Non-Riemannian metric emergent from scalar quantum field theory

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A Non-Riemannian Metric Emergent From Scalar Quantum Field Theory

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We show that the two-point function $\sigma(x,x')=\sqrt{<[\phi(x)-\phi(x')]^2>}$ of a scalar quantum field theory is a metric (i.e., a symmetric positive function satisfying the triangle inequality) on space-time (with imaginary time). It is very different from the Euclidean metric $|x-x'|$: space-time has a finite diameter $\propto \frac{1}{a^2}$ which is not universal (i.e., depends on the UV cut-off a and the regularization method used). The Lipschitz equivalence class of the metric is independent of the cut-off. $\sigma(x,x')$ is not the length of the geodesic in any Riemannian metric. Nevertheless, it is possible to embed space-time in a higher dimensional space so that $\sigma(x,x')$ is the length of the geodesic in the ambient space. $\sigma(x,x')$ should be useful in constructing the continuum limit of quantum field theory with fundamental scalar particles.

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I. INTRODUCTION

In view of the anticipated discovery of the Higgs et al. boson [1], it is timely to reconsider the fundamental implications of a plain scalar field of the standard model: one that is not composite, associated to supersymmetry, or to extra dimensions of space-time. It is of interest to study a quantum theory of scalar fields in any case, as it describes many other phenomena as well; such as phase transitions.

Such a field is not expected to have an effect on the geometry of space-time. This is unlike the other bosonic fields: according to General Relativity, gravity modifies the metric of space-time from Euclidean to Riemannian. And gauge fields have a geometric meaning in terms of parallel transport. A fundamental scalar field is thought to have a geometrical meaning only as a remnant of dimensional reduction: if space time has extra dimensions (whether continuous or discrete, as in non-commutative geometry), the extra components of the gauge or gravitational field would be scalars in four dimensional space-time.

In this paper, we will show that a scalar quantum field defines a metric on space-time as well. But to understand this metric we must go beyond Riemannian geometry. In recent years, the study of such general metric spaces has emerged [2–4] as a fundamental branch of mathematics, touching on topology, geometry and analysis.

In classical mechanics, a free particle moves along a straight-line. The length of this line is the distance between points. In quantum mechanics, the propagator is the sum over all paths[5], with a weight proportional to e^{-S}, where $S=\frac{1}{2}\int \dot{x}^2dt$ is the action. Thus, a quantum notion of distance should involve the propagator itself rather than a property of a particular path.

In quantum field theory, we should seek a notion of distance based on the correlations of fields at two points, which is the analogue of the propagator. We will show that the quantity

$$\sigma(x,x')=\sqrt{<[\phi(x)-\phi(x')]^2>}$$

(defined with a regularization) in a scalar quantum field theory satisfies the axioms of a metric [2] (see the Appendix for a summary of metric geometry). In particular, the triangle inequality

$$\sigma(x,x')\leq\sigma(x,x'')+\sigma(x'',x')$$

holds. This notion of distance between two points is very different though, from the Riemannian notion. It is worth exploring on its own right, even if we continue to use the Euclidean length for most physical purposes.

For example, the triangle inequality above cannot be saturated if x,x',x'' are all distinct. By contrast, for the Euclidean distance, as long as x'' lies on the straight line connecting x to x', we would have $|x-x'|=|x-x''|+|x''-x'|$. More generally, in Riemannian geometry, we can saturate the triangle inequality by choosing x'' to be any point on a shortest geodesic connecting x to x'.

This means that $\sigma(x,x')$ is not the length of geodesics in any Riemannian geometry: it defines a non-Riemannian metric geometry.

So why would space-time look Euclidean classically? It turns out that the length of a curve defined by $\sigma(x,x')$ (in a super-renormalizable or asymptotically free theory; we do not know the general answer yet) is the same as the Euclidean length. The reason is that for small distances and small interactions, $\sigma(x,x')\propto|x-x'|$. (The proportionality constant the depends on the regularization.) The length of a curve is defined by breaking it up into small segments (see the Appendix). For small

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enough segments we will get (up to a constant) their Euclidean length. Classical measurements of distance always involve lengths of curves. So even σ will give the Euclidean answer (up to the proportionality constant) in these measurements.

According to σ, the shortest curve connecting two points is still the straight line. But this shortest length is not the same as $\sigma(x,x')$. We will see that $\sigma(x,x')$ is a monotonically increasing function of $|x-x'|$ which, for dimensions $n=3,4$ tend to a constant for large $|x-x'|$: space-time has *finite diameter* according to σ. For the case of a massless free field in four dimensions using the heat kernel regularization, we have an explicit formula:

$$\sigma(x,x') = \left[\frac{1}{8\pi^2 a^2} - \frac{1}{2\pi^2(x-x')^2} + \frac{e^{-\frac{(x-x')^2}{4a^2}}}{2\pi^2(x-x')^2}\right]^{\frac{1}{2}}$$

The story can be different for a scalar quantum field theory that does not tend to free field theory at short distances. For a $\lambda\phi^4$ theory in four dimensions, perturbation theory breaks down at short distances. We are therefore not able to determine analytically the relationship of $\sigma(x,x')$ to $|x-x'|$ for short distances. This case is of great importance, as it describes the Higgs boson of the standard model. Large scale computer simulations are needed to study this relationship. Even if the Higgs boson turns out to be a fundamental particle and there are no indications of supersymmetry, compositeness or extra dimensions, the LHC will be exciting as a probe of this non-Riemannian metric of space-time.

If the metric depends on the cut-off a, can it still have physical significance? We will see in the continuum regularization schemes, a change of the cut-off does not change the Lipschitz equivalence class of the metric. Thus the Lipschitz class of space-time should have a physical significance: instead of differentiable functions we would talk of Lipschitz functions for example. This equivalence class does change if we let the cut-off go to zero: it is different from that of Euclidean space. Thus, the functions that are Lipschitz with respect to σ are not the same as those with respect to $|x-x'|$.

If we use discrete regularization schemes (e.g., lattice) the correct notion of equivalence of metrics might be quasi-isometry. Gromov [3] used such a notion to show that groups of polynomials are discrete approximations to Euclidean space.

When non-Euclidean geometry was still new, it was useful to understand a curved metric in terms of an embedding into Euclidean space. In the same way, it is useful to understand a non-Riemannian metric such as ours by embedding into a Riemannian manifold. We will show that our metric $\sigma(x,x')$ can be thought of as the length of the geodesic in a Riemannian manifold with one extra dimension: the geodesic does not lie in the submanifold, so has a shorter length than any curve that stays within the submanifold (in particular the Euclidean straight line).

In the section II, we describe scalar quantum field theory on a lattice and in the section III, how $\sigma(x,x')$ can be defined for it. We then calculate the metric explicitly for the case of free massless scalar field on a lattice in terms of certain discrete Fourier series. This special case is related to the resistance metric of Kigami. We show in section IV that the metric cannot be induced by any Riemannian geometry. In section V we consider other regularization schemes, in particular the heat kernel method. This allows us to get an explicit form in terms of elementary functions for free field theory. It is shown that the Lipschitz equivalence class of the metric is independent of the cut-off. A first step towards understanding interacting theories is taken in section VI where we calculate the metric in the ϵ-expansion of Wilson and Fischer fixed point of critical phenomena. In section VII, we show that it is possible to embed space-time in a Riemannian manifold of one higher dimension such that our σ is the induced metric. In the last section we summarize our results and give some directions of further research. And finally, in an Appendix, we collect together some facts about metric geometry that are known in the mathematics literature, but are rarely used by physicists.

II. LATTICE SCALAR FIELD THEORY

A scalar field on a graph Γ is a function $\phi : \Gamma \to \mathbb{R}^N$. The action (or energy, depending on the physical application) is a function of the scalar field

$$S(\phi) = \frac{a^{n-2}}{2}\sum_{x\sim x'}\left[\phi(x)-\phi(x')\right]^2 + a^n\sum_x V(\phi(x)) \quad (1)$$

The first sum is over nearest neighbors in the graph and a is the distance between them[6]. V is a polynomial whose coefficients are the "bare coupling constants". Free field theory is the special case where V is a quadratic function. Massless free field theory is the case $V=0$. The expectation value of any function of the field is defined to be

$$\langle f\rangle = \frac{\int f(\phi)e^{-S(\phi)}d\phi}{\int e^{-S(\phi)}d\phi} \quad (2)$$

In particular, the correlation functions are the expectation values

$$G(x_1,\cdots x_n) = \langle\phi(x_1)\cdots\phi(x_n)\rangle$$

Sometimes it will be more convenient to work with quantities such as

$$R(x,x') = \langle[\phi(x)-\phi(x')]^2\rangle$$

related to the correlation function.

$$R(x,x') = G(x,x) + G(x',x') - 2G(x,x'). \quad (3)$$

The case of greatest interest [7] is a cubic lattice $\Omega_{a,L} = a\left(\mathbb{Z}/\Lambda\mathbb{Z}\right)^n$ with period $L = \Lambda a$. The aim of quantum field theory is to construct the continuum limit $a \to 0, L \to \infty$ such that the correlation functions have a sensible limit. In taking this limit, the coefficients of the polynomial V are to be varied as functions of the cut-off a. This program is essentially complete [8] in the case $n = 2$. Constructing non-trivial examples (i.e., with a V of degree higher than two) is very difficult in the physically interesting cases of dimensions three (for the theory of phase transitions) and four (for particle physics). Wilson and Fischer used ingenious approximations [9, 10] to understand the three dimensional case. In dimensions higher than four, such a continuum limit does not exist except for the case of a free field [11]. The case of four dimensions is marginal and a non-trivial continuum limit cannot be constructed by standard methods [12].

III. STANDARD DEVIATION METRIC

The mean deviation

$$D(x, x') = \langle|\phi(x) - \phi(x')|\rangle$$

satisfies the triangle inequality since, for each instance of ϕ.

$$|\phi(x) - \phi(x')| \leq |\phi(x) - \phi(x'')| + |\phi(x'') - \phi(x')|$$

holds. So it will hold in the average as well.

More generally

$$D_p(x, x') = [\langle|\phi(x) - \phi(x')|^p\rangle]^{\frac{1}{p}}$$

for $p \geq 1$ is a metric on space-time[13]. The most interesting is the case $p = 2$ of the standard deviation

$$\sigma(x, x') = \left[\langle|\phi(x) - \phi(x')|^2\rangle\right]^{\frac{1}{2}}. \quad (4)$$

$\sigma(x, x')$ is obviously positive and symmetric. Also, $\sigma(x, x') > 0$ if $x \neq x'$ because otherwise, every instance of a scalar field would have to take the same value at x and x'.

Each theory of matter field will define a metric on space-time. The distance is a simple concept for scalar fields. For gauge fields, it is more subtle, but gauge invariant notions do exist [14]. Reflection positivity seems to imply such a metric even for fermion fields. When the scalar field takes values in a curved target space $\phi : \Omega^n \to M$ (e.g., the nonlinear sigma model [10]) we would use the metric d_M of the target to define $\sigma(x, x') = \sqrt{\langle|\left[d_M(\phi(x), \phi(x'))\right]^2|\rangle}$.

Why could we not have defined a metric using the average of the square of the distance

$$R(x, x') = \langle(\phi(x) - \phi(x'))^2\rangle$$

itself? It is more closely related to the correlation functions (3). The point is that the square of a metric, such as $(\phi(x) - \phi(x'))^2$ does not in general satisfy the triangle inequality. (By contrast, the square root of a metric always does.) In some special cases (e.g., massless free field) the expectation value $R(x, x')$ itself is a metric. This particular case was discovered by Kigami in the context of fractals [4, 15]. But there are other probability distributions (that are not Gaussians) for which $R(x, x')$ does not satisfy the triangle inequality. Also, with regularizations other than the lattice (e.g., heat kernel method, see section below) $R(x, x')$ does not satisfy the triangle inequality. But $D_1(x, x')$ and $\sigma(x, x')$ always do. Of the two, the standard deviation is $\sigma(x, x')$ is easier to calculate, as usual.

IV. FREE SCALAR FIELD

In the special case(1), of a free field

$$V(\phi) = \frac{1}{2}m^2 \sum_x \phi(x)^2, \quad (5)$$

$\phi(x) - \phi(x')$ is a Gaussian random variable with variance $R(x, x')$ and zero mean. So we have the relation

$$\sigma = \sqrt{\frac{\pi}{2}}D. \quad (6)$$

This relation is universal for free fields: it does not depend on the cut-off procedure used (e.g., square vs triangular lattice). But the range of values of σ will depend on the cut-off [16].

A. Explicit Formula for R

The expectation values are Gaussian integrals which we can evaluate explicitly in terms of the Green's function

$$G(x, x') = \langle\phi(x)\phi(x')\rangle$$

It is the solution of the lattice Helmholtz equation,

$$[\Delta_x + m^2]G(x, x') = \delta_{\Omega_{a,L}}(x, x').$$

The lattice laplacian is the sum over nearest neighbors y for fixed x of the difference:

$$\Delta\psi(x) = \frac{1}{a^2}\sum_{y\sim x}[\psi(x) - \psi(y)]$$

(According to this definition, the eigenvalues of the operator are positive.)

The lattice delta function depends on L through periodicity;

$$\delta_{\Omega_{a,L}}(x, x') = \begin{cases} a^{-n}, & \text{if } x = x' \bmod L \\ 0 & \text{otherwise} \end{cases}$$

The discrete Fourier transform of a function is given by

$$\tilde{\psi}(p) = a^n \sum_{x\in\Omega_{a,L}} e^{-ip\cdot x}\psi(x),$$

where the wavenumber p belongs to the dual lattice

$$p \in \tilde{\Omega}_{a,L} = \Omega_{\frac{2\pi}{L},\frac{2\pi}{a}}$$

for which a, L are exchanged for their reciprocals. Note the identity

$$L^{-n} \sum_{p\in\tilde{\Omega}_{a,L}} e^{ip\cdot(x-x')} = \delta_{\Omega_{a,L}}(x,x')$$

The inverse discrete Fourier transform is then

$$\psi(x) = L^{-n}\sum_p e^{ip\cdot x}\tilde{\psi}(p)$$

Then

$$\widetilde{\Delta\psi}(p) = \tilde{\Delta}(p)\tilde{\psi}(p), \quad \tilde{\Delta}(p) = \frac{4}{a^2}\sum_{r=1}^{n}\sin^2\frac{ap_r}{2}$$

Thus

$$G(x,x') = L^{-n}\sum_{p\in\tilde{\Omega}_{a,L}} \frac{e^{ip\cdot(x-x')}}{\tilde{\Delta}(p)+m^2}$$

It follows that

$$R(x,x') = L^{-n}\sum_{p\in\tilde{\Omega}_{a,L}} \frac{4\sin^2\frac{p\cdot(x-x')}{2}}{\tilde{\Delta}(p)+m^2} \tag{7}$$

B. The Resistance Metric

In the limit $m \to 0$ of a free massless scalar field, $R(x,x')$ itself (and not only its square root) is a metric. Explicitly

$$R(x,x') = L^{-n}\sideset{}{'}\sum_{p\in\tilde{\Omega}_{a,L}} \frac{4\sin^2\frac{p\cdot(x-x')}{2}}{\tilde{\Delta}(p)}$$

the sum being over terms with $\tilde{\Delta}(p) \neq 0$. This is the resistance metric of Kigami [15], evaluated for the cubic lattice. There is a simple physical interpretation for this quantity. Imagine a network, each pair of nearest neighbors being connected by a resistor of same resistance. Then $R(x,x')$ is the effective resistance between the pair of points x, x' after all the others have been eliminated using Kirchoff's laws of current conduction. It is obvious that $R(x,x')$ is positive and symmetric. See Ref. [4] for an ingenious proof that it satisfies the triangle inequality. However, we will not use this fact.

C. The Infinite Lattice

The resistance metric is often studied [4] on sequences of graphs that tend to a fractal. It has not been studied as a metric on the more familiar graph of a cubic lattice. Perhaps the reason is that it is totally different from the Euclidean metric. Physicists [17] have already calculated the properties of resistance on cubic lattices without noting that it satisfies the triangle inequality. In the limit of a lattice of infinite period $L \to \infty$, the momentum space becomes a torus of period $\frac{2\pi}{a}$. Then we have the integral representation[18] with $m = 0$,

$$R(x,x') = \left(\frac{1}{2\pi}\right)^n \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} d^np \frac{4\sin^2\frac{p\cdot(x-x')}{2}}{\frac{4}{a^2}\sum_{r=1}^{n}\sin^2\frac{p_r a}{2}} \tag{8}$$

$$R(x,x') = \frac{a^{2-n}}{n}, \quad |x-x'| = a, \quad m = 0$$

If we consider $x - x' = (0, \cdots a, \cdots 0)$ to be along the i th direction alone and consider the sum over all the n integrals, the trigonometric terms in the numerator and denominator would cancel and the integral evaluates to a^{2-n}. Also, the nearest neighbors have the same resistance.

In the opposite limit of large Euclidean distance, the answer depends more dramatically on the dimension.

For $n = 1$, it is easy to see that the resistance metric is simply the Euclidean distance

$$R(x,x') = |x-x'|$$

We have

$$R(x,x') = \frac{1}{2\pi}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} dp \frac{\sin^2\frac{p(x-x')}{2}}{a^{-2}\sin^2\frac{pa}{2}}$$

Let $p \to \frac{2\pi}{a}p$ and $r = \frac{|x-x'|}{a}$

$$R(x,x') = a\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sin^2\pi r}{\sin^2\pi p}dp = ra = |x-x'|.$$

For $n = 2$,

$$R(x,x') \to \frac{1}{2\pi}\left[\log\frac{|x-x'|}{a} + \gamma + \frac{1}{2}\log 8 + \cdots\right], \quad |x-x'| \to \infty$$

For $n > 2$, and $|x - x'| >> a$ we can approximate

$$R(x,x') \approx \frac{C_n}{a^{n-2}} - 2G_n(x-x')$$

where $G_n(x)$ is the continuum Green's function of the Laplace operator. The constant C_n is independent of a

but depends on the method of regularization. For the lattice regularization,

$$C_n = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{4\left[\sin^2 \pi p_1 + \sin^2 \pi p_2 + \cdots \sin^2 \pi p_n\right]} d^n p$$

$$R(x,x') \approx \frac{C_n}{a^{n-2}} - \frac{C_n'}{|x-x'|^{n-2}} + \mathrm{O}\left(|x-x'|^{n-3}\right)$$

The constant C_n' is universal: it is the same in every regularization scheme, being simply related to the continuum Green's function.

$$C_n' = \frac{1}{2}\pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}-1\right)$$

When $n=3$,

$$C_3 \approx 0.505462, \quad C_3' = \frac{1}{2\pi}$$

For $n=4$ also

$$C_4 = 0.309867, \quad C_4' = \frac{1}{2\pi^2}$$

The plot shows that $\sigma(x,x') = \sqrt{R(x,x')}$ always grows with the Euclidean distance.

But the rate of growth is very slow for large distances. For nearest neighbors

$$\sigma(x,x') = a^{1-\frac{n}{2}}\sqrt{\frac{1}{n}}, \quad |x-x'| = a.$$

And for large distances

$$\sigma(x,x') \approx \sigma_n a^{1-\frac{n}{2}} - \frac{\sigma_n'}{|x-x'|^{n-2}} + \mathrm{O}\left(|x-x'|^{n-3}\right)$$

$$\sigma_n = \sqrt{C_n}, \quad \sigma_n' = a^{\frac{n}{2}-1}\frac{C_n'}{2\sqrt{C_n}}.$$

Note that the above equations is consistent with $C_n > \frac{1}{n}$. We plot (Fig. 1) the distance in dimension three, in units of the lattice spacing a. Note that the length of a path is the sum of distances between nearest neighbors along the path. Hence the geodesic distance (the length of the shortest path connecting two points) is proportional to the Euclidean distance

$$\sigma_l(x,x') = a^{-\frac{n}{2}}\sqrt{\frac{1}{n}}|x-x'| \geq \sigma(x,x')$$

The equality holds only for nearest neighbors. For large $|x-x'|$, the geodesic distance is much greater: $\sigma_l(x,x') >> \sigma(x,x')$. It is easy to understand why using the resistance model: the shortest curve is just one among possibly many paths that connect the pair of points. When resistances corresponding to the paths are combined in parallel, the effective resistance obtained is smaller than all of them.

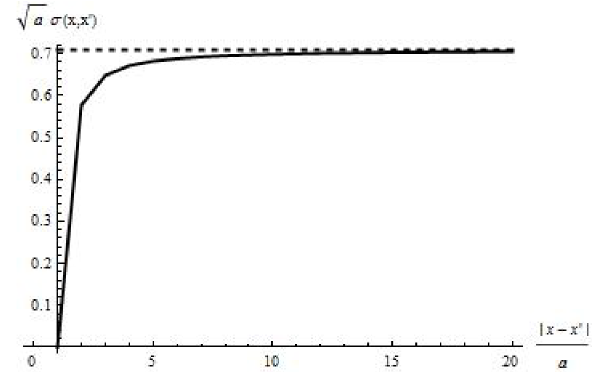

FIG. 1. $\sqrt{a}\sigma(x,x')$ is plotted as a function of Euclidean distance scaled by lattice length.

D. No Intermediate Point

We now show that when $n \geq 3$, the triangle inequality

$$\sigma(x,x') \leq \sigma(x,x'') + \sigma(x'',x')$$

can **never** be saturated unless one of the distances is zero. The distance between nearest neighbors is

$$\frac{1}{\sqrt{n}}a^{1-\frac{n}{2}}$$

So the smallest value for the r.h.s., being the sum of two non-zero distances, is twice this:

$$\frac{2}{\sqrt{n}}a^{1-\frac{n}{2}}.$$

On the other hand, the distance between any pair of points is bounded:

$$\sigma(x,x') < \sqrt{C_n}a^{1-\frac{n}{2}}$$

Now, its easy to check that

$$C_n < \frac{4}{n}.$$

Instead of an analytic proof, we can simply calculate the values in the two interesting cases numerically:

$$C_3 \approx 0.505462 < \frac{4}{3}$$
$$C_4 \approx 0.309867 < 1$$

Thus, the minimum value of the r.h.s. is greater than the maximum value of the l.h.s. and the inequality can never be saturated.

It follows that $\sigma(x,x')$ cannot be induced by any Riemannian metric.

E. Massive Scalar Field

In the case of a massive free scalar field, the asymptotic dependence on the Euclidean distance is given by the Yukawa potential between points charges exchanging a massive particle: it vanishes exponentially.

$$\sigma(x,x') \approx \sigma_n a^{1-\frac{n}{2}} - \sigma'_n \frac{e^{-m|x-x'|}}{|x-x'|^{n-2}} + \mathrm{O}\left(|x-x'|^{n-3}\right), \quad n>2 \tag{9}$$

The constants σ_n, σ'_n are as above.

V. OTHER REGULARIZATION SCHEMES

Although we have used the lattice definition of scalar field theories, universality implies that other regularization schemes will suffice. For example, we could use a sharp momentum cut-off or a smooth momentum cut-off or a heat kernel method.

We saw that in the limit $L \to \infty$, the momentum variables takes values on a torus of period $\frac{2\pi}{a}$ in each direction. Thus, the lattice regularization amounts to replacing a potentially divergent integral by

$$\int d^n p f(p) \to \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^n p}{(2\pi)^n} f'(p)$$

where $f'(p)$ agrees with $f(p)$ for $|p| << \frac{1}{a}$. A typical example we encountered above is

$$\frac{1}{p^2} \to \frac{1}{a^{-2}\sum_{r=1}^{n} 4\sin^2\frac{ap_r}{2}}$$

Another method, commonly used in scalar QFT is

$$\int d^n p f(p) \to \int d^n p K\left(a|p|\right) f(p)$$

where $K(\xi) \approx 1$ for $\xi << 1$ and is zero for $\xi >> 1$. The advantage is that this preserves rotation invariance which the lattice breaks. Examples are

$$K(a|p|) = \frac{1}{1+a^2p^2}, \quad \text{Pauli}-\text{Villars}$$
$$K(a|p|) = \Theta(a|p|<1), \ \text{Sharp Cut}-\text{off}$$
$$K(a|p|) = e^{-a^2p^2}, \quad \text{Heat Kernel}$$

Polchinski [19], among others, has advocated for a smooth function that is one for $|p| < a^{-1}$ and zero for $|p| > a^{-1}$.

The advantage of these schemes is that the underlying space-time continues to be $\mathbb{R}^n$, but with a possibly different measure of integration on its dual space (momentum space). Our proposal would be to determine a metric on space-time from the standard deviation computed using this regularized measure in momentum space. Again we begin with the free field,

$$G(x,x') = \int \frac{d^n p}{(2\pi)^n} K(a|p|) \frac{1}{p^2+m^2} e^{ip\cdot(x-x')}$$
$$R(x,x') = \int \frac{d^n p}{(2\pi)^n} K(a|p|) \frac{4\sin^2\frac{p\cdot(x-x')}{2}}{p^2+m^2}$$

The explicit answer seems simplest for the heat kernel regularization. With $m=0$, we get an answer in terms of the incomplete Gamma function, $\Gamma(\nu,z) = \int_z^\infty t^{\nu-1}e^{-t}dt$:

$$G(x,x') = \frac{\pi^{\frac{n}{2}}}{4|x-x'|^{n-2}} \left[\Gamma\left(\frac{n-2}{2}\right) - \Gamma\left(\frac{n-2}{2}, \frac{(x-x')^2}{4a^2}\right)\right] \tag{10}$$

To see this,

$$\begin{aligned} G(x,x') &= \int \frac{d^n p}{(2\pi)^n} e^{-a^2p^2} \frac{1}{p^2} e^{ip\cdot(x-x')} \\ &= \int_{a^2}^\infty dt \int \frac{d^n p}{(2\pi)^n} e^{-tp^2} e^{ip\cdot(x-x')} \\ &= \int_{a^2}^\infty dt \frac{e^{-\frac{(x-x')^2}{4t}}}{(4\pi t)^{\frac{n}{2}}} \end{aligned}$$

which we evaluate. a plays the same role as the nearest neighbor distance in the lattice regularization. It is the short distance cut off.

In particular,

$$G(x,x) = \frac{2^{1-n}\pi^{-\frac{n}{2}}a^{2-n}}{n-2} \tag{11}$$

$$R(x,x') = \frac{2^{2-n}\pi^{-\frac{n}{2}}}{n-2}a^{2-n} - \frac{1}{2}\frac{|x-x'|^{2-n}}{\pi^{\frac{n}{2}}} \left[\Gamma\left(\frac{n-2}{2}\right) - \Gamma\left(\frac{n-2}{2}, \frac{(x-x')^2}{4a^2}\right)\right] \tag{12}$$

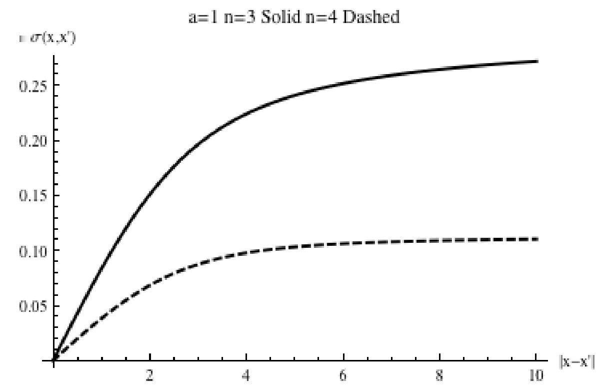

FIG. 2. This plot shows the dependence of σ with Euclidean distance in dimensions 3 and 4.

For even n the expression is more elementary:

$$R(x,x') = \frac{1}{8\pi^2 a^2} - \frac{1}{2\pi^2 (x-x')^2} + \frac{e^{-\frac{(x-x')^2}{4a^2}}}{2\pi^2 (x-x')^2}, \quad n=4. \tag{13}$$

It follows that our metric $\sigma(x,x') = \sqrt{R(x,x')}$ is proportional to the Euclidean distance for small $|x-x'|$:

$$R(x,x') = \frac{2^{-n}\pi^{-\frac{n}{2}}(x-x')^2}{na^n} + O\left(|x-x'|^4\right)$$
$$\sigma(x,x') = \frac{2^{-\frac{n}{2}}\pi^{-\frac{n}{4}}}{\sqrt{n}a^{\frac{n}{2}}}|x-x'| + O\left(|x-x'|^2\right)$$

We plot (Fig. 2) the metric as a function of Euclidean distance, in units with $a=1$ for $n=3,4$.

Note that $R(x,x')$ would not satisfy the triangle inequality, being proportional to the *square* of the Euclidean distance for small $|x-x'|$. This confirms that the correct choice of metric is the standard deviation, not the variance of the scalar field.

For large $|x-x'|$,

$$R(x,x') = \frac{2^{2-n}\pi^{-\frac{n}{2}}}{(n-2)a^{n-2}} - \frac{\pi^{-\frac{n}{2}}}{2}\Gamma\left(\frac{n}{2}-1\right)\frac{1}{|x-x'|^{n-2}} + \cdots \tag{14}$$

Thus, in the heat kernel regularization

$$C_n = \frac{2^{2-n}\pi^{-\frac{n}{2}}}{n-2}, \quad C'_n = \frac{1}{2}\pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}-1\right)$$

As noted earlier C_n is not universal but C'_n is. To compare the numerical values

$C_3 \approx 0.0897936$ vs 0.505462 for lattice
$C_4 \approx 0.0126651$ vs 0.309867 for lattice

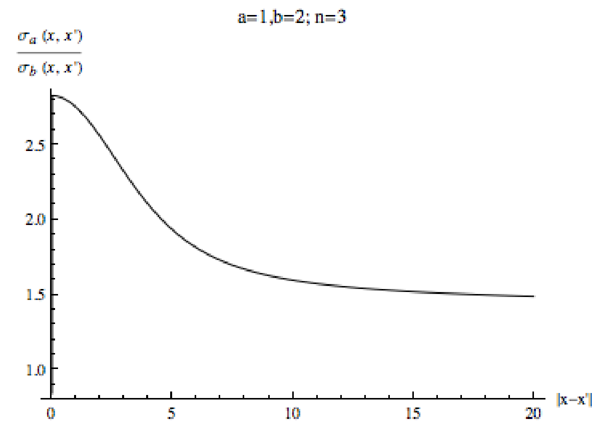

FIG. 3. This plot shows the ratio of distance using two cut offs on the Euclidean distance.

A. Lipschitz Equivalence

How does the change of the cut-off affect the geometry defined by σ? We now show that for the free field, there are constants k_1, k_2 such that

$$0 < k_1(a,b) \le \frac{\sigma_a(x,x')}{\sigma_b(x,x')} \le k_2(a,b) < \infty \tag{15}$$

That is, k_1, k_2 depend on the cutoffs a, b but not on the points $x, x' \in \mathbb{R}^n$. This means that the two metrics σ_a and σ_b on $\mathbb{R}^n$ are Lipschitz equivalent. The proof for the free massless theory uses the explicit formula to show that when $a < b$, the ratio $\frac{\sigma_a(x,x')}{\sigma_b(x,x')}$ takes its largest value for $x = x'$ and its smallest value as $|x-x'| \to \infty$. Fig. 3 illustrates this fact. Thus,

$$\left(\frac{b}{a}\right)^{\frac{n}{2}-1} \le \frac{\sigma_a(x,x')}{\sigma_b(x,x')} \le \left(\frac{b}{a}\right)^{\frac{n}{2}}, \quad a < b, \quad n > 2$$

We conjecture that this bi-Lipschitz inequality (15) holds for all renormalizable scalar QFT in the continuum regularization schemes. The actual values of the Lipschitz constants might change, however. Thus, we propose that although the metric itself depends on the cut-off, its Lipschitz equivalence class is universal. This makes some sense as Lipschitz equivalence for metric spaces is analogous to diffeomorphisms for manifolds.

VI. BEYOND FREE FIELDS: WILSON-FISHER

When $3 < n < 4$ the scalar field theory with potential

$$V(\phi) = \frac{1}{2}m^2|\phi|^2 + \frac{1}{4}\lambda|\phi|^4 \tag{16}$$

has a fixed point of the renormalization group. The momentum integrals defining the scalar theory make sense even for fractional values of n, even though the case, $n = 3$ is the case of physical interest. The case $n = 3, N = 1$ (N being the number of components of ϕ) for example, describes the critical point of a liquid and a gas. The connection of this to fractals remains mysterious. A more detailed study of quantum field theory on fractals is called for. We took a first step in this direction ourselves [20].

At this Wilson-Fisher fixed point, the Green's function is

$$G(x-x') = \int \frac{e^{ip\cdot(x-x')}}{p^{2-\eta}} K\left(a|p|\right) \frac{d^3p}{(2\pi)^3}$$

where the critical exponent η can be calculated in the ϵ-expansion (Section 25.5 of Ref. [10])

$$\eta = \frac{N+2}{2(N+8)^2}\epsilon^2 + \mathrm{O}\left(\epsilon^3\right), \quad \epsilon = 4-n. \tag{17}$$

This quantity is universal and has been calculated to much higher precision. (See [21] for the result up to order ϵ^5). In this example, the metric

$$\sigma^2(x,x') = 2G(0) - 2G(x-x')$$

becomes for $|x-x'| >> a$,

$$\sigma(x,x') = \frac{\sigma_3}{a^{\frac{1+\eta}{2}}} - \frac{\sigma_3'}{|x-x'|^{1+\eta}} + \cdots \tag{18}$$

That is, even when $n = 3$, it scales as if the dimension of space were a little bit higher than three. Again, the diameter of space is finite and the next-to-leading order correction contains the Green's function of physical interest. Also, for small $|x-x'|$ our $\sigma(x,x')$ is proportional to the Euclidean metric.

VII. EMBEDDING IN A RIEMANNIAN MANIFOLD

Our standard deviation is *not* a geodesic metric. The length metric of σ is proportional to the Euclidean metric, which is typically larger than σ. In the lattice regularization, the length of any curve is just the number of edges along it: the supremum above is achieved when each segment connects nearest neighbors. In the heat kernel regularization, we saw that when x, x' are close enough, $\sigma(x,x')$ is proportional to $|x-x'|$. So the length of any curve according to σ will be, up to a constant multiple, its Euclidean length.

Thus, the distance perceived by a quantum model of propagation is drastically different from the classical model. Classically, the particle simply takes the shortest path (which is also the path of least action). Quantum mechanically, we must sum over all the paths; longer paths are simply less probable. Long paths can dominate the sum, if the sheer number of long path makes up for their smaller probability. This is what happens on cubic

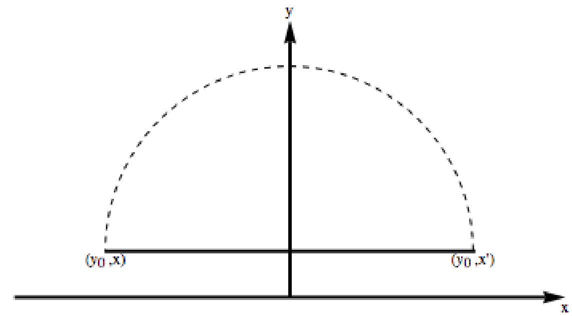

FIG. 4. The trajectory in the $x\,y$ plane is shown by dotted line.

lattices of dimension $n \geq 2$ as is easy to verify using the explicit form we found above. Again, this illustrates how far from being a geodesic metric σ is.

We will now show that we can embed $\mathbb{R}^n$ into a Riemannian manifold of one dimension higher such that the length of the geodesic connecting (x, x') in the ambient space is equal to $\sigma(x,x')$. The extra dimension provides a "short-cut" that allows us to realize our metric as a geodesic distance. In the Appendix we give an example involving the chord length of circles that illustrates this situation.

Consider a Riemannian metric on $\mathbb{R}^n \times \mathbb{R}^+$

$$ds^2 = h^2(\rho)d\rho^2 + \rho^2 dx^i dx^i \tag{19}$$

These co-ordinates are chosen to make later expressions simpler.

As an example, $n = 1$ and $h(\rho) = \frac{1}{\rho}$ is one description of the metric of constant negative curvature on a hyperboloid. The substitution $y = \frac{1}{\rho}$ turns this into the familiar Poincare' metric

$$ds^2 = \frac{dy^2 + dx^2}{y^2}.$$

To continue with this example, (which we will not use directly, but is similar enough to those we will use) the real line is a sub-manifold, the line of constant $y = y_0$. There are two induced distances on this sub-manifold: we can ask for the minimum length of curves that lie on the sub-manifold: this is just the Euclidean metric on the real line. Or the minimum over all curves that start and end at points (y_0, x) and (y_0, x') on the line, but in between can lie anywhere on the plane (Fig. 4). It is clear that the latter can be smaller than the Euclidean distance. It is in fact a non-geodesic metric on the real line, whose length metric is the Euclidean metric.

The geodesic (19) minimizes

$$\int \sqrt{h^2(\rho) + \rho^2 \left(\frac{dx}{d\rho}\right)^2} d\rho \tag{20}$$

We need to find the geodesic that connects the (ρ_0, x') with (ρ_0, x). By a rotation and translation we can choose $x = (\frac{r}{2}, 0, \cdots 0)$ and $x' = (-\frac{r}{2}, 0, \cdots 0)$, where $r = |x - x'|$ is the Euclidean distance. The geodesic will then lie in the (ρ, x^1) plane. The Euler-Lagrange equation implies that

$$\frac{\rho^2}{\sqrt{h^2(\rho) + \rho^2 \left(\frac{dx}{d\rho}\right)^2}} \left(\frac{dx}{d\rho}\right) = \rho_1$$

for some constant ρ_1. A moments thought will show that the geodesic is reflection symmetric around $x = 0$ and that $\frac{d\rho}{dx} = 0$ at $x = 0$. Thus $\frac{dx}{d\rho} = \infty$ at $x = 0$ and we conclude from the above equation that ρ_1 is simply the point at which $x = 0$.

$$\frac{dx}{d\rho} = \pm \frac{\rho_1 h(\rho)}{\rho\sqrt{\rho^2 - \rho_1^2}} \tag{21}$$

Thus

$$x(\rho) = \pm \int_{\rho_1}^{\rho} \frac{\rho_1 h(\rho) d\rho}{\rho\sqrt{\rho^2 - \rho_1^2}}$$

It follows from

$$h^2(\rho) + \rho^2 \left(\frac{dx}{d\rho}\right)^2 = \frac{h^2(\rho)\rho^2}{\rho^2 - \rho_1^2}$$

that

$$r = 2 \int_{\rho_1}^{\rho_0} \frac{\rho_1 h(\rho) d\rho}{\rho\sqrt{\rho^2 - \rho_1^2}} \tag{22}$$

$$L = 2 \int_{\rho_1}^{\rho_0} \frac{\rho h(\rho) d\rho}{\sqrt{\rho^2 - \rho_1^2}} \tag{23}$$

Together eqns. (22,23) give parametrically via ρ_1, the length of the geodesic in terms of the Euclidean distance r. When $r \to 0$, we have also $\rho_0 \approx \rho \approx \rho_1$ and

$$L \approx \rho_0 r \tag{24}$$

The inverse problem of determining $h(\rho)$ given $L(r)$ looks hard. But recall that only the asymptotic behavior of $\sigma(r)$ as $r \to \infty$ is universal. So we should be able to find an $h(\rho)$ within the same universality class by looking at the asymptotic behavior. What should $h(\rho)$ be in order that

$$L(r) \sim C - \frac{C'}{r^{n-2+\eta}} + \dots$$

This is the behavior of the metric $\sigma(x, x')$ in the continuum regularization we derived in the last section.

The equation (24) determines ρ_0 in terms of the cut-off.

$$\rho_0 = \frac{2^{\frac{-n}{2}} \pi^{-\frac{n}{4}}}{\sqrt{n} a^{\frac{n}{2}}} \tag{25}$$

Suppose $h(\rho) \sim h_1 \rho^{-\mu}$ as $\rho \to 0$. Then we get

$$r \approx 2h_1 \rho_1 \frac{\rho_1^{-1-\mu} - \rho_0^{-1-\mu}}{\mu + 1}$$
$$r \approx \frac{2h_1}{\mu + 1} \rho_1^{-\mu}$$

Also,

$$L \approx 2h_1 \frac{\rho_0^{1-\mu} - \rho_1^{1-\mu}}{1 - \mu}$$

so that

$$L \approx \frac{2h_1}{1-\mu} \rho_0^{1-\mu} - \frac{2h_1}{1-\mu} \left[\frac{\mu + 1}{2h_1}\right]^{\frac{\mu-1}{\mu}} r^{1-\frac{1}{\mu}}$$

This gives us what we want if

$$1 - \frac{1}{\mu} = -(n - 2 + \eta)$$
$$\mu = \frac{1}{n - 1 + \eta}$$

The resulting metric

$$ds^2 \approx h_1^2 \rho^{-\frac{2}{n-1+\eta}} d\rho^2 + \rho^2 dx^i dx^i$$

has curvature going to $-\infty$ as $\rho \to 0$ when $(n+\eta-1) \geq 1$. The case $n = 2, \eta = 0$ is marginal in that we get a metric of constant negative curvature asymptotically.

VIII. CONCLUSIONS AND FURTHER DIRECTIONS

Our main point is that a non-Riemannian metric on space-time emerges from scalar quantum field theory. In dimensions $n > 2$ even the free field induces a very different metric from Euclidean space: space-time has finite diameter for example. Yet, the length of any curve as defined by this metric is (up to a constant) the usual Euclidean length. Thus, classical measurements are unaffected. We calculated the metric explicitly in free field theory and also took a step towards understanding the interactions by calculating it for the Wilson-Fischer fixed point. It would be of interest to also study the case of $\lambda\phi^6$ interactions, as they describe multi-critical points and marginal perturbations.

It is of great interest to calculate the metric (perhaps exactly) for the case of asymptotically free scalar quantum field theories in two dimension (e.g., the nonlinear sigma model). We would expect that there are logarithmic corrections to the length of a curve, a first indication of non-Riemannian geometry. Also, there are many two dimensional scalar field theories that are exactly solvable; can we get an exact formula for the metric in some of them?

But by far the question of greatest interest is that of $\lambda\phi^4$ theory in four dimensions. Since perturbation theory breaks down at short distances, it is not possible to study this question analytically. A numerical simulation is needed to understand how $\sigma(x, x')$ depends on, or differs from, the Euclidean distance $|x - x'|$. Do they even define the same topology? Is the Euclidean length still the length induced by σ? How does the embedding in section VII of two dimensional space-time into three dimensional hyperbolic space change in the presence of interactions? Is there a connection to the AdS-CFT correspondence?

These questions are especially urgent in view of the expected discovery of the Higgs boson at the LHC. If such a fundamental scalar field exists, and there is no evidence of supersymmetry, the metric geometry induced by it might play a role in understanding the hierarchy problem of the standard model. In any case, as a natural property of the scalar quantum field, it is of interest to study σ in numerical simulation of lattice scalar field theory.

IX. ACKNOWLEDGEMENT

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X. APPENDIX : METRIC GEOMETRY

A metric on a set X is a function $d : X \times X \to \mathbb{R}$ such that

- $d(x, x) = 0$
- $d(x, x') > 0$ if $x \neq x'$, separation
- $d(x, x') = d(x', x)$, symmetry
- $d(x, x') \leq d(x, x'') + d(x'', x')$, the triangle inequality.

The most familiar example is the Euclidean metric on $\mathbb{R}^n$.

$$|x - x'| = \sqrt{\sum_{i=1}^{n} (x^i - x'^i)^2}$$

This metric is so ingrained in us that we might forget that the actual metric of space-time should be deduced by physical measurements and is not self-evidently Euclidean. Often (e.g., numerical simulations of scalar field theory, solution of PDEs) we have to approximate space by a discrete lattice $\Omega^n_{a,L} = a\,(\mathbb{Z}/\Lambda\mathbb{Z})^n$ with nearest neighbor spacing a and period $L = \Lambda a$ in each direction. Then the Euclidean metric is approximated by the length of the shortest path connecting two points on the lattice

$$l(x, x') = \sqrt{\sum_{i=1}^{n} (x^i - x'^i \bmod L)^2}. \tag{26}$$

The square root of a metric is again a metric. More generally, if $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a concave function $f''(x) < 0$ with $f(0) = 0$, then we can construct from a given metric d a new one $\tilde{d}(x, x') = f((d(x, x'))$. But the square of a metric is not always a metric. For example, the square of the Euclidean metric is not a metric: the sum of the sides of an obtuse triangle is not greater than the square of the side opposite.

The length of the shortest curve (geodesic) connecting two points in a Riemannian manifold is a metric. Any metric arising as the length of geodesics has the intermediate property: given any pair of points (x, x') there is another x'' (a midpoint) that saturates the triangle inequality:

$$d(x, x') = d(x, x'') + d(x'', x')$$

Any point x'' lying along the shortest geodesic connecting x, x' would suffice. There are metrics that do not have this property. According to them, the distance between two points can be shorter than the length of every curve connecting them. Obviously, such metrics are non-Riemannian. This is precisely the case of interest to us.

Although the concept of derivative does not make sense in general on a metric space (for that we would need a differential manifold), Lipschitz functions are the analogue of differentiable functions. A function $f : X \to Y$ between metric spaces is said to be k-Lipschitz if

$$\frac{d_Y(f(x), f(x'))}{d_X(x, x')} < k$$

for all $x, x' \in X$. Roughly speaking, the magnitude of the derivative is less than k. Two metric spaces are Lipschitz equivalent if there are continuous, one-to-one Lipschitz maps in each direction which are inverses of each other. Lipschitz equivalence is roughly analogous to diffeomorphisms between manifolds.

A. Length of Curves

Given a metric we can define the length of a curve as the largest sum of the length of line segments. In more detail, a curve $\gamma : [0, T] \to X$ can be broken up into segments

$$0 \equiv t_0 < t_1 < t_2 < \cdots t_k < T \equiv t_{k+1}$$

The sum of the chord lengths

$$\sum_{i=1}^{k+1} d(\gamma(t_{i-1}), \gamma(t_i))$$

can be thought of as an approximation to its length. The actual length of the curve is the least upper bound of all such approximations; i.e., avoiding all the "short-cuts" made by the chords:

$$l[\gamma] = \sup_{0<t_1<\cdots t_k<T} \sum_{i=1}^{k+1} d(\gamma(t_{i-1}), \gamma(t_i))$$

The length of a continuous curve can be infinite. There are well known examples of continuous curves (e.g., Koch curve) with an infinite length in the Euclidean metric. Also, suppose we define $d(x, x') = |x - x'|^{\frac{1}{2}}$, the square root of the Euclidean distance. Then the length of every straight-line segment is infinite!

For the familiar case of a differentiable curve in a Riemannian manifold, it is not hard to verify that this agrees with the usual definition

$$l[\gamma] = \int_0^T \sqrt{g_{\gamma_t}\left(\dot{\gamma(t)}, \dot{\gamma(t)}\right)} dt \tag{27}$$

The reason is that, for short enough segments, the chord length is approximated by the length of the tangent vector. It is not hard to come up with continuous but not differentiable curves of infinite length.

B. Length Metric

Given a metric d we can often construct from it a (possibly distinct) length metric $d_l(x, x')$ as the greater lower bound of the lengths of all the curves that connect x to x'. (This construction could fail if the length of every continuous curve is infinite, or if there is no greater lower bound.)

A metric is said to be *geodesic* (also called a interior space or intrinsic metric) if this the one we started with: $d_l(x, x') = d(x, x')$.

The Euclidean distance is an example of a geodesic metric. Any length metric is itself a geodesic metric. That is, $(d_l)_l = d_l$ for any d. For more on these matters see Ref. [2].

An example of a non-geodesic metric (Fig. 5) is the length of a chord connecting two points on a circle:

$$d(\theta, \theta') = 2 \sin \frac{|\theta - \theta'|}{2}$$

where $0 \leq \theta \leq 2\pi$ is the usual polar co-ordinate. For small angles this agrees with the arc-length

$$d(\theta, \theta') \approx |\theta - \theta'|.$$

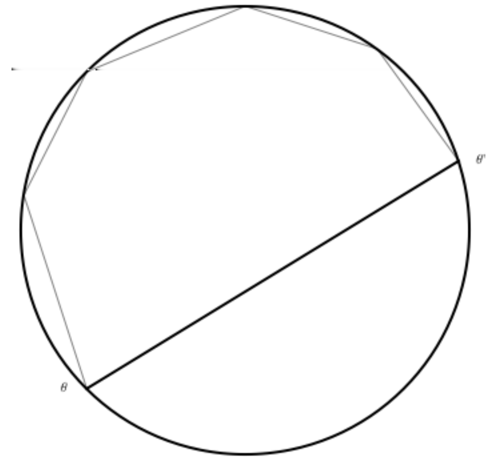

FIG. 5. The diagram shows how tangents are approximated by chord lengths.

Therefore if we break-up an arc into small segments and add up the lengths we will get the arc-length; i.e., the length metric of d is just the arc-length

$$d_l(\theta, \theta') = |\theta - \theta'|.$$

But in general

$$d(\theta, \theta') < d_l(\theta, \theta')$$

Note that although the chord length is not a geodesic metric on the circle, it is the length of a geodesic in the plane in which the circle is embedded. We showed that the standard deviation metric of a scalar field theory can be similarly realized as the geodesic length in a space of one dimension higher.

C. Triangle Inequality for Standard Deviation Metric

Suppose that a_i, b_i, c_i (for some finite range of the index i) are positive numbers satisfying the inequality $a_i \leq b_i + c_i$. Then it is obvious that the weighted averages $\langle a \rangle = \frac{\sum_i a_i w_i}{\sum_i w_i}$ also satisfy $\langle a \rangle \leq \langle b \rangle + \langle c \rangle$. More generally, the L^p-averages for $p \geq 1$,

$$\langle a \rangle_p = \left[\frac{\sum_i a_i^p w_i}{\sum_i w_i} \right]^{\frac{1}{p}} \tag{28}$$

satisfy

$$\langle a \rangle_p \leq \langle b \rangle_p + \langle c \rangle_p.$$

To see this, simply note that $\langle a \rangle_p \leq \langle b + c \rangle_p$ by monotonicity; the rest follows by the fact that the L^p-norm satisfies the triangle inequality.

If we replace the discrete average above by an integral with respect to a probability measure $e^{-S(\phi)}d\phi$, the inequality continues to hold. For positive functions,

$$a(\phi) \leq b(\phi) + c(\phi) \implies \langle a\rangle_p \leq \langle b\rangle_p + \langle c\rangle_p.$$

$$\langle a\rangle_p = \left[\frac{\int e^{-S(\phi)} a^p(\phi) d\phi}{\int e^{-S(\phi)} d\phi}\right]^{\frac{1}{p}} \tag{29}$$

These facts are useful for us because they show that the L^p average of a metric is also a metric. We just have to choose

$$a(\phi) = |\phi(x) - \phi(x')|, b(\phi) = |\phi(x) - \phi(x'')|, c(\phi) = |\phi(x'') - \phi(x')|$$

Thus $\sigma(x, x') = \sqrt{\langle(\phi(x) - \phi(x'))^2\rangle}$ satisfies the triangle inequality.

We thank L. Gross for illuminating this point.

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