will take to infinity).

Any other realization of quantum matter whose density can be varied continuously by an applied chemical potential can generically be referred to as a ‘strange metal.’ A very promising candidate of a strange metal is a model of fermions at non-zero density coupled to an Abelian or non-Abelian gauge field. The non-Fermi liquid effects are strongest in $d = 2$, and this model has been the focus of much study in the condensed matter literature [1–13]. The theory scales to strong coupling, and a perturbative expansion in the gauge coupling constant cannot be used to analyze the leading infrared behavior. The flavor large N_f expansion also leads to difficulty: an expansion in the inverse number of fermion flavors cannot be reduced to counting fermion loops because of infrared divergences [9, 10].

Another possible approach is to take the gauge-charged fermions in the *adjoint* representation of the gauge group, and to then take the ‘t Hooft large N limit for the $SU(N)$ gauge group. In this case, infrared divergences do not spoil

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the naive counting in powers of N , and so even the non-zero density case has a $1/N$ expansion controlled by the genus of the surface defined by a Feynman graph, as in all matrix models [14]. However, one is then left with the generally intractable task of summing all graphs with a given genus. For certain supersymmetric gauge theories, such matrix field theories can, in principle, be solved in the large N limit by the AdS/CFT correspondence [15–17]. Studies of such finite density models by the AdS/CFT correspondence [18–35] have so far only provided a rather incomplete picture of the non-zero density quantum state. The boundary theory has density, Q/L^d , which scales as N^2 , and essentially all of this density is associated in the bulk with degrees of freedom which are beyond the infrared horizon, with an unknown fate. Under appropriate parameter regimes, gauge-invariant probe fermions (‘mesinos’) can acquire a Fermi surface; however such a Fermi surface is only associated with a density of order unity, and is incidental to the physics of the non-Fermi liquid state [36–41]. These probe Fermi surfaces are analogous to conduction electron Fermi surfaces in the ‘fractionalized Fermi liquid’ state of Kondo lattice [42, 43], and do not yield much information on the underlying non-Fermi liquid state. In certain uncontrolled computations, all of the boundary density Q/L^d can be associated with visible Fermi surfaces in the bulk [44–49], but then the resulting state is a Fermi liquid, although an interesting non-Fermi liquid state seems to have been obtained in recent work [49].

In an attempt to shed light on the difficult question of the fate of the ‘hidden’ matter of density proportional to N^2 , this paper will examine the problem of adjoint Dirac fermions, at non-zero density, coupled to a $SU(N)$ gauge field in $1+1$ dimensions, i.e. for $d=1$. We will show that a number of exact results can be obtained for general N , which we hope will help elucidate the structure of the large N limit of such matrix field theories.

We consider the theory with Lagrangian

$$\mathcal{L} = \text{Tr} \left[\bar{\Psi} (i\gamma^\mu D_\mu - m - \mu\gamma^0) \Psi \right] - \frac{1}{2g_{YM}^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} \quad (1.1)$$

with a $SU(N)$ gauge field A_μ , gauge field strength $F_{\mu\nu}$, gauge coupling g_{YM} , and adjoint 2-component complex Dirac fermions Ψ with mass m . The chemical potential μ couples to a global $U(1)$ charge which is distinct from all the $SU(N)$ gauge charges. Note that this $U(1)$ is the *only* global symmetry of this Lagrangian. An analogous $d=1$ model was examined earlier for adjoint Majorana fermions [50–54]: in that case there is no global $U(1)$ that can be coupled to a chemical potential; it was found that the ground state had an energy gap to all excitations, even at $m=0$. As we will see here, just introducing a global $U(1)$ by making the fermions complex is sufficient to transform the physics, and a gapless compressible state is obtained provided μ is large enough, or when $\mu=m=0$.

This theory is characterized by three energy scales, m , $g_{YM}\sqrt{N}$, and μ . We will consider first the “high density” limit $\mu \gg m, g_{YM}\sqrt{N}$, where we can begin the analysis with a Fermi sea of free gauge-charged fermions. Next, we will consider the opposite “low density” limit, where $m \ll g_{YM}\sqrt{N}$ while μ is comparable with m . Here we have to begin with an analysis of the spectrum of the $SU(N)$ singlet excitations of the zero density vacuum: this will be carried out via the discrete light-cone quantization (DLCQ) [55, 56].

We begin with a description of our results for the high density theory. The theory of free fermions has a Fermi wavevector related to the variable $U(1)$ density Q/L by

$$\frac{Q}{L} = (N^2 - 1) \frac{k_F}{\pi} . \quad (1.2)$$

The $N^2 - 1$ prefactor is a characteristic signature identifying this Fermi surface as that of gauge-charged fermions; this Fermi surface is ‘hidden’ [39] because the single fermion Green’s function is not gauge-invariant. The Luttinger relation implies that this value of k_F will not be renormalized [57]. We can analyze the infrared singular effects of the gauge interactions by writing down a continuum theory obtained by linearizing the fermions about the Fermi wavevector. Then, following a procedure standard in the condensed matter physics literature, we can express the Dirac fermion in terms of its right-moving and left-moving components at the Fermi surface

$$\Psi(x, t) \sim \frac{1}{\sqrt{2E_F}} \left(u(-k_F) \psi_R(x, t) e^{ik_F x} + u(k_F) \psi_L(x, t) e^{-ik_F x} \right); \quad (1.3)$$

where $u(\pm k_F)$ are the standard Dirac basis spinors at the Fermi wavevectors; see (2.2) and Appendix C for a more detailed explanation, and see figure 1 for an illustration of this field redefinition. We will assume that $\psi_{L,R}$, and the gauge-fields, are slowly varying on the spatial scale k_F^{-1} , and so all spatial integrals of fields multiplied by non-zero integral powers of $e^{\pm ik_F x}$ vanish. (In condensed matter models, such theories are obtained in the continuum limit of a lattice Hamiltonian, and in this context we are assuming that density is incommensurate, and so there is no ‘umklapp’ scattering.) An immediate consequence is that the resulting low energy theory has an emergent global $U(1)$ conservation law: the total number of left-moving and right-moving fermions are *separately* conserved. We will denote these two $U(1)$ charges as Q_L and Q_R respectively, with $Q = Q_L + Q_R$. This $U(1) \times U(1)$ global symmetry

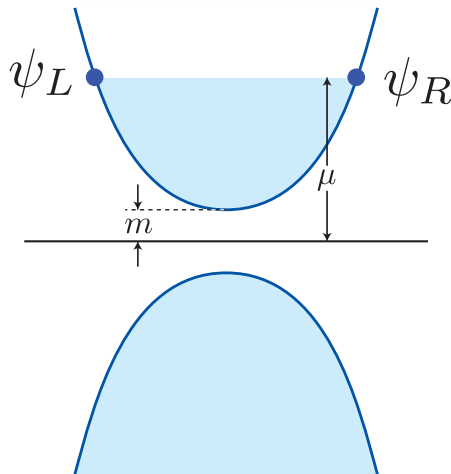


FIG. 1: Energy dispersion of the Dirac fermions as a function of momentum. The full line is the zero of energy. The shaded region represents the occupied states. The filled circles are at $\pm k_F$.

will be crucial to our analysis. All operators appearing in the effective low-energy Lagrangian must have *both* $Q_L = 0$ and $Q_R = 0$.

As we will describe in section II, the high-density, low energy theory so obtained is a two dimensional conformal field theory (CFT), associated with the coset

$$\frac{SU(N)_N \otimes SU(N)_N}{SU(N)_{2N}} \quad (1.4)$$

of central charge

$$c = \frac{N^2 - 1}{3}. \quad (1.5)$$

The two dimensions of the CFT are the euclidean continuation of the original 1+1 dimensions. With the requirement that the coset CFT have a global $U(1) \times U(1)$ symmetry, it actually has the $\mathcal{N} = (2, 2)$ supersymmetry [59]. For $N = 2, 3$ the central charges are $c = 1, 8/3$, and then the theories coincide with the $\mathcal{N} = 2$ superconformal minimal models [58] with $c = 3k/(k+2)$ for $k = 1, 16$ (in the $k = 16$ case we actually find a certain consistent truncation of the minimal model); the $N \geq 4$ theories were only briefly considered earlier [59]. For all N , the R -charge symmetry of these $\mathcal{N} = (2, 2)$ superconformal field theories (SCFTs) is $U(1) \times U(1)$, and this provides the needed global symmetry; the SCFT has no other global flavor symmetries. Note that this supersymmetry is an emergent symmetry at low energies and high densities; it is not a symmetry of the underlying Lagrangian. It is also remarkable that the diagonal R -charge is conjugate to the chemical potential, μ , as has been assumed by fiat in many earlier higher dimensional studies of non-zero density quantum matter.

These two dimensional SCFTs are our ‘strange metals.’ They are $T = 0$ phases with variable density in models with only a global $U(1)$ symmetry, but with a central charge which can become much greater than unity. The density fluctuations associated with the $U(1)$ symmetry cannot be represented by a gapless scalar field which is decoupled from all other sectors, as is the case for Luttinger liquids [60] (exceptions for the $N = 2, c = 1$ case will be discussed in detail below). We note that the ‘Bose metal’ phases found in multi-leg ladder models in Refs. [61] also have $c > 1$ and only a global $U(1)$ symmetry, although they are not expected to be described by our SCFTs.

Armed with this construction of the SCFTs, we will compute exact scaling dimensions of gauge-invariant operators. An important observable which is sensitive to the presence of the underlying Fermi surface of the deconfined fermions is the Friedel oscillation in response to a localized perturbation coupling to the density. Upon perturbing the Lagrangian (1.1) via $\mathcal{L}_{\text{imp}} = \mathcal{L} + \lambda \delta(x) \rho(x, t)$, where the density operator

$$\rho \equiv \text{Tr}(\bar{\Psi} \gamma^0 \Psi) = \text{Tr} \left(\psi_L^\dagger \psi_L + \psi_R^\dagger \psi_R + \frac{m}{\mu} e^{-2ik_F x} \psi_L^\dagger \psi_R + \frac{m}{\mu} e^{2ik_F x} \psi_R^\dagger \psi_L \right), \quad (1.6)$$

the Friedel oscillation response is

$$\langle \rho(x) \rangle_{\text{imp}} \propto \lambda \frac{\cos(2k_F |x| + \delta)}{|x|^{2\Delta_F}} + \dots, \quad (1.7)$$

where δ is a possible phase offset, Δ_F is the scaling dimension of the operator $\text{Tr}(\psi_L^\dagger \psi_R)$, which we will call the Friedel operator in the CFT. Equivalently, we can relate the Friedel oscillation to an oscillatory term in the density-density correlator in the original system without an impurity:

$$\langle \rho(x) \rho(x') \rangle \propto \frac{\cos(2k_F|x - x'| + \delta)}{|x - x'|^{2\Delta_F}} + \dots \quad (1.8)$$

Our exact result for Δ_F , obtained by finding the smallest scaling dimension of operators with $Q_L = 1$ and $Q_R = -1$, is

$$\Delta_F = 1/3 \quad \text{for all } N \geq 2. \quad (1.9)$$

Observation of the oscillatory terms in (1.7) and (1.8) constitutes a measurement of the k_F in (1.2), and is a direct signature of the gauge-charged Fermi surfaces in these strange metals. Unfortunately, we do not determine the N dependence of the missing proportionality constants in (1.7) and (1.8); the vanishing of this proportionality constant in the $N \rightarrow \infty$ limit is the presumed reason for the absence of such Friedel oscillations in existing studies via the AdS/CFT correspondence [62].

A second important observable is the fermion pair operator $\text{Tr}(\psi_L \psi_R)$. Condensation of this operator leads to a superfluid ground state. The spatial decay of its two-point correlations is determined by its scaling dimension Δ_P . Our result for Δ_P was obtained by finding the smallest scaling dimension of operators with $Q_L = Q_R = -1$:

$$\Delta_P = 1/3 \quad \text{for all } N \geq 3. \quad (1.10)$$

For $N = 2$ there is no fermion pair operator $\text{Tr}(\psi_L \psi_R)$ in the CFT, and so in the original gauge theory we expect the two-point functions of $Q = 2$ and $Q = -2$ operators to decay exponentially fast. Instead, the lowest CFT operator with $Q_L = Q_R$ appears for $N = 2$ at $Q_L = Q_R = -3$ and has scaling dimension 3 (see section II A). In Appendix A we review the Luttinger liquid of fermions with short-range interactions (*e.g.* the Thirring model at non-zero density), and find that it obeys $\Delta_F = 1/\Delta_P$; this identity is clearly not obeyed by the present adjoint matter theory. Appendix A also computes the values of $\Delta_{F,P}$ in models of fundamental Dirac fermions coupled to a $SU(N)$ gauge field.

Finally, to assess the stability of the theory in (1.4) as a description of the low energy limit of (1.1), we have to determine the scaling dimensions of all perturbations allowed by symmetry: these are all operators with $Q_L = Q_R = 0$. Here we again find a distinction between $N = 2$ and $N \geq 3$. For $N \geq 3$ the smallest scaling dimension of such an operator is

$$\dim \left[\text{Tr}(\psi_L^\dagger \psi_L \psi_R^\dagger \psi_R) \right] = \frac{2(N-2)}{3N}, \quad (1.11)$$

which is smaller than 2, and so always relevant. So the $N \geq 3$ SCFT₂ is unstable to such a perturbation. We are not able to assess the N dependence of the coefficient of such a perturbation, or the ultimate fate of the ground state. A natural conjecture is that this is an instability to a paired superfluid. In contrast, for $N = 2$ there are no relevant perturbations, and only a marginal perturbation.

The low density limit will be considered in section III. Here we determine the spectrum of $SU(N)$ singlet excitations above the zero density vacuum with the aim of using this as input to describe the finite μ state as a dilute gas of such states. We determine the mass M of the lightest state for a series of values of Q ; we will obtain a dilute gas of such states for $\mu > M/Q$. So we need to determine the value of Q for which M/Q is a minimum. Our numerical analysis, carried out in the limit $N = \infty$, suggests that M/Q may accumulate to a dense set of decreasing values as Q becomes larger (see figure 3). This suggests that M/Q becomes degenerate at some value in the limit of large Q (however, more extensive numerical work is needed to decide if the degeneracy is actually present). This degeneracy would indicate that even in the low density limit it is not appropriate to use a description of a dilute gas of gauge-neutral particles, and suggests the possibility that the gauge-charged-Fermi-sea description of the high density limit applies even at low densities.

II. HIGH DENSITY

In this section we will analyze the regime $\mu \gg m, g_Y M \sqrt{N}$. We will derive the SCFT₂ in (1.4), and then analyze its properties in the subsequent subsections.

We begin by writing the Hamiltonian for the free Dirac fermion in (1.1) in terms of particle, p , and hole, h , creation and annihilation operators introduced via (C6):

$$H_0 = \int \frac{dk}{2\pi} \text{Tr} \left[(\sqrt{k^2 + m^2} - \mu) p^\dagger(k) p(k) - (\sqrt{k^2 + m^2} + \mu) h^\dagger(k) h(k) \right] \quad (2.1)$$

where k is spatial momentum. This defines k_F by $\mu = \sqrt{k_F^2 + m^2}$. Now we introduce the left and right movers by

$$\psi_R(k) = p(k_F + k) \quad , \quad \psi_L(k) = p(-k_F + k), \quad (2.2)$$

and then linearize about k_F by approximating

$$H_0 = \int \frac{dk}{2\pi} v k \text{Tr} \left[\psi_R^\dagger(k) \psi_R(k) - \psi_L^\dagger(k) \psi_L(k) \right] \quad (2.3)$$

where the velocity $v = k_F / \sqrt{k_F^2 + m^2}$. We will henceforth set $v = 1$. Carrying out the same mapping to low energy degrees of freedom in the presence of the gauge field, we obtain the effective Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & \text{Tr} \left[\psi_R^\dagger (\partial_\tau - \partial_x) \psi_R + \psi_L^\dagger (\partial_\tau + \partial_x) \psi_L + (A_\tau - A_x) [\psi_L^\dagger, \psi_L] + (A_\tau + A_x) [\psi_R^\dagger, \psi_R] \right] \\ & - \frac{1}{2g_{YM}^2} \text{Tr} F^2 . \end{aligned} \quad (2.4)$$

Note that this theory is of the same form as (1.1), but after setting $\mu = m = 0$. The CFT structure of this theory becomes clearer upon writing the complex Dirac fields in terms of a pair of Majorana fields $\psi_{L,R}^a$, and $a = 1, 2$:

$$\psi_{L,R} = \frac{1}{\sqrt{2}} (\psi_{L,R}^1 + i\psi_{L,R}^2) \quad (2.5)$$

and then the Lagrangian becomes

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & \frac{1}{2} \text{Tr} [\psi_R^a (\partial_\tau - \partial_x) \psi_R^a + \psi_L^a (\partial_\tau + \partial_x) \psi_L^a + (A_\tau - A_x) \psi_L^a \psi_L^a + (A_\tau + A_x) \psi_R^a \psi_R^a] \\ & - \frac{1}{2g_{YM}^2} \text{Tr} F^2 . \end{aligned} \quad (2.6)$$

As is well-known, each adjoint Majorana fermion is equivalent to a $SU(N)$ WZW model at level N [63–67], each with central charge $(N^2 - 1)/2$. We assume the gauge theory is in the strong coupling limit ($g_{YM} \rightarrow \infty$), and then the integral over the gauge field reduces to a constraint: the vanishing of the currents $J_L = \psi_L^a \psi_L^a$ and $J_R = \psi_R^a \psi_R^a$. It is easily verified that these currents obey a $SU(N)$ Kac-Moody algebra at level $2N$ [68], and central charge $2(N^2 - 1)/3$. The standard coset construction [64] then leads to the CFT in (1.4).

The fact that the low energy CFT₂ of the gauged adjoint Dirac fermion system has $\mathcal{N} = (2, 2)$ supersymmetry was demonstrated explicitly in (12) of [59]. This fact can also be seen to follow easily from an extension of earlier arguments. It is useful to write the CFT in (1.4) in the compact notation $(N, N; 2N)$ as a special case of the general diagonal coset model $(k, \ell; k + \ell)$ of $SU(N)$, which is $SU(N)_k \otimes SU(N)_\ell / SU(N)_{k+\ell}$. Section 3 of [64] considered the coset model $(N, \ell; N + \ell)$ and established that it had $\mathcal{N} = (1, 1)$ supersymmetry. We can obtain the second pair of supercharges by applying the same argument to the coset $(k, N; k + N)$, and so conclude that the coset $(N, N; 2N)$ has $\mathcal{N} = (2, 2)$ supersymmetry. This argument also shows that the R -charge symmetry rotates between the two $SU(N)_N$ components i.e. between the two a components of the Majorana fermions. From (2.5) we then see that the R -charge symmetry is the global $U(1)$ which is conjugate to the chemical potential.

In the following subsections we will describe the structure of these theories, including their modular-invariant partition functions and operator scaling dimensions.

A. $N = 2$

Let us first discuss the simplest non-trivial CFT, corresponding to the $SU(2)$ gauge theory coupled to an adjoint Dirac fermion. This $c = 1$ CFT may be described by the coset

$$\frac{SU(2)_2 \otimes SU(2)_2}{SU(2)_4}. \quad (2.7)$$

The primary fields of this coset theory are therefore labeled by three $SU(2)$ spins j_1, j_2, j where $j = |j_1 - j_2|, \dots, j_1 + j_2$. Their conformal weights are

$$h(j_1, j_2; j) = \frac{j_1(j_1 + 1)}{4} + \frac{j_2(j_2 + 1)}{4} - \frac{j(j + 1)}{6} + n . \quad (2.8)$$

The $\mathcal{N} = 2$ superconformal symmetry fixes which values of (j_1, j_2, j) appear in the spectrum. Here n is a non-negative integer determined in terms of (j_1, j_2, j) . It will be zero for the cases of interest below.

An important quantity characterizing a CFT₂ is its modular invariant partition function on a torus:

$$Z(\tau, \bar{\tau}) = \sum_j e^{2\pi i \tau (h_j - c/24)} e^{-2\pi i \bar{\tau} (\tilde{h}_j - c/24)} , \quad (2.9)$$

where the sum runs over the entire spectrum, and (h_j, \tilde{h}_j) are the holomorphic and anti-holomorphic conformal weights of the state j . Once the modular invariant is known, it is not hard to read off the spectrum of the theory. While there is a variety of possible modular invariants at $c = 1$, the $\mathcal{N} = 2$ superconformal invariance turns out to fix $Z(\tau, \bar{\tau})$ completely, up to an additive constant.

The $c = 1$ CFT turns out to be the simplest member of the series of $\mathcal{N} = 2$ superconformal minimal models [58, 59] with central charges $c = 3k/(k+2)$: it is its $k = 1$ member. The dimensions of the $\mathcal{N} = 2$ superconformal primary fields, and their $U(1)$ charges are in general given by [58, 59]

$$h(p, s, r) = \frac{p^2 - 1 - (s - r)^2}{4(k+2)} + \frac{|r|}{8} , \quad q = \frac{s - r}{2(k+2)} + \frac{r}{4} , \quad (2.10)$$

where $1 \leq p \leq k+1$, $|s| \leq p-1$, and $p-s$ must be odd. In the NS sector $r = 0$, while in the R sector $r = \pm 1$. The NS sector operators with $p = s+1$ have $h = q$; these are the special operators that form the chiral ring [69]. We should note that each primary field of the extended $\mathcal{N} = 2$ algebra gives rise to various Virasoro primary fields obtained by acting with the supercharges and $U(1)_R$ current oscillators.

The $k = 1$ theory has the following NS-sector $\mathcal{N} = 2$ primaries: the identity operator, and the operators with $h = \frac{1}{6}$ and $U(1)_R$ charge $q = \pm \frac{1}{6}$. In the R sector the primary fields are $(h = \frac{1}{24}, q = \pm \frac{1}{12})$ and $(h = \frac{3}{8}, q = \pm \frac{1}{4})$. The former are the R ground states with $h = c/24$.

Now, let us recall the $c = 1$ CFT of a compact massless scalar field ϕ of radius r , i.e. ϕ is identified with $\phi + 2\pi r$. The torus partition function of such a theory has the simple explicit form

$$Z_{\text{scalar}}(\tau, \bar{\tau}) = |\eta(\tau)|^{-2} \sum_{n=-\infty}^{\infty} \sum_{w=-\infty}^{\infty} e^{\pi i \tau k_L^2} e^{-\pi i \bar{\tau} k_R^2} . \quad (2.11)$$

The spectrum of left and right momenta is

$$k_L = \frac{n}{r} - wr/2 , \quad k_R = \frac{n}{r} + wr/2 , \quad (2.12)$$

where n and w are the integer momentum and winding numbers, respectively. The left and right dimensions of the Virasoro primary fields,

$$\exp(ik_L \phi_L + ik_R \phi_R) , \quad (2.13)$$

are $h = k_L^2/2$, $\tilde{h} = k_R^2/2$. In addition, for a generic radius r , this CFT has certain primary fields with $k_L = k_R = 0$ which occur for $h = \tilde{h} = n^2$ where n is an integer. The simplest of such discrete primary fields is the exactly marginal operator $\partial\phi\bar{\partial}\phi$ which changes the radius r .

It is well-known that the compact scalar theory at radius $r = 2\sqrt{3}$ (in units where the self-dual radius is $\sqrt{2}$) has $\mathcal{N} = 2$ supersymmetry. This theory is the bosonization of the above $\mathcal{N} = 2$ minimal model which provides the correct modular invariant. Let us consider some of the simplest operators in the bosonized theory and translate them into the original adjoint fermion language. The marginal operator that changes the radius, $\partial\phi\bar{\partial}\phi$, corresponds to $J_L(z)J_R(\bar{z})$, where

$$J_L \propto \text{Tr}(\psi_L^\dagger \psi_L) , \quad J_R \propto \text{Tr}(\psi_R^\dagger \psi_R) \quad (2.14)$$

are the $U(1) \times U(1)$ currents. We identify the $U(1) \times U(1)$ charges in the $k = 1$ $\mathcal{N} = 2$ minimal model as $q_L = \frac{k_L}{2\sqrt{3}}$ and $q_R = -\frac{k_R}{2\sqrt{3}}$. It follows that the relation between n and w in the compact boson model and the integer charges Q_L and Q_R of the fermions in the gauge theory is

$$n = Q_L - Q_R , \quad w = -(Q_L + Q_R)/6 , \quad (2.15)$$

where $Q_L = 6q_L$ and $Q_R = 6q_R$.

For $n = \pm 1, w = 0$ we get $h = \tilde{h} = \frac{1}{24}$. This corresponds to $h(\frac{1}{2}, \frac{1}{2}; 1)$ in (2.8). These two spin zero operators are products of the R-sector $\mathcal{N} = 2$ superconformal primaries with $h = \frac{1}{24}$.

For $n = \pm 2, w = 0$ we get $h_{n=\pm 2} = \tilde{h}_{n=\pm 2} = \frac{1}{6}$ corresponding to $h(1, 0; 1)$ or $h(0, 1; 1)$. These two spin zero operators are products of the NS-sector $\mathcal{N} = 2$ superconformal primaries with dimension $\frac{1}{6}$. In the gauge theory these operators are $\text{Tr}(\psi_L^\dagger \psi_R)$ with charges $Q_L = 1, Q_R = -1$, and $\text{Tr}(\psi_L \psi_R^\dagger)$ with charges $Q_L = -1, Q_R = 1$. Their sum is simply the fermion mass term. The total dimension of these operators is $\Delta_F = h + \tilde{h} = \frac{1}{3}$; this is the scaling exponent for decay of the Friedel oscillations.

For $(n, w) = \pm(3, 0)$ we find $h = \tilde{h} = \frac{3}{8}$. The holomorphic part of this operator is the $(h = \frac{3}{8}, q = \pm \frac{1}{4})$ $\mathcal{N} = 2$ superconformal primary from the R sector (in the coset theory, it is $h(\frac{1}{2}, \frac{1}{2}; 0) = \frac{3}{8}$).

We could also consider $n = \pm 4, w = 0$ operators with $h_{n=\pm 4} = \tilde{h}_{n=\pm 4} = \frac{2}{3}$. These operators are not superconformal primary fields, but they are Virasoro primary. We note that $h_{\pm 4} = h_{\pm 2} + \frac{1}{2}$. This suggests that the $n = \pm 4$ operators are obtained from $n = \pm 2$ by acting with a holomorphic and an anti-holomorphic supercharge.

For $n = 0, w = \pm 1$ we get $h = \tilde{h} = \frac{3}{2}$. According to (2.15), these are the operators with $Q_L = Q_R = \pm 3$ responsible for superconducting correlations. Since $e^{i\sqrt{3}\phi_L}$ is the supercurrent, we identify the $n = 0, w = \pm 1$ operators with double-trace operators which are products of two super-currents, each having dimension $3/2$. Their net dimension is 3. We do not find, therefore, separate fermion pair operators $\text{Tr}(\psi_L \psi_R)$ and $\text{Tr}(\psi_L^\dagger \psi_R^\dagger)$. This is a special feature of the $N = 2$ case; we will see that such distinct operators appear for $N \geq 3$.

B. $N = 3$

In appendix B we outline a general approach to obtaining the modular invariant partition sum for the $(N, N; 2N)$ cosets and, thereby, the operator content of the theory. We first show how for $N = 2$ this approach reproduces the result given in (2.11) and then proceed to the case $N = 3$.

The proper starting point for the $N = 2$ gauged fermion model is the $SO(6)_1$ invariant partition sum, broken down to $SO(3)_1 \times SO(3)_1$,

$$\begin{aligned} Z^{SO(6)_1} &= |\chi_1^{SO(6)_1}|^2 + |\chi_v^{SO(6)_1}|^2 + 2|\chi_{sp}^{SO(6)_1}|^2 \\ &= |\chi_1^{SO(3)_1} \chi_1^{SO(3)_1} + \chi_v^{SO(3)_1} \chi_v^{SO(3)_1}|^2 + 4|\chi_1^{SO(3)_1} \chi_v^{SO(3)_1}|^2 + 2|\chi_{sp}^{SO(3)_1} \chi_{sp}^{SO(3)_1}|^2. \end{aligned} \quad (2.16)$$

We consider the branching rules of the relevant combinations of $SO(3)_1$ characters. The $\mathcal{N} = (2, 2)$ superconformal symmetry of the coset CFT_2 guarantees that chiral branching functions will organize into characters of $\mathcal{N} = 2$ SCFT, defined as

$$\text{ch}_h^{\text{R,NS}} = \text{Tr}[e^{2\pi i \tau (L_0 - c/24)}] \quad (2.17)$$

(the complete characters of the $\mathcal{N} = 2$ superconformal symmetry would keep track the $U(1)$ charges as well - for simplicity we suppress those in our notations). Considering the explicit leading terms in the various characters, we established the following relations

$$\begin{aligned} &\chi_1^{SO(3)_1} \chi_1^{SO(3)_1} + \chi_v^{SO(3)_1} \chi_v^{SO(3)_1} + 2\chi_1^{SO(3)_1} \chi_v^{SO(3)_1} \\ &= \text{ch}_0^{\text{NS}} (\chi_{(0)}^{SU(2)_4} + \chi_{(4)}^{SU(2)_4}) + 2\text{ch}_{1/6}^{\text{NS}} \chi_{(2)}^{SU(2)_4} \\ &\chi_1^{SO(3)_1} \chi_1^{SO(3)_1} + \chi_v^{SO(3)_1} \chi_v^{SO(3)_1} - 2\chi_1^{SO(3)_1} \chi_v^{SO(3)_1} \\ &= \tilde{\text{ch}}_0^{\text{NS}} (\chi_{(0)}^{SU(2)_4} + \chi_{(4)}^{SU(2)_4}) + 2\tilde{\text{ch}}_{1/6}^{\text{NS}} \chi_{(2)}^{SU(2)_4} \\ &\chi_{sp}^{SO(3)_1} \chi_{sp}^{SO(3)_1} \\ &= \text{ch}_{1/24}^{\text{R}} (\chi_{(0)}^{SU(2)_4} + \chi_{(4)}^{SU(2)_4}) + \text{ch}_{3/8}^{\text{R}} \chi_{(2)}^{SU(2)_4}. \end{aligned} \quad (2.18)$$

Note that the characters $\tilde{\text{ch}}^{\text{NS}}$ are obtained by inserting a $(-1)^F$, with F the fermion parity operator. In the R-sector the characters $\tilde{\text{ch}}^{\text{R}}$ are constants known as the Witten index of the sector.

For $SU(2)_4$ there exists an exceptional invariant (labeled as D_4 in the literature [70]), which groups the characters according to the automorphism (B7)

$$Z^{SU(2)_4} = |\chi_{(0)}^{SU(2)_4} + \chi_{(4)}^{SU(2)_4}|^2 + 2|\chi_{(2)}^{SU(2)_4}|^2. \quad (2.19)$$

This invariant sets a modular invariant metric on the $SU(2)_4$ characters. Using the metric we can project out the level-4 characters in the various product terms in the $SO(6)_1$ partition sum. The final result is the following modular invariant coset partition sum

$$\begin{aligned} Z^{\text{coset}(2,2;4)} &= \frac{1}{2} [|\text{ch}_0^{\text{NS}}|^2 + 2|\text{ch}_{1/6}^{\text{NS}}|^2 + (\text{ch}^{\text{NS}} \rightarrow \tilde{\text{ch}}^{\text{NS}})] \\ &\quad + \frac{1}{2} [|\text{ch}_{3/8}^{\text{R}}|^2 + 2|\text{ch}_{1/24}^{\text{R}}|^2] . \end{aligned} \quad (2.20)$$

Comparing with the expression (2.11), evaluated at the $\mathcal{N} = 2$ radius $r = 2\sqrt{3}$, one checks that the two expressions agree up to a constant which we can write as the sum of the R-sector Witten indices and which is by itself modular invariant

$$Z^{\text{coset}(2,2;4)} = Z_{\text{scalar}} - \frac{1}{2} [2|\tilde{\text{ch}}_{1/24}^{\text{R}}|^2] = Z_{\text{scalar}} - 1 . \quad (2.21)$$

We are now ready to take on the case $N = 3$. The central charge $c = 8/3$ corresponds to the $k = 16$ entry in the minimal series of unitary $\mathcal{N} = (2, 2)$ SCFT₂. One thus expects that the partition sum can be expressed as a finite sum of terms of the form $\text{ch}_{h,q}^{\mathcal{N}=2} \overline{\text{ch}}_{h',q'}^{\mathcal{N}=2}$. Modular invariant partition sums of this type have been completely classified [71] - our challenge is thus to identify the correct entry from the (rather extensive) list.

The $N = 3$ strange metal starts from 16 fermions with partition sum

$$\begin{aligned} Z^{SO(16)_1} &= |\chi_1^{SO(16)_1}|^2 + |\chi_v^{SO(16)_1}|^2 + 2|\chi_{\text{sp}}^{SO(16)_1}|^2 \\ &= |\chi_1^{SO(8)_1} \chi_1^{SO(8)_1} + \chi_v^{SO(8)_1} \chi_v^{SO(8)_1}|^2 + 4|\chi_1^{SO(8)_1} \chi_v^{SO(8)_1}|^2 + 8|\chi_{\text{sp}}^{SO(8)_1} \chi_{\text{sp}}^{SO(8)_1}|^2 . \end{aligned} \quad (2.22)$$

A curiosity specific to $N = 3$ is that the vector representation of $SO(8)$ is isomorphic to the spinors - in the coset theory this leads to degeneracies between NS and R sector dimensions.

As before we now study the branching into characters of $SU(3)_6$ times branching functions which we expect to be characters of a W -algebra extension of $\mathcal{N} = (2, 2)$ superconformal symmetry.¹ Working through explicit details, one observes that in the r.h.s. of the branching rules, the $SU(3)_6$ characters consistently show up in the combinations

$$\begin{aligned} &\chi_{(00)}^{SU(3)_6} + \chi_{(60)}^{SU(3)_6} + \chi_{(06)}^{SU(3)_6} , \quad \chi_{(11)}^{SU(3)_6} + \chi_{(41)}^{SU(3)_6} + \chi_{(14)}^{SU(3)_6} , \\ &\chi_{(33)}^{SU(3)_6} + \chi_{(30)}^{SU(3)_6} + \chi_{(03)}^{SU(3)_6} , \quad \chi_{(22)}^{SU(3)_6} . \end{aligned} \quad (2.23)$$

A modular invariant projector for these terms is provided by the \mathcal{D}_6 invariant of $SU(3)_6$ [75]

$$Z^{SU(3)_6} = |\chi_{(00)} + \chi_{(60)} + \chi_{(06)}|^2 + |\chi_{(11)} + \chi_{(41)} + \chi_{(14)}|^2 + |\chi_{(33)} + \chi_{(30)} + \chi_{(03)}|^2 + 3|\chi_{(22)}|^2 . \quad (2.24)$$

Completing this analysis, we have identified (up to a constant) the partition sum for the $(3, 3; 6)$ strange metal with the exceptional invariant of $\mathcal{N} = (2, 2)$ superconformal symmetry at $c = 8/3$ which in the classification [71] is labeled as $\widetilde{M}^{4,2}$ with parameters $v = 3$, $z = 1$, $x = 1$. This invariant involves a subset of all NS and R characters at $k = 16$; furthermore, those that survive are grouped into groups of 6 (4 groups in each sector) and per sector one group of 3. So in total, 54 fields survive. In the NS sector the extended characters are ($\mathcal{N} = 2$ characters labeled as $\text{ch}_{l=p-1,s}$ with $r = 0$ for NS and $r = -1$ for R)

$$\begin{aligned} \text{ch}_0^{\text{NS,ext}} &= \text{ch}_{0,0}^{\text{NS}} + \text{ch}_{16,0}^{\text{NS}} + \text{ch}_{16,6}^{\text{NS}} + \text{ch}_{16,-6}^{\text{NS}} + \text{ch}_{16,12}^{\text{NS}} + \text{ch}_{16,-12}^{\text{NS}} \\ \text{ch}_{1/9}^{\text{NS,ext}} &= \text{ch}_{2,0}^{\text{NS}} + \text{ch}_{14,0}^{\text{NS}} + \text{ch}_{14,6}^{\text{NS}} + \text{ch}_{14,-6}^{\text{NS}} + \text{ch}_{14,12}^{\text{NS}} + \text{ch}_{14,-12}^{\text{NS}} \\ \text{ch}_{1/3}^{\text{NS,ext}} &= \text{ch}_{4,0}^{\text{NS}} + \text{ch}_{12,0}^{\text{NS}} + \text{ch}_{12,6}^{\text{NS}} + \text{ch}_{12,-6}^{\text{NS}} + \text{ch}_{12,12}^{\text{NS}} + \text{ch}_{12,-12}^{\text{NS}} \\ \text{ch}_{1/6}^{\text{NS,ext}} &= \text{ch}_{6,0}^{\text{NS}} + \text{ch}_{10,0}^{\text{NS}} + \text{ch}_{6,6}^{\text{NS}} + \text{ch}_{6,-6}^{\text{NS}} + \text{ch}_{10,6}^{\text{NS}} + \text{ch}_{10,-6}^{\text{NS}} \\ \text{ch}_{11/18}^{\text{NS,ext}} &= \text{ch}_{8,0}^{\text{NS}} + \text{ch}_{8,6}^{\text{NS}} + \text{ch}_{8,-6}^{\text{NS}} \end{aligned} \quad (2.25)$$

¹ See [72] for a general review of W -symmetry and [73, 74] for W -extensions of $\mathcal{N} = 1$ superconformal symmetry in general cosets involving a $SU(N)_N$ factor.

and in the R sector

$$\begin{aligned}
\text{ch}_1^{\text{R,ext}} &= \text{ch}_{16,-16}^{\text{R}} + \text{ch}_{16,14}^{\text{R}} + \text{ch}_{16,-10}^{\text{R}} + \text{ch}_{16,8}^{\text{R}} + \text{ch}_{16,-4}^{\text{R}} + \text{ch}_{16,2}^{\text{R}} \\
\text{ch}_{1/9}^{\text{R,ext}} &= \text{ch}_{14,14}^{\text{R}} + \text{ch}_{14,-10}^{\text{R}} + \text{ch}_{14,8}^{\text{R}} + \text{ch}_{14,-4}^{\text{R}} + \text{ch}_{14,2}^{\text{R}} + \text{ch}_{2,2}^{\text{R}} \\
\text{ch}_{1/3}^{\text{R,ext}} &= \text{ch}_{12,-10}^{\text{R}} + \text{ch}_{12,8}^{\text{R}} + \text{ch}_{12,-4}^{\text{R}} + \text{ch}_{12,2}^{\text{R}} + \text{ch}_{4,-4}^{\text{R}} + \text{ch}_{4,2}^{\text{R}} \\
\text{ch}_{2/3}^{\text{R,ext}} &= \text{ch}_{10,-10}^{\text{R}} + \text{ch}_{10,8}^{\text{R}} + \text{ch}_{10,-4}^{\text{R}} + \text{ch}_{10,2}^{\text{R}} + \text{ch}_{6,-4}^{\text{R}} + \text{ch}_{6,2}^{\text{R}} \\
\text{ch}_{1/9'}^{\text{R,ext}} &= \text{ch}_{8,-4}^{\text{R}} + \text{ch}_{8,2}^{\text{R}} + \text{ch}_{8,8}^{\text{R}} .
\end{aligned} \tag{2.26}$$

The extended vacuum character has the following content (returning to the notation $\text{ch}_{h,q}$ for the $\mathcal{N} = 2$ characters)

$$\begin{aligned}
\text{ch}_0^{\text{NS,ext}} &= \\
&\text{ch}_{h=0,q=0}^{\text{NS}} + \text{ch}_{h=2,q=1/3}^{\text{NS}} + \text{ch}_{h=2,q=-1/3}^{\text{NS}} + \text{ch}_{h=7/2,q=1/6}^{\text{NS}} + \text{ch}_{h=7/2,q=-1/6}^{\text{NS}} + \text{ch}_{h=4,q=0}^{\text{NS}} .
\end{aligned} \tag{2.27}$$

These fields span a W -algebra extension of $\mathcal{N} = 2$ superconformal symmetry, with extra currents of dimension 2, 2, $7/2$, $7/2$, 4. The W -currents at $h = 2$ are given by

$$W_L = \text{Tr}(\psi_L^\dagger \psi_L^3) , \quad W_L^\dagger = \text{Tr}(\psi_L \psi_L^{\dagger 3}) . \tag{2.28}$$

These currents exist for all $N \geq 3$. For $N = 2$ the triple products ψ_L^3 and $\psi_L^{\dagger 3}$ are $SU(2)$ singlets and the trace vanishes. For $N \geq 3$ explicit expressions for W_L and W_L^\dagger in terms of component fields $(\psi_L)_A$ and $(\psi_L^\dagger)_A$, with $A = 1, 2, \dots, N^2 - 1$ an adjoint index, involve the 3-index d -symbols d_{ABC} .

The $\widetilde{M}^{4,2}$ invariant reads

$$\begin{aligned}
Z^{\widetilde{M}^{4,2}} &= \frac{1}{2} \left[\sum_{h=0,1/9,1/6,1/3} |\text{ch}_h^{\text{NS,ext}}|^2 + 2|\text{ch}_{11/18}^{\text{NS,ext}}|^2 + (\text{ch}^{\text{NS}} \rightarrow \widetilde{\text{ch}}^{\text{NS}}) \right. \\
&\quad \left. + \sum_{h=1/9,1/3,2/3,1} |\text{ch}_h^{\text{R,ext}}|^2 + 2|\text{ch}_{1/9'}^{\text{R,ext}}|^2 \right] .
\end{aligned} \tag{2.29}$$

and the claim is

$$Z^{\text{coset}(3,3;6)} = Z^{\widetilde{M}^{4,2}} - 1/3 . \tag{2.30}$$

The $U(1)_R$ charges of the fields surviving in the partition sum are easily extracted from the field labels in the partition sum. In both the NS and the R sectors, the $U(1)_R$ charges are multiples of $\pm 1/6$.

Comparing with $N = 2$, we observe, at $N = 3$, the presence of fermion pair operators $\text{Tr}(\psi_L \psi_R)$ and $\text{Tr}(\psi_L^\dagger \psi_R^\dagger)$ which were absent from the operator spectrum for $N = 2$. From (2.25) we read off that these operators are in the same extended symmetry multiplet as the mass (Friedel) operators $\text{Tr}(\psi_L^\dagger \psi_R)$ and $\text{Tr}(\psi_L \psi_R^\dagger)$. Both these sets of operators have scaling dimension $1/3$ - the degeneracy is due to the fact that the zero mode of the charged spin two currents in (2.28) relates the two sets of operators.

We also observe the presence of a number of charge neutral relevant operators. The most relevant operator is the $(l = 2, s = 0)$ field in the NS sector, with scaling dimension $\Delta = 2/9$, as in (1.11).

C. General N

We have also obtained several explicit results for $N = 4$ and higher, confirming the general structure outlined above and in appendix B. We defer the details of the description of the general N case to a forthcoming publication. Here we briefly indicate some of the findings, paying special attention to the extrapolation to large N .

current	(q_L, q_R)	(h, \tilde{h})
$J_L \propto \text{Tr}(\psi_L \psi_L^\dagger)$	$(0, 0)$	$(1, 0)$ $N \geq 2$
$G_L \propto \text{Tr}(\psi_L^3)$	$(-\frac{1}{2}, 0)$	$(\frac{3}{2}, 0)$ $N \geq 2$
$G_L^\dagger \propto \text{Tr}(\psi_L^{\dagger 3})$	$(\frac{1}{2}, 0)$	$(\frac{3}{2}, 0)$ $N \geq 2$
$T_L \propto \text{Tr}(\psi_L \partial \psi_L^\dagger + \psi_L^\dagger \partial \psi_L)$	$(0, 0)$	$(2, 0)$ $N \geq 2$
$W_L \propto \text{Tr}(\psi_L^\dagger \psi_L^3)$	$(-\frac{1}{3}, 0)$	$(2, 0)$ $N \geq 3$
$W_L^\dagger \propto \text{Tr}(\psi_L \psi_L^{\dagger 3})$	$(\frac{1}{3}, 0)$	$(2, 0)$ $N \geq 3$
...		

TABLE I: This table lists some of the (left) chiral fields (currents) in the $(N, N; 2N)$ coset model. The currents $J_L, G_L, G_L^\dagger, T_L$ constitute an $\mathcal{N} = 2$ superconformal algebra. The dimension-2 currents W_L, W_L^\dagger are the first of an extensive set of extra primary symmetry generators that exist for $N \geq 3$.

operator type	(q_L, q_R)	(h, \tilde{h})	channel	
$\text{Tr}(\psi_L^\dagger \psi_R)$	$(\frac{1}{6}, -\frac{1}{6})$	$(\frac{1}{6}, \frac{1}{6})$	$(10 \dots 01)$	$N \geq 2$
$\text{Tr}(\psi_L \psi_R^\dagger)$	$(-\frac{1}{6}, \frac{1}{6})$	$(\frac{1}{6}, \frac{1}{6})$	$(10 \dots 01)$	$N \geq 2$
$\text{Tr}(\psi_L \psi_R)$	$(-\frac{1}{6}, -\frac{1}{6})$	$(\frac{1}{6}, \frac{1}{6})$	$(10 \dots 01)$	$N \geq 3$
$\text{Tr}(\psi_L^\dagger \psi_R^\dagger)$	$(\frac{1}{6}, \frac{1}{6})$	$(\frac{1}{6}, \frac{1}{6})$	$(10 \dots 01)$	$N \geq 3$
$\text{Tr}(\psi_L \psi_L \psi_R \psi_R)$	$(-\frac{1}{3}, -\frac{1}{3})$	$(\frac{1}{3}, \frac{1}{3})$	$(20 \dots 10)$	$N \geq 3$
$\text{Tr}(\psi_L \psi_L \psi_R^\dagger \psi_R)$	$(-\frac{1}{3}, 0)$	$(\frac{1}{3}, \frac{1}{3})$	$(20 \dots 10)$	$N \geq 3$
$\text{Tr}(\psi_L \psi_L \psi_R^\dagger \psi_R^\dagger)$	$(-\frac{1}{3}, \frac{1}{3})$	$(\frac{1}{3}, \frac{1}{3})$	$(20 \dots 10)$	$N \geq 3$
$\text{Tr}(\psi_L^\dagger \psi_L \psi_R \psi_R)$	$(0, -\frac{1}{3})$	$(\frac{1}{3}, \frac{1}{3})$	$(20 \dots 10)$	$N \geq 3$
$\text{Tr}(\psi_L^\dagger \psi_L \psi_R^\dagger \psi_R)$	$(0, 0)$	$(\frac{1}{3}, \frac{1}{3})$	$(20 \dots 10)$	$N \geq 3$
$\text{Tr}(\psi_L^\dagger \psi_L \psi_R^\dagger \psi_R^\dagger)$	$(0, \frac{1}{3})$	$(\frac{1}{3}, \frac{1}{3})$	$(20 \dots 10)$	$N \geq 3$
$\text{Tr}(\psi_L^\dagger \psi_L^\dagger \psi_R \psi_R)$	$(\frac{1}{3}, -\frac{1}{3})$	$(\frac{1}{3}, \frac{1}{3})$	$(20 \dots 10)$	$N \geq 3$
$\text{Tr}(\psi_L^\dagger \psi_L^\dagger \psi_R^\dagger \psi_R)$	$(\frac{1}{3}, 0)$	$(\frac{1}{3}, \frac{1}{3})$	$(20 \dots 10)$	$N \geq 3$
$\text{Tr}(\psi_L^\dagger \psi_L^\dagger \psi_R^\dagger \psi_R^\dagger)$	$(\frac{1}{3}, \frac{1}{3})$	$(\frac{1}{3}, \frac{1}{3})$	$(20 \dots 10)$	$N \geq 3$
same 9 terms		$(\frac{1}{3}, \frac{1}{3})$	$(010 \dots 02)$	$N \geq 3$
same 9 terms		$(\frac{2}{3}, \frac{2}{3})$	$(10 \dots 01)$	$N \geq 3$
$\text{Tr}(\psi_L^\dagger \psi_L \psi_R^\dagger \psi_R)$	$(0, 0)$	$(\frac{N-2}{3N}, \frac{N-2}{3N})$	$(20 \dots 02)$	$N \geq 3$
$\text{Tr}(\psi_L^\dagger \psi_L \psi_R^\dagger \psi_R)$	$(0, 0)$	$(\frac{N+2}{3N}, \frac{N+2}{3N})$	$(010 \dots 010)$	$N \geq 4$
$\text{Tr}(\psi_L^\dagger \psi_L^2 \psi_R \psi_R^\dagger)$	$(-\frac{1}{6}, \frac{1}{6})$	$(\frac{N-2}{2N}, \frac{N-2}{2N})$	$(30 \dots 011)$	$N \geq 4$
...				
$\text{Tr}(\psi_L^\dagger \psi_L \psi_R^\dagger \psi_R)^2$	$(0, 0)$	$(\frac{2(N-2)}{3N}, \frac{2(N-2)}{3N})$	$(210 \dots 012)$	$N \geq 4$
$\text{Tr}(\psi_L^\dagger \psi_L \psi_R^\dagger \psi_R)^2$	$(0, 0)$	$(\frac{2(N-3)}{3N}, \frac{2(N-3)}{3N})$	$(40 \dots 020)$	$N \geq 4$
...				
$\text{Tr}(\psi_L^\dagger \psi_L \psi_R^\dagger \psi_R)^n$	$(0, 0)$	$(n \frac{N-1-n}{3N}, n \frac{N-1-n}{3N})$	$(2n0 \dots 02n)$	$N \geq n+2$
...				

TABLE II: Operators in the $(N, N; 2N)$ coset model. The operators listed (all in the NS sector) are primaries with respect to the $\mathcal{N} = (2, 2)$ superconformal algebra. The channel indicated is the $SU(N)$ representation of the gauged diagonal $SU(N)_{2N}$ subalgebra. We give the complete list of 2- and 4-fermion primaries and in addition list some of the higher operators.

- In tables I and II we list the general form of the leading currents and primary field operators in the $(N, N; 2N)$ coset. These operators are all in the NS sector. We see that their scaling dimensions are either N -independent or are such that h (\tilde{h}) converge in the large N limit to $n_{L(R)}/6$ if the operator involves $n_{L(R)}$ left (right) moving fermions. In particular, we see that both fermion pair operators and the Friedel operators are present in the spectrum and have a scaling dimension $\Delta_F = \Delta_P = 1/3$ independent of N . Their degeneracy is due to the presence of the charged spin two currents (2.28) which are part of the extended W -symmetry for any $N \geq 3$.
- One can also quickly verify that the scaling dimensions in the R sector satisfy

$$h^R \geq \frac{c}{24} = \frac{N^2 - 1}{72} \quad (2.31)$$

so that for large N all R-sector operators are irrelevant.

- The algebraic structure of the general- N coset appears to be highly involved. The cosets of the form

$$\frac{SU(N)_k \otimes SU(N)_1}{SU(N)_{k+1}} \quad (2.32)$$

are known to have an extended W_N symmetry with one chiral current for each spin $s = 2, \dots, N$ [72, 76]. For a general diagonal coset (which includes our coset as a special case)

$$\frac{SU(N)_k \otimes SU(N)_\ell}{SU(N)_{k+\ell}} \quad (2.33)$$

we can construct higher spin- s currents from polynomial (of degree s) combinations of the individual numerator $SU(N)$ currents which commute with the diagonal $SU(N)$. See (7.42) and (7.43) of [72] for an explicit expression for the spin-3 current. These general cosets are expected to have many additional currents as well. In the case of $N = 3$ we found two charged spin two fields, see (2.28), in addition to the stress tensor. Since these are primaries of the $\mathcal{N} = 2$ supersymmetry, this then implies the existence of charged spin $\frac{5}{2}$ and spin-3 currents as well. Similarly, we see from (2.27) that there are charged spin $\frac{7}{2}$ currents and their $\mathcal{N} = 2$ descendants.

For general N we expect to be able to build many chiral operators from ψ_L, ψ_L^\dagger and holomorphic derivatives. In the large N limit we do not expect any trace relations for such operators of spin $s \ll N$. Then by the usual counting arguments for words built from matrix valued fields one expects to roughly see a Hagedorn growth in the number of these currents (as a function of the dimension, which is the same as the spin). Thus we might expect an algebra much larger than the conventional W_N symmetry algebra. It would be interesting to work out the consequences of this larger symmetry algebra.

There are a number of issues that would be important to explore further for this class of coset models.

A key algebraic structure of $\mathcal{N} = 2$ SCFT₂ is the chiral ring: the collection of NS sector primaries with $q = \pm h$, which form a closed algebra under fusion [69]. In the $N = 3$ case, we read off from the NS sector primaries in (2.25) that the only chiral primaries are those with $p - 1 = s = 6, 12$ in (2.10). The chiral ring in this minimal model case is generated by one generator x obeying the relation $x^{k+2} = x^{18} = 0$. The chiral primaries present in the coset correspond to the elements x^6 and x^{12} which form a consistent subring of the original chiral ring. The large N structure of the chiral ring can, we believe, be exploited to study the large N physical characteristics of the gauged fermion model.

Finally, it would be very interesting to uncover the AdS dual to this interesting class of CFTs (in the large N limit). The simpler coset models in (2.32) have been identified [77, 78] with a class of higher spin Vasiliev theories on AdS_3 [79]. These have a single bulk gauge field of every spin $s \geq 2$. Roughly speaking there is a single Regge trajectory in the bulk and no Hagedorn growth of states. This is related to the fact, mentioned above, that there is a single conserved current in these coset CFTs for a given spin. The W -symmetry in the coset considered here is very different having many more fields. In general, we expect there to be a Hagedorn density of states in this theory corresponding to single trace words built from $\psi_L, \psi_L^\dagger, \psi_R, \psi_R^\dagger$ together with derivatives (modulo the projection by the diagonal $SU(N)$ currents). Thus the bulk dual is presumably a string theory on AdS_3 . This fits with the fact that these cosets have a central charge proportional to N^2 as opposed to the cosets in (2.32) which have $c \propto N$ and behave like vector models with fewer gauge invariant states [80][77].

Though the coset theory is a strongly interacting CFT as evidenced by the anomalous dimensions of different operators, it is interesting that the dimensions of the operators in the table II are like those of a free theory in the large N limit. They are integer multiples of $\frac{1}{3}$. In that sense the spectrum is similar to that of free Yang-Mills theory or that of the D1-D5 CFT [81, 82] at a symmetric orbifold point. Note that these are the points where we expect a dual tensionless string theory with an unbroken higher spin symmetry. This is consistent with the fact that these cosets have a large chiral W -algebra as discussed above.

III. LOW DENSITY

In this section, we will study the behavior of the $SU(N)$ gauge theory coupled to two adjoint multiplets of Majorana fermions for small $U(1)$ chemical potential μ . We will work in the regime where m , the mass of the adjoint, is much smaller than the scale set by the 't Hooft coupling $g_{YM}\sqrt{N}$. We are interested in the low energy dynamics of the system as μ is increased from zero to values comparable to the scale set by m . Some features of this low energy theory can be inferred from the spectrum of gauge invariant hadronic states at $\mu = 0$. Let us describe how this works for our model.

Generally, it is extremely difficult to compute the spectrum of hadronic bound states in gauge field theories such as QCD. In $1+1$ dimensions, however, the light cone quantization can make this problem tractable. Compactification of the light-like coordinate on a circle: $x^- \sim x^- + L$, a formal regularization procedure known as the discrete light cone quantization (DLCQ) [55, 56], typically reduces the problem to matrix diagonalization. The problem is further simplified in the planar (large N) limit where the $SU(N)$ singlet states are non-interacting. The physical bound state spectrum can be inferred by taking the decompactification limit for the light-cone coordinate x^- . This can be computationally expensive for certain models, but is nonetheless a well controlled approximation scheme. Often, this continuum limit is presented by taking the limit $K \rightarrow \infty$, where the integer ‘harmonic resolution parameter’ K enters the definition light cone momentum:

$$P^+ = \frac{\pi K}{L} \quad (3.1)$$

After diagonalization of the light cone hamiltonian, P^- , the spectrum of bound state masses M is read off from

$$M^2 = 2P^+P^- . \quad (3.2)$$

In this paper we will be primarily concerned with the spectrum of bound state masses in the regime where m^2 is kept fixed while the 't Hooft coupling $g_{YM}^2 N$ is sent to infinity. We find a rich spectrum of bound states whose masses divided by m approach constants in this limit. These bound states originate from the large N coset CFT discussed in section II, perturbed by the operator $m(\text{Tr}(\psi_L \psi_R^\dagger) + \text{h.c.})$ of dimension $1/3$. This mass operator breaks the $Q_L - Q_R$ symmetry of the CFT, but the overall $U(1)$ charge symmetry, $Q = Q_L + Q_R$, remains unbroken. Therefore, the bound states break up into sectors labeled by the integer charge Q .

The DLCQ of a closely related system consisting of a single adjoint multiplet of Majorana fermions has been analyzed in the past [50, 52, 54]. However, in the same limit of sending the 't Hooft coupling to infinity while keeping m^2 fixed, all bound states acquire masses of order $\sim g_{YM}\sqrt{N}$. The fact that there are no light bound states with masses of order m is due to the triviality of the CFT arising from the $SU(N)$ gauge theory coupled to an adjoint Majorana fermion: in that case, instead of (1.5) one finds $c = 0$ [52].

We will summarize the details of the DLCQ computation in Appendix C. In this section, we will focus on presenting the results of the computation.

First, let us describe the basic physics of the model. In $1+1$ dimensions, gauge fields are non-dynamical but serve as agents binding the colored matter fields. The hadronic bound states will then be superpositions of traces of products of the adjoint creation operators (C22). The problem can be separated into boson and fermion sectors depending on whether the number of the creation operators in the trace is odd or even.²

When there are two adjoint Majorana fermions, there is a $U(1) = SO(2)$ global flavor symmetry which provides an additional quantum number to label the states in the spectrum. One way to make this manifest is to combine the two adjoint Majorana fermions into an adjoint Dirac fermion, and count the difference between the number of fermions and anti-fermions for each state.

Suppose for $\mu = 0$ we succeed in computing the masses M and charges Q of all the bound states. Suppose also that all states have $M > 0$ and as a result the theory is gapped in the far IR. What happens when μ is increased?

We expect that some of the particles will condense when

$$\zeta = M - \mu Q \quad (3.3)$$

² The CFT arising in the $m = 0$ limit exhibits the $\mathcal{N} = 2$ supersymmetry relating the bosonic and the fermionic operators. Since the light cone quantization is unreliable in the presence of massless states, we will keep m non-vanishing. Then the supersymmetry is broken; so, the boson and fermion bound state spectra are not identical. However, the bound states with $m \ll M \ll g_{YM}\sqrt{N}$ may exhibit approximate supersymmetry. Such highly excited states are difficult to access numerically, but it would be very interesting to look for the emergent supersymmetry in this region of bound state masses.

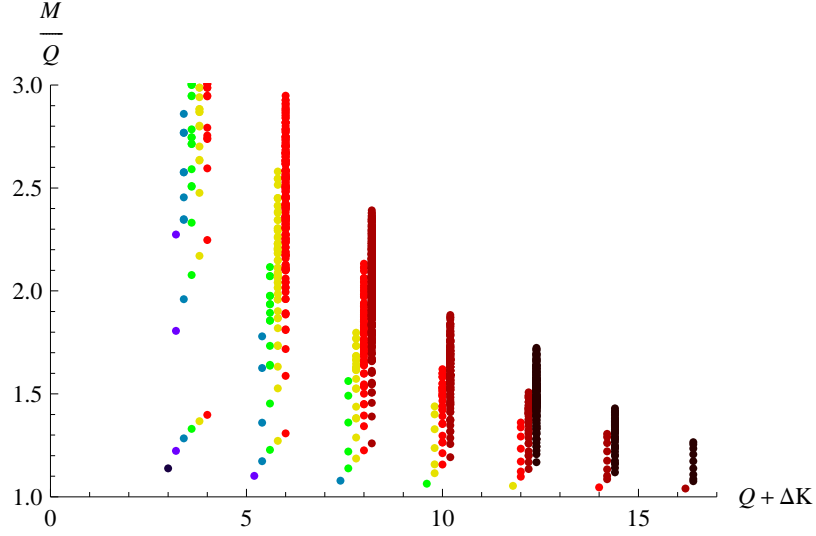


FIG. 2: The spectrum of fermionic bound states for $K = 5, 7, 9, 11, 13, 15, 17, 19$. The states with the same K are shown using the same color. Increasing K for fixed Q is illustrated by a gradual shift to the right in each of the columns.

becomes negative.

If we were working in dimensions greater than $1 + 1$, then depending on whether the first state for which ζ becomes negative is a boson or a fermion, the system would exhibit the universal behavior of either the Bose-Einstein condensation or formation of a degenerate Fermi gas. In $1 + 1$ dimensions, the distinction is somewhat blurred by the fact that bosons and fermions are related by bosonization. If the spectrum is sufficiently generic so that there is precisely one state for which ζ is going to zero at the minimal critical μ , then one expects the system to behave as a Luttinger liquid.

Additional subtleties can arise from the fact that there might be a degeneracy causing more than one state to hit $\zeta = 0$ at the same time. Logically, there are three distinct possibilities.

I The value of ζ goes to zero for exactly one state.

II The value of ζ goes to zero for several, but a finite number of states.

III The value of ζ goes to zero for an infinite set of degenerate states.

The DLCQ computation of the bound state spectrum can help distinguish among these three possibilities.

The details regarding the implementation of the DLCQ procedure are summarized in the appendix. Here we will merely present the result, where we display the full spectrum in figure 2.

The points illustrated in figure 2 are to be interpreted as follows.

1. These points correspond to the spectrum of fermions for which the harmonic resolution parameter K takes odd integer values. One expects to recover the exact spectrum in the limit $K \rightarrow \infty$.
2. The 't Hooft coupling $g_{YM}^2 N$ is taken, for the sake of definitiveness, to be $2\pi \cdot 10^3$ times the bare mass-squared of the fermions, m^2 . As long as this number is very large, the spectrum in the range illustrated, presented in units where $m^2 = 1$, is insensitive to its precise value. The idea is to extract the behavior of this system in the limit of large $g_{YM}^2 N$.
3. M is the mass of the hadronic bound state. Here we are displaying M/Q . The state with lowest M/Q is the one whose ζ will hit zero first as μ is increased.
4. For each K , Q in the range from $Q = 1$ to $Q = K$ are possible. However, the $Q = 1$ states are heavier than the range of M illustrated in figure 2 and as a result are not displayed in this figure. The state in the $Q = K$ sector is decoupled from the rest of the dynamics. In general, taking the large K limit with Q fixed will give rise to a reliable extrapolation of the spectrum of that fixed Q sector. The spectrum with Q of the order of K , however, is sensitive to the DLCQ artifacts and does not effectively approximate the spectrum in the continuum limit. Here we have computed and presented the states for Q in the range $3 \leq Q \leq K - 2$ with the exception of some small Q states for large values of K for which the computations became numerically intractable.

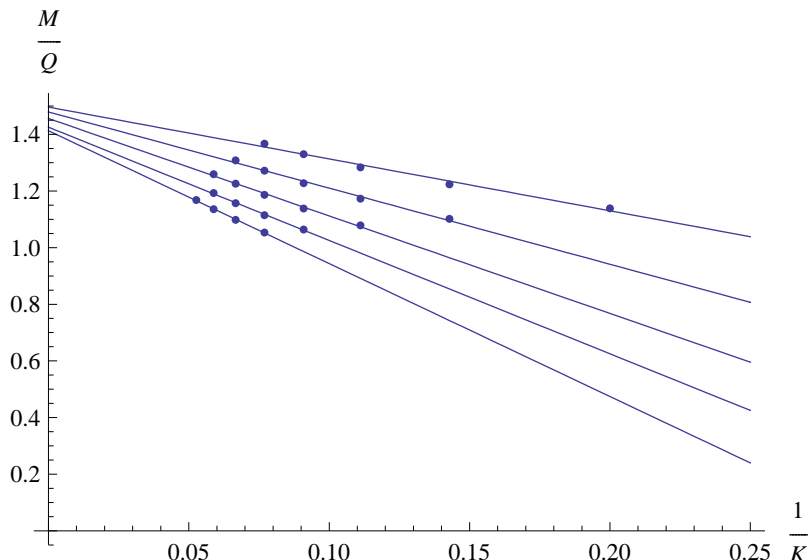


FIG. 3: The spectrum M/Q of lightest states as a function of $1/K$ for $Q = 3, 5, 7, 9, 11$. The lines are linear fits to the available data. The points for $Q = 3$ have the largest M/Q . For each successive Q , the M/Q is decreasing. The intercept of the linear fit at $1/K = 0$ is the extrapolated value of the M/Q for the lightest state for fixed value of Q .

5. For each value of K , states with different charges Q are displayed in separate columns. States with the same K , however, can be identified by the fact that they are plotted using the same color. For each Q , increase in K is indicated by gradual shift in the column of points to the right.

In order to identify the states with smallest M/Q , one must, for each Q , track the lowest mass state and extrapolate to large values of K . The spectrum illustrated in figure 2 suggests that, for each Q , the masses are gradually increasing in a similar manner as K is increased. It also suggests that these increasing masses are converging to the large K limit.

In order to analyze the asymptotic behavior of the masses of the low-lying states, it is useful to plot their masses as a function of $1/K$. Since we are interested in how these states are affected by the chemical potential, we will actually plot M/Q as a function of $1/K$ for different values of K . This is illustrated in figure 3 for $Q = 3, 5, 7, 9, 11$.

We have also superposed a line indicating the linear extrapolation available from the set of data available. These lines cross $1/K = 0$ at finite values. This can be viewed as a crude method to extract the extrapolated value for the large K limit.

What we see in figure 3 is the tendency for the large K limit of the M/Q of the lightest states to become dense with increasing Q . Since Q can get arbitrarily large, this suggests that in the large K limit, states with arbitrarily large Q are converging to the same value of M/Q . In other words, we seem to be finding out that our model is exhibiting scenario **III** enumerated earlier in this section. It should be kept in mind, however, that one must send $K \rightarrow \infty$ first, and then look for a trend as Q is increased. The order of these two limits cannot be exchanged.

It is not easy to determine the critical value μ_{crit} of M/Q . To settle this question, a higher precision computation is required. Unfortunately, for the reason given above, one must explore very large values of K in order to explore large values of Q . This is computationally very expensive.

The fact that there may be infinitely many states with degenerate $\mu_{crit} = M/Q$ suggests that, when μ approaches μ_{crit} , the system may undergo a transition from a gapped phase into a phase with a non-trivial interacting conformal field theory with $c > 1$ in the IR.

The DLCQ results presented in this section focused mainly on fermionic bound states. We have also carried out the analogous computation for bosonic bound states and found a similar behavior of M/Q . We will not present the detailed results of our analysis for the bosonic spectrum here. Some technical issues which arise for the DLCQ computation in the bosonic sector will be described briefly in Appendix C.

Let us, in closing, mention that since all of these results were presented for the case where N was taken to infinity first, it is certainly possible for the seemingly dense spectrum of M/Q to be discretized by a fine structure of order $N^{-\alpha}$ for some $\alpha > 0$. Such a structure can change the basic feature of our model from scenario **III** to **II** or **I**. This may also be related to the fate, following the relevant flow, e.g. (1.11), of the coset CFT discussed in the earlier sections.

IV. CONCLUSIONS

In this paper, we described a novel metallic state of matter in one spatial dimension, with a continuously variable density.

The well-studied one-dimensional metallic state is the Luttinger liquid, and in many respects this state can be considered to be the natural limiting case of the Fermi liquid state of higher dimensions. Indeed, the Luttinger liquid reduces to a free fermion model at a specific value of the Luttinger parameter, and other parameter values are continuously connected to this one. It has central charge $c = 1$, and this is directly linked to the massless scalar representing fluctuations of the globally conserved $U(1)$ charge density. We note that some Luttinger liquids have additional gapless modes associated with other global symmetries (and so a larger central charge), such as the models described in Appendix A.

Our state was obtained by considering a non-zero density of Dirac fermions carrying *adjoint* color charges of a $SU(N)$ gauge field. We found a ‘strange metal’ state described by two-dimensional superconformal field theories with central charges $c = (N^2 - 1)/3$. For large N the central charge becomes large, while the global symmetry remains only the $U(1)$ associated with fermion number conservation. The strange metal has a Fermi surface with a Fermi wavevector, given by (1.2), which is equal to that of non-interacting color-charged particles; the Fermi wavevector changes continuously as the density is varied. This Fermi surface is ‘hidden’ [39], because the single fermion Green’s function is not a gauge-invariant observable. Nevertheless, the Fermi surface and the value of the Fermi wavevector are detectable in the Friedel oscillations of (1.7). We propose that this variable density state, with its large phase space of low energy excitations linked to its large central charge and Fermi surface of color-charged fermions, can serve as a paradigm for non-Fermi liquid states in two and higher spatial dimensions.

The structure of our $d = 1$ strange metal is quite analogous to the ‘hidden’ Fermi surface states obtained recently for general d in Refs. [37–39, 41] via the AdS/CFT correspondence. In Refs. [38, 39] it was postulated that the Fermi surfaces of gauge-charged particles could be detected by the hyperscaling violation of the thermal entropy density, and by a logarithmic violation of the ‘boundary law’ of entanglement entropy. The coefficient of the entanglement entropy logarithm was used to deduce a value for the Fermi wavevector which depended upon ultraviolet details only through the value of the density Q/L^d , in just the manner expected from the Luttinger relation [39]. In our $d = 1$ strange metal here, analogous properties of the entropy and entanglement entropy are trivially satisfied, because the hyperscaling violation exponent $\theta = d - 1 = 0$ (in the notation of Ref. [39]), and every CFT_2 has a logarithmic violation of the boundary law of entanglement entropy [83, 84]. However, here we were able to detect the Fermi surface, and determine the Fermi wavevector, from the Friedel oscillations in the density correlations. Obtaining the Friedel oscillations in the AdS/CFT description of strange metals in general d is clearly one of the important challenges for future work. In this direction, it would be useful to understand the large N dependence of the proportionality constant in (1.7): this should shed light on how Friedel oscillations of gauge-charged particles appear in holographic theories. We also note the interesting recent computation of [41] which detected Friedel oscillations in an anisotropic quantum liquid of strings in $5 + 1$ dimensions.

In section III we studied the bound state spectrum of the large N gauge theory in the limit $g_{YM}^2 N \gg m^2$ using the numerical DLCQ approximation. This approach sheds light on the properties of the $SU(N)$ gauge theory at low density. A very interesting phenomenon that we have uncovered is the emergent $\mathcal{N} = 2$ supersymmetry of the gauge theory in the limit $m \rightarrow 0$. A useful direction for future work would be to obtain some numerical DLCQ evidence for the emergent supersymmetry by studying the masses of highly excited bound states with $g_{YM}\sqrt{N} \gg M \gg m$. Hopefully, this spectrum will exhibit approximate supersymmetry.

Another intriguing direction for future work, which was discussed at the end of section II.C, is the possibility of a dual description of the large N CFT in terms of a theory with higher-spin gauge symmetry in AdS_3 . The existence of such a dual description is suggested by the large W -symmetry, and by the fact all the operator dimensions appear to approach constants in the large N limit that are quantized in units of $1/3$. A useful $3 + 1$ dimensional analogue of the $\mathcal{N} = 2$ supersymmetric large N CFT we are considering may be the $\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory, which is dual to the $AdS_5 \times S^5$ background of type IIB string theory [15–17]. Both theories have anomaly coefficients of order N^2 . When the $\mathcal{N} = 4$ gauge theory has a large ‘t Hooft coupling $g_{YM}^2 N$, the dual $AdS_5 \times S^5$ background becomes weakly curved. In this limit the dimensions of all operators not protected by supersymmetry become very large. On the other hand, at vanishing ‘t Hooft coupling all the operators have integer dimensions, and their number exhibits the exponential Hagedorn growth. In this limit the curvature of the dual string background becomes very large; this is often referred to as the “tensionless string limit.” It has been suggested [85] that a useful dual description of the free $\mathcal{N} = 4$ SYM theory may involve higher-spin gauge theory in AdS_5 [86] coupled to an infinite number of additional fields.

Similarly, if an AdS_3 string dual of our $\mathcal{N} = 2$ supersymmetric coset CFT is found, we expect it to be strongly curved. It is therefore interesting to ask if the coset CFT has an exactly marginal operator which could correspond to increasing the radius of the dual background. In fact, the CFT has an exactly marginal double-trace operator which is

a product of the left and right $U(1)$ currents, $J_L(z)J_R(\bar{z})$. This operator breaks the $\mathcal{N} = 2$ supersymmetry as well as some of the extended W -symmetries. For the $N = 2$ coset CFT (2.7), this marginal operator changes the radius of the compact scalar in the bosonized formulation. In the limit of large radius, a large gap develops between the dimensions of typical momentum and winding operators. It would be interesting to study the effect of the marginal deformation on the operator dimensions for $N > 2$, and to see if deforming the CFT along this marginal direction could also create a large gap in the spectrum of operator dimensions. The presence of such a gap would suggest [87] a weakly curved AdS_3 dual of the large N CFT.

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Appendix A: Fundamental matter

This appendix briefly describes the high density physics of the theory in (1.1), but for the case where Ψ transforms as a fundamental of the $SU(N)$ gauge group. For completeness, we include the case where Ψ has a flavor index which takes N_f values, and then the model has a $U(N_f)$ global symmetry. It is convenient to decompose the global symmetries into a $U(1)$ symmetry associated with the 'charge' density, and a $SU(N_f)$ flavor symmetry (the latter is absent for $N_f = 1$). We proceed just as in section II. The high density limit is characterized by a Fermi wavevector of gauge-charged fermions given by

$$\frac{Q}{L} = NN_f \frac{k_F}{\pi}; \quad (\text{A1})$$

note that the r.h.s. has a prefactor of N , rather than the $(N^2 - 1)$ in (1.2). The fluctuations near this wavevector map onto the $m = \mu = 0$ theory, which was considered in [88, 89]. Now we bosonize the NN_f complex Dirac fermions differently. Rather than considering them as $2NN_f$ Majorana fermions, we note that they can be used generate WZW currents of $SU(N)_{N_f}$, $SU(N_f)_N$, and $U(1)$, and these fully span the Hilbert space [88–90]. The $SU(N)_{N_f}$ currents are projected out by the gauge field, and so the low energy theory is made up of two decoupled sectors: a $c = 1$ free gapless scalar associated with the $U(1)$, and a $SU(N_f)_N$ WZW model associated with the $SU(N_f)$ global flavor symmetry with $c = N(N_f^2 - 1)/(N + N_f)$. This decoupling of the $U(1)$ density mode is the key simplifying feature of the fundamental matter case, and was absent for the adjoint matter case considered in the body of the paper. Consequently, the $U(1)$ sector here is similar to an ordinary Luttinger liquid, while the spectator $SU(N_f)_N$ WZW model is directly linked to the additional global flavor symmetries of the model.

In the notation of [91], we can write the Hamiltonian of the $U(1)$ sector as

$$H = \frac{1}{2\pi} \int dx \left[\frac{1}{K} (\partial_x \phi)^2 + K (\partial_x \theta)^2 \right] \quad (\text{A2})$$

where K is the Luttinger parameter, θ and ϕ are scalar fields obeying the commutation relations

$$[\partial_x \phi(x), \theta(x')] = [\partial_x \theta(x), \phi(x')] = i\pi \delta(x - x'), \quad (\text{A3})$$

and the $U(1)$ charge is

$$Q = \frac{1}{\pi} \int dx \partial_x \phi. \quad (\text{A4})$$

The variable K is related to the exactly marginal perturbation to the $U(1)$ theory, the analog of the ‘radius’ of the scalar in the string theory notation. The fermion fields are related to these scalar fields via [92]

$$\psi_{R,L} = e^{-i(\theta \pm \phi)/\sqrt{NN_f}} \varphi_{R,L}^c \varphi_{R,L}^f \quad (\text{A5})$$

where $\varphi_{R,L}^c$ is the primary field the $SU(N)_{N_f}$ WZW model transforming as a fundamental of $SU(N)$, $\varphi_{R,L}^f$ is the primary field the $SU(N_f)_N$ WZW model transforming as a fundamental of $SU(N_f)$. The exponential factor has been chosen so that the free fermion theory without the $SU(N)$ gauge field is properly bosonized at $K = 1$ with both WZW models conformal so that $\dim[\varphi_{R,L}^c] = (N^2 - 1)/(2N(N + N_f))$ and $\dim[\varphi_{R,L}^f] = (N_f^2 - 1)/(2N_f(N + N_f))$.

To determine the Friedel exponent, we need the smallest scaling dimension operator with $Q_L = 1$ and $Q_R = -1$. Applying (A5) to the operator $\text{Tr}(\psi_L^\dagger \psi_R)$, we can set the trace over the $SU(N)_{N_f}$ WZW fields to constants [89, 90], and the scaling dimensions of the remaining sectors yield

$$\Delta_F = \dim[e^{2i\phi/\sqrt{NN_f}}] + 2 \dim[\varphi_L^f] = \frac{K}{NN_f} + \frac{(N_f^2 - 1)}{N_f(N + N_f)}, \quad (\text{A6})$$

where K is now allowed to be not equal to unity because, in general, there will be an exactly marginal interaction in the $c = 1$ sector. We note that the $\Delta_F \rightarrow 0$ corresponds to a crystalline state with broken translational symmetry; such continuous symmetry breaking is not possible in 1+1 dimensions, but a crystal was discussed as a mean field theory of baryons valid in the formal large N limit [93].

In the fundamental matter model, the pairing operator $\psi_L \psi_R$ can be reduced to a gauge singlet only for $N = 2$. Extending our Friedel operator argument to a gauge singlet pairing operator implies that we need the simplest operator with $Q_L = Q_R = -1$, and this yields

$$\Delta_P = \dim[e^{2i\theta/\sqrt{NN_f}}] + 2 \dim[\varphi_L^f] = \frac{1}{NN_f K} + \frac{(N_f^2 - 1)}{N_f(N + N_f)}, \quad N = 2. \quad (\text{A7})$$

Finally, we note that the case $N = 1$ and $N_f = 1$ corresponds to the finite density phase of the massive Thirring model, which realizes the simplest Luttinger liquid.

Appendix B: Modular invariants for coset CFT_2

Modular invariant partition functions for coset CFT_2 can be constructed once invariants for both the numerator and denominator CFT_2 ’s have been specified [94, 95]. Rather than describing the general construction, we focus on the specific example of the $(N, N; 2N)$ cosets for gauged adjoint fermions.

The numerator CFT_2 has two copies of the $SU(N)_N$ theory, each describing $(N^2 - 1)$ adjoint fermions. There are several options for a modular invariant, including a simple product of the diagonal modular invariant of each of the $SU(N)_N$ factors. However, we should here select the invariant that describes the situation where boundary conditions on the combined fermions are such that a global $U(1)$ symmetry arises. The appropriate modular invariant turns out to be the diagonal modular invariant of an $SO(2N^2 - 2)_1$ symmetry, written as

$$Z^{SO(2N^2-2)_1} = |\chi_1^{SO(2N^2-2)_1}|^2 + |\chi_v^{SO(2N^2-2)_1}|^2 + 2|\chi_{\text{sp}}^{SO(2N^2-2)_1}|^2 \quad (\text{B1})$$

with ‘1’, ‘v,’ and ‘sp’ denoting the identity, vector and spinor representations of $SO(2N^2 - 2)_1$. This result arises from the well known result, known as non-Abelian bosonization [96], that the CFT_2 based on $SO(M)_1$, at central charge $c = M/2$, describes M real fermions.

The partition sum can be re-expressed in terms of characters of two copies of $SO(N^2 - 1)_1$, one for each of the groups of $N^2 - 1$ fermions

$$\begin{aligned} Z^{SO(2N^2-2)_1} &= |\chi_1^{SO(N^2-1)_1} \tilde{\chi}_1^{SO(N^2-1)_1} + \chi_v^{SO(N^2-1)_1} \tilde{\chi}_v^{SO(N^2-1)_1}|^2 \\ &+ |\chi_v^{SO(N^2-1)_1} \tilde{\chi}_1^{SO(N^2-1)_1} + \chi_1^{SO(N^2-1)_1} \tilde{\chi}_v^{SO(N^2-1)_1}|^2 + 2\lambda |\chi_{\text{sp}}^{SO(N^2-1)_1} \tilde{\chi}_{\text{sp}}^{SO(N^2-1)_1}|^2, \end{aligned} \quad (\text{B2})$$

with $\lambda = 4(1)$ for N odd(even). The $SO(N^2 - 1)_1$ characters can each be branched into characters of $SU(N)_N$. For the NS sector characters (labels ‘1’ and ‘v’) the results are

$$\begin{aligned} \chi_1^{SO(N^2-1)_1} &= \chi_{(00\dots 00)}^{SU(N)_N} + \chi_{(20\dots 10)}^{SU(N)_N} + \chi_{(01\dots 02)}^{SU(N)_N} + \dots \\ \chi_v^{SO(N^2-1)_1} &= \chi_{(10\dots 01)}^{SU(N)_N} + \chi_{(110\dots 011)}^{SU(N)_N} + \dots \end{aligned} \quad (\text{B3})$$

We use Dynkin labels $(l_1 l_2 \dots l_{N-1})$ to tag the $SU(N)$ representations: $(00 \dots 00)$ is the identity, $(10 \dots 01)$ the adjoint, etc. (a useful reference for the group theory is [97]). We remark that the $SU(N)$ representations featuring in the NS sector satisfy the N -ality condition

$$l_1 + 2l_2 + \dots (N-1)l_{N-1} \equiv 0 \pmod{N} . \quad (\text{B4})$$

To obtain a partition sum for the coset CFT_2 the following steps are taken. First one writes branching rules for the $SO(2N^2 - 2)_1$ characters into products of branching functions times characters of the affine algebra $SU(N)_{2N}$ that features in the denominator of the coset. Schematically

$$\chi_A^{SO(2N^2-2)_1} = \sum_a b_A^a \times \chi_a^{SU(N)_{2N}} . \quad (\text{B5})$$

The labels a take values in the integral dominant weights of $SU(N)_{2N}$, which are written as Dynkin labels $(l_1 l_2 \dots l_{N-1})$ satisfying $\sum_i l_i \leq 2N$. We find that, for general N , the terms on the r.h.s. of (B5) are grouped into combinations of the form

$$\chi_a^{SU(N)_{2N}} + \chi_{\lambda(a)}^{SU(N)_{2N}} + \chi_{\lambda^2(a)}^{SU(N)_{2N}} + \dots \quad (\text{B6})$$

where λ is the automorphism

$$\lambda : (l_1 l_2 \dots l_{N-1}) \rightarrow ([2N - \sum_i l_i] l_1 l_2 \dots l_{N-2}) . \quad (\text{B7})$$

Again, the $SU(N)$ representations featuring in the NS sector satisfy the N -ality condition (B4).

Writing the modular invariants for the denominator (d) and numerator (n) as

$$\begin{aligned} Z^d &= \sum_{AB} N_{AB}^d \chi_A^{SO(2N^2-2)_1} \bar{\chi}_B^{SO(2N^2-2)_1} , \\ Z^n &= \sum_{ab} N_{ab}^n \chi_a^{SU(N)_{2N}} \bar{\chi}_b^{SU(N)_{2N}} \end{aligned} \quad (\text{B8})$$

the coset invariant is obtained as

$$Z^{\text{coset}} = \sum_{ABab} N_{AB}^d N_{ab}^n b_A^a \bar{b}_B^b . \quad (\text{B9})$$

For the $(N, N; 2N)$ cosets, the natural choice for the denominator modular invariant is the $SU(N)_{2N}$ invariant that displays the same grouping of characters, according to the automorphism (B7), that we observed in the branching rules (B5). Such an invariant exists for general N [98]; we display explicit examples for $N = 2, 3$ in the main text of this paper.

The $\mathcal{N} = (2, 2)$ superconformal symmetry of the $(N, N; 2N)$ coset guarantees that the branching functions b_A^a are characters of (an extension) of the $\mathcal{N} = 2$ superconformal algebra. For $N \geq 3$ we find (see again the main text) that the vacuum character of this extended symmetry takes the form

$$\text{ch}_{q=0, h=0}^{\mathcal{N}=2, \text{ext}} = \text{ch}_{q=0, h=0}^{\text{NS}} + \text{ch}_{q=1/3, h=2}^{\text{NS}} + \text{ch}_{q=-1/3, h=2}^{\text{NS}} + \dots \quad (\text{B10})$$

The $(N, N; 2N)$ coset modular invariant can be viewed as a diagonal modular invariant for this W -extension of $\mathcal{N} = 2$ superconformal symmetry.

Appendix C: DLCQ Quantization of the gauged adjoint Dirac fermions in 1+1 dimensions

In this appendix, we will summarize the computation of the discretized light cone quantization spectrum of the 1 + 1 dimensional $SU(N)$ gauge theory coupled to adjoint Dirac fermions in the large N limit.

The first step is to write down the Lagrangian which follows the general pattern of [50, 52, 54] except that the fermions are now complex. Here we follow II.B of [50]. We start with (1) of [54] but treat

$$\Psi = 2^{1/4} \begin{pmatrix} \psi \\ \chi \end{pmatrix} \quad (\text{C1})$$

as Dirac fermions.

The light cone coordinates are defined by

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1) \quad (\text{C2})$$

so that

$$A_\pm = \frac{1}{\sqrt{2}}(A_0 \pm A_1) . \quad (\text{C3})$$

We will use for the Dirac matrices

$$\gamma^0 = i\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} . \quad (\text{C4})$$

The Lagrangian is normalized to take the form

$$\text{Tr} \{ \bar{\Psi}(i\cancel{\partial} - m)\Psi \} = \text{Tr} \left\{ 2i\psi^\dagger \partial_+ \psi + 2i\chi^\dagger \partial_- \chi - i\sqrt{2}m(\psi^\dagger \chi + \chi^\dagger \psi) \right\} . \quad (\text{C5})$$

To compare this Lagrangian with the Hamiltonian (2.1), simply note that the fermion field Ψ can be expressed in the standard mode expansion

$$\Psi(t, x) = \int \frac{dk_1}{2\pi} \frac{1}{\sqrt{2k_0}} \left(u(k)p(k)e^{-ik_\mu x^\mu} + v(k)h(k)e^{ik_\mu x^\mu} \right) \quad (\text{C6})$$

where $u(k)$ and $v(k)$ is the standard 2 component spinor basis satisfying

$$(\not{k} - m)u(k) = (\not{k} + m)v(k) = 0, \quad \bar{u}(k)u(k) = -\bar{v}(k)v(k) = 2m . \quad (\text{C7})$$

Then, in terms of $p(k)$ and $h(k)$ we recover (2.1) for the Hamiltonian.

Returning to (C5), we gauge the free fermion theory by introducing covariant derivatives

$$D\Psi = \partial_\mu \Psi + i[A_\mu, \Psi] . \quad (\text{C8})$$

It is customary in light cone quantization to use the gauge

$$A_- = 0 \quad (\text{C9})$$

so that the gauge kinetic term takes the form

$$-\frac{1}{2g_{YM}^2} \text{Tr} F^2 = \frac{1}{g_{YM}^2} \text{Tr} (\partial_- A_+)^2 \quad (\text{C10})$$

and the Lagrangian reads

$$\mathcal{L} = \text{Tr} \left\{ 2i\psi^\dagger \partial_+ \psi + 2i\chi^\dagger \partial_- \chi - \sqrt{2}im(\psi^\dagger \chi + \chi^\dagger \psi) + A_+ J^+ + \frac{1}{g_{YM}^2} (\partial_- A_+)^2 \right\} \quad (\text{C11})$$

with

$$J^+ = 2(\psi\psi^\dagger + \psi^\dagger\psi) . \quad (\text{C12})$$

If we take x^+ as the time direction, χ and A_+ are non-dynamical and can be integrated out, giving rise to the light-cone momentum and Hamiltonian

$$P^+ = \int dx^- \text{Tr} \{ 2i\psi^\dagger \partial_- \psi \} , \quad (\text{C13})$$

$$P^- = \int dx^- \text{Tr} \left\{ -im^2 \psi^\dagger \frac{1}{\partial_-} \psi - \frac{1}{4} g_{YM}^2 J^+ \frac{1}{\partial_-^2} J^+ \right\} . \quad (\text{C14})$$

Imposing canonical quantization on the fermions gives rise to relation

$$\{\psi_{ij}^\dagger(x^-), \psi_{kl}(\tilde{x}^-)\} = \frac{1}{2}\delta(x^- - \tilde{x}^-) \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right) \quad (\text{C15})$$

with

$$\{\psi_{ij}(x^-), \psi_{kl}(\tilde{x}^-)\} = \{\psi_{ij}^\dagger(x^-), \psi_{kl}^\dagger(\tilde{x}^-)\} = 0 . \quad (\text{C16})$$

The Dirac fermions are expanded in modes

$$\psi(x^-) = \frac{1}{\sqrt{2L}} \sum_{n \in \text{odd} > 0} \left\{ A_{ij}(n) e^{-i\pi n x^- / L} + B_{ji}^\dagger(n) e^{i\pi n x^- / L} \right\} \quad (\text{C17})$$

where we have compactified the x^- direction and imposed the anti-periodic boundary condition on the $\psi(x^-)$ field; this typically leads to a better DLCQ computation than choosing the periodic boundary condition and removing the zero mode by hand. The choice of boundary condition should not matter in the decompactification limit $K \rightarrow \infty$.

The anti-commutation relation for the modes are is

$$\{A_{ij}(m), A_{kl}(n)\} = \delta(m+n) \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right), \quad (\text{C18})$$

$$\{B_{ij}(m), B_{kl}(n)\} = \delta(m+n) \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right) \quad (\text{C19})$$

where n takes odd integer values, and

$$A_{ij}(-n) = A_{ji}^\dagger(n), \quad B_{ij}(-n) = B_{ji}^\dagger(n) . \quad (\text{C20})$$

We can now set up the light cone vacuum

$$A_{ij}(n)|0\rangle = B_{ij}(n)|0\rangle = 0, \quad (n > 0) . \quad (\text{C21})$$

The states are then generated by acting by a string of “letters” $A(-n)$ and $B(-n)$ in a single trace state, i.e.

$$|\psi\rangle = \# \text{Tr}(B(-n_1)A(-n_2)\dots B(-n_k))|0\rangle \quad (\text{C22})$$

where $\#$ is a symmetry factor to ensure that the norm of each state is one.

Our next step is to write the light cone momentum and Hamiltonian operators in terms of the fermion oscillators. It is clear that the light cone momentum operator (C13) can be written in the form

$$P^+ = \sum_{n \geq 1} \left\{ \frac{\pi n}{L} A_{ij}^\dagger(n) A_{ji}(n) + \frac{\pi n}{L} B_{ij}^\dagger(n) B_{ji}(n) \right\} = \frac{\pi K}{L} , \quad (\text{C23})$$

when acting on a state with fixed K .

Instead of writing the light cone Hamiltonian P^- in terms of fermion oscillators, let us consider the Lorentz invariant mass operator

$$2P^+P^- = V + T \quad (\text{C24})$$

where V corresponds to terms associated with the term proportional to m^2 and T corresponds to the term proportional to $(J^+)^2$ in (C14). Then, it is easy to show that

$$V = Km^2 \sum_{n \geq 1} \left\{ \left(\frac{1}{n} \right) A_{ij}^\dagger(n) A_{ji}(n) + \left(\frac{1}{n} \right) B_{ij}^\dagger(n) B_{ji}(n) \right\} . \quad (\text{C25})$$

Note that the dependence on L drops out, but there is still a dependence on K .

Computation of T involves a somewhat tedious exercise of normal ordering the $(J^+)^2$ written in terms of fermion oscillator operators. One can organize T in accordance to the number of oscillators destroyed and created.

$$T = T_{1 \rightarrow 1} + T_{1 \rightarrow 3} + T_{2 \rightarrow 2} + T_{3 \rightarrow 1} . \quad (\text{C26})$$

In this form, one finds after some algebra, that

$$T_{1 \rightarrow 1} = \frac{2g_{YM}^2 NK}{\pi} \sum_n \sum_{m=1}^{n-2} \left\{ \left(\frac{1}{n-m} \right)^2 A_{ji}^\dagger(n) A_{ji}(n) + \left(\frac{1}{n-m} \right)^2 B_{ji}^\dagger(n) B_{ji}(n) \right\} \quad (C27)$$

$$\begin{aligned} T_{1 \rightarrow 3} = & \left(\frac{g_{YM}^2 NK}{2\pi} \right) \sum \delta(n_1 + n_2 + n_3 - m_1) \times \\ & \left\{ \frac{2}{(n_1 - m_1)^2} B_{ik}^\dagger(n_3) A_{kl}^\dagger(n_2) A_{lj}^\dagger(n_1) A_{ij}(m_1) \right. \\ & \left(\frac{2}{(n_1 - m_1)^2} - \frac{2}{(n_3 - m_1)^2} \right) A_{ik}^\dagger(n_3) B_{kl}^\dagger(n_2) A_{lj}^\dagger(n_1) A_{ij}(m_1) \\ & - \frac{2}{(n_3 - m_1)^2} A_{kl}^\dagger(n_3) A_{lj}^\dagger(n_2) B_{ji}^\dagger(n_1) A_{ki}(m_1) \\ & \frac{2}{(n_1 - m_1)^2} A_{lj}^\dagger(n_3) B_{ji}^\dagger(n_2) B_{ik}^\dagger(n_1) B_{lk}(m_1) \\ & \left(\frac{2}{(n_1 - m_1)^2} - \frac{2}{(n_3 - m_1)^2} \right) B_{ji}^\dagger(n_3) A_{ik}^\dagger(n_2) B_{kl}^\dagger(n_1) B_{jl}(m_1) \\ & \left. - \frac{2}{(n_3 - m_1)^2} B_{ji}^\dagger(n_3) B_{ik}^\dagger(n_2) A_{kl}^\dagger(n_1) B_{jl}(m_1) \right\} \end{aligned} \quad (C28)$$

$$\begin{aligned} T_{2 \rightarrow 2} = & \left(\frac{g_{YM}^2 NK}{2\pi} \right) \sum \delta(n_1 + n_2 - m_1 - m_2) \times \\ & \left\{ \frac{2}{(n_1 - m_1)^2} A_{kl}^\dagger(n_2) A_{lj}^\dagger(n_1) A_{ki}(m_2) A_{ij}(m_1) \right. \\ & \left(\frac{2}{(n_1 - m_1)^2} - \frac{2}{(m_1 + m_2)^2} \right) A_{ik}^\dagger(n_2) B_{kl}^\dagger(n_1) A_{ij}(m_2) B_{jl}(m_1) \\ & - \frac{2}{(m_1 + m_2)^2} B_{ik}^\dagger(n_2) A_{kl}^\dagger(n_1) A_{ij}(m_2) B_{jl}(m_1) \\ & - \frac{2}{(m_1 + m_2)^2} A_{lj}^\dagger(n_2) B_{ji}^\dagger(n_1) B_{lk}(m_2) A_{ki}(m_1) \\ & \left(\frac{2}{(n_1 - m_1)^2} - \frac{2}{(m_1 + m_2)^2} \right) B_{kl}^\dagger(n_2) A_{lj}^\dagger(n_1) B_{ki}(m_2) A_{ij}(m_1) \\ & \left. \frac{2}{(n_1 - m_1)^2} B_{ji}^\dagger(n_2) B_{ik}^\dagger(n_1) B_{jl}(m_2) B_{lk}(m_1) \right\} \end{aligned} \quad (C29)$$

$$\begin{aligned} T_{3 \rightarrow 1} = & \left(\frac{g_{YM}^2 NK}{2\pi} \right) \sum \delta(n_1 - m_1 - m_2 - m_3) \times \\ & \left\{ - \frac{2}{(n_1 - m_1)^2} A_{lj}^\dagger(n_1) B_{lk}(m_3) A_{ki}(m_2) A_{ij}(m_1) \right. \\ & + \left(\frac{2}{(m_2 + m_1)^2} - \frac{2}{(n_1 - m_1)^2} \right) A_{lj}^\dagger(n_1) A_{lk}(m_3) B_{ki}(m_2) A_{ij}(m_1) \\ & + \frac{2}{(m_1 + m_2)^2} A_{kl}^\dagger(n_1) A_{ki}(m_3) A_{ij}(m_2) B_{jl}(m_1) \\ & - \frac{2}{(n_1 - m_1)^2} B_{ik}^\dagger(n_1) A_{ij}(m_3) B_{jl}(m_2) B_{lk}(m_1) \\ & + \left(\frac{2}{(m_1 + m_2)^2} - \frac{2}{(n_1 - m_1)^2} \right) B_{kl}^\dagger(n_1) B_{ki}(m_3) A_{ij}(m_2) B_{jl}(m_1) \\ & \left. + \frac{2}{(m_1 + m_2)^2} B_{ji}^\dagger(n_1) B_{jl}(m_3) B_{lk}(m_2) A_{ki}(m_1) \right\} . \end{aligned} \quad (C30)$$

At this point, a computer program must be written to generate the set of states and the elements of the mass operator $M^2 = 2P^+ P^-$.

	$K = 5$	$K = 7$	$K = 9$	$K = 11$	$K = 13$	$K = 15$	$K = 17$	$K = 19$	$K = \infty$	M/Q
$Q = 1$	4.60	5.08	5.42	6.42	6.42
$Q = 3$	3.42	3.67	3.85	4.00	4.10	4.49	1.50
$Q = 5$	-	5.51	5.87	6.14	6.36	6.54	7.40	1.47
$Q = 7$	-	-	7.55	7.97	8.30	8.58	8.82	...	10.19	1.46
$Q = 9$	-	-	-	9.57	10.03	10.041	10.73	...	12.83	1.43
$Q = 11$	-	-	-	-	11.59	12.08	12.49	12.85	15.54	1.41

TABLE III: Mass M of the lightest fermionic bound state in the fixed Q sector for various K . The “-” indicates entries which are not defined. The “...” indicate entries which are well defined but have not been computed due to limits in computational resources. These are the data presented in figure 3. Note that M as a function of Q is minimized at $Q = 3$. However, M/Q as a function of Q appears to slowly be decreasing monotonically.

	$K = 4$	$K = 6$	$K = 8$	$K = 10$	$K = \infty$	M/Q
$Q = 0$	2.31	2.47	2.58	2.67	2.88	∞

TABLE IV: Mass M of the bosonic bound state in the $Q = 0$ sector for $K = 4, 6, 8, 10$.

As an example, for $K = 5$ and $Q = 1$, we find

$$T = \frac{g_{YM}^2 N K}{2\pi} \begin{pmatrix} \frac{13}{8} & \frac{1}{8} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{8} & \frac{13}{8} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & 1 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 & 3 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & \frac{3}{2} \end{pmatrix} \quad (C31)$$

whose eigenvalues in units of $g_{YM}^2 N/2\pi$ are

$$\{0, 6.25, 10, 10, 17.5\} . \quad (C32)$$

Strictly speaking, eigenvalues of just the T without the contribution from V are unreliable since they correspond to taking massless fermions as the matter fields. Nonetheless, it is encouraging to find an exactly massless state in the spectrum which would survive the limit of strong gauge coupling. The massless states for these $m^2 = 0$ cases continue to be present as the values of K are increased.

The actual computation reported in section III is the computation of the spectrum of

$$\frac{M^2}{m^2} = \frac{1}{m^2}(V + T) \quad (C33)$$

where, for definitiveness, we set the dimensionless parameter

$$x = \frac{2\pi m^2}{g_{YM}^2 N} = 10^{-3} . \quad (C34)$$

The result of carrying out the calculation for $K = 5, 7, 9, 11, 13, 15, 17, 19$, is summarized in table III. There, we tabulate the calculated value of the hadronic bound state mass M for fixed Q as K is increased. The $K = \infty$ is a result of linear extrapolation illustrated in figure 3. We observe that the lightest bound state appears to be in the $Q = 3$ sector, at least among the fermionic states. However, the quantity M/Q which determines μ_{crit} appears to be slowly decreasing as Q is increased.

Similar computations can be carried out for the bosonic bound states when the values of K are taken to be even. We did not perform the computation at the same level of precision for the bosonic bound states as we did for the fermions. The plot analogous to figure 2 is in figure 4 below. From figure 4, it is quite apparent that the pattern of states with smallest M/Q is becoming degenerate at around $\mu_{crit} = M/Q \approx 1$ as Q is increased.

There is one additional feature which is notable regarding the bosonic spectrum: the state with smallest M is in the $Q = 0$ sector. For $K = 4, 6, 8, 10$, we find the masses presented in table IV.

A closer examination of the wavefunction indicates that the lightest state is mostly a mixture of the “two bit” states of the form

$$\sum_p c_p \text{Tr} A^\dagger(p) B^\dagger(K - p) |0\rangle . \quad (C35)$$

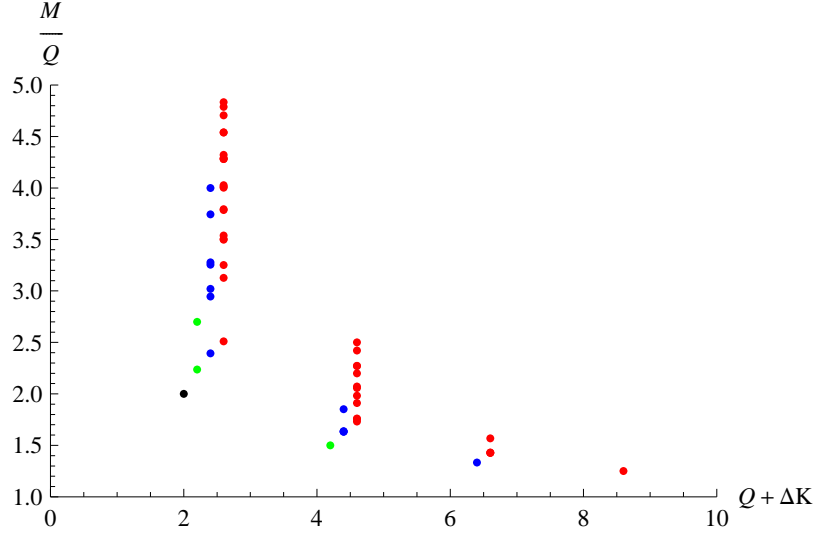


FIG. 4: Spectrum of bosonic bound states for $K = 4, 6, 8, 10$. This figure is the analogue of figure 2 for the fermionic bound states.

This is analogous to what was found for the adjoint Majorana model [52]. These are interesting features of our model from the point of view of its dynamics at vanishing chemical potential. As should be clear from the right most column in table IV, however, the $Q = 0$ states do not have any impact on the physics at finite chemical potential μ .

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