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# Confronting Seiberg's Duality with $r$ Duality in $\mathcal{N} = 1$ Supersymmetric QCD

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## Abstract

Systematizing our results on  $r$  duality obtained previously we focus on comparing  $r$  duality with the generalized Seiberg duality in the  $r$  vacua of  $\mathcal{N} = 2$  and  $\mathcal{N} = 1$  super-Yang–Mills theories with the  $U(N)$  gauge group and  $N_f$  matter flavors ( $N_f > N$ ). The number of condensed (s)quarks  $r$  is assumed to be in the interval  $\frac{2}{3}N_f < r \leq N$ . To pass to  $\mathcal{N} = 1$  we introduce an  $\mathcal{N} = 2$  -breaking deformation, a mass term  $\mu$  for the adjoint matter, eventually decoupling the adjoint matter in the limit of large  $\mu$ . If one starts from a large value of the parameter  $\xi \sim \mu m$ , where the original theory is at weak coupling, and let  $\xi$  decrease one hits a crossover transition from weak to strong coupling (here  $m$  is a typical value of the quark masses). Below this transition the original theory is described in terms of a weakly coupled infrared-free  $r$  dual theory with the  $U(N_f - r)$  gauge group and  $N_f$  light quark-like dyon flavors. Dyon condensation leads to confinement of monopoles, defying a naive expectation of quark confinement. The quarks and gauge bosons of the original theory are in an “instead-of-confinement” phase. The  $r$  and Seiberg dualities are demonstrated to coincide in the  $r = N$  vacua. In the  $\frac{2}{3}N_f < r < N$  vacua two dualities do not match. In this window Seiberg's dual is at strong coupling while our  $r$ -dual model is at weak coupling. Thus, we can speak of triality. Seiberg's dual solution at weak coupling reappears again at  $r < N_f - N < \frac{1}{3}N_f$ .

# 1 Introduction

The discovery of the Seiberg duality [1, 2, 3] was a major breakthrough in  $\mathcal{N} = 1$  Yang–Mills theories at strong coupling, with far reaching consequences both in field theory and string theory. In this paper we will explore interrelations between the Seiberg duality and a novel, recently discovered  $r$  duality, in those situations where they overlap.

The original Seiberg duality in Yang–Mills theories with matter was most useful inside the conformal window, in the conformal regime. Our prime interest is in theories with confinement. The initial impetus for explorations of confinement in supersymmetric Yang–Mills was given by the Seiberg–Witten solution [4, 5] revealing condensation of monopoles [6] in the monopole vacua of  $\mathcal{N} = 2$  supersymmetric QCD. The mechanism of string formation and confinement obtained in [4, 5] is essentially Abelian [7, 8, 9, 10].

The non-Abelian gauge group (say,  $SU(2)$ ) is broken down to an Abelian subgroup at a high scale by condensation of the adjoint scalars. An effective Abelian low-energy theory ensues. The monopole condensation and formation of confining flux tubes (strings) occurs in this effective Abelian theory.

Within Seiberg–Witten solution it remained unclear in which way a confining scenario could work in  $\mathcal{N} = 1$  QCD, where there are no adjoint scalars and no dynamical Abelianization. Attempts to extrapolate the line of reasoning of [4, 5] to  $\mathcal{N} = 1$  QCD were hindered due to the fact that the low-energy theory in the monopole vacua becomes strongly coupled and untreatable by known methods.

In a bid to uncover a non-Abelian implementation of confinement we passed to the quark vacua of  $\mathcal{N} = 2$  supersymmetric QCD with the  $U(N)$  gauge group and  $N_f$  flavors ( $N_f > N$ ). In this setting not only non-Abelian strings were constructed [11, 12, 13] but, as an additional bonus, continuation to  $\mathcal{N} = 1$  SQCD became possible [14, 15, 16]. To this end we deformed  $\mathcal{N} = 2$  SQCD by adding a mass term  $\mu$  for the adjoint matter. On the way from small to large  $\mu$  an “instead-of-confinement” phase sets in. We found a crossover (in the Fayet–Iliopoulos [17] parameter  $\xi$ ), a transition that takes us from weak to strong coupling in  $\mathcal{N} = 1$  SQCD, and established a dual (weakly coupled) theory in the regime where the original  $\mathcal{N} = 1$  SQCD is strongly coupled. Thus, we observed what can be called  $r$  duality in  $\mathcal{N} = 1$ .

To be more exact, in our previous paper [16] in which all necessary technical work was carried out, we explored the  $r$  vacua of the theory, with<sup>1</sup>

$$\frac{2}{3}N_f < r \leq N. \quad (1.1)$$

This explains the origin of the term,  $r$  duality. At the same time, the original Seiberg duality (formulated in [1, 2] in the monopole  $r = 0$  vacua) can be generalized to  $r$  vacua [18] which survive in passing from  $\mathcal{N} = 2$  to our basic  $\mathcal{N} = 1$  model at large but finite

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<sup>1</sup>Our definition of  $r$  refers to the domain of large quark masses, see Sec. 2. It will become clear in Sec. 7 that effectively  $r$  depends on the quark masses.

$\mu$ .<sup>2</sup> Further explorations of the generalized Seiberg duality were undertaken in [19]. In these works classical vacua were identified – the vacua that correspond to Seiberg dual description.

Thus, our  $r$  duality in the  $r$  vacua can (and should) be compared with Seiberg’s duality. Are they identical or complementary? How can they coexist?

These are the questions we address here building on the technical work carried out previously. We will prove that at  $r = N$  both dualities present one and the same description. This is not always the case, however. In the window  $\frac{2}{3}N_f < r < N$  Seiberg’s dual is a model at strong coupling and, thus, is of a limited use from the standpoint of description of low-energy physics. At the same time, our  $r$ -dual model is at weak coupling (in fact, infrared free), and thus fully describes low-energy physics. In this window we can speak of detection of a triality, conceptually similar to that found in [20] in  $SO(N)$  model: two of the dual models in a triplet are strongly coupled while the third one is weakly coupled.

We will argue that among the Seiberg dual solutions found in [19] the ones that are at weak coupling refer to the following domain of  $r$ :

$$\begin{aligned} r &< \tilde{N}, \\ \tilde{N} &\equiv N_f - N. \end{aligned} \tag{1.2}$$

To explain the interrelation between our  $r$  duality and Seiberg’s duality it is instructive to look at Fig. 1. Our derivation [16] based on exact results [4, 5] for  $\mathcal{N} = 2$ , with the

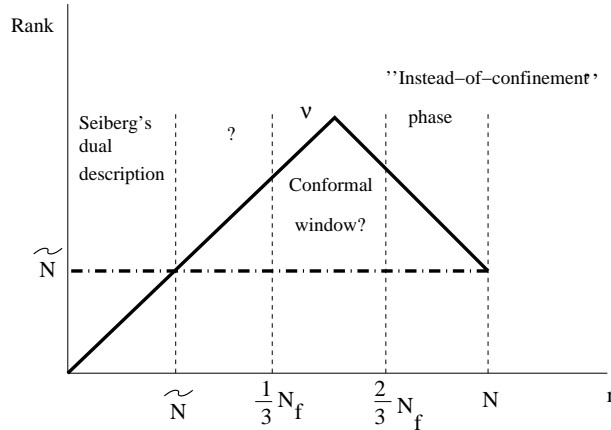


Figure 1: Rank of the dual gauge groups in the triality triplet is plotted as a function of  $r$ . For  $r$  duality this rank =  $\nu$  (solid line, see Eq. (2.1)) while for the Seiberg duality the rank of the dual gauge group is  $\tilde{N}$  for all  $r$ , dashed-dotted line. The domains of the weakly coupled Seiberg’s dual (the leftmost strip) and so far established  $r$  duality, i.e. “instead-of-confinement” phase (the rightmost strip) do not overlap except the fact that the identical coincidence between Seiberg’s dual and ours occurs at  $r = N$ .

subsequent (theoretically controlled) continuation to  $\mathcal{N} = 1$ , refers to the rightmost strip.

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<sup>2</sup>At  $\mu = \infty$  only the  $r = 0$  monopole vacua remain, while others become run-away vacua.

Our solution is fully controllable, and the dual model we get is at weak coupling, while Seiberg’s duality in this domain is not at weak coupling (except its very boundary,  $r = N$ ). Weak coupling regime in the Seiberg’s dual theory refers to the leftmost strip. What lies between these two strips?

Strictly speaking, today we do not know for certain. One can present certain speculations reflected in the central area in Fig. 1, see also [16]. These speculations go beyond the scope of the present paper. One can consider this question as a task for a future investigation.

Another problem for a future analysis is interpreting  $r$  duality in the framework of strings/branes, in the spirit it had been done with the Seiberg duality.

The paper is organized as follows. Section 2 highlights the main points of our analysis. In Sec. 3 we briefly describe the basic theory we work with,  $\mu$ -deformed  $\mathcal{N} = 2$  SQCD. In Sec. 4 we review  $r$  duality and “instead-of-confinement” mechanism in the  $r$  vacua. In Sec. 5 we review  $r$  duality at large  $\mu$  in the  $\mathcal{N} = 1$  SQCD limit. In Sec. 6 we describe the generalized Seiberg duality and compare it with our  $r$  duality in the  $r = N$  vacuum. Section 7 presents an analysis of the results obtained in [19]. In this section we establish that the Seiberg dual solutions found in [19] are at weak coupling only in the domain (1.2), i.e. at  $r < N$ . In Sec. 8 we confront our  $r$  duality with that of Seiberg in the  $r < N$  vacua. We argue that Seiberg’s duality is not implemented at weak coupling at  $\frac{2}{3}N_f < r < N$ , while the  $r$  duality is. Finally, in Appendix a more general  $\mu$  deformation is considered.

## 2 Analysis outline and main statements

As was, mentioned, our starting point is  $\mathcal{N} = 2$  SQCD, with the  $U(N)$  gauge group, in which we choose vacua in a judicious way. First we treat it at large values of an effective Fayet–Iliopoulos (FI) parameter  $\xi$ , namely,  $\xi \sim \mu m$ , where  $m$  is a generic quark mass. At large  $m$  we arrange  $r$  quark flavors to condense. This is our definition of the parameter  $r$ . In fact, the number of condensed quarks can depend on  $m$ , see Sec. 7 for details.

In the large- $m$  vacuum with  $r$  condensed quarks the effective low energy-theory with the gauge group  $U(r) \times U(1)^{N-r}$  is at weak coupling.

A global color-flavor locked symmetry survives in the limit of the equal quark masses. At large  $\xi$  this theory supports non-Abelian flux tubes (strings) [11, 12, 13, 21] (see also [22, 23, 24, 25] for reviews). It is the formation of these strings that ensures confinement of monopoles. Monopoles manifest themselves as two-string junctions. The distinction between the  $r < N$  vacua and that with  $r = N$  is that for  $r < N$  one  $U(1)$  factor of the  $U(N)$  gauge group always remains unbroken [26]. Thus, in this case, long-range forces are always present.

Exploring these vacua we established an  $r$ -duality. Upon reducing the  $\xi$  parameter the theory under consideration goes through a crossover transition [14, 15, 16] into a strongly coupled regime which can be described in terms of a *weakly coupled dual* infrared-free  $\mathcal{N} =$

2 SQCD. The gauge group of the dual theory is

$$U(\nu) \times U(1)^{N-\nu}, \quad \nu = \begin{cases} r, & r \leq \frac{N_f}{2} \\ N_f - r, & r > \frac{N_f}{2}, \end{cases} \quad (2.1)$$

So far we limited ourselves to the case  $\nu = N_f - r$ . For  $r = N$  vacuum our  $r$  dual gauge group reduces to that of Seiberg’s duality, in which the first factor in the second line of (2.1) is  $U(N_f - N)$ . The coincidence does not extend to the case  $r < N$  [16]: instead of  $U(N_f - N)$  we get  $U(N_f - r)$ .

It is worth noting that the presence of the non-Abelian  $SU(\nu) \times U(1)^{N_f-\nu}$  gauge group at the roots of the nonbaryonic branches in massless ( $\xi = 0$ )  $\mathcal{N} = 2$   $SU(N)$  SQCD was first observed in [27]. Moreover, in this paper the  $SU(N_f - N)$  dual gauge group was identified at the root of the baryonic Higgs branch in the  $SU(N)$  theory. The relation between  $r$  and  $\nu$  given by (2.1) was noted in [28, 29], where it was interpreted as a correspondence between the “classical and quantum  $r$  vacua.” We interpret it as a duality occurring upon reducing  $\xi$  below the crossover transition line.

The dual theory supports non-Abelian strings due to condensation of light dyons in much the same way as non-Abelian strings in the original theory which are due to condensation of quarks. The strings of the dual theory confine *monopoles* too, rather than quarks [14, 16]. This is explained by the fact that the light dyons condensing in the dual theory carry weight-like chromoelectric charges (in addition to chromomagnetic charges). In other words, they carry the quark charges. The strings formed through condensation of these dyons can confine only the states with the root-like magnetic charges, i.e. the monopoles [14].

Thus, our  $r$  duality is *not* electromagnetic. There is no confinement of the chromoelectric charges in the dual theory; on the contrary, they are Higgs-screened.

At strong coupling, when the  $r$  dual description sets in, the gauge bosons and quarks of the original theory are in what we call “instead-of-confinement” phase. Namely, the quarks and gauge bosons of the original theory decay into monopole-antimonopole pairs on the curves of marginal stability (CMS) [14, 30]. The (anti)monopoles forming the pair are confined. In other words, the original quarks and gauge bosons evolve in the strong coupling domain of small  $\xi$  into stringy mesons with two constituents being connected by two strings as shown in Fig. 2. These mesons are expected to lie on the Regge trajectories.

Moreover, deep in the non-Abelian quantum regime the confined monopoles were demonstrated [30] to belong to the *fundamental representation* of the global (color-flavor locked) group. Therefore, the monopole-antimonopole mesons can be both, in the adjoint and singlet representation of this group. This pattern seems to be similar to what we have in the real world. The role of the “constituent quarks” inside the above mesons is played by the monopoles.

At this stage we are still not far away from the  $\mathcal{N} = 2$  limit. Then we increased the deformation parameter  $\mu$  decoupling the adjoint fields thus sending the original theory to the limit of  $\mathcal{N} = 1$  SQCD [15, 16]. In the passage from  $\mathcal{N} = 2$  to  $\mathcal{N} = 1$  we observed no dramatic qualitative changes. At large  $\mu$  the dual theory was demonstrated to be weakly coupled and

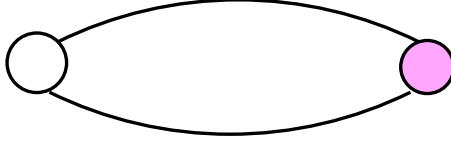


Figure 2: Mesons built from the monopole-antimonopole pairs connected by two strings. Open and closed circles denote the monopole and antimonopole, respectively.

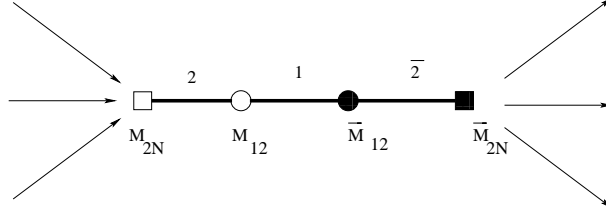


Figure 3: Example of the dipole meson formed as result of breaking of the second string by pair creation of the monopole  $M_{2N}$  (shown by boxes) interpolating between the second string and the would-be  $N$ -th string, which is absent. Arrows denote unconfined flux. Circles denote the monopoles  $M_{KK'}$ ,  $K, K' = 1, \dots, \nu$ . Open and closed circles/boxes denote the monopoles and antimonopoles, respectively.

infrared free, with the  $U(\nu)$  gauge group and  $N_f$  light dyons  $D^{lA}$ , (here  $l = 1, \dots, \nu$  is the color index in the dual gauge group, while  $A = 1, \dots, N_f$  is the flavor index). Non-Abelian strings still confine monopoles. “Instead-of-confinement” mechanism works at large  $\mu$  as follows. In the  $r = N$  vacuum the quarks and gauge bosons of the original  $\mathcal{N} = 1$  SQCD continue to be presented by stringy mesons built from the monopole-antimonopoles pairs connected by two non-Abelian strings, see Fig. 2.

In the  $r < N$  vacua (but  $r > \frac{2}{3}N_f$ ) there is a novel feature: one (say,  $N$ -th)  $Z_N$  string is absent in  $r < N$  vacua and the associated flux of the unbroken  $U(1)^{\text{unbr}}$  gauge factor is not squeezed into a flux tube. It is spread out in space via the Coulomb law.

As a result, non-Abelian strings become metastable in the  $r < N$  vacua: they can be broken by a monopole-antimonopole pair creation. The monopoles in the produced pair are junctions of one of the first  $r$   $Z_N$ -strings with the would-be  $N$ -th string (which is in fact absent). An example of the meson resulting in this way is shown in Fig. 3. The endpoints emit fluxes of the unbroken  $U(1)$  gauge field. This makes this meson a dipole-like configuration.

Note, that the non-Abelian fluxes of the  $SU(\nu)$  gauge group are always squeezed in the non-Abelian strings. Long-range forces are associated only with the unbroken  $U(1)^{\text{unbr}}$  gauge factor. The monopoles inside the dipole meson cannot annihilate if the overall flavor charge of the meson is nontrivial, say, the meson is in the adjoint.

Armed with the knowledge of the confining dynamics in the dual pair of  $\mathcal{N} = 1$  theories, we move on to compare our  $r$  duality with Seiberg’s duality. The simplest case is  $r = N$ . In the  $r = N$  vacuum our dual gauge group  $U(\nu = N_f - r)$  coincides with Seiberg’s dual group

$U(\tilde{N})$ , where

$$\tilde{N} = N_f - N. \quad (2.2)$$

Moreover, in this case the generalized Seiberg dual superpotential has a classical vacuum. We show that, upon integrating out heavy mesonic  $M$ -fields, this superpotential coincides with our  $r$ -dual superpotential obtained in [15]. Seiberg’s “dual quarks” are found to reduce to quark-like dyons  $D^{IA}$ , up to a normalization. Both dualities perfectly match in the  $r = N$  vacuum. This entails, in particular, that in the  $r = N$  vacuum Seiberg’s “dual quarks” are quark-like dyons, rather than monopole-like states. Their condensation leads to confinement of monopoles, while the quarks are in the “instead-of-confinement” phase [15].

For  $\frac{2}{3}N_f < r < N$  the generalized Seiberg superpotential has no supersymmetric classical vacua provided that the quark masses are generic. However, there are so-called “quantum vacua” which can be found by integrating out Seiberg’s “dual quarks” [2, 31] which, in turn, leads to an effective superpotential in terms of mesonic  $M$  fields.

In doing so one obtains an extension of the Affleck–Dine–Seiberg (ADS) superpotential [32] to  $N_f > N$ . The latter correctly reproduces the quark and gaugino condensates. We explicitly check that it gives the same results for the chiral condensates as the exact analysis of the chiral rings carried out in [26].

At the same time, in the  $\frac{2}{3}N_f < r < N$  vacua our  $r$  duality does not match Seiberg’s duality. We demonstrate our dual theory to have the  $U(\nu)$  gauge group instead of  $U(\tilde{N})$  and a different superpotential for light matter. Our dual theory does have a supersymmetric classical vacuum and, in a certain regime (with small  $\xi$ ), stays at weak coupling. Our interpretation of this is as follows. In the range  $\frac{2}{3}N_f < r < N$  generalized Seiberg dual theory does *not* describe low-energy physics in its entirety in the  $r$  vacua. However, it does describe the chiral sector in the sense of the Veneziano–Yankielowicz effective superpotential [33] (which is not a genuine low-energy superpotential). The spectrum of excitations is not reproduced correctly.

Low-energy physics in the  $r$  vacua is described (in the range  $\frac{2}{3}N_f < r < N$ ) by  $r$  duality, with the dual gauge group  $U(\nu = N_f - r)$  replacing Seiberg’s  $U(\tilde{N} = N_f - N)$ .

We also show that for smaller  $r$ , namely for  $r < \tilde{N}$ , Seiberg’s dual theory has supersymmetric classical vacua and in fact describes low-energy physics. This range, however, is beyond the scope of the present paper.

### 3 $\mu$ -Deformed $\mathcal{N} = 2$ SQCD and its vacuum structure at large $\xi$

The model we start from has the  $U(N) = SU(N) \times U(1)$  gauge symmetry and  $N_f$  massive quark hypermultiplets. In the absence of the  $\mu$  deformation the model is  $\mathcal{N} = 2$  supersymmetric. We assume that  $N_f > N$  but  $N_f < \frac{3}{2}N$ . The latter inequality ensures infrared freedom of the dual theory.



In addition, we will introduce the mass term  $\mu$  for the adjoint matter breaking  $\mathcal{N} = 2$  supersymmetry down to  $\mathcal{N} = 1$ .

The field content is as follows. The  $\mathcal{N} = 2$  vector multiplet consists of the U(1) gauge field  $A_\mu$  and the SU( $N$ ) gauge field  $A_\mu^a$ , where  $a = 1, \dots, N^2 - 1$ , and their Weyl fermion superpartners plus complex scalar fields  $a$ , and  $a^a$  and their Weyl superpartners, respectively. The  $N_f$  quark multiplets of the U( $N$ ) theory consist of the complex scalar fields  $q^{kA}$  and  $\tilde{q}_{Ak}$  (squarks) and their fermion superpartners — all in the fundamental representation of the SU( $N$ ) gauge group. Here  $k = 1, \dots, N$  is the color index while  $A$  is the flavor index,  $A = 1, \dots, N_f$ . We will treat  $q^{kA}$  and  $\tilde{q}_{Ak}$  as rectangular matrices with  $N$  rows and  $N_f$  columns.

Let us first discuss the undeformed  $\mathcal{N} = 2$  theory. The superpotential has the form

$$\mathcal{W}_{\mathcal{N}=2} = \sqrt{2} \sum_{A=1}^{N_f} \left( \frac{1}{2} \tilde{q}_A \mathcal{A} q^A + \tilde{q}_A \mathcal{A}^a T^a q^A + m_A \tilde{q}_A q^A \right), \quad (3.1)$$

where  $\mathcal{A}$  and  $\mathcal{A}^a$  are chiral superfields, the  $\mathcal{N} = 2$  superpartners of the gauge bosons of U(1) and SU( $N$ ), respectively, while  $m_A$  are the quark masses. Next, we add the mass term for the adjoint fields which breaks  $\mathcal{N} = 2$  supersymmetry down to  $\mathcal{N} = 1$ ,

$$\mathcal{W}_{\text{br}} = \sqrt{\frac{N}{2}} \frac{\mu_0}{2} \mathcal{A}^2 + \frac{\mu}{2} (\mathcal{A}^a)^2, \quad (3.2)$$

where  $\mu_0$  and  $\mu$  are mass parameters for the chiral superfields in  $\mathcal{N} = 2$  gauge supermultiplets, U(1) and SU( $N$ ), respectively. In the bulk of the paper we will consider the single trace perturbation which amounts to choosing  $\mu_0$  in such a way that the parameter

$$\gamma = 1 - \sqrt{\frac{2}{N}} \frac{\mu_0}{\mu} \quad (3.3)$$

vanishes. In this case the deformation superpotential (3.2) reduces to a single trace,

$$\mathcal{W}_{\text{br}} = \mu \text{Tr } \Phi^2, \quad (3.4)$$

where

$$\Phi = \frac{1}{2} \mathcal{A} + T^a \mathcal{A}^a. \quad (3.5)$$

Non-single trace deformation is discussed in the Appendix.

The mass term (3.4) splits the  $\mathcal{N} = 2$  supermultiplets, breaking  $\mathcal{N} = 2$  supersymmetry down to  $\mathcal{N} = 1$ . Our strategy is as follows. First we assume that deformation to be weak,

$$|\mu| \ll \Lambda_{\mathcal{N}=2}, \quad (3.6)$$

where  $\Lambda_{\mathcal{N}=2}$  is the scale of the  $\mathcal{N} = 2$  theory, so the theory is close to the  $\mathcal{N} = 2$  limit. We reduce the parameter  $\xi$  and describe  $r$  duality at small  $\xi$  [14, 16]. Finally, we make  $\mu$  large sending the theory to  $\mathcal{N} = 1$  SQCD, and discuss how this affects the dual theory [15, 16].

### 3.1 The $r = N$ vacuum

With generic values of the quark masses we have

$$\mathcal{N}_{r=N} = C_{N_f}^N = \frac{N_f!}{N!(N_f - N)!} \quad (3.7)$$

isolated vacua in which  $r = N$  quarks (out of  $N_f$ ) develop vacuum expectation values (VEVs). Following [14] consider, say, the vacuum in which the first  $N$  flavors develop VEVs, to be denoted as  $(1, 2 \dots, N)$ . In this vacuum the adjoint fields develop VEVs too, namely,

$$\langle \Phi \rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & m_N \end{pmatrix}. \quad (3.8)$$

For generic values of the quark masses, the  $SU(N)$  subgroup of the gauge group is broken down to  $U(1)^{N-1}$ . However, in the *special limit*

$$m_1 = m_2 = \dots = m_{N_f}, \quad (3.9)$$

the adjoint field VEVs do not break the  $SU(N) \times U(1)$  gauge group. In this limit the theory acquires a global flavor  $SU(N_f)$  symmetry.

With all quark masses equal (and to the leading order in  $\mu$ ) the mass term for the adjoint matter (3.4) reduces to the Fayet–Iliopoulos  $F$ -term of the  $U(1)$  factor of the  $SU(N) \times U(1)$  gauge group, which does *not* break  $\mathcal{N} = 2$  supersymmetry [8, 10]. Higher orders in the parameter  $\mu$  break  $\mathcal{N} = 2$  supersymmetry by splitting all  $\mathcal{N} = 2$  multiplets. If the quark masses are unequal the  $U(N)$  gauge group is broken down to  $U(1)^N$  by the adjoint field VEVs (3.8).

Using (3.4) and (3.8) it is not difficult to obtain the quark field VEVs from Eq. (3.1) supplemented by  $D$ -term conditions. By virtue of a gauge rotation they can be written as [34]

$$\begin{aligned} \langle q^{kA} \rangle &= \langle \bar{q}^{kA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1} & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \sqrt{\xi_N} & 0 & \dots & 0 \end{pmatrix}, \\ k &= 1, \dots, N, \quad A = 1, \dots, N_f, \end{aligned} \quad (3.10)$$

where we present the quark fields as matrices in color ( $k$ ) and flavor ( $A$ ) indices. The Fayet–Iliopoulos  $F$ -term parameters for each  $U(1)$  gauge factor are given (in the quasiclassical approximation) by the following expressions:

$$\xi_P \approx 2 \mu m_P, \quad P = 1, \dots, N. \quad (3.11)$$

While the adjoint VEVs do not break the  $SU(N) \times U(1)$  gauge group in the limit (3.9), the quark condensate (3.10) does result in the spontaneous breaking of both gauge and flavor

symmetries. A diagonal global  $SU(N)$  combining the gauge  $SU(N)$  and an  $SU(N)$  subgroup of the flavor  $SU(N_f)$  group survives, however.

Thus, the pattern of the color and flavor symmetry breaking is

$$U(N)_{\text{gauge}} \times SU(N_f)_{\text{flavor}} \rightarrow SU(N)_{C+F} \times SU(\tilde{N})_F \times U(1), \quad (3.12)$$

where  $\tilde{N}$  is given by (2.2). Here  $SU(N)_{C+F}$  is a global unbroken color-flavor rotation, which involves the first  $N$  flavors, while the  $SU(\tilde{N})_F$  factor stands for the flavor rotation of the last  $\tilde{N}$  quarks. The presence of the global  $SU(N)_{C+F}$  group is instrumental for formation of the non-Abelian strings [11, 12, 13, 21, 34]. Tensions of  $N$  elementary strings are determined by parameters  $\xi_P$  via [34]

$$T_P = 2\pi\xi_P. \quad (3.13)$$

These strings confine monopoles, in fact elementary monopoles become junctions of two distinct elementary strings [35, 13, 21].

Since the global (flavor)  $SU(N_f)$  group is broken by the quark VEVs anyway, it will be helpful for our purposes to consider the following mass splitting:

$$m_P = m_{P'}, \quad m_K = m_{K'}, \quad m_P - m_K = \Delta m \quad (3.14)$$

where

$$P, P' = 1, \dots, N \quad \text{and} \quad K, K' = N + 1, \dots, N_f. \quad (3.15)$$

This mass splitting respects the global group (3.12) in the  $(1, 2, \dots, N)$  vacuum. Moreover, this vacuum becomes isolated. No Higgs branch develops. We will often assume this limit below.

Now let us briefly discuss the perturbative excitation spectrum. Since both  $U(1)$  and  $SU(N)$  gauge groups are broken by the squark condensation, all gauge bosons become massive. To the leading order in  $\mu$ ,  $\mathcal{N} = 2$  supersymmetry is unbroken. In fact, with nonvanishing  $\xi_P$ 's (see Eq. (3.11)), both the quarks and adjoint scalars combine with the gauge bosons to form long  $\mathcal{N} = 2$  supermultiplets [10], for a review see [24]. In the limit (3.14)  $\xi_P \equiv \xi$ , and all states come in representations of the unbroken global group (3.12), namely, in the singlet and adjoint representations of  $SU(N)_{C+F}$ ,

$$(1, 1), \quad (N^2 - 1, 1), \quad (3.16)$$

and in the bifundamental representations

$$(\bar{N}, \tilde{N}), \quad (N, \tilde{\tilde{N}}). \quad (3.17)$$

We mark representations in (3.16) and (3.17) with respect to two non-Abelian factors in (3.12). The singlet and adjoint fields are (i) the gauge bosons, and (ii) the first  $N$  flavors of the squarks  $q^{kP}$  ( $P = 1, \dots, N$ ), together with their fermion superpartners. The bifundamental fields are the quarks  $q^{kK}$  with  $K = N + 1, \dots, N_f$ . These quarks transform in the two-index

representations of the global group (3.12) due to the color-flavor locking. Singlet and adjoint fields have masses of order  $g\sqrt{\xi}$ , while masses of bifundamental fields are  $\Delta m$ .

The above quasiclassical analysis is valid if the theory is at weak coupling. This is the case if the quark VEVs are sufficiently large so that the gauge coupling constant is frozen at a large scale. From (3.10) we see that the quark condensates are of order of  $\sqrt{\mu m}$  (see also [4, 5, 27, 18]). The weak coupling condition is

$$|\sqrt{\mu m}| \gg \Lambda_{N=2}, \quad (3.18)$$

where we assume all quark masses to be of the same order  $m_A \sim m$ . In particular, the condition (3.18), combined with the condition (3.6) of smallness of  $\mu$ , implies that the average quark mass  $m$  is very large.

### 3.2 The $r < N$ vacua

At large  $\xi$  the quark sector of the theory in the  $r$  vacua is at weak coupling and can be analyzed semiclassically. The number of the  $r$  vacua with  $r < N$  is [18]

$$\mathcal{N}_{r < N} = \sum_{r=0}^{N-1} (N-r) C_{N_f}^r = \sum_{r=0}^{N-1} (N-r) \frac{N_f!}{r!(N_f-r)!}. \quad (3.19)$$

It is equal to the number of choices one can pick up  $r$  quarks which develop VEVs (out of  $N_f$  quarks) times the Witten index (number of vacua) in the classically unbroken  $SU(N-r)$  pure gauge theory.

Below we will consider a particular vacuum in which the first  $r$  quarks develop VEVs. We denote it as  $(1, \dots, r)$ . Quasiclassically, with large mass differences, the adjoint scalar VEVs are

$$\left\langle \text{diag} \left( \frac{1}{2} a + T^a a^a \right) \right\rangle \approx -\frac{1}{\sqrt{2}} [m_1, \dots, m_r, 0, \dots, 0], \quad (3.20)$$

where the first  $r$  diagonal elements are proportional to the quark masses, while the last  $(N-r)$  entries classically vanish. In quantum theory they become of order of  $\Lambda_{N=2}$ . The classically unbroken  $U(N-r)$  pure gauge sector gets broken through the Seiberg–Witten mechanism [4] first down to  $U(1)^{N-r}$  and then almost completely by condensation of  $(N-r-1)$  monopoles. A single  $U(1)$  factor remains unbroken, because monopoles are charged only with respect to the Cartan generators of the  $SU(N-r)$  group. The presence of the unbroken  $U(1)^{\text{unbr}}$  symmetry in all  $r < N$  vacua makes them different from the  $r = N$  vacuum where there are no long-range forces. In the terminology of [26] these sets of vacua belong to two different “branches.”

Following [16] we consider the  $r = (N-1)$  vacuum as an example. The low energy theory in the  $r = N-1$  vacuum at large  $\xi$  has non-Abelian gauge fields  $A_\mu^n$ ,  $n = 1, \dots, (r^2-1)$  as well as Abelian ones  $A_\mu$  and  $A_\mu^{(N^2-1)}$ . The last field is associated with the last Cartan generator of  $SU(N)$ . These fields have scalar superpartners  $a^n$ ,  $a$  and  $a^{(N^2-1)}$ . Light matter consists of

the  $q^{kA}$  quarks,  $k = 1, \dots, r$ . Note, that all non-Abelian gauge fields from the  $SU(N)/SU(r)$  sector are heavy and decouple in the large mass limit due to the structure of the adjoint VEVs (see (3.20)). Also the  $q^{NA}$  quarks are heavy and not included in the low-energy theory.

The vacuum structure in the  $r = N - 1$  vacuum is as follows. The adjoint VEVs have the form

$$\langle \text{diag}(\Phi) \rangle \approx -\frac{1}{\sqrt{2}} [m_1, \dots, m_{N-1}, 0], \quad (3.21)$$

while the (s)quark VEVs are

$$\begin{aligned} \langle q^{kA} \rangle &= \langle \bar{q}^{kA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1} & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \sqrt{\xi_{N-1}} & 0 & \dots & 0 \end{pmatrix}, \\ k &= 1, \dots, (N-1), \quad A = 1, \dots, N_f. \end{aligned} \quad (3.22)$$

The first  $(N-1)$  parameters  $\xi$  are given quasiclassically by (3.11) while

$$\xi_N = 0. \quad (3.23)$$

In the  $r = N$  vacuum the last entry in (3.11) is  $m_N$  while now we have zero. The condition (3.23) reflects the fact that the  $N$ -th quark is heavy and develops no VEV. This is also valid in quantum theory [16].

Quarks interact with a particular linear combination of the  $U(1)$  gauge fields  $A_\mu$  and  $A_\mu^{N^2-1}$ , namely,

$$A_\mu + \sqrt{\frac{2}{N(N-1)}} A_\mu^{N^2-1}. \quad (3.24)$$

Quark condensation makes this combination massive. The orthogonal combination

$$\sqrt{\frac{2}{N(N-1)}} A_\mu - A_\mu^{N^2-1}. \quad (3.25)$$

remains massless and corresponds to the unbroken  $U(1)^{\text{unbr}}$  gauge group.

In the equal mass limit the global flavor symmetry  $SU(N_f)$  in the  $r$  vacuum is broken down to

$$SU(r)_{C+F} \times SU(\nu = N_f - r)_F \times U(1). \quad (3.26)$$

Now  $SU(r)_{C+F}$  is a global unbroken color-flavor rotation, which involves only the first  $r$  flavors, while the  $SU(\nu = N_f - r)_F$  factor stands for the flavor rotation of the remainder of the quark set.

Since the global (flavor)  $SU(N_f)$  group is broken by the quark VEVs anyway, it is useful to consider the split quark masses, as in (3.14), with (3.15) replaced by

$$P, P' = 1, \dots, r \quad \text{and} \quad K, K' = r + 1, \dots, N_f. \quad (3.27)$$

This mass splitting respects the global group (3.26) in the  $(1, 2, \dots, r)$  vacuum. This vacuum becomes isolated.

In much the same way as in the  $r = N$  vacuum in the  $r < N$  vacua all states in the limit (3.27) come in representations of the unbroken global group (3.26), namely, in the singlet and adjoint representations of  $SU(r)_{C+F}$ ,

$$(1, 1), \quad (r^2 - 1, 1), \quad (3.28)$$

and in the bifundamental representations

$$(\bar{r}, \nu), \quad (r, \bar{\nu}). \quad (3.29)$$

The singlet and adjoint fields are the gauge bosons, and the first  $r$  flavors of the quarks  $q^{kP}$  ( $P = 1, \dots, r$ ). The bifundamental fields are the  $q^{kK}$  quarks with  $K = r + 1, \dots, N_f$ . The singlet and adjoint fields have masses of order  $g\sqrt{\xi}$ , where  $\xi$  is the common value of the first  $r$   $\xi$ 's in the limit (3.14), (3.27). The masses of bifundamental fields are  $\Delta m$ .

Quasiclassical analysis is valid if the theory is at weak coupling. The weak coupling condition in the asymptotically free  $SU(r)$  sector reduces to

$$|\sqrt{\xi}| \sim |\sqrt{\mu m}| \gg \Lambda_{\mathcal{N}=2}^{\text{LE}}, \quad (3.30)$$

where  $\Lambda_{\mathcal{N}=2}^{\text{LE}}$  is the scale of the low energy theory determined by

$$\Lambda_{\mathcal{N}=2}^{2N-N_f} = m^2 (\Lambda_{\mathcal{N}=2}^{\text{LE}})^{2(N-1)-N_f}. \quad (3.31)$$

Quarks in  $r = N - 1$  vacuum develop VEVs; therefore monopoles should be confined, in much the same way as in the  $r = N$  vacuum. The distinction is that one  $U(1)$  factor of the gauge group remains unbroken, therefore the associated magnetic flux is unconfined. In fact one of  $Z_N$  strings (say, the  $N$ -th string) is absent due to the condition (3.23).

Therefore  $r$  strings associated with windings of  $r$  quarks can terminate on the monopoles  $M_{PN}$ ,  $P = 1, \dots, r$  interpolating between one of these string and the spurious  $N$ -th string. The endpoint is a magnetic source of unbroken  $U(1)^{\text{unbr}}$  gauge field. All other monopole fluxes, in particular, all non-Abelian fluxes from the  $SU(r)$  subgroup are absorbed by confining strings, see [16] for details.

## 4 $r$ Duality

What happens if we relax the condition (3.18) or (3.30) and pass to the strong coupling domain at

$$|\sqrt{\xi_P}| \ll \Lambda_{\mathcal{N}=2}, \quad |m_A - m_B| \ll \Lambda_{\mathcal{N}=2} \quad (4.1)$$

still keeping  $\mu$  small?

As was shown in [14, 16], our theory in the  $r$  vacuum undergoes a crossover transition on the way from large to small  $\xi$ . The domain (4.1) can be described in terms of weakly coupled (infrared free) dual theory with the gauge group

$$U(\nu) \times U(1)^{N-\nu}, \quad (4.2)$$

and  $N_f$  light dyon flavors ( $r$  is assumed to be in the range  $\frac{1}{2}N_f < r \leq N$ ).

The quark-like dyons  $D^{lA}$ , ( $l = 1, \dots, \nu$ ,  $A = 1, \dots, N_f$ ) are in the fundamental representation of the  $SU(\nu)$  gauge group and are charged under the Abelian factors indicated in Eq. (4.2). In addition, there are  $(N - \tilde{N})$  or  $(r - \nu)$  light quark-like dyons  $D^J$ , neutral under the  $SU(\nu)$  group, but charged under the  $U(1)$  factors in the  $r = N$  and  $r < N$  vacua, respectively. In the  $r < N - 1$  vacua there are also  $(N - r - 1)$  light monopoles charged under the  $U(1)$  factors.<sup>3</sup>

The dyon condensates are

$$\begin{aligned} \langle D^{lA} \rangle &= \langle \tilde{D}^{lA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \dots & 0 & \sqrt{\xi_1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & \sqrt{\xi_\nu} \end{pmatrix}, \\ \langle D^J \rangle &= \langle \tilde{D}^J \rangle = \sqrt{\frac{\xi_J}{2}}, \end{aligned} \quad (4.3)$$

where  $J = (\tilde{N} + 1), \dots, N$  in the  $r = N$  vacuum and  $J = (\nu + 1), \dots, r$  in the  $r < N$  vacua.

The most important feature apparent in (4.3), as compared to the squark VEVs of the original theory (3.10), is a “vacuum leap” [14],

$$(1, \dots, r)_{\sqrt{\xi} \gg \Lambda_{N=2}} \rightarrow (r + 1, \dots, N_f, (\nu + 1), \dots, r)_{\sqrt{\xi} \ll \Lambda_{N=2}}. \quad (4.4)$$

In other words, if we pick up the vacuum with nonvanishing VEVs of the first  $r$  quark flavors in the original theory at large  $\xi$  and then reduce  $\xi$  below  $\Lambda_{N=2}$ , the system goes through a crossover transition and ends up in the vacuum of the dual theory with the nonvanishing VEVs of the last  $\nu$  dyons (plus VEVs of the  $SU(\nu)$  singlets).

The Fayet–Iliopoulos parameters  $\xi_P$  in (4.3) are determined by the quantum version of the classical expressions (3.11) [34, 16]. Defining

$$u_k = \left\langle \text{Tr} \left( \frac{1}{2} a + T^a a^a \right)^k \right\rangle, \quad k = 1, \dots, N, \quad (4.5)$$

we obtain [34]

$$\xi_P = -2\sqrt{2} \mu E_P, \quad (4.6)$$

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<sup>3</sup>We collectively refer to all dyons carrying root-like electric charges as “monopoles.” This is to avoid confusion with the dyons which appear in Eq. (4.3). The latter dyons carry weight-like electric charges and, roughly speaking, behave as quarks, see [14, 16] for further details.

where  $E_P$  ( $P = 1, \dots, N$ ) are the diagonal elements of the  $N \times N$  matrix

$$E = \frac{1}{N} \frac{\partial u_2}{\partial a} + T^{\tilde{a}} \frac{\partial u_2}{\partial a^{\tilde{a}}}, \quad (4.7)$$

and  $T^{\tilde{a}}$  are the Cartan generators of the  $SU(N)$  gauge group ( $\tilde{a} = 1, \dots, (N-1)$ ). The  $E_P$  parameters are expressible via the roots of the Seiberg–Witten curve (see below).

The Seiberg–Witten curve in our theory takes the form [27]

$$y^2 = \prod_{P=1}^N (x - \phi_P)^2 - 4 \left( \frac{\Lambda_{N=2}}{\sqrt{2}} \right)^{2N-N_f} \prod_{A=1}^{N_f} \left( x + \frac{m_A}{\sqrt{2}} \right). \quad (4.8)$$

Here  $\phi_P$  are gauge invariant parameters on the Coulomb branch. Semiclassically,

$$\text{diag} \left( \frac{1}{2} a + T^a a^a \right) \approx [\phi_1, \dots, \phi_N]. \quad (4.9)$$

In the  $r = N$  vacuum the curve (4.8) has  $N$  double roots associated with condensation of  $N$  quarks. It reduces to

$$y^2 = \prod_{P=1}^N (x - e_P)^2, \quad (4.10)$$

where quasiclassically (at large masses)  $e_P$ 's and  $\phi_P$ 's are given by the mass parameters,  $\sqrt{2}e_P \approx \sqrt{2}\phi_P \approx -m_P$ ,  $P = 1, \dots, N$ . In the  $r < N$  quark vacuum (i.e. the  $(1, \dots, r)$  vacuum) we have

$$\phi_P \approx -\frac{m_P}{\sqrt{2}}, \quad P = 1, \dots, r, \quad \phi_P \sim \Lambda_{N=2}, \quad P = r+1, \dots, N \quad (4.11)$$

in the large  $m_A$  limit, see (3.20).

To identify the  $r < N$  vacuum in terms of the curve (4.8) it is necessary to find such values of  $\phi_P$  which ensure that the curve has  $N-1$  double roots, and  $r$  parameters  $\phi_P$  are determined by the quark masses in the semiclassical limit, see (4.11).  $(N-1)$  double roots are associated with  $r$  condensed quarks and  $(N-r-1)$  condensed monopoles – altogether  $N-1$  condensed states. In contrast, in the  $r = N$  vacuum the maximal possible number of condensed states (quarks) in the  $U(N)$  theory is  $N$ . As was already mentioned, this difference is related to the unbroken  $U(1)^{\text{unbr}}$  gauge group in the  $r < N$  vacua [26]. In the classically unbroken (after quark condensation)  $U(N-r)$  gauge group,  $N-r-1$  monopoles condense at the quantum level breaking the non-Abelian  $SU(N-r)$  subgroup. One  $U(1)$  factor remains unbroken because monopoles do not interact with it.

Thus in the  $r < N$  vacua the Seiberg–Witten curve factorizes [36],

$$y^2 = \prod_{P=1}^{N-1} (x - e_P)^2 (x - e_N^+) (x - e_N^-). \quad (4.12)$$



The last two roots (and  $\phi_N$ ) are of order of  $\Lambda_{N=2}$ . For the single-trace deformation superpotential (3.4) corresponding to  $\gamma = 0$  (see (3.3)) their sum vanishes [36],

$$e_N^+ + e_N^- = 0. \quad (4.13)$$

This condition is equivalent to a physical condition

$$\xi_N = -2\sqrt{2}\mu E_N = 0, \quad (4.14)$$

which ensures that the  $N$ -th quark is heavy and develops no VEV [16]. The root  $e_N^+$  determines the value of the gaugino condensate [26], see (6.19) in Sec. 6.3.

The parameters  $E_P$  in the  $r = N$  vacuum are given by double roots of the Seiberg–Witten curve [34], namely,

$$E_P = e_P, \quad P = 1, \dots, N. \quad (4.15)$$

This implies, in turn, that the dyon condensates at small  $\xi$  in the  $r = N$  vacuum are

$$\xi_P = -2\sqrt{2}\mu e_P. \quad (4.16)$$

As long as we keep  $\xi_P$  small (i.e. in the domain (4.1)) the coupling constants of the infrared-free dual theory (frozen at the scale of the dyon VEVs) are small; the dual theory is at weak coupling.

At small  $m_A - m_B \equiv \Delta m_{AB}$ , in the domain (4.1), the double roots of the Seiberg–Witten curve are

$$\sqrt{2}e_I = -m_{I+N}, \quad \sqrt{2}e_J = \Lambda_{N=2} \exp\left(\frac{2\pi i}{N - \tilde{N}}J\right) \quad (4.17)$$

for  $N - \tilde{N} > 1$ , where

$$I = 1, \dots, \tilde{N} \quad \text{and} \quad J = \tilde{N} + 1, \dots, N. \quad (4.18)$$

In particular, the first  $\tilde{N}$  roots are determined by the masses of the last  $\tilde{N}$  quarks — a reflection of the fact that the non-Abelian sector of the dual theory is not asymptotically free and is at weak coupling in the domain (4.1).

In the  $r < N$  vacua the relation between the parameters  $E_P$  which determine the dyon condensates and the roots of the Seiberg–Witten curve changes. Namely, we have [16]

$$E_P = \sqrt{(e_P - e_N^+)(e_P - e_N^-)}, \quad P = 1, \dots, (N - 1), \quad E_N = 0. \quad (4.19)$$

In terms of the roots of the Seiberg–Witten curve this implies for the dyon VEVs

$$\xi_P = -2\sqrt{2}\mu \sqrt{(e_P - e_N^+)(e_P - e_N^-)}, \quad P = 1, \dots, (N - 1), \quad \xi_N = 0. \quad (4.20)$$

In much the same way as in the  $r = N$  vacuum, the first  $\nu$  roots are determined by the masses of the last  $\nu$  quarks at small  $\Delta m_{AB}$ , i.e. in the domain (4.1),

$$\sqrt{2}e_I = -m_{I+r}, \quad I = 1, \dots, \nu. \quad (4.21)$$

This is because the non-Abelian sector of the dual theory is infrared free and is at weak coupling in the domain (4.1).

## 4.1 “Instead-of-confinement” mechanism in the $r$ vacua

The “vacuum leap” (4.4) ensures that in the  $r$ -vacua we have “instead-of-confinement” mechanism for quarks and gauge bosons [14, 16] (assuming  $\frac{1}{2}N_f < r \leq N$ ). Consider the mass choice (3.14), (3.27). Both, the gauge group and the global flavor  $SU(N_f)$  group, are broken in the vacuum. However, the color-flavor locked form of (4.3) shows that the unbroken global group of the dual theory is

$$SU(r)_F \times SU(\nu)_{C+F} \times U(1). \quad (4.22)$$

The  $SU(\nu)_{C+F}$  factor in (4.22) is a global unbroken color-flavor rotation, which involves the last  $\nu$  flavors, while the  $SU(r)_F$  factor stands for the flavor rotation of the first  $r$  dyons.

In the equal mass limit, or given the mass choice (3.14), (3.27), the global unbroken symmetry (4.22) of the dual theory at small  $\xi$  coincides with the global group (3.26) (or (3.12) for  $r = N$ ) in the original theory at large  $\xi$ . However, this global symmetry is realized in two different ways in the dual pair at hand. The quarks and gauge bosons of the original theory at large  $\xi$  come in the  $(1, 1)$ ,  $(r^2 - 1, 1)$ ,  $(\bar{r}, \nu)$ , and  $(r, \bar{\nu})$  representations (see (3.28), (3.29) or (3.16), (3.17)), while the dyons and  $U(\nu)$  gauge bosons form

$$(1, 1), \quad (1, \nu^2 - 1) \quad (4.23)$$

and

$$(r, \bar{\nu}), \quad (\bar{r}, \nu) \quad (4.24)$$

representations of (4.22). We see that the adjoint representations of the  $(C + F)$  subgroup are different in two theories.

This means that quarks and gauge bosons which form the adjoint  $(r^2 - 1)$  representation of  $SU(r)$  at large  $\xi$  and the dyons and dual gauge bosons which form the adjoint  $(\nu^2 - 1)$  representation of  $SU(\nu)$  at small  $\xi$  are different states. What happens with quarks and gauge bosons at small  $\xi$ ?

Screened quarks and gauge bosons which exist in the  $r$  vacuum in the large- $\xi$  domain decay into the monopole-antimonopole pairs on the CMS.<sup>4</sup> This is in accordance with the results obtained in  $\mathcal{N} = 2$   $SU(2)$  gauge theories [4, 5, 37] on the Coulomb branch at vanishing  $\xi$ ; for the theory under consideration such a behavior was established in [30]. The general rule is that the only states which exist at strong coupling inside CMS are those which can become massless on the Coulomb branch [4, 5, 37]. For our theory these are light dyons shown in Eq. (4.3), gauge bosons of the dual gauge group and monopoles.

As shown in [14, 16], at small nonvanishing  $\xi$  the monopoles and antimonopoles produced in the decay process of the adjoint  $(r^2 - 1, 1)$  states are confined. Therefore, the (screened)

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<sup>4</sup>Strictly speaking, such pairs can be formed by monopole-antidions and dyon-antidions as well, the dyons carrying root-like electric charges. As was already explained above, we collectively refer to all such states as “monopoles” to avoid confusion with quark-like dyons which appear in Eq. (4.3). The latter dyons carry weight-like electric charges, see [14, 16] for details.

quarks or gauge bosons evolve into stringy mesons in the strong coupling domain of small  $\xi$  – the monopole-antimonopole pairs connected by two strings, as shown in Fig. 2.

The distinction between the “instead-of-confinement” phase in the  $r < N$  vacua and that in the  $r = N$  vacuum is that in the  $r < N$  vacua the strings can be broken by  $M_{PN}$ -monopole-antimonopole pairs,  $P = 1, \dots, r$ . Here  $M_{PN}$  denote the monopoles that are junctions of the  $P$ -th and  $N$ -th strings (the latter is spurious, see [16] for details). As a result, the dipole stringy states emitting unbroken  $U(1)^{\text{unbr}}$  magnetic gauge fields are formed, see Fig. 3. Non-Abelian  $SU(\nu)$  fluxes are confined in these stringy dipoles.

## 5 $r$ Duality at large $\mu$

In this section we discuss continuation of  $r$  duality to the domain of large but finite  $\mu$ , i.e.  $\mathcal{N} = 1$  SQCD. We consider separately two cases:  $r = N$  [15] and  $r < N$  [16].

### 5.1 The $r = N$ vacuum

From Eqs. (4.3), (4.6) and (4.17) we see that the VEVs of the non-Abelian dyons  $D^{lA}$  are determined by  $\sqrt{\mu m}$  and are much smaller than the VEVs of the Abelian dyons  $D^J$  at small  $m$ . The latter are of order of  $\sqrt{\mu \Lambda_{\mathcal{N}=2}}$ . This circumstance is most crucial for us. It allows us to increase  $\mu$  and decouple the adjoint fields without ruining the weak coupling condition in the dual theory [15].

Now we assume that

$$|\mu| \gg |m_A|, \quad A = 1, \dots, N_f. \quad (5.1)$$

The Abelian dyon VEVs become large at large  $\mu$ . This makes heavy the  $U(1)$  gauge fields of the dual group (4.2). Decoupling these gauge factors together with adjoint matter and the Abelian dyons themselves we get the low-energy theory with the gauge group

$$U(\tilde{N}) \quad (5.2)$$

and non-Abelian dyons  $D^{lA}$  ( $l = 1, \dots, \tilde{N}$  and  $A = 1, \dots, N_f$ ). For the single-trace  $\gamma = 0$  perturbation (see (3.4)) the superpotential for  $D^{lA}$  has the form [15]

$$\mathcal{W} = -\frac{1}{2\mu} (\tilde{D}_A D^B)(\tilde{D}_B D^A) + m_A (\tilde{D}_A D^A), \quad (5.3)$$

where the color indices are contracted inside each parentheses.

Minimization of this superpotential leads to the dyon VEVs shown in the first line of Eq. (4.3). Note, that  $\xi$ 's which determine the non-Abelian dyon VEVs are of order  $\mu m$ , see (4.17).

Below the scale  $\mu$  our theory becomes dual to  $\mathcal{N} = 1$  SQCD with the scale

$$\tilde{\Lambda}_{r=N}^{N-2\tilde{N}} = \frac{\Lambda_{\mathcal{N}=2}^{N-\tilde{N}}}{\mu^{\tilde{N}}}. \quad (5.4)$$

The only condition we impose to keep this infrared-free theory at the weak coupling is

$$|\sqrt{\mu m}| \ll \tilde{\Lambda}_{r=N}. \quad (5.5)$$

This means that at large  $\mu$  we must keep the quark masses sufficiently small, which is always achievable.

We would like to stress that if all dyon VEVs were of order of  $\sqrt{\mu \Lambda_{N=2}}$ , it would not be possible to decouple the adjoint matter keeping the dual theory at weak coupling. As soon as we increase  $\mu$  beyond the above scale, we will break the weak coupling condition in the dual theory.

## 5.2 The $r < N$ vacua

In order to keep our dual theory at weak coupling we need to constrain the parameters  $\xi$  (at least  $\nu$  of them) from above. At large  $\mu$  this creates a problem. This problem was overcome in [16] as follows. Equation (4.20) shows that if we assume the mass differences to be very small and fine-tune the average value of  $\nu$  double roots (determined by masses, which are almost equal) to be close to one of the roots  $e_N^\pm$ , we guarantee  $\nu$  parameters  $\xi$  to be small. Say, we tune the quark masses to ensure

$$e_P \rightarrow e_N^+, \quad \Delta m_{KK'} \ll \Lambda_{N=2}, \quad P = 1, \dots, \nu, \quad K, K' = (r+1), \dots, N_f. \quad (5.6)$$

Note, that it is possible to make all  $\nu$  double roots close to  $e_N^+$  because it is the quark masses rather than  $\Lambda_{N=2}$  that determine the “non-Abelian” roots of the Seiberg–Witten curve and VEVs of the non-Abelian dyons, see (4.21).

The above limit means moving towards the Argyres–Douglas (AD) regime [38]. Indeed, on the Coulomb branch the masses of  $\nu$  monopoles  $M_{PN}$  ( $P = 1, \dots, \nu$ ) are determined by the differences  $(e_P - e_N^+) \rightarrow 0$ ; the corresponding  $\beta$ -cycles shrink. Thus, in addition to the light dyons  $D^{lA}$  and  $D^J$  which are always present in our  $r$  vacuum we get extra light monopoles mutually non-local with dyons. If we were on the Coulomb branch (at  $\xi_P = 0$ ) this would definitely mean moving into strong coupling.

However, at small but nonvanishing  $\xi$  we are *not* on the Coulomb branch. In fact, monopoles are confined. As shown in [16], this ensures the theory to stay at weak coupling. Basically the reason is that  $\nu$  monopoles  $M_{PN}$ ,  $P = 1, \dots, \nu$  in question form stringy dipole states shown in Fig. 3. Although the  $M_{PN}$  masses themselves tend to zero in the limit (5.6) the mass of the stringy dipole state formed by these monopole and antimonopole is determined by the string tension. It is of order of  $\sqrt{\xi_P}$  and, therefore, is much larger. This ensures the smallness of the renormalized coupling constant, provided we keep  $\xi$ ’s small enough. The fact that the light matter VEVs tend to zero in the AD point was first recognized in [39] in the Abelian case.

Now we can proceed in much the same way as in the  $r = N$  vacuum. We let  $\mu$  grow passing to the limit (5.6). The VEVs of the non-Abelian dyons  $D^{lA}$  become much smaller than the VEVs of the Abelian dyons  $D^J$ , see (4.3), (4.20) and (4.21). As a result,  $(N - \nu - 1)$

U(1) gauge fields of the dual gauge group (4.2), together with  $D^J$  dyons themselves, acquire large masses ( $\sim \sqrt{\mu\Lambda_{\mathcal{N}=2}}$ ) and decouple. At large  $\mu$ ,

$$|\mu| \gg |\sqrt{\xi}| \quad (5.7)$$

adjoint matter decouples too.

In order to keep the dual theory at weak coupling we go to the AD limit (5.6) and require

$$|\sqrt{\xi_P}| \ll \tilde{\Lambda}_r, \quad P = 1, \dots, \nu, \quad (5.8)$$

where

$$\tilde{\Lambda}_r^{r-2\nu} = \frac{\Lambda_{\mathcal{N}=2}^{r-\nu}}{\mu^\nu}. \quad (5.9)$$

We also assume that quark mass differences are very small, even smaller than  $E_P$ , namely,

$$\Delta m_{KK'} \ll E_P = \sqrt{(e_P^2 - e_N^2)}, \quad P = 1, \dots, \nu, \quad K, K' = (r+1), \dots, N_f. \quad (5.10)$$

At low energies our dual theory has the gauge group

$$\mathrm{U}(\nu) \times \mathrm{U}(1)^{\mathrm{unbr}}, \quad (5.11)$$

while light matter is represented by the non-Abelian dyons  $D^{lA}$  ( $l = 1, \dots, \nu$  and  $A = 1, \dots, N_f$ ). The superpotential is [16]

$$\begin{aligned} \mathcal{W} = & \frac{\hat{E}}{\sqrt{2}\hat{m}\mu} (\tilde{D}_A D^B)(\tilde{D}_B D^A) + \left[ (m_A - \hat{m}) + \frac{(\sqrt{2}\hat{E})^2}{\hat{m}} \right] (\tilde{D}_A D^A) \\ & + c \left[ \frac{1}{2\mu} (\tilde{D}_A D^A)^2 + \sqrt{2}\nu \hat{E} (\tilde{D}_A D^A) \right], \end{aligned} \quad (5.12)$$

where  $c$  is a constant,  $c \sim 1$ . Here

$$\hat{m} = \frac{1}{\nu} \sum_{P=1}^{\nu} m_{r+P}, \quad \hat{E} = \frac{1}{\nu} \sum_{P=1}^{\nu} E_P = \frac{1}{\sqrt{2}} \sqrt{\hat{m}^2 - \frac{4S}{\mu}}, \quad (5.13)$$

where  $S$  is the gaugino condensate, see (6.19). The non-Abelian dyon VEVs obtained from this superpotential are given by the first line in (4.3). They are small, corresponding  $\xi$ 's are of order of  $\mu\hat{E}$ .

### 5.3 Summary

Systematizing the overall picture behind  $r$  duality (in  $\mathcal{N} = 1$ , i.e. at large  $\mu$ , and under the condition  $\frac{2}{3}N_f < r \leq N$ ) upon reducing  $\xi$  the original theory undergoes a crossover

transition at strong coupling. In the region (5.8) at small quark masses in the  $r = N$  vacuum (or close to the AD points (5.6) in the  $r < N$  vacua) the strongly coupled theory is described by a dual weakly coupled infrared-free theory,  $U(\tilde{N})$  or  $U(\nu) \times U(1)^{\text{unbr}}$  SQCD, with  $N_f$  light dyon flavors. Condensation of light dyons  $D^{lA}$  in the dual theory leads to formation of non-Abelian strings and confinement of monopoles. Quarks and gauge bosons of the original  $\mathcal{N} = 1$  SQCD are in the “instead-of-confinement” phase: they decay into the monopole-antimonopole pairs on CMS and form stringy mesons. In the  $r < N$  vacua in the AD-regime (5.6), the  $M_{PN}$  monopoles ( $P = 1, \dots, \nu$ ) become very light and, therefore, strings are highly unstable. As a result, the stringy mesons shown in Fig. 2 decay into stringy dipoles, see Fig 3. Stringy dipoles with nontrivial charges with respect to the  $SU(r)$  part of the global group (for example, adjoint) are stable.

## 6 Generalized Seiberg’s duality

Now we would like to compare  $r$  duality we established with Seiberg’s duality [1, 2]. Originally Seiberg’s duality was formulated for  $\mathcal{N} = 1$  SQCD corresponding to the limit  $\mu \rightarrow \infty$ . Therefore, in the original formulation Seiberg’s duality referred to the monopole vacua with  $r = 0$ . Other vacua, with  $r \neq 0$ , have condensates of  $r$  quark flavors  $\langle \tilde{q}q \rangle_A \sim \mu m_A$  and, therefore, become runaway vacua in the limit  $\mu \rightarrow \infty$ .

At the same time, the  $r$  duality [16] can be continued to large but finite  $\mu$  in the  $r$  vacua ( $\frac{2}{3}N_f < r \leq N$ ), see Sec. 5. In order to compare both dualities with each other we rely on a generalization of Seiberg’s duality to the  $r$  vacua [18].<sup>5</sup>

At large  $\mu$  one can integrate out adjoint matter in superpotentials (3.1), (3.4). For the single-trace deformation with  $\gamma = 0$  this gives the superpotential

$$-\frac{1}{2\mu} (\tilde{q}_A q^B)(\tilde{q}_B q^A) + m_A (\tilde{q}_A q^A), \quad (6.1)$$

where color indices inside the brackets are contracted. This suggests that the Seiberg dual theory for our  $\mu$ -deformed  $U(N)$   $\mathcal{N} = 2$  SQCD at large but finite  $\mu$  has the gauge group (5.2) and  $N_f$  flavors of Seiberg’s “dual quarks”  $h^{lA}$  ( $l = 1, \dots, \tilde{N}$  and  $A = 1, \dots, N_f$ ) and (being  $\mathcal{N} = 1$  supersymmetric) possesses superpotential

$$\mathcal{W}_S = -\frac{\kappa^2}{2\mu} \text{Tr} (M^2) + \kappa m_A M_A^A + \tilde{h}_{Al} h^{lB} M_B^A, \quad (6.2)$$

where  $M_A^B$  is the Seiberg neutral mesonic  $M$  field defined as

$$(\tilde{q}_A q^B) = \kappa M_A^B. \quad (6.3)$$

Here  $\kappa$  is a parameter of dimension of mass needed to formulate Seiberg’s duality [1, 2].

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<sup>5</sup>It was suggested in [18] for  $SU(N)$  gauge theories. We use here a similar formulation for  $U(N)$  gauge theories. For a later development see [19].

From the definition (6.3) it is clear that in the  $r$  vacuum the number of eigenvalues of the matrix  $\tilde{q}q = \kappa M$  which scales as  $\mu m$  at large  $\mu m$  is  $r$ . Moreover,

$$r \leq N, \quad (6.4)$$

since classically the rank of the  $(\tilde{q}_B q^A)$  matrix cannot exceed  $N$ .

Now let us discuss the vacuum structure of the Seiberg dual theory (6.2). We do it separately in the  $r = N$  and  $r < N$  vacua.

## 6.1 The $r = N$ vacuum

Let us minimize superpotential (6.2) to find the classical vacua of the generalized Seiberg dual theory. Assuming that  $\langle M_A^B \rangle = \delta_A^B M_A$  we obtain the equations

$$\begin{aligned} -\frac{\kappa^2}{\mu} M_A + \kappa m_A + \tilde{h}_{Al} h^{lA} &= 0, \\ M_A h^{lA} = \tilde{h}_{Al} M_A &= 0, \end{aligned} \quad (6.5)$$

which should be valid for any  $A$ .

To solve these equations we note that the rank of the  $\tilde{h}_{Ak} h^{kB}$  matrix cannot exceed  $\tilde{N}$ . In particular, for  $r = N$  vacuum we have the maximal number of condensed  $h$ -fields equal to  $\tilde{N}$ . In this case we can choose the  $(1, \dots, N)$  vacuum as follows

$$\begin{aligned} M_A &= \frac{\mu}{\kappa} m_A, & (\tilde{h}h)_A &= 0, & A &= 1, \dots, N, \\ (\tilde{h}h)_A &= -\kappa m_A, & M_A &= 0, & A &= (N+1), \dots, N_f, \end{aligned} \quad (6.6)$$

where  $(\tilde{h}h)_A$  are diagonal elements of the matrix  $\tilde{h}_{Ak} h^{kB}$ . The number of  $r = N$  vacua is given in (3.7). It is equal to the number of possibilities of choosing  $N$  nonvanishing elements  $M_A$  out of  $N_f$ . This is also the number of the  $r = N$  vacua in the original theory at small  $\mu$ , i.e. close to the  $\mathcal{N} = 2$  limit.

Now we assume the fields  $M_A^B$  to be heavy and integrate them out. This implies that  $\kappa$  is large.<sup>6</sup> Integrating out the  $M$  fields in (6.2) we get

$$\mathcal{W}_S^{\text{LE}} = \frac{\mu}{2\kappa^2} (\tilde{h}_A h^B)(\tilde{h}_B h^A) + \frac{\mu}{\kappa} m_A (\tilde{h}_A h^A). \quad (6.7)$$

The change of variables

$$D^{lA} = \sqrt{-\frac{\mu}{\kappa}} h^{lA}, \quad l = 1, \dots, \tilde{N}, \quad A = 1, \dots, N_f \quad (6.8)$$

brings this superpotential to the form

$$\mathcal{W}_S^{\text{LE}} = \frac{1}{2\mu} (\tilde{D}_A D^B)(\tilde{D}_B D^A) - m_A (\tilde{D}_A D^A). \quad (6.9)$$

---

<sup>6</sup>We will see that the parameter  $\kappa$  does not enter the low-energy theory.

We see that (up to a sign) this superpotential coincides with the superpotential of our  $r$  dual theory (5.3). As was already mentioned, the dual gauge groups also coincide for Seiberg's and  $r$  dualities in the  $r = N$  vacuum. Note, that kinetic terms are not known in the Seiberg's dual theory, thus, normalization of the  $h$  fields is not fixed.

This leads us to the conclusion that in the  $r = N$  vacua both dual theories are identical. In Appendix A we show that this coincidence remains valid for more generic (non-single-trace) deformations of  $\mathcal{N} = 2$  SQCD. Of course, upon identification (6.8) the  $h^{lA}$  VEVs (6.6) coincide with the VEVs of the  $D^{lA}$  dyons in (4.3) in the  $r = N$  vacuum, see (4.16) and (4.17).

The identification (6.8) reveals the physical nature of Seiberg's "dual quarks". They are not monopoles as naive duality suggests. Instead, they are quark-like dyons appearing in the  $r$ -dual theory below crossover. Their condensation leads to confinement of monopoles and "instead-of-confinement" phase for the quarks and gauge bosons of the original theory.

## 6.2 The $r < N$ vacua

As we will show in Sec. 7 there are no classical supersymmetric vacua in the Seiberg dual theory with superpotential (6.2) for  $r$  vacua in the range  $\frac{2}{3}N_f < r < N$ . However, one can look for quantum vacua. Following [2, 31], we assume that the  $M_A^B$  fields develop VEVs making "dual quarks" heavy and then integrate  $h^{lA}$  out. The gluino condensation in the  $U(\tilde{N})$  gauge theory with no matter induces the superpotential

$$\mathcal{W}_S^{\text{eff}} = -\frac{\kappa^2}{2\mu} \text{Tr}(M^2) + \kappa m_A M_A^A + \tilde{N} \tilde{\Lambda}_S^{\frac{2\tilde{N}-N}{N}} (\det M)^{\frac{1}{N}}, \quad (6.10)$$

where  $\tilde{\Lambda}_S$  is the scale of Seiberg's dual theory defined via [1, 2]

$$\tilde{\Lambda}_S^{N-2\tilde{N}} \Lambda^{2N-\tilde{N}} = (-1)^{\tilde{N}} \kappa^{N_f}, \quad (6.11)$$

while  $\Lambda$  is the scale of the original  $\mathcal{N} = 1$  theory. It is related to  $\Lambda_{\mathcal{N}=2}$  as follows:

$$\Lambda^{2N-\tilde{N}} = \mu^N \Lambda_{\mathcal{N}=2}^{N-\tilde{N}} \quad (6.12)$$

Substituting the definition of the  $M$  fields (6.3) in (6.10) we arrive at

$$\mathcal{W}_S^{\text{eff}} = -\frac{1}{2\mu} \text{Tr}(\tilde{q}q)^2 + m_A \text{Tr}(\tilde{q}q) + (N - N_f) \frac{\Lambda^{\frac{3N-N_f}{N-N_f}}}{(\det \tilde{q}q)^{\frac{1}{N-N_f}}}. \quad (6.13)$$

The last quantum term is nothing other than the continuation of the Affleck–Dine–Seiberg (ADS) superpotential [32] to  $N_f > N$ . As was explained in the beginning of this section, the first term is obtained by integrating out adjoint fields in the original theory. Note, that in much the same way as in the  $r = N$  vacuum the dependence on the  $\kappa$  parameter disappeared.



Assuming, as before, that the matrix  $(\tilde{q}_A q^B)$  is diagonal we present the vacuum equation in the form

$$\frac{1}{\mu} (\tilde{q}q)_A = m_A - \frac{1}{\Lambda^{\frac{2N-\tilde{N}}{N}}} \frac{\prod_B [(\tilde{q}q)_B]^{\frac{1}{N}}}{(\tilde{q}q)_A}. \quad (6.14)$$

The superpotential (6.13) is exact and we can use it in any domain in the parameter space. In particular, for large masses,  $m_A \gg \Lambda_{N=2}$ , the solution of Eq. (6.14) for the  $(1, \dots, r)$  vacuum is

$$\begin{aligned} (\tilde{q}q)_A &\approx \mu m, & A = 1, \dots, r \\ (\tilde{q}q)_A &\approx \mu \Lambda_{N=2}^{\frac{N-\tilde{N}}{N-r}} m^{\frac{\tilde{N}-r}{N-r}} e^{\frac{2\pi k}{N-r} i}, & A = (r+1), \dots, N_f, \\ k &= 1, \dots, (N-r), \end{aligned} \quad (6.15)$$

where we assume the equal mass limit for simplicity. We have  $r$  large classical VEVs and  $(N_f - r)$  small “quantum” VEVs.

The linear dependence of  $(\tilde{q}q)$  on  $\mu$  is exact and is fixed by U(1) symmetries [39] after condensates are expressed in terms of  $\Lambda_{N=2}$ . The presence of  $(N - r)$  solutions ensures that the total number of the  $r < N$  vacua in our theory is

$$\mathcal{N}_{r < N} = \sum_{r=0}^{N-1} (N-r) C_{N_f}^r = \sum_{r=0}^{N-1} (N-r) \frac{N_f!}{r!(N_f - r)!}, \quad (6.16)$$

where the upper limit for  $r$  is implemented by the condition (6.4). This number coincides with the result (3.19) obtained in the  $\mathcal{N} = 2$  limit and, therefore, matches number of the  $r < N$  vacua in our  $r$ -dual theory.<sup>7</sup>

### 6.3 Generalized Seiberg’s duality and exact chiral rings

In this section we would like to make sure that generalized Seiberg’s duality gives correct values of the chiral condensates in the  $r < N$  vacua. To this end we compare quark condensates determined by Eq. (6.13) with the exact results obtained in [26]. This section overlaps with what is already known, and we include it here mostly for the sake of completeness. For example, a somewhat similar analysis for SU( $N$ ) gauge theory can be found in [40].

All chiral condensates in our theory can be encoded in the following functions [26]

$$\begin{aligned} T(x) &= \left\langle \text{Tr} \frac{1}{x - \Phi} \right\rangle, \\ R(x) &= \frac{1}{32\pi^2} \left\langle \text{Tr} \frac{W_\alpha W^\alpha}{x - \Phi} \right\rangle, \\ M(x)_A^B &= \left\langle \tilde{q}_A \frac{1}{x - \Phi} q^B \right\rangle, \end{aligned} \quad (6.17)$$

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<sup>7</sup>Our large- $m$  counting (6.16) also agrees with the final result in [18], see also Sec. 7.

where  $W_\alpha$  is the gauge field strength superfield. For the quadratic single-trace deformation (3.4) (“one-cut” model) the function  $R(x)$  has the form

$$R(x) = \frac{1}{2} \left( W'_{\text{br}}(x) - \sqrt{W'_{\text{br}}(x) + f(x)} \right) = \mu \left( x - \sqrt{x^2 - e_N^2} \right), \quad (6.18)$$

where the undoubled root of the Seiberg–Witten curve  $e_N = e_N^+$  (see (4.12)) is related to the gaugino condensate,

$$e_N^2 = \frac{2S}{\mu}, \quad S = \frac{1}{32\pi^2} \langle \text{Tr } W_\alpha W^\alpha \rangle. \quad (6.19)$$

The solutions for the chiral rings were obtained in [26] in the  $r < N$  vacua. In the  $r = N$  vacuum the gaugino condensate vanishes, all roots of the Seiberg–Witten curve are doubled, and there are no cuts in the  $x$ -plane. As we already mentioned, all  $r < N$  vacua belong to a single “branch” with a single  $U(1)$  gauge factor unbroken, while in the  $r = N$  vacuum the gauge group is fully Higgsed.

From the solution for the function  $M_A^B(x)$  in [26] one can obtain the values of the quark VEVs in terms of the gaugino condensate  $S$ . In the  $r$  vacuum  $(1, \dots, r)$  (when the function  $M_A^B(x)$  has  $r$  poles on the first sheet) we have

$$\begin{aligned} (\tilde{q}q)_A &= \frac{\mu}{2} \left( m_A + \sqrt{m_A^2 - \frac{4S}{\mu}} \right), \quad A = 1, \dots, r \\ (\tilde{q}q)_A &= \frac{\mu}{2} \left( m_A - \sqrt{m_A^2 - \frac{4S}{\mu}} \right), \quad A = (r+1), \dots, N_f, \end{aligned} \quad (6.20)$$

Now, to find the gaugino condensate  $S$  we use the glueball superpotential calculated in [26] from a matrix model [41]. For our theory with the quadratic single-trace deformation (3.4) it has the form [28]

$$\begin{aligned} \mathcal{W}_{\text{glueball}} &= S \left[ N + \log \frac{\mu^N \Lambda_{N=2}^{N-\tilde{N}} \prod_A m_A}{S^N} \right] \\ &- \sum_{A=r+1}^{N_f} S \left[ -\log \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4S}{\mu m_A^2}} \right) + \frac{\mu m_A^2}{4S} \left( \sqrt{1 - \frac{4S}{\mu m_A^2}} - 1 \right) + \frac{1}{2} \right] \\ &- \sum_{A=1}^r S \left[ -\log \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4S}{\mu m_A^2}} \right) \right. \\ &\left. + \frac{\mu m_A^2}{4S} \left( -\sqrt{1 - \frac{4S}{\mu m_A^2}} - 1 \right) + \frac{1}{2} \right]. \end{aligned} \quad (6.21)$$

Minimization of this superpotential gives the equation for  $S$  from which we obtain

$$S^N = \mu^N \Lambda_{\mathcal{N}=2}^{N-\tilde{N}} \left( \frac{m}{2} - \frac{1}{2} \sqrt{m^2 - \frac{4S}{\mu}} \right)^r \left( \frac{m}{2} + \frac{1}{2} \sqrt{m^2 - \frac{4S}{\mu}} \right)^{N_f-r}, \quad (6.22)$$

where we assume the equal-mass limit for simplicity.

Now let us derive equations for the quark VEVs using the Cachazo–Seiberg–Witten expressions (6.20) and equation (6.22). To this end we first express the right-hand side of (6.22) in terms of the quark VEVs using (6.20). Solving this equation for  $S$  we get

$$S = \frac{(\det \tilde{q}q)^{\frac{1}{N}}}{\mu^{\frac{N}{N}} \Lambda_{\mathcal{N}=2}^{\frac{N-\tilde{N}}{N}}}. \quad (6.23)$$

Substituting  $S$  from (6.23) in the right-hand side of (6.20) we derive the following equation for the quark VEVs:

$$\frac{1}{\mu} (\tilde{q}q)_A = m - \frac{1}{\mu^{\frac{N}{N}} \Lambda_{\mathcal{N}=2}^{\frac{N-\tilde{N}}{N}}} \frac{(\det \tilde{q}q)^{\frac{1}{N}}}{(\tilde{q}q)_A}. \quad (6.24)$$

These equations coincide with those in (6.14) (for equal quark masses). We see that the Cachazo–Seiberg–Witten exact solution [26] produces the same equations for the quark condensates as the continuation of the ADS superpotential to  $N_f > N$  in Eq. (6.13). This justifies the latter superpotential.

## 7 Classical and quantum $r$ vacua in Seiberg’s dual theory

As was mentioned in Sec. 1, generalized Seiberg’s duality suggested in [18] was later studied in [19] in the  $\mathcal{N} = 1$  theories with the  $SU(N)$  gauge group. The numbers of classical and quantum vacua corresponding to the superpotentials (6.2) and (6.13) were analyzed. In particular, a certain number of classical vacua was detected.

In this section we show that there are no classical vacua in the Seiberg’s dual theory in the range  $\frac{2}{3}N_f < r < N$  we explore in this paper. For smaller values of  $r$ , namely for  $r < \tilde{N}$ , the generalized Seiberg superpotential (6.2) does have classical vacua.

First, we briefly review the analysis carried out in [19]. The solution for (6.5) was written as (cf. (6.6))

$$\begin{aligned} (\tilde{q}q)_A &= \kappa M_A = \mu m_A, & (\tilde{h}h)_A &= 0, & A &= 1, \dots, p, \\ (\tilde{h}h)_A &= -\kappa m_A, & M_A &= 0, & A &= (p+1), \dots, N_f, \end{aligned} \quad (7.1)$$

where now  $p$  should obey the constraint  $p > N$ , since the rank of the matrix  $(\tilde{h}h)$  cannot exceed  $\tilde{N}$ , and we do not consider the  $r = N$  vacua in this section.

This solution can describe low-energy physics if the infrared-free Seiberg dual theory is at weak coupling. To ensure this we assume the quark masses to be small,

$$m_A \ll \Lambda_{N=2}. \quad (7.2)$$

As we will see below, in the case at hand  $p$  does not coincide with  $r$ , the latter parameter being defined at large masses. Therefore, the condition (6.4) does not apply for  $p$ . The number of the above vacuum solutions is

$$\mathcal{N}_{\text{class}}^S = \sum_{p=N+1}^{N_f} (p - N) C_{N_f}^p, \quad (7.3)$$

where  $(p - N)$  is the rank of the gauge group unbroken by the  $h$ -condensation, and we modify the results of [19] to include the combinatorial factor  $C_{N_f}^p$ , see [18].<sup>8</sup> The number of these classical vacua is less than the total number of the  $r < N$  vacua (6.16). The missing vacua are in fact quantum vacua which are not seen in Seiberg's superpotential (6.2) even at small  $m_A$ . They can be recovered from (6.13), however [18, 19].

The  $(\tilde{q}q)$  matrix in Eq. (6.14) has two different eigenvalues (in the limit of equal quark masses), namely

$$(\tilde{q}q)_B^A = \text{diag}(z, \dots, z, y, \dots, y), \quad (7.4)$$

where (at small  $m$ )  $z$  appears  $p$  times while  $y$  appears  $(N_f - p)$  times, and, in addition,

$$z + y = \mu m. \quad (7.5)$$

From (6.14) we can write the following equation [19]:

$$z^{p-\tilde{N}} = (\mu \Lambda_{N=2})^{N-\tilde{N}} y^{p-N} \quad (7.6)$$

which, in combination with (7.5), allows us to determine both  $z$  and  $y$ .

Following [19] we note that for  $p \geq N$  Eq. (7.6) is a polynomial of degree  $(p - \tilde{N})$  with respect to  $z$  and, therefore, has  $(p - \tilde{N})$  solutions for  $z$ . For  $\frac{1}{2}N_f \leq p < N$  this equation has  $(N - \tilde{N})$  solutions. Summing up all solutions together we get the number of vacua in the form

$$\begin{aligned} \mathcal{N}_{r < N} &= \sum_{p=N+1}^{N_f} (p - \tilde{N}) C_{N_f}^p + (N - \tilde{N}) \left( \sum_{p=\frac{N_f}{2}+1}^N C_{N_f}^p + \frac{1}{2} C_{N_f}^{\frac{1}{2}N_f} \right) \\ &= \sum_{p=N}^{N_f} (p - N) C_{N_f}^p + \frac{1}{2} (N - \tilde{N}) \sum_{p=0}^{N_f} C_{N_f}^p. \end{aligned} \quad (7.7)$$

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<sup>8</sup>At this point we keep the quark masses slightly different so that all vacua are isolated and can be counted. The calculation of [19] refers to the equal-mass limit and, in fact, corresponds to counting the number of the Higgs branches which are continuously degenerate in the equal-mass limit (vacuum moduli).

The last sum here reduces to  $(N - \tilde{N}) 2^{N_f - 1}$ ; and then Eq. (7.7) can be rewritten as [18]

$$\mathcal{N}_{r < N} = \sum_{p=N}^{N_f} (p - N) C_{N_f}^p + \sum_{p=0}^{N_f} (N - p) C_{N_f}^p = \sum_{r=0}^N (N - r) C_{N_f}^r. \quad (7.8)$$

This calculation refers to the small-mass limit. The first term corresponds to the number of the classical vacua (7.3), while the second one counts the *missing quantum* vacua. The total number of vacua obviously coincides with Eq. (6.16) obtained at large  $m$ .

Now, we can solve Eqs. (7.5) and (7.6) at small  $m$ . In addition to “large” solutions with  $z \approx -y \sim \Lambda_{\mathcal{N}=2}$ , we also get “small” solutions

$$\begin{aligned} (\tilde{q}q)_A &= z \approx \mu m, & A = 1, \dots, p \\ (\tilde{q}q)_A &= y \approx \mu \frac{m^{\frac{p-\tilde{N}}{p-N}}}{\Lambda_{\mathcal{N}=2}^{\frac{N-\tilde{N}}{p-N}}} e^{\frac{2\pi k}{p-N} i}, & A = (p+1), \dots, N_f, \\ k &= 1, \dots, (p - N). \end{aligned} \quad (7.9)$$

These solutions should be compared with the classical solutions (7.1). We see that the  $m$  dependence of  $(\tilde{q}q)$  matches; thus, these solutions corerespond to the classical vacua of Seiberg’s dual theory. In order to have  $y$  much smaller than  $z$  in the small-mass limit we impose the condition  $p > N$ . (This is to be contrasted with the condition  $r < N$  in (6.15) for large  $m$ ). Given the multiplicity of these solutions equal to  $(p - N)$  we see that the number of these vacua precisely matches the number of the classical Seiberg vacua (7.3).

The behavior of  $(\tilde{q}q)$  in (7.9) ensures that the gaugino condensate is very small in these vacua, see (6.20). Namely,

$$S \approx \mu \frac{m^{\frac{2p-N_f}{p-N}}}{\Lambda_{\mathcal{N}=2}^{\frac{N-\tilde{N}}{p-N}}} e^{\frac{2\pi k}{p-N} i}, \quad k = 1, \dots, (p - N). \quad (7.10)$$

What is the relation between  $r$  and  $p$ ?

At large  $m_A$  we start from an  $r$  vacuum, with  $r$  quarks (classically) condensed, hence  $r \leq N$ . On the other hand,  $p$  is defined as the number of “plus” signs in Eq. (6.20) for

$$(\tilde{q}q)_A = z.$$

(Then  $(N_f - p)$  is the number of “minus” signs). In fact,  $p$  depends on  $m_A$ . At large  $m_A$  we have  $p(\infty) = r$ . As we reduce  $m_A$  certain poles can pass through the cut from the first sheet to the second or vice versa [26]. When it happens  $p(m_A)$  reduces by one unit or increases by one unit.

In Eqs. (7.9) and (7.10)  $p$  is  $p(m_A)$  in the small mass limit,  $p = p(0)$ . Clearly  $p$  can differ from  $r$ , and the condition (6.4) does not apply for  $p$ . In fact,  $(p - r)$  is the net number of

poles which pass through the cut from the second sheet to the first one as we reduce the quark masses from infinity to zero.

At large  $m$  we start in the  $r$  vacuum, with  $r < N$ , and the quark condensate given by (6.15). This solution corresponds to

$$S \approx \mu \Lambda_{\tilde{N}=2}^{\frac{N-\tilde{N}}{N-r}} m^{\frac{N_f-2r}{N-r}} e^{\frac{2\pi k}{N-r} i}, \quad k = 1, \dots, (N-r). \quad (7.11)$$

This behavior can be seen in Eq. (6.22) as follows. We expand the square roots in  $S/\mu m^2$  in the right-hand side of (6.22). The second factor tends to a constant while the first factor gives  $S^r$ , which reproduces the behavior in (7.11).

Now, to determine the relation between  $r$  and  $p$  in the vacua which are described by Seiberg's duality we must find the solution of Eq. (6.22) which approaches (7.11) at large  $m$  and has the behavior (7.10) at  $m \rightarrow 0$ .

There is only one possibility for this to happen. As  $m$  reduces all poles should pass thorough the cut, so that the signs of the square roots in (6.22) change. In other words, as we reduce  $m$  from large to small values, all  $r$  poles from the first sheet pass to the second one and, simultaneously, all  $(N_f - r)$  poles from the second sheet pass to the first one. Then at small  $m$  the first factor in the right-hand side of Eq. (6.22) tends to a constant, while the second one gives  $S^{N_f-r}$ . This gives the behavior (7.10) where

$$p = N_f - r. \quad (7.12)$$

We stress that there are other solutions to (6.22) which have different behavior at small  $m$  ( $S \sim \mu \Lambda_{\tilde{N}=2}^2$ ). We are interested in the behavior (7.10) with  $p > N$  because these solutions correspond to the vacua seen classically in the Seiberg dual theory. Other vacua are “quantum” vacua (see (7.8)) which remain classically invisible.

For Seiberg's classical vacua we need  $p > N$ . This translates into the constraint

$$r < \tilde{N}. \quad (7.13)$$

In this paper we study  $r$ -duality in the range  $\frac{2}{3}N_f < r \leq N$ ; thus, the above vacua are beyond the range of our analysis. This means that the  $r$  vacua described by our  $r$ -duality should be interpreted as “missing quantum” vacua from the standpoint of Seiberg's duality.

## 8 $r$ Duality versus Seiberg's duality for the $\frac{2}{3}N_f < r < N$ vacua

For  $\frac{2}{3}N_f < r < N$  vacua our  $r$ -dual theory does not agree with the generalized Seiberg dual theory. First, we have the  $U(\nu)$  gauge group, while the Seiberg dual has the gauge group  $U(\tilde{N})$ . Light matter sectors and effective superpotentials are also different in two theories: the  $D^{lA}$  dyons ( $l = 1, \dots, \nu$ ) with superpotential (5.12) versus “dual quarks”  $h^{lA}$  ( $l = 1, \dots, \tilde{N}$ ) plus  $M$  fields with superpotential (6.2).

Both dual theories are well justified and verified. On the one hand, the  $r$ -dual theory is derived from the  $\mathcal{N} = 2$  limit by increasing  $\mu$  and keeping the theory at weak coupling at all intermediate stages. On the other hand, as was checked in Sec. 6.3, the generalized Seiberg dual theory (more exactly, the generalized ADS superpotential (6.13)) matches the exact solution of [26]. What is going on?

Our interpretation is as follows. In the  $r$ -vacua (in the range  $\frac{2}{3}N_f < r < N$ ) the generalized Seiberg dual theory is at strong coupling and, therefore, cannot describe low-energy physics in its entirety. However, it does describe the chiral sector in the sense of the Veneziano–Yankielowicz effective superpotential [33]. Namely, condensates from the chiral ring are correctly reproduced. The spectrum of excitations is not.

As an example consider superpotentials (6.13) or (6.21). Although these superpotentials correctly reproduce the chiral condensates, taken at their face value they do not describe low energy-spectrum. Namely, it is clear that neither the quark mesonic field ( $\tilde{q}q$ ) nor the glueball field  $S$  are light degrees of freedom at strong coupling.

We believe that the superpotential (6.2) is of the same kind in the window  $\frac{2}{3}N_f < r < N$ . This assertion is supported by the fact that supersymmetric vacua are not seen at the classical level in the superpotential (6.2) for  $\frac{2}{3}N_f < r < N$ . In order to find supersymmetric vacua we have to integrate out the  $h$  fields and search for solutions in the effective quantum superpotential (6.13). This suggests that the “dual quarks”  $h$  are not the low-energy degrees of freedom and, in fact, Seiberg’s dual theory (6.2) is strongly coupled at small  $\xi$ ’s in this window.

Instead, the  $r$ -dual theory *is* the low-energy description at small  $\xi$ , where the original  $\mathcal{N} = 1$  SQCD is at strong coupling. As long as we keep the parameters  $\xi$  small (see (5.8)) the  $r$ -dual infrared-free theory is at weak coupling and under control. Condensation of the quark-like dyons  $D^{IA}$  in this theory leads to confinement of monopoles and “instead-of-confinement” phase for the quarks and gauge bosons.

As was shown in Sec. 7, in the range  $r < \tilde{N}$  the generalized Seiberg’s dual theory has supersymmetric classical vacua and, being infrared-free, is at weak coupling (the same applies to the  $r = N$  vacuum where it matches  $r$ -duality). Therefore, it does describe low-energy physics in the  $r$  vacua with  $r < \tilde{N}$ . The schematic picture of both dual descriptions versus  $r$  is shown in Fig. 1.

*A very important problem for future studies is extrapolating  $r$  duality to  $r \leq \frac{2}{3}N_f$  and comparing it in this range with Seiberg’s duality. Another problem is understanding  $r$  duality in the framework of strings/branes, in the spirit it had been done with the Seiberg duality.*

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## Appendix: Non-single-trace deformations in the $r = N$ vacuum

In this Appendix we show the matching of the effective superpotentials for  $r$ -dual and generalized Seiberg's dual theories for generic deformation (3.2) with  $\gamma \neq 0$ . For this case the superpotential of  $r$ -dual theory obtained in [15] has the form

$$\begin{aligned} \mathcal{W} = & -\frac{1}{2\mu} \left[ (\tilde{D}_A D^B)(\tilde{D}_B D^A) - \frac{\alpha_D}{\tilde{N}} (\tilde{D}_A D^A)^2 \right] \\ & + \left[ m_A - \frac{\gamma(1 + \frac{\tilde{N}}{N})}{1 + \gamma \frac{\tilde{N}}{N}} m \right] (\tilde{D}_A D^A), \end{aligned} \quad (\text{A.1})$$

where

$$\alpha_D = \frac{\gamma \frac{\tilde{N}}{N}}{1 + \gamma \frac{\tilde{N}}{N}}, \quad m = \frac{1}{N_f} \sum_{A=1}^{N_f} m_A \quad (\text{A.2})$$

and  $\gamma$  is given by (3.3).

On the other hand the generalized Seiberg's superpotential for  $\gamma \neq 0$  is

$$\mathcal{W}_S = -\frac{\kappa^2}{2\mu} \text{Tr}(M^2) + \frac{\kappa^2}{2N\mu} \alpha (\text{Tr} M)^2 + \kappa m_A M_A^A + \tilde{h}_{Al} h^{lB} M_B^A, \quad (\text{A.3})$$

where

$$\alpha = 1 - \sqrt{\frac{N}{2}} \frac{\mu}{\mu_0} = -\frac{\gamma}{1 - \gamma}. \quad (\text{A.4})$$

Upon integrating out the  $M$  fields we get

$$\begin{aligned} \mathcal{W}_S^{\text{LE}} = & \frac{\mu}{2\kappa^2} \left[ (\tilde{h}_A h^B)(\tilde{h}_B h^A) - \frac{\alpha_D}{\tilde{N}} (\tilde{h}_A h^A)^2 \right] \\ & + \frac{\mu}{\kappa} \left[ m_A - \frac{\gamma(1 + \frac{\tilde{N}}{N})}{1 + \gamma \frac{\tilde{N}}{N}} m \right] (\tilde{h}_A h^A). \end{aligned} \quad (\text{A.5})$$

We see again that upon change of variables (6.8) two superpotentials (A.5) and (A.1) coincide (up to a sign).

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