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Jet mass with a jet veto at two loops and the universality of non-global structure

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Abstract: We investigate the exclusive jet mass distribution in e^+e^- events, defined with a veto on the out-of-jet radiation, at two-loop order. In particular, we calculate the two-loop soft function, which is required to describe this distribution in the region of small jet mass. When combined with other ingredients using soft-collinear effective theory, this generates the complete singular distribution for jet thrust, the sum of the jet masses, at two-loop order. The result is in excellent agreement with full QCD. The integrated jet thrust distribution is found to depend in an intricate way on both the finite jet cone size, R, and the jet veto scale. The result clarifies the structure of the potentially large logarithms (both global and non-global) which arise in jet observables for the first time at this order. Somewhat surprisingly, we find that, in the small R limit, there is a precise and simple correspondence between the non-global contribution to the integrated jet thrust distribution and the previously calculated non-global contribution to the integrated hemisphere soft function, including subleading terms. This suggests that the small R limit may provide a useful expansion for studying other exclusive jet substructure observables.

1. Introduction

High energy scattering processes in Quantum Chromodynamics (QCD) often involve the production of jets. A QCD observable of particular interest is the jet mass. Jet mass plays an important role in many applications, such as in the identification of a boosted W boson [1, 2] or top quark [3, 4]. Jet mass is also important because many event shapes studied at e^+e^- colliders reduce to some measure of jet mass in particular limits. For example, thrust in the dijet limit reduces to the sum of the masses of two hemisphere jets, and heavy jet mass reduces to the larger of the two hemisphere jet masses. Thrust and heavy jet mass are two of the most well-studied e^+e^- event shapes and both have been measured with extraordinary precision [5]. On the theoretical side, next-to-next-to-leading order (NNLO) QCD calculations of these observables have been carried out for jets in e^+e^- events [6, 7, 8, 9]. Unfortunately, jet physics at hadron colliders is much more difficult, both experimentally and theoretically. It is therefore still worth studying e^+e^- jets, where the analytical calculations can be done, in the hope that universality properties will be found which may ultimately make the hadron collider calculations tractable, at least in some regime.

In the limit of small jet mass, one cannot straightforwardly make accurate predictions for the thrust or heavy jet mass using QCD perturbation theory. In this limit, the cross section is dominated by the soft and collinear interactions of massless partons which produce large logarithmic contributions of the form $\alpha_s^i \ln^j(m/Q)$ at each order in perturbation theory. Without a correct resummation of these terms to all orders in α_s, perturbative predictions are not reliable when $m/Q \to 0$. In recent years, much progress has been made on this front and NNNLL resummations of both thrust and heavy jet mass are now available [10, 11, 12]. These resummations led to a measurement of the strong coupling constant, α_s, which is competitive with the world average [13]. Soft-collinear effective theory (SCET) is the theoretical tool that facilitated many of these important developments [14, 15, 16, 17]. Indeed, before the simplifications brought about by SCET, thrust and heavy jet mass had only been resummed at the NLL level [18].

Although these studies have shown the power of resummation, the results for thrust and heavy jet mass are not directly applicable to the study of observables at hadron colliders. These event shapes assign a single number to each event and so any large logarithms will involve only a single scale of the event shape. At a hadron collider, the same strategy does not work since there is soft and collinear activity in the initial state, which complicates the calculation of jet mass using effective field theory techniques [19]. To account for initial state radiation, experiments at hadrons colliders use inclusive jet definitions and an explicit veto procedure to reduce the huge QCD background. The calculation of the rate of these events necessarily requires at least two scale – the jet observable and the veto scale. Therefore, it is interesting and non-trivial to try and carry out the resummation of a QCD observable in the presence of a jet veto. In fact, some work in this direction has already been done and it is known that, in this case, a new class of so-called non-global logarithms (NGLs), qualitatively different from the class of logarithms encountered in the resummation of thrust and heavy jet mass, appear that also need to be resummed in certain kinematic limits [20, 21, 22, 23]. Over the last couple of years, there has been renewed theoretical interest in exclusive jet mass distributions and NGLs, both with and without SCET [24, 25, 26, 27, 28, 29, 30, 31].

Unfortunately, the resummation of non-global logarithms is not straightforward (see Refs [20, 22, 28, 32, 33]). One strategy is to find a class of jet observables for which non-global logarithms are not present, i.e. global observables. Global hadronic events shapes were resummed to NLL in [34] and an updated discussion can be found [35]. Some global hadronic event shapes, such as beam thrust [36] or threshold thrust [19] have been resummed to the NNLL level using SCET. This paper will not attempt to resum non-global logarithms, rather it will calculate the full non-global structure for a specific e^+e^- (dijet) event shape with an energy veto. The complications of the initial state for hadronic event shapes is avoided in order to focus on how multiple scales come into play.

In a previous publication, Ref. [27], a simple observable called jet thrust was introduced. It was subsequently studied in Refs. [31, 37]. Jet thrust, denoted τ_ω, is defined as the sum of the masses of the two hardest jets in an event with a veto that restricts the total out-of-jet energy to be less than ω. This observable, being a single variable (once the jet size R and the veto scale ω are fixed) is easy to study qualitatively with one-dimensional plots. It is also free of a subset of non-global logarithms of the form $\ln(m_1/m_2)$. In Ref. [27], it was observed that the soft function for jet thrust seems to refactorize at small R. Heuristic arguments for this refactorization were given, as was a convincing numerical comparison of predictions derived from this refactorization to the singular part of the exact $\mathcal{O}(\alpha_s^2)$ results in QCD derived using the program EVENT2 [38, 39]. The leading non-global logarithms for jet thrust, which were ignored in Ref. [27], were subsequently studied in Refs. [31, 37]. Understanding analytically how the refactorization conjectured in Ref. [27] works, as well as the structure of the subleading non-global contributions, requires the two-loop result. The two-loop result is the subject of the present paper.

In Ref. [28], the exact $\mathcal{O}\left(\alpha_s^2\right)$ hemisphere soft function was calculated. This facilitated an analytical study of the hemisphere mass distribution at NLO. In particular, subleading non-global logarithms were uncovered for the first time, as well as an intricate non-global functional dependence, in sharp contrast to the simple powers of logarithms suggested by [40]. Parts of this result were independently confirmed by two other groups [29, 41].

In this paper, we go one step further and perform the calculation of the soft function for jet mass with a jet veto at $\mathcal{O}(\alpha_s^2)$ using jets of cone size R. This produces an exact form for the singular part of the differential jet thrust distribution. In fact, we present the complete scale-dependent contribution to the integrated jet thrust distribution in the limit of small jet mass. Our result is exact up to the addition of a τ_ω- and ω-independent function of R. This function is calculable, but does not affect the differential distribution, and is therefore not essential for our present purposes. Our results allow us to both test the refactorization ansatz of Ref. [27] and determine the precise form of the non-global terms which arise in the integrated jet thrust distribution.

A number of authors have already observed that there are similarities between the leading NGLs in jet observable distributions defined using narrow anti-k_T jets and the NGLs that appear in the hemisphere mass distribution [26, 27, 29, 31]. The R-dependent coefficient of the leading non-global logarithm

$$f(R) \ln^2 \left(\frac{\omega}{\tau_\omega Q} \right) \tag{1.1}$$

has been calculated for various algorithms, for arbitrary R. However, all existing NGL studies with R dependence focus on the extraction of the leading non-global logarithm, which can be done without a complete two-loop calculation.

In this work, we calculate the *complete* non-global contribution to the integrated jet thrust distribution, up to a scale-independent function of R. Remarkably, in the small R limit, we find that this non-global contribution to the integrated τ_ω distribution can be expressed in terms of the non-global contribution to the integrated hemisphere soft function calculated in Ref. [28]. The correspondence actually reproduces the entire non-global functional form, not just the non-global large logarithms. We also confirm that all of the singular terms in R in the integrated jet thrust distribution can be reproduced by the refactorization formula of Ref. [27], provided that one takes into account the natural R dependence in the argument of the non-global function.

This paper is organized as follows. In Section 2, we briefly review the factorization theorem for the τ_ω distribution and recall the refactorization ansatz for the τ_ω soft function proposed in Ref. [27]. In Section 3, we summarize the calculation of the τ_ω soft function, at times focusing only on those moments of the integrals which ultimately contribute to the scale-dependent terms in the integrated τ_ω distribution. In Section 4, we compare the prediction of SCET to full QCD and use the difference of our results and the prediction of the integrated refactorization ansatz to define the (μ-independent) non-global contribution to the integrated τ_ω distribution. In Section 5, we take the small R limit of the non-global contribution to the integrated τ_ω distribution. We then make precise the correspondence alluded to above between this non-global contribution and the non-global contribution to the integrated hemisphere soft function. Finally, we analyze the NGLs in detail, discuss their R dependence, and present a refined refactorization formula for the τ_ω soft function consistent with our two-loop results. In Section 6, we present our conclusions and discuss some interesting open problems. In Appendix A we present some of the technical details of our calculation of the integrated τ_ω distribution and in Appendix B we collect two rather lengthy analytical expressions that would have been awkward to define in the text where they first appear. A `Mathematica` notebook with these expressions is available upon request.

2. Jet Thrust in SCET

Soft functions are squared matrix elements of Wilson lines integrated against some measurement operator. They are integral to QCD resummation studies and have appeared in the literature in many different contexts [42, 43, 44, 45]. In Ref. [27], an inclusive jet mass observable called jet thrust, τ_ω, was discussed, and after making a conjecture for part of the two-loop soft function, the $\mathcal{O}(\alpha_s^2)$ prediction from SCET for the τ_ω distribution was compared with the output of EVENT2. In this section, we briefly review the definition of

τ_ω and recapitulate the main results of Ref. [27] which sets the stage for the exact results calculated in the present paper.

Jet thrust τ_ω is defined as follows. First, for some multi-jet event at an e^+e^- collider, cluster the particles into jets using some jet algorithm with size parameter R. Define λ to be the energy of the radiation not going into the two hardest jets.[1] Then jet thrust is defined as the sum of the squared masses of the two hardest jets normalized to the center of mass energy if $\lambda < \omega$ and as 0 otherwise. In symbols,

$$\tau_\omega = \frac{M_1^2 + M_2^2}{Q^2}\,\Theta(\omega - \lambda)\,, \tag{2.1}$$

where M_1 and M_2 are the masses of the two hardest jets and $\Theta(x)$ is the unit step function.

In the limit $\tau_\omega \ll 1$ and $\omega/Q \ll 1$, the jet thrust distribution factorizes [27]

$$\begin{aligned}\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_\omega} &= H(Q^2,\mu)\int \mathrm{d}k_L\,\mathrm{d}k_R\,\mathrm{d}M_L^2\,\mathrm{d}M_R^2\,J(M_L^2 - Qk_L,\mu)J(M_R^2 - Qk_R,\mu)\\ &\quad\times\int_0^\omega \mathrm{d}\lambda\,S_R(k_L,k_R,\lambda,\mu)\delta\left(\tau_\omega - \frac{M_L^2+M_R^2}{Q^2}\right)\,,\end{aligned} \tag{2.2}$$

up to power corrections in τ_ω and ω/Q. In Eq. (2.2), the hard function, $H(Q^2,\mu)$, is the same as the one that appears in the factorization theorems for the thrust and heavy jet mass distributions. In principle, one could use R-dependent jet functions [24, 25, 46]; however, the R dependence of such jet functions shows up only in terms non-singular in the jet masses. Therefore, provided we work in a regime that $M_{L,R}^2 \ll Q^2R$ (or $\tau_\omega \ll R$), we are free to use the simpler inclusive jet functions in [25, 46] – we will make this choice. It is worth emphasizing that the soft function in Eq. (2.2), the subject of this paper, is valid for any R, and thus any $\tau_\omega \lesssim R$, and does not depend on whether we use inclusive or R-dependent jet functions.

The distribution of τ_ω depends on the precise jet definition used. In the QCD calculation, the jet definition affects the whole distribution. In the SCET calculation, it only affects the soft function. In Ref. [27], the Cambridge/Aachen (C/A) algorithm was used. The C/A algorithm first calculates the distances between all particles i and j

$$R_{ij} = \frac{1}{2}\Big(1 - \cos\theta_{ij}\Big) \tag{2.3}$$

and then merges the two closest particles into a single particle by adding their four-momenta. The algorithm stops when no two objects are closer than a given jet size, R, to one another. The Cambridge/Aachen algorithm has many appealing qualities and is infrared-safe. The corresponding τ_ω soft function, however, appears challenging to calculate at two loops.

Fortunately, there is a simple cone algorithm one can use instead. We define cone algorithm jet thrust as follows. First, find the thrust axis in the event. Then define the distance $R_{i\,\mathbf{n}}$ ($R_{i\,\bar{\mathbf{n}}}$) between particle i in the event and the thrust axis $\mathbf{n}\,(\bar{\mathbf{n}})$ as

$$R_{i\,\mathbf{n}} = \frac{1}{2}(1 - \cos\theta_{i\,\mathbf{n}}) \qquad\qquad R_{i\,\bar{\mathbf{n}}} = \frac{1}{2}(1 - \cos\theta_{i\,\bar{\mathbf{n}}}), \tag{2.4}$$

[1] λ can also be defined as the energy of the third hardest jet, as it was in [27]. The definitions are equivalent definition as far as the differential τ_ω distribution at two loops is concerned.

where $\theta_{i\,\mathbf{n}}$ ($\theta_{i\,\bar{\mathbf{n}}}$) is the angle between particle i and $\mathbf{n}\,(\bar{\mathbf{n}})$. If $R_{i\,\mathbf{n}}$ ($R_{i\,\bar{\mathbf{n}}}$) is less than R, then cluster particle i into the right (left) jet. Because we use the thrust axis, this algorithm is infrared-safe at e^+e^- colliders.

The jet thrust factorization formula bears a close resemblance to the factorization formula for thrust [10, 17]. The only difference is in the soft function. In the case of thrust, the soft function is just the hemisphere one, $S_{\text{hemi}}(k_L, k_R, \mu)$, and depends on two scales, k_L and k_R. The τ_ω soft function depends on the scales λ and R as well and it is therefore significantly more complicated. The soft function is defined as

$$S_R(k_L, k_R, \lambda, \mu) = \frac{1}{N_c} \sum_{X_s} \langle 0|\bar{Y}_{\bar{n}} Y_n \widehat{\mathcal{M}}(k_L, k_R, \lambda)|X_s\rangle\langle X_s|Y_n^\dagger \bar{Y}_{\bar{n}}^\dagger|0\rangle, \tag{2.5}$$

where $\widehat{\mathcal{M}}(k_L, k_R, \lambda)$ implements the measurement via

$$\widehat{\mathcal{M}}(k_L, k_R, \lambda)|X_s\rangle = \delta\Big(k_R - n\cdot P^R_{X_s}\Big)\delta\Big(k_L - \bar{n}\cdot P^L_{X_s}\Big)\delta\Big(\lambda - E_{X_s}\Big)|X_s\rangle, \tag{2.6}$$

and $P^R_{X_s}$ $\big(P^L_{X_s}\big)$ is the four-momentum of the soft radiation clustered into the right (left) jet, and E_{X_s} is the total energy of the out-of-jet radiation. At zeroth order, the soft function is simply a delta function

$$S_R(k_L, k_R, \lambda, \mu) = \delta(k_L)\delta(k_R)\delta(\lambda), \tag{2.7}$$

reflecting the fact that there is no soft radiation. The first non-trivial corrections come at $\mathcal{O}(\alpha_s)$ (what is traditionally called leading-order (LO)) and were calculated in Refs. [25] and [27].

Using the renormalization-group (RG) invariance of the factorization formula, one can show that the soft function must have the form

$$\begin{aligned} S_R(k_L, k_R, \lambda, \mu) &= S_\mu(k_L, \mu) S_\mu(k_R, \mu) \otimes S_f(\lambda, k_L, k_R) \\ &= \int dk'_L dk'_R\; S_\mu(k'_L, \mu) S_\mu(k'_R, \mu) S_f(\lambda, k_L - k'_L, k_R - k'_R) \end{aligned} \tag{2.8}$$

where $\otimes$ denotes a convolution (it is a product in Laplace space). Here $S_\mu(k,\mu)$ is the soft evolution kernel that precisely cancels the RG evolution of the jet and hard functions and explicit expressions can be found in [12, 17, 40]. An equally valid form can be written as

$$\begin{aligned} S_R(k_L, k_R, \lambda, \mu) &= S_R^{\text{in}}(k_L, \mu) S_R^{\text{in}}(k_R, \mu) S_R^{\text{out}}(\lambda, \mu) \otimes S_R^f(\lambda, k_L, k_R) \\ &= \int dk'_L dk'_R d\lambda'\; S_R^{\text{in}}(k'_L, \mu) S_R^{\text{in}}(k'_R, \mu) S_R^{\text{out}}(\lambda', \mu) S_R^f(\lambda - \lambda', k_L - k'_L, k_R - k'_R) \end{aligned} \tag{2.9}$$

There is not actually much content in this separation until the objects involved are given precise definitions. For example, we can take $S_R^{\text{out}}(\lambda,\mu) = \delta(\lambda)$, thus reducing Eq. (2.9) to Eq. (2.8). A refactorization formula like this has appeared multiple times in the literature [24, 25, 27, 31]. However, without a precise statement about $S_R^{\text{out}}(\lambda,\mu)$, Eq. (2.9) actually has less content than Eq. (2.8). One choice of a definition for these objects were given

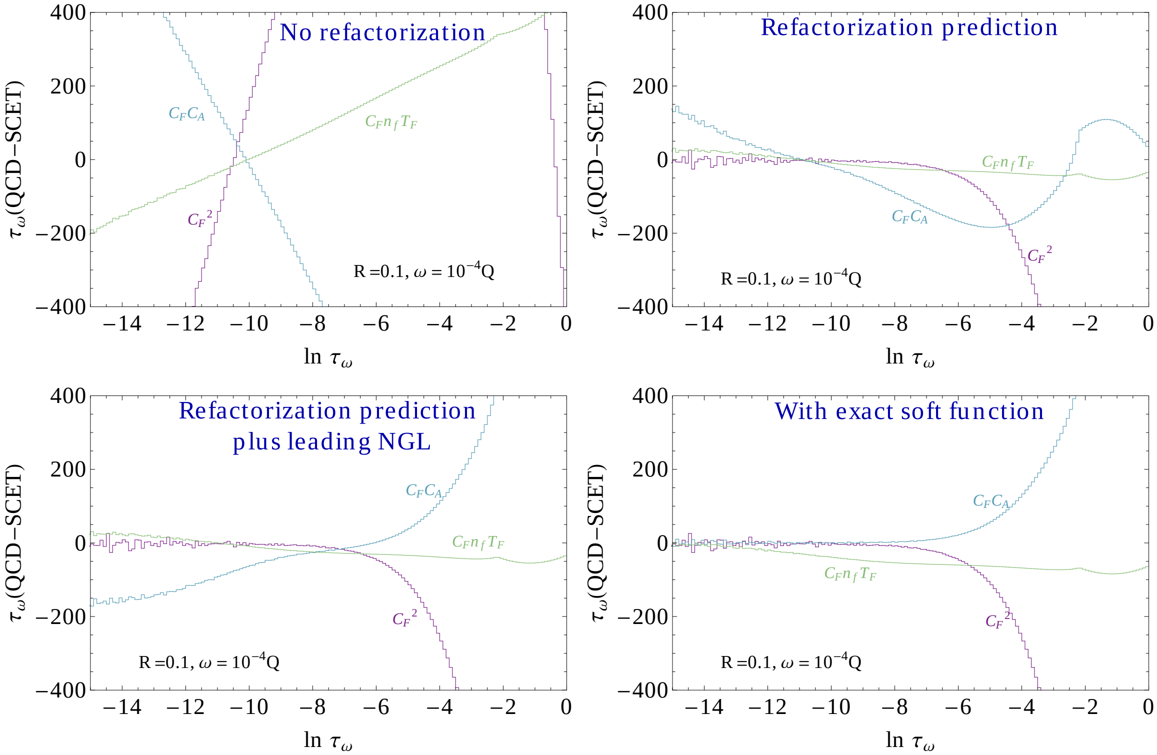

Figure 1: The difference between coefficient of $\left(\frac{\alpha_s}{2\pi}\right)^2$ in $d\sigma/d\ln(\tau_\omega)$ in full QCD and in SCET. The first panel shows the difference using only the NNLL pediction with SCET, which clearly does not get correct all of the singular terms. The second panel shows the improvement using the refactorization and Γ_{cusp} ansatz of Ref. [27]. The third panel adds the leading non-global double-logarithmic term, which was also known since [27]. The final panel uses the the precise $\mathcal{O}(\alpha_s^2)$ results derived in this paper. All plots have $R = 0.1$ and $\omega = 10^{-4}Q$.

in [27]. $S_R^{\text{in}}(k,\mu)$ was defined to account for all radiation inside a given jet of size R, whereas $S_R^{\text{out}}(\lambda,\mu)$ accounted for the radiation outside of all jets. The explicit definitions for $S_R^{\text{in}}(k,\mu)$ and $S_R^{\text{out}}(\lambda,\mu)$ are given by replacing the measurement function, $\widehat{\mathcal{M}}(k_L,k_R,\lambda)$, in Eq. (2.6) with the following operators,

$$\widehat{\mathcal{M}}^{\text{in}}(k,\mu)|X_s\rangle = \delta(k - n\cdot P_{X_s})|X_s\rangle\,, \tag{2.10}$$

and

$$\widehat{\mathcal{M}}^{\text{out}}(\lambda,\mu)|X_s\rangle = \delta(\lambda - E_{X_s})|X_s\rangle\,. \tag{2.11}$$

In Eq. (2.10) above, P_{X_s} is the four-momentum of the soft radiation clustered into the jet under consideration. The mixed "in-out" radiation is accounted for by the function $S_R^f(\lambda,k_L,k_R)$.

In Ref. [27] it was argued that, in the small R limit, $S_R^{\text{in}}(k,\mu)$ and $S_R^{\text{out}}(\lambda,\mu)$ should act like soft functions in their own right, with anomalous dimensions. Only the function $S_R^{\text{in}}(k,\mu)$ can have a μ-dependent anomalous dimension, since the soft-collinear region of phase-space lies within the jet. Thus, the anomalous dimension of $S_R^{\rm in}(k,\mu)$ has a cusp part which agrees with that of $S_\mu(k,\mu)$. In Ref. [27] it was conjectured that the regular anomalous dimensions of the in and out soft functions, at small R, should be

$$\gamma_S^{\rm in} = \frac{1}{2}\left(\gamma_S + \Gamma_{\rm cusp}\ln\left(\frac{R}{1-R}\right)\right) \qquad \text{and} \qquad \gamma_S^{\rm out} = -\Gamma_{\rm cusp}\ln\left(\frac{R}{1-R}\right), \tag{2.12}$$

where $\Gamma_{\rm cusp}$ is the usual cusp anomalous dimension and γ_S is the regular anomalous dimension of the thrust soft function [11]. Note that this ansatz has the cusp anomalous dimension generating something other than $\ln(\mu)$ in the anomalous dimension. The connection of the cusp anomalous dimension to the R-dependence at LO is also implicit in Refs. [24, 25]. Having $S_R^{\rm in}(k_L,\mu)$, $S_R^{\rm in}(k_R,\mu)$, and $S_R^{\rm out}(\lambda,\mu)$ together with their anomalous dimensions allows us define $S_R^f(\lambda,k_L,k_R)$ precisely: $S_R^f(\lambda,k_L,k_R)$ is everything that is left over from refactorization with the anomalous dimensions in Eqs. (2.12).

Using the above ingredients, and setting $S_R^f(\lambda,k_L,k_R)=\delta(k_L)\delta(k_R)\delta(\lambda)$, Ref. [27] predicted the differential τ_ω distribution to $\mathcal{O}(\alpha_s^2)$ and explicitly compared the result to the output of EVENT2. The predictions made from refactorization come very close to reproducing the output of EVENT2 in the regime where the factorization formula of Eq. (2.2) is valid. In the left panel of Figure 1, we reproduce Figure 6 from that paper.

Figure 1 shows the differences between the NNLL prediction from SCET, the prediction from refactorization, and the precise $\mathcal{O}(\alpha_s^2)$ results, containing the full non-global structure, derived in this paper. The plots displayed in Figure 1 take $R=0.1$, $\omega=10^{-4}Q$, and are separated by color structure. If the singular terms in the QCD distribution are correctly reproduced by the effective theory, the plots in Figure 1 should tend to zero as $\tau_\omega\to 0$. In both panels this vanishing is qualitatively visible. However, the fact that the lower right panel vanishes best is a sign that the non-global structure is an important (but subleading) effect. For example, the leading non-global logarithm, which was set to zero in Ref. [27], should contribute a term of the schematic form $\ln(\tau_\omega)/\tau_\omega$ to the differential distribution. Two papers, Refs. [31] and [37], have pointed out the importance of these NGLs. Such terms, however, were beyond the scope of the analyses carried out in Ref. [27].

To gain a better understanding of the refactorization conjecture and the NGLs, we explicitly calculate the τ_ω soft function to $\mathcal{O}(\alpha_s^2)$ in the next section, with the understanding that our ultimate goal will be to study the scale-dependent contributions to the integrated τ_ω distribution. As was observed in Ref. [31], the advantage of approaching the calculation in this way is that both the μ-dependent terms and NGLs of the soft contributions are completely transparent at the level of the integrated distribution. There are three color structures at $\mathcal{O}(\alpha_s^2)$: C_F^2, C_FC_A, and $C_Fn_fT_F$. The C_F^2 terms are uniquely determined by the Abelian exponentiation theorem [47, 48] and we therefore only present results for the C_FC_A and $C_Fn_fT_F$ color structures.

3. Calculation of the τ_ω Soft Function and Moments Thereof

The soft-function is usually expanded in α_s as

$$S_R(k_L, k_R, \lambda, \mu) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n S_R^{(n)}(k_L, k_R, \lambda, \mu). \tag{3.1}$$

In this section we summarize the calculation of the two-loop τ_ω soft function, $S_R^{(2)}$, at times focusing only on those moments of the soft function which will be needed later on when we discuss the jet thrust distribution and its non-global structure. We begin with a review of the one-loop soft function, establishing our notation, and then describe the NLO results For readers who are only interested in the interpretation of the results, this section can be skipped.

3.1 LO (one-loop) soft-function

The tree-level result is trivially given as a product of delta functions,

$$S_R^{(0)}(k_L, k_R, \lambda, \mu) = \delta(k_L)\delta(k_R)\delta(\lambda)\,. \tag{3.2}$$

At first order in α_s, the τ_ω soft function in $d = 4 - 2\epsilon$ dimensions is given by the integral[2]

$$S_R^{(1)}(k_L, k_R, \lambda, \mu) = (4\pi)^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{4iF_1(k_L, k_R, \lambda)}{q^+ q^-}\,. \tag{3.3}$$

In Eq. (3.3), $q^+ = n \cdot q$, $q^- = \bar{n} \cdot q$, and $F_1(k_L, k_R, \lambda)$ is the function that implements the one-particle phase-space cuts [49]:

$$\begin{aligned} F_1(k_L, k_R, \lambda) \,=\, -2\pi i\, \delta\left(q^2\right) \Theta\left(q^0\right) \Big[& \Theta_n^q \delta(k_R - q^+)\delta(k_L)\delta(\lambda) + \Theta_{\bar{n}}^q \delta(k_L - q^-)\delta(k_R)\delta(\lambda) \\ & + \Theta_{\mathrm{out}}^q \delta\left(\lambda - \frac{q^+ + q^-}{2}\right) \delta(k_L)\delta(k_R) \Big]. \end{aligned} \tag{3.4}$$

In Eq. (3.4),

$$\begin{aligned} \Theta_n^q &= \Theta\left(rq^- - q^+\right) \\ \Theta_{\bar{n}}^q &= \Theta\left(rq^+ - q^-\right) \\ \Theta_{\mathrm{out}}^q &= \Theta\left(q^- - rq^+\right) \Theta\left(q^+ - rq^-\right)\,, \end{aligned} \tag{3.5}$$

where

$$r \equiv \frac{R}{1 - R}\,. \tag{3.6}$$

This last abbreviation occurs frequently throughout the rest of this paper.

[2] In this paper we suppress the $i\varepsilon$ prescription. This is appropriate since we are dealing only with real radiation.

Carrying out the integral in Eq. (3.3), we find that, up to and including terms of $\mathcal{O}(\alpha_s)$, the τ_ω soft function can be written as

$$S_R(k_L, k_R, \lambda, \mu) = S_R^{\rm in}(k_L, \mu) S_R^{\rm in}(k_R, \mu) S_R^{\rm out}(\lambda, \mu), \tag{3.7}$$

where

$$S_R^{\rm in}(k, \mu) = \delta(k) + \left(\frac{\alpha_s}{4\pi}\right) \frac{4C_F e^{\gamma_E \epsilon} \mu^{2\epsilon} r^\epsilon}{k^{1+2\epsilon}\epsilon\,\Gamma(1-\epsilon)} \tag{3.8}$$

and

$$\begin{aligned} S_R^{\rm out}(\lambda, \mu) &= \delta(\lambda) + \left(\tfrac{\alpha_s}{4\pi}\right) \left(\tfrac{8C_F e^{\gamma_E \epsilon}\mu^{2\epsilon}}{(1-\epsilon)\epsilon\,\Gamma(1-\epsilon)(1+r)(2\lambda)^{1+2\epsilon}}\right) \left(\tfrac{r}{(1+r)^2}\right)^{-\epsilon} \\ &\times \Bigg\{ \left(\tfrac{r}{1+r}\right)^\epsilon \left[\epsilon\, {}_2F_1\left(1-\epsilon, 1+\epsilon, 2-\epsilon; \tfrac{1}{1+r}\right) + (\epsilon - 1)(1+r)\, {}_2F_1\left(-\epsilon, \epsilon, 1-\epsilon; \tfrac{1}{1+r}\right)\right] \\ &- \left(\tfrac{1}{1+r}\right)^\epsilon \left[\epsilon\, r\, {}_2F_1\left(1-\epsilon, 1+\epsilon, 2-\epsilon; \tfrac{r}{1+r}\right) + (\epsilon-1)(1+r)\, {}_2F_1\left(-\epsilon, \epsilon, 1-\epsilon; \tfrac{r}{1+r}\right)\right]\Bigg\} \end{aligned} \tag{3.9}$$

are the relevant expressions calculated through $\mathcal{O}(\alpha_s)$ and to all orders in ϵ. In Eq. (3.9), ${}_2F_1(a, b, c; z)$ is the Gauss hypergeometric function. Expanding the above expressions in ϵ reproduces the results published in Ref. [25, 27].

The form of Eq. (3.9) suggests that the R dependence of the out-of-jet soft function is not simple, even at $\mathcal{O}(\alpha_s)$. On the other hand, we note that the $\mathcal{O}(\alpha_s)$ R dependence of $S_R^{\rm in}(k, \mu)$ is just r^ϵ; it factorizes from the rest of the $\mathcal{O}(\alpha_s)$ expression. It turns out that the full answer is simply r^ϵ times the result obtained several years ago [10, 17] for the LO hemisphere soft function. As will be shown below, a similar phenomenon occurs at NLO for the part of the in-jet soft function that dominates in the small R limit. Stated more precisely, at $\mathcal{O}(\alpha_s^n)$ the contribution with n soft partons clustered into the same jet is equal to the analogous contribution to the hemisphere soft function multiplied by a factor of $r^{n\epsilon}$. This simple relation has its origin in the fact that the integral representation of this part of the $\mathcal{O}(\alpha_s^n)$ τ_ω soft function has nice transformation properties under the rescaling

$$n_\mu = \sqrt{r} n'_\mu \qquad\qquad \bar{n}_\mu = \frac{1}{\sqrt{r}} \bar{n}'_\mu\,. \tag{3.10}$$

This observation will be discussed in more detail in Section 5.

3.2 NLO (two-loop) soft function

In this section we present the calculation of the NLO τ_ω soft function, focusing at times on those moments of the soft function which we will need in subsequent sections. A moment's thought reveals that the squared matrix elements written down for the hemisphere soft function in Ref. [28] can be used for the τ_ω soft function as well. The two-particle phase-space cuts, however, are significantly more complicated. As in the hemisphere case, it is convenient to split the calculation up according to how many partons cross the final state cut.

3.2.1 One-parton contributions

There are two different classes of contributions which have a single gluon crossing the final state cut, the real-virtual contributions and the contributions proportional to the LO τ_ω soft function, derived by expanding the charge renormalization constant to $\mathcal{O}(\alpha_s)$. In what follows, $S_R^{\text{in},(1)}(k,\mu)$ and $S_R^{\text{out},(1)}(\lambda,\mu)$ are simply the $\mathcal{O}(\alpha_s)$ terms in Eqs. (3.8) and (3.9) respectively.

The real-virtual interference terms can be derived by judiciously combining the analogous result derived in Ref. [28] for the hemisphere soft function and the all-orders-in-ϵ LO results collected in Section 3.1. We find that the result can be written as

$$S_R^{R-V}(k_L,k_R,\lambda,\mu) = S^V_{C_A}(k_L,k_R)\,r^{2\epsilon}\,\delta(\lambda) + C_A\, S_R^{\text{out},(1)}(\lambda,\mu)\Big|_{\epsilon\to 2\epsilon}\delta(k_L)\delta(k_R) \times\frac{\pi\Gamma(2+\epsilon)\Gamma(1-2\epsilon)\cot(\pi\epsilon)\Gamma(-\epsilon)^2}{\Gamma(1-\epsilon)\Gamma(-2\epsilon)(1+\epsilon)}\,, \tag{3.11}$$

where $S^V_{C_A}(k_L,k_R)$ is defined in Eqs. (20) and (21) of Ref. [28]. It is worth pointing out that the in-jet terms, $S^V_{C_A}(k_L,k_R)\,r^{2\epsilon}\,\delta(\lambda)$, are simply derived by appropriately rescaling the real-virtual hemisphere integrals. For the sake of completeness, we recapitulate the result for $S^V_{C_A}(k_L,k_R)$ derived in Ref. [28]:

$$S^{\text{V}}_{C_A}(k_L,k_R) = C_F\,C_A\,\mu^{4\epsilon}\left(\frac{\delta(k_R)}{k_L^{1+4\epsilon}}+\frac{\delta(k_L)}{k_R^{1+4\epsilon}}\right)\left(-\frac{4}{\epsilon^3}+\frac{2\pi^2}{\epsilon}+\frac{32\zeta_3}{3}-\epsilon\frac{\pi^4}{30}+\mathcal{O}\left(\epsilon^2\right)\right)\,. \tag{3.12}$$

Given the results of Section 3.1, it is trivial to write down the charge renormalization contributions to the NLO τ_ω soft function:

$$S_R^{\text{Ren}}(k_L,k_R,\lambda,\mu) = -\frac{\beta_0}{\epsilon}\Bigg[\left(S_R^{\text{in},(1)}(k_L,\mu)\delta(k_R)+S_R^{\text{in},(1)}(k_R,\mu)\delta(k_L)\right)\delta(\lambda) +S_R^{\text{out},(1)}(\lambda,\mu)\delta(k_L)\delta(k_R)\Bigg], \tag{3.13}$$

where $\beta_0=\frac{11}{3}C_A-\frac{4}{3}n_fT_F$ is the leading order β-function.

3.2.2 Two-parton contributions

We now turn to the contributions with two partons crossing the final state cut. The sum of all such contributions to the τ_ω soft function has the form:

$$S_R^{\text{Real}}(k_L,k_R,\lambda,\mu) = (4\pi)^4\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{2\epsilon}\int\frac{\mathrm{d}^dq}{(2\pi)^d}\frac{\mathrm{d}^dk}{(2\pi)^d}\left(\frac{1}{2!}I_{C_A}+I_{n_f}\right)F_2(k_L,k_R,\lambda)\,, \tag{3.14}$$

where $\frac{1}{2!}I_{C_A}$ and I_{n_f} are the integrands for the C_FC_A and $C_Fn_fT_F$ color structures, for which explicit formulas can be found in Eqs. (16) and (27) of Ref. [28]. The statistical factor of $1/2!$ in front of I_{C_A} has its origin in the fact that, for the C_FC_A color structure, the two-parton contributions have two indistinguishable gluons in the final state. As mentioned above, the two-parton phase-space cuts are much more complicated than the one-parton ones:

$$
\begin{aligned}
&F_2(k_L,k_R,\lambda) = (-2\pi i)^2\delta\left(k^2\right)\Theta\left(k^0\right)\delta\left(q^2\right)\Theta\left(q^0\right)\\
&\times\Bigg[\Theta_n^q\Theta_n^k\delta(k_R-k^+-q^+)\delta(k_L)\delta(\lambda)+\Theta_{\bar n}^q\Theta_{\bar n}^k\delta(k_L-k^--q^-)\delta(k_R)\delta(\lambda)\\
&+\Theta_n^k\Theta_{\bar n}^q\delta(k_R-k^+)\delta(k_L-q^-)\delta(\lambda)+\Theta_n^q\Theta_{\bar n}^k\delta(k_L-k^-)\delta(k_R-q^+)\delta(\lambda)\\
&+\Theta_n^k\Theta_{\rm out}^q\delta(k_R-k^+)\delta\left(\lambda-\frac{q^++q^-}{2}\right)\delta(k_L)+\Theta_n^q\Theta_{\rm out}^k\delta(k_R-q^+)\delta\left(\lambda-\frac{k^++k^-}{2}\right)\delta(k_L)\\
&+\Theta_{\bar n}^k\Theta_{\rm out}^q\delta(k_L-k^-)\delta\left(\lambda-\frac{q^++q^-}{2}\right)\delta(k_R)+\Theta_{\bar n}^q\Theta_{\rm out}^k\delta(k_L-q^-)\delta\left(\lambda-\frac{k^++k^-}{2}\right)\delta(k_R)\\
&+\Theta_{\rm out}^k\Theta_{\rm out}^q\delta\left(\lambda-\frac{q^++k^++q^-+k^-}{2}\right)\delta(k_L)\delta(k_R)\Bigg]. \qquad (3.15)
\end{aligned}
$$

There are four independent cases to consider:

1. The same-side in-in contributions, where both partons are in the same jet (the second line of Eq. (3.15)).
2. The opposite-side in-in contributions, where one parton is in the $\mathbf{n}$ jet and the other is in the $\bar{\mathbf{n}}$ jet (the third line of Eq. (3.15)).
3. The in-out contributions, where one parton is in a jet and the other is outside of both jets (the fourth and fifth lines of Eq. (3.15)).
4. The out-out contribution, where both partons are outside of all jets (the last line of Eq. (3.15)).

We consider each class of contributions in turn.

Naïvely, one might expect the calculation of the same-side in-in contributions to be challenging since the evaluation of the analogous hemisphere integrals was by far the most technically demanding part of the calculation described in Ref. [28]. Fortunately, it turns out that we can recycle the same-side in-in contributions to the hemisphere soft function and obtain the desired result for free. The result of interest can be derived by appropriately rescaling the analogous hemisphere integrals. For arbitrary R we find

$$
\begin{aligned}
&S_R^{r_1}(k_L,k_R,\lambda,\mu) = \left(\frac{\delta(k_R)}{k_L^{1+4\epsilon}}+\frac{\delta(k_L)}{k_R^{1+4\epsilon}}\right)\mu^{4\epsilon}r^{2\epsilon}\delta(\lambda)\Bigg\{C_FC_A\Bigg[\frac{4}{\epsilon^3}+\frac{22}{3\epsilon^2}+\Bigg(\frac{134}{9}\\
&-\frac{4\pi^2}{3}\Bigg)\frac{1}{\epsilon}+\frac{772}{27}+\frac{11\pi^2}{9}-\frac{116\zeta_3}{3}+\epsilon\left(\frac{4784}{81}+\frac{67\pi^2}{27}-\frac{137\pi^4}{90}+\frac{484\zeta_3}{9}\right)+\mathcal{O}\left(\epsilon^2\right)\Bigg]\\
&+C_Fn_fT_F\left[-\frac{8}{3\epsilon^2}-\frac{40}{9\epsilon}-\frac{152}{27}-\frac{4\pi^2}{9}-\epsilon\left(\frac{952}{81}+\frac{20\pi^2}{27}+\frac{176\zeta_3}{9}\right)+\mathcal{O}\left(\epsilon^2\right)\right]\Bigg\}, \qquad (3.16)
\end{aligned}
$$

where we have made use of Eqs. (18) and (29) in Ref. [28].

For the opposite-side in-in contributions, the result can be expressed as

$$S_R^{r_2}(k_L, k_R, \lambda, \mu) = \frac{\mu^{4\epsilon}}{(k_L k_R)^{1+2\epsilon}}\,\delta(\lambda) \times \left[C_F C_A\, f_{C_A}\left(\frac{k_L}{k_R}, r\right) + C_F n_f T_F\, f_{n_f}\left(\frac{k_L}{k_R}, r\right)\right], \tag{3.17}$$

where $f_{C_A}(z,r)$ and $f_{n_f}(z,r)$ are functions with ϵ expansions that begin at $\mathcal{O}\left(\epsilon^0\right)$. In other words,

$$\begin{aligned} f_{C_A}(z,r) &= f^{(0)}_{C_A}(z,r) + f^{(1)}_{C_A}(z,r)\,\epsilon + \mathcal{O}\left(\epsilon^2\right) \\ f_{n_f}(z,r) &= f^{(0)}_{n_f}(z,r) + f^{(1)}_{n_f}(z,r)\,\epsilon + \mathcal{O}\left(\epsilon^2\right)\,. \end{aligned} \tag{3.18}$$

It is worth pointing out that, in the limit $R \to 0$, these functions are suppressed relative to the other contributions discussed in this section. In order to obtain exact results for the scale-dependent contributions to the integrated τ_ω distribution, we need several moments of the functions defined implicitly above:

$$\begin{aligned} f^{(0)}_{C_A}(0,r) &= 16\mathrm{Li}_2\left(r^2\right)\,, \\ f^{(0)}_{n_f}(0,r) &= 0\,, \\ f^{(1)}_{C_A}(0,r) &= -16\mathrm{Li}_3\left(1-r^2\right) + 16\mathrm{Li}_3\left(\frac{r^2}{r^2-1}\right) + 32\mathrm{Li}_2\left(r^2\right)\ln\left(\frac{r}{1-r^2}\right) \\ &\quad -\frac{8}{3}\ln^3\left(1-r^2\right) - 16\ln(r)\ln^2\left(1-r^2\right) + \frac{8}{3}\pi^2\ln\left(1-r^2\right) + 16\zeta_3\,, \\ f^{(1)}_{n_f}(0,r) &= 0\,, \end{aligned}$$

$$\int_0^1 \frac{\mathrm{d}z}{z}\left(\frac{2f^{(0)}_{C_A}\left(\frac{z}{2-z}, r\right)}{2-z} - f^{(0)}_{C_A}(0,r)\right) = -\frac{88\mathrm{Li}_2\left(r^2\right)}{3} + 8\mathrm{Li}_3\left(r^2\right)$$
$$-8\mathrm{Li}_2\left(r^2\right)\ln(r) + 16\ln(2)\mathrm{Li}_2\left(r^2\right) + \frac{8r^2}{3\left(r^2-1\right)} - \frac{16r^2\ln(r)}{3\left(r^2-1\right)^2} - \frac{88}{3}\ln(r)\ln\left(1-r^2\right)\,,$$

and

$$\int_0^1 \frac{\mathrm{d}z}{z}\left(\frac{2f^{(0)}_{n_f}\left(\frac{z}{2-z}, r\right)}{2-z} - f^{(0)}_{n_f}(0,r)\right) = \frac{32\mathrm{Li}_2\left(r^2\right)}{3} - \frac{16r^2}{3\left(r^2-1\right)} + \frac{32r^2\ln(r)}{3\left(r^2-1\right)^2} + \frac{32}{3}\ln(r)\ln\left(1-r^2\right)\,. \tag{3.19}$$

The calculation of the in-out contributions proceeds similarly. However, in this case, the results are significantly more complicated. As we shall see, these contributions lead to terms in the integrated τ_ω distribution that depend on $\frac{\tau_\omega Q}{2\omega}$ in a highly non-trivial way.

The result can be written as

$$
\begin{aligned}
S_R^{r_3}(k_L,k_R,\lambda,\mu) \;=\; & \frac{\mu^{4\epsilon}}{(2\lambda)^{1+2\epsilon}}\Bigg\{\frac{\delta(k_R)}{k_L^{1+2\epsilon}}\left[C_F C_A\, g_{C_A}\left(\frac{k_L}{2\lambda},r\right)\right.\\
&\left.+C_F n_f T_F\, g_{n_f}\left(\frac{k_L}{2\lambda},r\right)\right]+(k_L\leftrightarrow k_R)\Bigg\}\,, \qquad (3.20)
\end{aligned}
$$

where $g_{C_A}(z,r)$ and $g_{n_f}(z,r)$ are functions with ϵ expansions that begin at $\mathcal{O}(\epsilon^0)$. In other words,

$$
\begin{aligned}
g_{C_A}(z,r) &= g_{C_A}^{(0)}(z,r)+g_{C_A}^{(1)}(z,r)\,\epsilon+\mathcal{O}(\epsilon^2)\\
g_{n_f}(z,r) &= g_{n_f}^{(0)}(z,r)+g_{n_f}^{(1)}(z,r)\,\epsilon+\mathcal{O}(\epsilon^2)\,. \qquad (3.21)
\end{aligned}
$$

In order to obtain exact results for the scale-dependent contributions to the integrated τ_ω distribution, we need several moments of the functions defined implicitly above:

$$
\begin{aligned}
g_{C_A}^{(0)}(0,r) &= \frac{16\pi^2}{3}-64\mathrm{Li}_2(r)-64\mathrm{Li}_2(-r)\,,\\
g_{n_f}^{(0)}(0,r) &= 0\,,\\
g_{C_A}^{(1)}(0,r) &= -16\mathrm{Li}_3(r^2)+128\mathrm{Li}_3(1-r)+128\mathrm{Li}_3\left(\frac{r}{r+1}\right)-128\mathrm{Li}_2(-r)\ln(r+1)\\
&\quad+64\mathrm{Li}_2(-r)\ln(r)+128\mathrm{Li}_2(r)\ln(1-r)-\frac{64}{3}\ln^3(r+1)+64\ln^2(1-r)\ln(r)\\
&\quad-\frac{64}{3}\pi^2\ln(1-r)-96\zeta_3\,,\\
g_{n_f}^{(1)}(0,r) &= 0\,,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1\frac{\mathrm{d}z}{z}\left(g_{C_A}^{(0)}(z,r)+g_{C_A}^{(0)}(1/z,r)-2g_{C_A}^{(0)}(0,r)\right)=\frac{352\mathrm{Li}_2(r^2)}{3}-16\mathrm{Li}_3(r^2)\\
&-64\mathrm{Li}_3(r)+64\mathrm{Li}_3\left(\frac{1}{r+1}\right)+64\mathrm{Li}_3\left(\frac{r}{r+1}\right)+64\mathrm{Li}_2(-r)\ln(r)+64\mathrm{Li}_2(r)\ln(r)\\
&-\frac{16(r^2+1)}{3(r^2-1)}+\frac{64r^2\ln(r)}{3(r^2-1)^2}-\frac{64}{3}\ln^3(r+1)+32\ln(r)\ln^2(r+1)+\frac{352}{3}\ln(r)\ln(r+1)\\
&\qquad+\frac{32}{3}\pi^2\ln(r+1)+\frac{352}{3}\ln(1-r)\ln(r)-32\zeta_3-\frac{176\pi^2}{9}\,,\\
&\int_0^1\frac{\mathrm{d}z}{z}\left(g_{n_f}^{(0)}(z,r)+g_{n_f}^{(0)}(1/z,r)-2g_{n_f}^{(0)}(0,r)\right)=-\frac{128\mathrm{Li}_2(r^2)}{3}-\frac{128r^2\ln(r)}{3(r^2-1)^2}\\
&\qquad+\frac{64\pi^2}{9}+\frac{32(r^2+1)}{3(r^2-1)}-\frac{128}{3}\ln(r)\ln(1-r^2)\,,\\
&\int_1^x\frac{\mathrm{d}z_2}{z_2}\int_0^1\frac{\mathrm{d}z_1}{z_1}\left(g_{C_A}^{(0)}\left(\frac{z_2}{z_1},r\right)-g_{C_A}^{(0)}(0,r)\right)=\chi_{C_A}(x,r)-\chi_{C_A}(1,r)\,,
\end{aligned}
$$

and

$$
\int_1^x\frac{\mathrm{d}z_2}{z_2}\int_0^1\frac{\mathrm{d}z_1}{z_1}\left(g_{n_f}^{(0)}\left(\frac{z_2}{z_1},r\right)-g_{n_f}^{(0)}(0,r)\right)=\chi_{n_f}(x,r)-\chi_{n_f}(1,r)\,. \qquad (3.22)
$$

The functions $\chi_{C_A}(x,r)$ and $\chi_{n_f}(x,r)$ are given in Appendix B in terms of an appropriate set of one- and two-dimensional harmonic polylogarithms.[3]

Finally, for the out-out contribution, the result is

$$
\begin{aligned}
S_R^{\mathrm{r}_4}(k_L,k_R,\lambda,\mu) &= \frac{\mu^{4\epsilon}}{(2\lambda)^{1+4\epsilon}}\delta(k_L)\delta(k_R)\Bigg\{C_F C_A\Bigg[-\frac{32\ln(r)}{\epsilon^2}+\frac{1}{\epsilon}\Bigg(16\pi^2\\
&-\frac{176\ln(r)}{3}+32\ln^2(r)+64\mathrm{Li}_2(-r)-64\mathrm{Li}_2(r)\Bigg)+64\mathrm{Li}_3\left(\frac{r^2}{r^2-1}\right)-\frac{704\mathrm{Li}_2(r)}{3}\\
&+256\mathrm{Li}_3(1-r)+128\mathrm{Li}_3(r)+128\mathrm{Li}_3\left(\frac{1}{r+1}\right)-128\mathrm{Li}_3\left(\frac{r}{r+1}\right)-128\mathrm{Li}_2(-r)\ln(1-r)\\
&+128\mathrm{Li}_2(r)\ln(1-r)-64\mathrm{Li}_2(-r)\ln(r)+128\mathrm{Li}_2(-r)\ln(r+1)+64\mathrm{Li}_2(r)\ln(r)\\
&-128\mathrm{Li}_2(r)\ln(r+1)-\frac{16\left(r^2+1\right)}{3-3r^2}-\frac{64r^2\ln(r)}{3\left(r^2-1\right)^2}-\frac{32}{3}\ln^3(1-r)-\frac{64\ln^3(r)}{3}\\
&-\frac{32}{3}\ln^3(r+1)+128\ln(r)\ln^2(1-r)-32\ln(r+1)\ln^2(1-r)-32\ln^2(r+1)\ln(1-r)\\
&+\frac{176\ln^2(r)}{3}+64\ln(r)\ln^2(r+1)-\frac{352}{3}\ln(r)\ln(1-r)-\frac{128}{3}\pi^2\ln(1-r)+\frac{32}{3}\pi^2\ln(r)\\
&-\frac{1072\ln(r)}{9}-\frac{352}{3}\ln(r)\ln(r+1)+\frac{64}{3}\pi^2\ln(r+1)-128\zeta_3+\frac{352\pi^2}{9}+\mathcal{O}(\epsilon)\Bigg]\\
&+C_F n_f T_F\Bigg[\frac{64\ln(r)}{3\epsilon}+\frac{256\mathrm{Li}_2(r)}{3}+\frac{32(1+r^2)}{3\left(1-r^2\right)}+\frac{128r^2\ln(r)}{3\left(1-r^2\right)^2}-\frac{64\ln^2(r)}{3}+\frac{320\ln(r)}{9}\\
&+\frac{128}{3}\ln(r)\ln\left(\frac{1-r}{r+1}\right)+\frac{256}{3}\ln(r)\ln(r+1)-\frac{128\pi^2}{9}+\mathcal{O}(\epsilon)\Bigg]\Bigg\}. \qquad (3.23)
\end{aligned}
$$

The calculation of this contribution is actually quite involved; in the integrals that lead to Eq. (3.23), there is explicit R dependence (which does not factorize from the rest of the expression) and one has to use sector decomposition [52] in a non-trivial way to deal with the phase-space singularities. As we shall see later on, integrating the out-out contribution to the τ_ω soft function leads to logarithms of $\frac{\mu}{2\omega}$. We expanded the non-trivial part of the right-hand side of Eq. (3.23) to $\mathcal{O}\left(\epsilon^0\right)$ because it turns out that this completely determines the logarithms of $\frac{\mu}{2\omega}$ which appear in the integrated distribution; the general arguments of [27] forbid terms that go like $\ln^3\left(\frac{\mu}{2\omega}\right)$ or $\ln^4\left(\frac{\mu}{2\omega}\right)$.

Let us now briefly summarize what we have accomplished. In this section, various contributions to the NLO τ_ω soft function were calculated. The full result, $S_R^{(2)}(k_L,k_R,\lambda,\mu)$, is naturally written as a sum of six contributions:

$$
\begin{aligned}
S_R^{(2)}(k_L,k_R,\lambda,\mu) \;=\; & S_R^{R-V}(k_L,k_R,\lambda,\mu)+S_R^{\mathrm{Ren}}(k_L,k_R,\lambda,\mu)+S_R^{\mathrm{r}_1}(k_L,k_R,\lambda,\mu)\\
&+S_R^{\mathrm{r}_2}(k_L,k_R,\lambda,\mu)+S_R^{\mathrm{r}_3}(k_L,k_R,\lambda,\mu)+S_R^{\mathrm{r}_4}(k_L,k_R,\lambda,\mu). \qquad (3.24)
\end{aligned}
$$

The three terms in the first line of Eq. (3.24) were easily calculated using known results (computed in Refs. [27] and [28]). The three terms in the last line were more involved.

[3] For the reader less familiar with harmonic polylogarithms, we recommend reading Ref. [50] and Appendix A of Ref. [51].

Instead of computing $S_R^{\mathrm{r}_2}(k_L, k_R, \lambda, \mu)$ and $S_R^{\mathrm{r}_3}(k_L, k_R, \lambda, \mu)$ directly, we instead computed the moments of those contributions relevant to the integrated jet mass distribution. We also chose not to compute the $\mathcal{O}(\epsilon)$ coefficient of $S_R^{\mathrm{r}_4}(k_L, k_R, \lambda, \mu)$, which only affects the constant part of the integrated distribution. The results given in this section determine the unknown scale-dependent terms in the integrated τ_ω distribution in the small jet thrust limit at NLO. In the next section, we examine and check these results.

4. The Jet Thrust Distribution at Two-Loops

In the previous section, the parts of the two-loop τ_ω soft function relevant for the differential jet thrust distribution were calculated in dimensional regularization. We have chosen not to compute the parts of the soft function that contribute only to the zero bin ($\tau_\omega = 0$) of the differential jet thrust distribution. For the hemisphere soft function, these terms were calculated (see Ref. [28]) because they are important for precision α_s measurements. However, in this paper, we are primarily interested in understanding the structure of the non-global contributions to the integrated τ_ω distribution and in testing the refactorization conjecture of Ref. [27]. We therefore followed the approach of Ref. [31] in Section 3 and calculated only those contributions to the soft function which depend on ratios of dimensionful scales.

The triply differential jet mass cross section is a complicated singular distribution which does not have a simple expression in momentum space. Although the two-loop integrated jet thrust distribution is not simple either for finite R, it is quite a bit easier to work with in practice due to the fact it is free of distributions. In terms of the various contributions to the τ_ω soft function studied in Section 3, we can write the integrated jet mass distribution as

$$\begin{aligned} K_R^{(2)}(\tau_\omega, \omega, \mu) &= \int_0^{\tau_\omega} \mathrm{d}\tau'_\omega \int_0^{\omega} \mathrm{d}\lambda \int_0^{\infty} \mathrm{d}k_L \mathrm{d}k_R S_R^{(2)}(k_L, k_R, \lambda, \mu) \delta\left(\tau'_\omega - \frac{k_L + k_R}{Q}\right) \\ &= \int_0^{\tau_\omega} \mathrm{d}\tau'_\omega \int_0^{\omega} \mathrm{d}\lambda \int_0^{\infty} \mathrm{d}k_L \mathrm{d}k_R \Big[S_R^{R-V}(k_L, k_R, \lambda, \mu) \\ &\quad + S_R^{\mathrm{Ren}}(k_L, k_R, \lambda, \mu) + S_R^{\mathrm{r}_1}(k_L, k_R, \lambda, \mu) + S_R^{\mathrm{r}_2}(k_L, k_R, \lambda, \mu) \\ &\quad + S_R^{\mathrm{r}_3}(k_L, k_R, \lambda, \mu) + S_R^{\mathrm{r}_4}(k_L, k_R, \lambda, \mu) \Big] \delta\left(\tau'_\omega - \frac{k_L + k_R}{Q}\right), \end{aligned} \tag{4.1}$$

where the superscript (2) reminds us that we are calculating the integrated distribution at NLO. The scale-dependent parts of the six terms in this expression are given explicitly in Appendix A. When these soft contributions to the integrated τ_ω distribution are appropriately combined with the contributions coming from the hard and jet functions, the result is a concrete prediction for the integrated τ_ω distribution in the small jet mass regime. This prediction should be valid for arbitrary R, up to power corrections in τ_ω and ω/Q.

4.1 Comparison to full QCD

We would like to compare our analytic results for the small jet thrust limit to full QCD, for which the distribution is only known numerically. To do so, we first differentiate

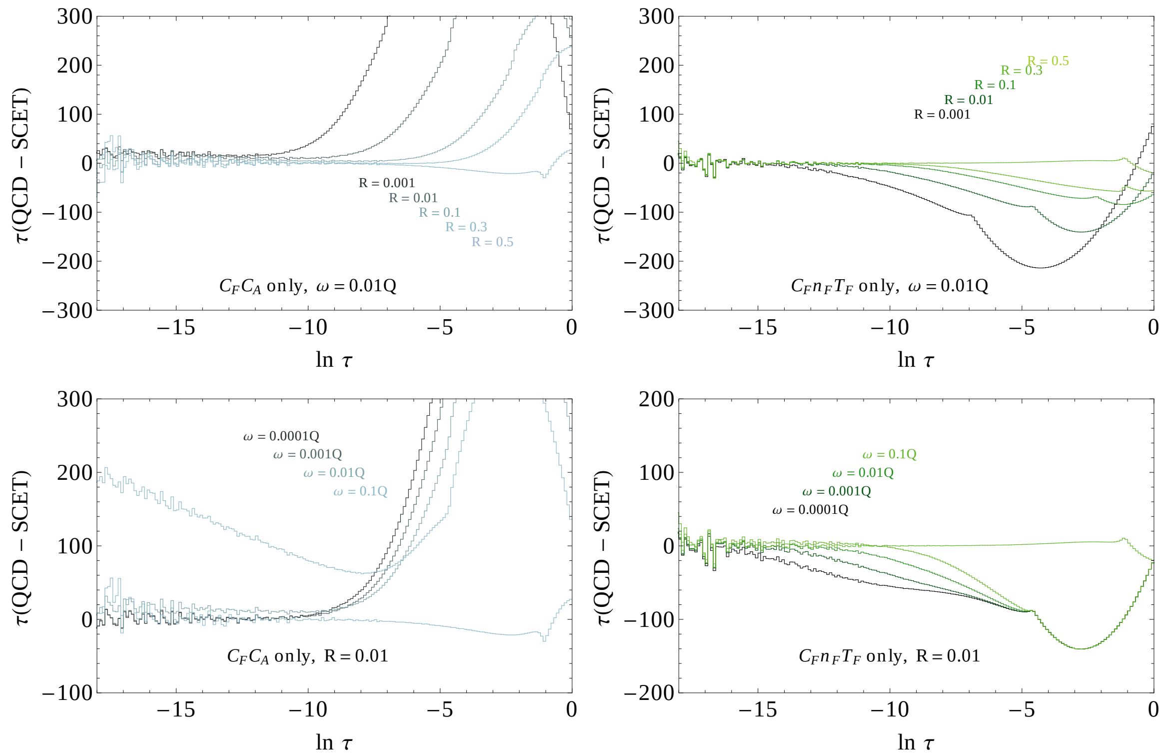

Figure 2: The difference between the SCET and the QCD calculations of $\frac{\tau_\omega}{\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_\omega}$ for various values of R and ω. The left plots show the $C_F C_A$ color structure and the right plots shows the $C_F n_f T_F$ color structure. The top plots show the variation with R at fixed $\omega = 10^{-2}Q$ and the bottom plots the variation with ω at fixed $R = 10^{-2}$. For the QCD calculation, a cone algorithm is used with the jet direction taken to be the thrust axis.

$K_R^{(2)}(\tau_\omega, \omega, \mu)$ with respect to τ_ω to determine the NLO part of the differential distribution. This result is then appropriately combined with the LO contributions coming from the hard and jet functions, as prescribed by the factorization formula, Eq. (2.2). The resulting expression is the small τ_ω prediction for $\frac{\tau_\omega}{\sigma_0}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_\omega}$. This prediction can then be expanded to $\mathcal{O}\left(\alpha_s^2\right)$ and numerically compared to QCD using the program EVENT2. The details of the α_s expansion can be found in *e.g.* Ref. [27].

The comparison shown in Figure 2 is the difference between the full QCD distribution calculated using EVENT2 and the two-loop prediction using SCET, for various values of R and ω. The plots show that SCET is reproducing all of the singular dependence in τ_ω, for all values of R up to power corrections in ω/Q and τ_ω. In the lower left plot, for $\omega = 0.1Q$, the power corrections of the form $\omega/Q \ln \tau_\omega$ are clearly present. A hint of the power correction is evident in the same plot for $\omega = 0.01Q$, but for smaller values of ω, the plots tend to zero as expected. There are also power corrections of the form $\tau_\omega \ln(\tau_\omega/2\omega R)$. These are responsible for the R and ω dependence in the plots, and only for sufficiently small τ_ω do the plots tend to zero. Furthermore, this structure can be seen as a linear shift when varying ω or R, while the other variable is held fixed. Note that for very small ω or

R, the region of validity migrates into a numerically unstable regime, which is apparent in the figure. When τ_ω is larger, the full τ_ω dependence becomes important as noted by kink in the graphs. This kink is due to 3 parton events transitioning between 2-jet and 3-jet kinematics at $\tau_\omega = -1/R(R-2+2\sqrt{1-R})$.[4] These plots serve as a non-trivial check that SCET, through factorization formula (2.2), is reproducing the correct singular behavior in τ_ω for any R and sufficiently small ω.

4.2 Analytic checks and definition of the non-global contribution

Next, we can compare the μ-dependent parts of $K_R^{(2)}(\tau_\omega, \omega, \mu)$ to the μ dependence predicted by the factorization theorem in SCET (Eq. (2.2)). With the one- and two-loop anomalous dimensions for the jet and hard functions, the one-loop QCD β-function coefficient, and $K_R^{(1)}(\tau_\omega, \omega, \mu)$ as inputs, SCET predicts the μ dependence of the integrated τ_ω distribution exactly. As it must, the μ dependence of $K_R^{(2)}(\tau_\omega, \omega, \mu)$ exactly matches the prediction of the factorization theorem.

In order to say more about the non-global contributions to the integrated jet thrust distribution, it is convenient to subtract off the μ-dependent part of $K_R^{(2)}(\tau_\omega, \omega, \mu)$. In fact, as we will argue in Section 5, the preferred way of doing this employs the integrated form of the refactorization ansatz reviewed in Section 2. In integrated form, the refactorization formula of Eq. (2.9) is

$$K_R(\tau_\omega, \omega, \mu) = K_R^{\rm in}(\tau_\omega, \mu) K_R^{\rm out}(\omega, \mu) K_R^f(\tau_\omega, \omega) \tag{4.2}$$

In order to make use of the above formula, one also needs the anomalous dimensions associated to $K_R^{\rm in}(\tau_\omega, \mu)$ and $K_R^{\rm out}(\omega, \mu)$,

$$\gamma_S^{\rm in} = \gamma_S + \Gamma_{\rm cusp} \ln\left(\frac{R}{1-R}\right) \qquad \text{and} \qquad \gamma_S^{\rm out} = -\Gamma_{\rm cusp} \ln\left(\frac{R}{1-R}\right). \tag{4.3}$$

In Eqs. (4.3) above, γ_S is the thrust soft function anomalous dimension. Eq. (4.2), Eqs. (4.3), and some of the ingredients mentioned in the previous paragraph determine a function $K_R^{\rm refac}(\tau_\omega, \omega, \mu)$ which one can use to define the non-global contributions to the integrated τ_ω distribution. Explicitly, we have

$$\begin{aligned}
&K_R^{\rm refac}(\tau_\omega, \omega, \mu) \equiv K_R^{\rm in}(\tau_\omega, \mu) K_R^{\rm out}(\omega, \mu)\Big|_{\rm two-loop} \\
&= \quad C_F C_A \Bigg[\ln\left(\frac{\mu}{2\omega}\right)\left(-\frac{176 {\rm Li}_2(-r)}{3} - \frac{44}{3}\ln^2(r) - \frac{8}{3}\pi^2\ln(r) + \frac{536\ln(r)}{9} - \frac{44\pi^2}{9}\right) \\
&+ \ln\left(\frac{\mu}{\tau_\omega Q}\right)\left(-\frac{44}{3}\ln^2(r) + \frac{8}{3}\pi^2\ln(r) - \frac{536\ln(r)}{9} + 56\zeta(3) + \frac{44\pi^2}{9} - \frac{1616}{27}\right) \\
&+ \frac{88}{3}\ln(r)\ln^2\left(\frac{\mu}{2\omega}\right) + \left(-\frac{88\ln(r)}{3} + \frac{8\pi^2}{3} - \frac{536}{9}\right)\ln^2\left(\frac{\mu}{\tau_\omega Q}\right) - \frac{176}{9}\ln^3\left(\frac{\mu}{\tau_\omega Q}\right) \Bigg]
\end{aligned}$$

[4] In 3-parton kinematics, the gluon contributes to τ_ω when it propagates within angle $\theta_R = \cos^{-1}(1-2R)$ of the quark or anti-quark. If it lies outside, the configuration is treated as 2-jets if the deposited energy is less that ω, otherwise it only contributes to the overall normalization.

$$+ C_F n_f T_F \Bigg[\ln\left(\frac{\mu}{2\omega}\right)\left(\frac{64\mathrm{Li}_2(-r)}{3} + \frac{16\ln^2(r)}{3} - \frac{160\ln(r)}{9} + \frac{16\pi^2}{9}\right)$$
$$+ \ln\left(\frac{\mu}{\tau_\omega Q}\right)\left(\frac{16\ln^2(r)}{3} + \frac{160\ln(r)}{9} - \frac{16\pi^2}{9} + \frac{448}{27}\right)$$
$$- \frac{32}{3}\ln(r)\ln^2\left(\frac{\mu}{2\omega}\right) + \left(\frac{32\ln(r)}{3} + \frac{160}{9}\right)\ln^2\left(\frac{\mu}{\tau_\omega Q}\right) + \frac{64}{9}\ln^3\left(\frac{\mu}{\tau_\omega Q}\right)\Bigg]. \tag{4.4}$$

The difference of the exact result and $K_R^{\text{refac}}(\tau_\omega, \omega, \mu)$ is

$$K_R^f(\tau_\omega, \omega) \equiv K_R^{(2)}(\tau_\omega, \omega, \mu) - K_R^{\text{refac}}(\tau_\omega, \omega, \mu)$$
$$= C_F C_A \Bigg[\chi_{C_A}\left(\frac{\tau_\omega Q}{2\omega}, r\right) - \chi_{C_A}(1, r) + \left(16\,\mathrm{Li}_2\left(r^2\right) - \frac{8}{3}\pi^2\right)\ln^2\left(\frac{\tau_\omega Q}{2\omega}\right)$$
$$+ \ln\left(\frac{\tau_\omega Q}{2\omega}\right)\Bigg(- \frac{176\mathrm{Li}_2\left(r^2\right)}{3} + 8\mathrm{Li}_3\left(r^2\right) - 32\mathrm{Li}_3\left(\frac{r^2}{r^2-1}\right) + 32\mathrm{Li}_2\left(r^2\right)\ln(r+1)$$
$$-64\mathrm{Li}_2(r)\ln\left(1-r^2\right) - 24\mathrm{Li}_2\left(r^2\right)\ln\left(r^2\right) + 32\mathrm{Li}_2\left(r^2\right)\ln\left(1-r^2\right) - 64\mathrm{Li}_3(1-r)$$
$$-64\mathrm{Li}_3\left(\frac{r}{r+1}\right) - 32\mathrm{Li}_2(-r)\ln(r) - \frac{16r^2}{3-3r^2} + \frac{16}{3}\ln^3\left(1-r^2\right) - \frac{32r^2\ln(r)}{3\left(r^2-1\right)^2}$$
$$-\frac{88}{3}\ln\left(r^2\right)\ln\left(1-r^2\right) + \frac{32}{3}\ln^3(r+1) - 32\ln^2(1-r)\ln(r) + \frac{32}{3}\pi^2\ln(1-r)$$
$$+64\zeta(3)\Bigg) + \ln\left(\frac{\tau_\omega Q}{2\omega}\right)\left(\frac{16}{3}\pi^2\ln(r) - 16\zeta(3) + \frac{88\pi^2}{9} - \frac{8}{3}\right)\Bigg]$$
$$+C_F n_f T_F \Bigg[\chi_{n_f}\left(\frac{\tau_\omega Q}{2\omega}, r\right) - \chi_{n_f}(1, r) + \Bigg(\frac{64\mathrm{Li}_2\left(r^2\right)}{3} + \frac{32r^2}{3\left(1-r^2\right)} + \frac{64r^2\ln(r)}{3\left(1-r^2\right)^2}$$
$$+\frac{64}{3}\ln(r)\ln\left(1-r^2\right)\Bigg)\ln\left(\frac{\tau_\omega Q}{2\omega}\right) + \left(\frac{16}{3} - \frac{32\pi^2}{9}\right)\ln\left(\frac{\tau_\omega Q}{2\omega}\right)\Bigg]. \tag{4.5}$$

As required, this difference is μ-independent and contains all of the non-global logarithms. The functions $\chi_{C_A}(x, r)$ and $\chi_{n_f}(x, r)$ in Eq. (4.5) are defined in Appendix B. There should be an additional τ_ω- and ω-independent function of R in Eq. (4.5), which has not been calculated. As mentioned above, this function would not affect the differential τ_ω distribution and is therefore of secondary importance.

In recent years, quite a bit of effort has been devoted to understanding the structure of NGLs. So far, most of the analytical calculations have focused on the leading logarithms. For example, in Eq. (4.5) above, the leading NGL is simply

$$C_F C_A\left(16\,\mathrm{Li}_2(r^2) - \frac{8}{3}\pi^2\right)\ln^2\left(\frac{\tau_\omega Q}{2\omega}\right). \tag{4.6}$$

which agrees with results from Refs. [27, 31]. Leading NGLs are certainly of interest but such computations are not, by themselves, likely to be of much use if our ultimate goal is to understand the resummation of NGLs. On the other hand, our exact result for the non-global contribution to the integrated τ_ω distribution is extremely complicated (both

$\chi_{C_A}(x,r)$ and $\chi_{n_f}(x,r)$ depend on two independent scales in a highly non-trivial way) and this makes it difficult to gain any further theoretical insight. If there is any hidden structure in our results, it should be significantly easier to identify in the small R limit.

5. The Small R Limit of the Non-Global Logarithms

It was observed in Ref. [27] that, in the small R limit, the jet thrust soft function simplifies. Having computed the exact scale-dependent contributions to the integrated jet thrust distribution at finite R, we can now understand more precisely what is happening. As we will see, in the small R limit, the non-global contribution to the integrated τ_ω distribution is determined completely by the non-global contribution to the integrated hemisphere soft function.

It has been noted by a number of authors that, for jet observables defined with the anti-k_T jet algorithm, one typically finds that the $R \to 0$ limit of the leading NGL is simply given by the result in the hemisphere case, up to possibly a factor of two depending on the observable in question [26, 27, 29, 31]. The jet algorithm employed in this paper is very closely related to the anti-k_T algorithm. In fact, the in-in and in-out contributions are exactly the same in both jet algorithms. The only difference between the algorithms at two loops is that they treat the out-of-jet radiation differently. We strongly suspect that the NGLs in the two jet algorithms actually coincide, at least in the small R limit. This implies that we should be able to check whether, for the integrated anti-k_T jet thrust distribution, one can relate the next-to-leading NGLs to analogous NGLs in the integrated hemisphere soft function calculated in Refs. [27] and [29]; since we have computed the exact expression, we are now in a position to go beyond the leading-logarithm approximation. In what follows, we assume some familiarity with the conventions and definitions that we established in Section 3.

5.1 In-in rescaling

Before discussing the in-out contributions and non-global logarithms, it is instructive to first consider the simpler case where both soft particles get clustered into the same jet. The same-side in-in contribution to the integrated τ_ω distribution is

$$K_R^{r_1}(\tau_\omega,\omega,\mu) = 2\int_0^{\tau_\omega} \mathrm{d}\tau'_\omega \int_0^{\omega} \mathrm{d}\lambda \int_0^{\infty} \mathrm{d}k_L \mathrm{d}k_R\, (4\pi)^4 \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{2\epsilon} \int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\mathrm{d}^d k}{(2\pi)^d}$$
$$\times\left(\frac{1}{2!}I_{C_A} + I_{n_f}\right)(-2\pi i)^2 \delta(q^2)\Theta\left(q^0\right)\delta(k^2)\Theta\left(k^0\right)\Theta(rk^- - k^+)$$
$$\times\Theta(rq^- - q^+)\delta(k_L)\delta(k_R - q^+ - k^+)\delta\left(\lambda\right)\delta\left(\tau'_\omega - \frac{k_L + k_R}{Q}\right), \quad (5.1)$$

where I_{C_A} and I_{n_f} are the two-loop, two-parton integrands for the $C_F C_A$ and $C_F n_f T_F$ color structures, given respectively by Eqs. (16) and (27) of Ref. [28].

To simplify the theta and delta functions, we can rescale the **n**-jet Wilson line. This turns the cone jet into a hemisphere jet, as illustrated in Figure 5.1. At the level of the

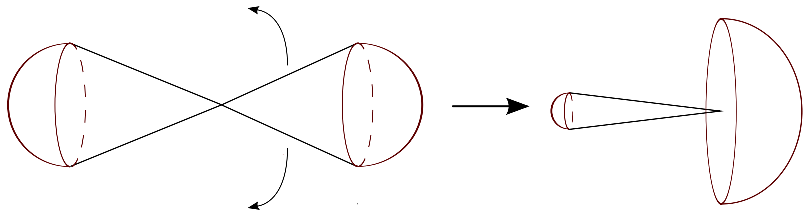

Figure 3: By rescaling one of the jet directions a small cone can be mapped to a hemisphere. Rescaling in this way makes some universal properties of non-global logarithms easier to identify.

integrand, this rescaling amounts to making the replacements:

$$k^- \to \frac{k^-}{r} \qquad \text{and} \qquad q^- \to \frac{q^-}{r}\,. \tag{5.2}$$

This change of variables induces the transformations

$$I_{C_A} \to r^2 I_{C_A} \qquad \text{and} \qquad I_{n_f} \to r^2 I_{n_f} \tag{5.3}$$

and the phase-space cuts in Eq. (5.1) that depend on the minus components of the light-cone momenta become

$$\Theta(k^- - k^+)\Theta(q^- - q^+). \tag{5.4}$$

Remarkably, the transformed integrand reduces to the $\mathcal{O}\left(\alpha_s^2\right)$ same-side hemisphere integrand times an overall factor of $r^{2\epsilon}$ A similar rescaling was observed in the context of beam functions in [53].

After performing the trivial integrations over k_L and k_R, the result is simply expressed in terms of the same-side contributions to the integrated hemisphere soft function:

$$\begin{aligned} K_R^{r_1}(\tau_\omega, \omega, \mu) = 2r^{2\epsilon} \int_0^{\tau_\omega} Q\,\mathrm{d}\tau'_\omega \int_0^\omega \mathrm{d}\lambda \frac{\delta(\lambda)\mu^{4\epsilon}}{\left(Q\,\tau'_\omega\right)^{1+4\epsilon}} \\ \times \left[C_F C_A g_{C_A}^{\mathrm{hemi}}(\epsilon) + C_F n_f T_F g_{n_f}^{\mathrm{hemi}}(\epsilon) \right], \end{aligned} \tag{5.5}$$

where $g_{C_A}^{\mathrm{hemi}}(\epsilon)$ and $g_{n_f}^{\mathrm{hemi}}(\epsilon)$ are the same-side contributions to the NLO hemisphere soft function, given in Eqs. (18) and (29) of Ref. [28].

We used this rescaling to compute the two-loop same-side in-in integrals in Section 3. In fact, this result is straightforward to generalize. At any order in perturbation theory, if all of the particles go into a single jet, the contribution to the integrated τ_ω soft function will be given by the analogous hemisphere calculation with the light-cone momenta appropriately rescaled in the manner described above.

Before moving on, we briefly consider the case where the two partons end up in separate jets and show that this contribution is suppressed in the small R limit. First, for any r, this contribution to $K_R^{(2)}(\tau_\omega, \omega, \mu)$ is given by

$$K_R^{r_2}(\tau_\omega, \omega, \mu) = 2 \int_0^{\tau_\omega} \mathrm{d}\tau'_\omega \int_0^\omega \mathrm{d}\lambda \int_0^\infty \mathrm{d}k_L \mathrm{d}k_R\, (4\pi)^4 \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{2\epsilon} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{d}^d k}{(2\pi)^d}$$

$$\times\left(\frac{1}{2!}I_{C_A}+I_{n_f}\right)(-2\pi i)^2\delta(q^2)\Theta\left(q^0\right)\delta(k^2)\Theta\left(k^0\right)\Theta(rk^- - k^+)$$
$$\times\Theta(rq^+ - q^-)\delta(k_R - k^+)\delta(k_L - q^-)\delta\left(\lambda\right)\delta\left(\tau'_\omega - \frac{k_L+k_R}{Q}\right)\,. \tag{5.6}$$

Let us try to proceed as above, this time rescaling k^- and q^+:

$$k^- \to \frac{k^-}{r} \qquad \text{and} \qquad q^+ \to \frac{q^+}{r}\,. \tag{5.7}$$

This time I_{C_A} and I_{n_f} do not transform simply under the rescaling. Instead we find that

$$I_{C_A} \to r^3\tilde{I}_{C_A}(r) \qquad \text{and} \qquad I_{n_f} \to r^5\tilde{I}_{n_f}(r)\,, \tag{5.8}$$

where $\tilde{I}_{C_A}(r)$ and $\tilde{I}_{n_f}(r)$ are regular but non-vanishing at $r=0$. Since, as before, the integration measure contributes a factor $r^{-2+2\epsilon}$, we see that this entire contribution is suppressed at small R relative to the one studied above. In fact, in the small R limit, the opposite-side in-in contributions are suppressed relative to all of the other contributions and can therefore be neglected.

5.2 In-out rescaling at small R

Next, we will look at the contributions where one soft parton gets clustered into a jet and the other does not. Here we will find a similar mapping to the hemisphere integrals but only at small R. At the level of the integrand, the in-out contribution to the integrated τ_ω distribution is

$$K_R^{r_3}(\tau_\omega,\omega,\mu) = 4\int_0^{\tau_\omega}\mathrm{d}\tau'_\omega\int_0^\omega\mathrm{d}\lambda\int_0^\infty\mathrm{d}k_L\mathrm{d}k_R\,(4\pi)^4\left(\frac{\mu^2e^{\gamma_E}}{4\pi}\right)^{2\epsilon}\int\frac{\mathrm{d}^dq}{(2\pi)^d}\frac{\mathrm{d}^dk}{(2\pi)^d}(-2\pi i)^2$$
$$\times\left(\frac{1}{2!}I_{C_A}+I_{n_f}\right)\delta(q^2)\Theta\left(q^0\right)\delta(k^2)\Theta\left(k^0\right)\Theta(rk^- - k^+)\Theta(q^- - rq^+)$$
$$\times\Theta(q^+ - rq^-)\delta(k_R - k^+)\delta\left(\lambda - \frac{q^+ + q^-}{2}\right)\delta(k_L)\delta\left(\tau'_\omega - \frac{k_L+k_R}{Q}\right)\,. \tag{5.9}$$

Changing variables in an attempt to map the cone jet to a hemisphere jet as above, we see that the phase-space cuts in Eq. (5.9) that depend on the minus components of the light-cone momenta can be written as

$$2r\Theta(k^- - k^+)\Theta(q^- - r^2q^+)\Theta(q^+ - q^-)\delta\left(q^+r + q^- - 2r\lambda\right)\,. \tag{5.10}$$

This is not exactly a hemisphere projection for general R. However, in the small R regime, the above expression simplifies to

$$2r\,\Theta(k^- - k^+)\Theta(q^-)\Theta(q^+ - q^-)\delta\left(q^- - 2r\lambda\right)\,. \tag{5.11}$$

The phase-space cuts enforced by (5.11) are identical to the ones which arise in the calculation of the opposite-side contributions to the two-loop hemisphere soft function, provided that one makes the replacement $k_L \to 2r\lambda$. To see this, after performing the trivial integrations over k_L and k_R in Eq. (5.9) above, replace the phase-spaces cuts that depend on k^- and q^- using (5.11) and then compare the resulting product of thetas and deltas in the integrand to the second line of Eq. (13) in Ref. [28] (with $k_L = 2r\lambda$ and $k_R = Q\tau'_\omega$). With this understanding, we find that the above expression can be written in terms of the opposite-side contributions to the integrated hemisphere soft function:

$$K^{r_3}_{R\to 0}(\tau_\omega,\omega,\mu) = 2r^{2\epsilon}\left(\frac{\alpha_s}{4\pi}\right)^2 \int_0^{\tau_\omega Q} \mathrm{d}x \int_0^{2\,r\,\omega} \mathrm{d}y \frac{\mu^{4\epsilon}}{(xy)^{1+2\epsilon}} \times \left[C_F C_A f^{\text{hemi}}_{C_A}\left(\frac{x}{y},\epsilon\right) + C_F n_f T_F f^{\text{hemi}}_{n_f}\left(\frac{x}{y},\epsilon\right)\right], \tag{5.12}$$

where $f^{\text{hemi}}_{C_A}(z,\epsilon)$ and $f^{\text{hemi}}_{n_f}(z,\epsilon)$ are the opposite-side contributions to the NLO hemisphere soft function, defined in Eqs. (17) and (28) and Appendix A of Ref. [28]. In deriving Eq. (5.12) we found it useful to make the change of variables

$$\tau'_\omega = \frac{x}{Q} \qquad\qquad \lambda = \frac{y}{2r}\,. \tag{5.13}$$

This correspondence between the in-out contributions to the integrated τ_ω distribution and the opposite-side contributions to the integrated hemisphere soft function is valid for sufficiently small R and arbitrary values of τ_ω and ω. The striking similarity between Eqs. (5.5) and (5.12) is suggestive. In fact, for sufficiently small R, we will show in the next section that it is possible to very accurately model *all* of the terms in the integrated τ_ω distribution not captured by the refactorization ansatz – including all of the NGLs – using the non-global contributions to the integrated hemisphere soft function.

5.3 Small R limit

We have shown that the in-out contributions at small R (and so $r \sim R$) are given by the opposite-side contributions to the integrated hemisphere soft function times an overall factor of $2R^{2\epsilon}$. We were inspired by this small R correspondence to take a closer look at the non-global contributions to the integrated hemisphere soft function, $\mathcal{R}_f\,(z)$.

Using the results of Sections 5.1 and 5.2, we are able to take the small R limit of the scale-dependent non-global contribution to the integrated τ_ω distribution that we presented in Eq.(4.5). We find

$$K^f_{R\to 0}(\tau_\omega,\omega) = 2\left[\mathcal{R}_f\left(\frac{\tau_\omega Q}{2R\omega}\right) - \mathcal{R}_f\left(\frac{1}{R}\right)\right], \tag{5.14}$$

where $\mathcal{R}_f(z)$ is the non-global contribution to the integrated hemisphere soft function, defined in Eq. (53) of Ref. [28]:

$$\mathcal{R}_f(z) = \Bigg[-88\mathrm{Li}_3(-z) - 16\mathrm{Li}_4\left(\frac{1}{z+1}\right) - 16\mathrm{Li}_4\left(\frac{z}{z+1}\right) + 16\mathrm{Li}_3(-z)\ln(z+1)$$
$$+\frac{88\mathrm{Li}_2(-z)\ln(z)}{3} - 8\mathrm{Li}_3(-z)\ln(z) - 16\zeta(3)\ln(z+1) + 8\zeta(3)\ln(z) - \frac{4}{3}\ln^4(z+1)$$

$$
+\frac{8}{3}\ln(z)\ln^3(z+1)+\frac{4}{3}\pi^2\ln^2(z+1)-\frac{4}{3}\pi^2\ln^2(z)-\frac{4\left(3(z-1)+11\pi^2(z+1)\right)\ln(z)}{9(z+1)}
$$

$$
-\frac{506\zeta(3)}{9}+\frac{16\pi^4}{9}-\frac{871\pi^2}{54}-\frac{2032}{81}\Bigg] C_F C_A+\Bigg[32\mathrm{Li}_3(-z)-\frac{32}{3}\mathrm{Li}_2(-z)\ln(z)
$$

$$
+\frac{8(z-1)\ln(z)}{3(z+1)}+\frac{16}{9}\pi^2\ln(z)+\frac{184\zeta(3)}{9}+\frac{154\pi^2}{27}-\frac{136}{81}\Bigg] C_F n_f T_F\,. \tag{5.15}
$$

In deriving this small R limit, the rescaling of the in-out integrals in particular was critical – it is certainly not straightforward to derive Eq. (5.14) directly from the expression for $K_R^f(\tau_\omega,\omega)$ given in Eq. (4.5). Again, there may be additional contributions to $K_{R\to 0}^f(\tau_\omega,\omega)$ which are τ_ω- and ω-independent and have not been computed, and we have arbitrarily chosen the normalization of $K_{R\to 0}^f(\tau_\omega,\omega)$ such that $K_{R\to 0}^f(2\omega/Q,\omega)=0$.

In Figure 4, we compare the τ_ω distributions coming from the exact expression, $K_R^f(\tau_\omega,\omega)$, and from the small R approximation, $K_{R\to 0}^f(\tau_\omega,\omega)$, for both non-trivial color structures. We see that the approximation works spectacularly well and remains valid for small but experimentally relevant values of the jet cone size ($R\sim 0.1$).

5.4 Extraction of the non-global logarithms

Traditionally, one would define the non-global logarithms as those terms in $K_R^f(\tau_\omega,\omega)$ which diverge in the $\left|\ln\left(\frac{\tau_\omega Q}{2\omega}\right)\right|\gg 1$ limit. However, the non-global logarithms in our integrated jet mass distribution are somewhat more subtle since there is a term in $K_R^f(\tau_\omega,\omega)$ which diverges in the $R\to 0$ limit as well. To sensibly extract the NGLs in our case, we actually need to consider the double limits $1\gg R\gg \frac{\tau_\omega Q}{2\omega}$ and $\frac{\tau_\omega Q}{2\omega}\gg 1\gg R$. We found in Section 5.3 that

$$
K_R^f(\tau_\omega,\omega)=2\mathcal{R}_f\left(\frac{\tau_\omega Q}{2R\omega}\right)+f(R)+\mathcal{O}(R)\,, \tag{5.16}
$$

where $f(R)$ is a ω-independent and τ_ω-independent function of R which we have not calculated. The appearance of the function $\mathcal{R}_f(z)$ in Eq. (5.16) suggests that the leading and next-to-leading NGLs in the integrated τ_ω distribution ought to be simply related to the leading and next-to-leading hemisphere NGLs derived in Refs. [28, 29].

We can easily extract the NGLs in the integrated τ_ω distribution by first taking the $\left|\ln\left(\frac{\tau_\omega Q}{2\omega}\right)\right|\gg 1$ limit of the non-global function $K_R^f(\tau_\omega,\omega)$ and then taking the small R limit of the expression that results. In this approximation, we find that the function $K_R^f(\tau_\omega,\omega)$ reduces to

$$
K_{R\to 0}^{f,\,\mathrm{NGL}}(\tau_\omega,\omega)=C_F C_A\left[-\frac{8\pi^2}{3}\ln^2\left(\frac{\tau_\omega Q}{2R\omega}\right)+\left(-\frac{8}{3}+\frac{88\pi^2}{9}-16\zeta_3\right)\left|\ln\left(\frac{\tau_\omega Q}{2R\omega}\right)\right|\right]
$$

$$
+C_F n_f T_F\left(\frac{16}{3}-\frac{32\pi^2}{9}\right)\left|\ln\left(\frac{\tau_\omega Q}{2R\omega}\right)\right|+\cdots\,. \tag{5.17}
$$

Strictly speaking, Eq. (5.17) is only correct up to terms independent of τ_ω and ω (*i.e.* functions unconstrained by the calculations performed in this paper). However, although it remains to be conclusively proven, we strongly suspect that Eq. (5.17) actually captures *all*

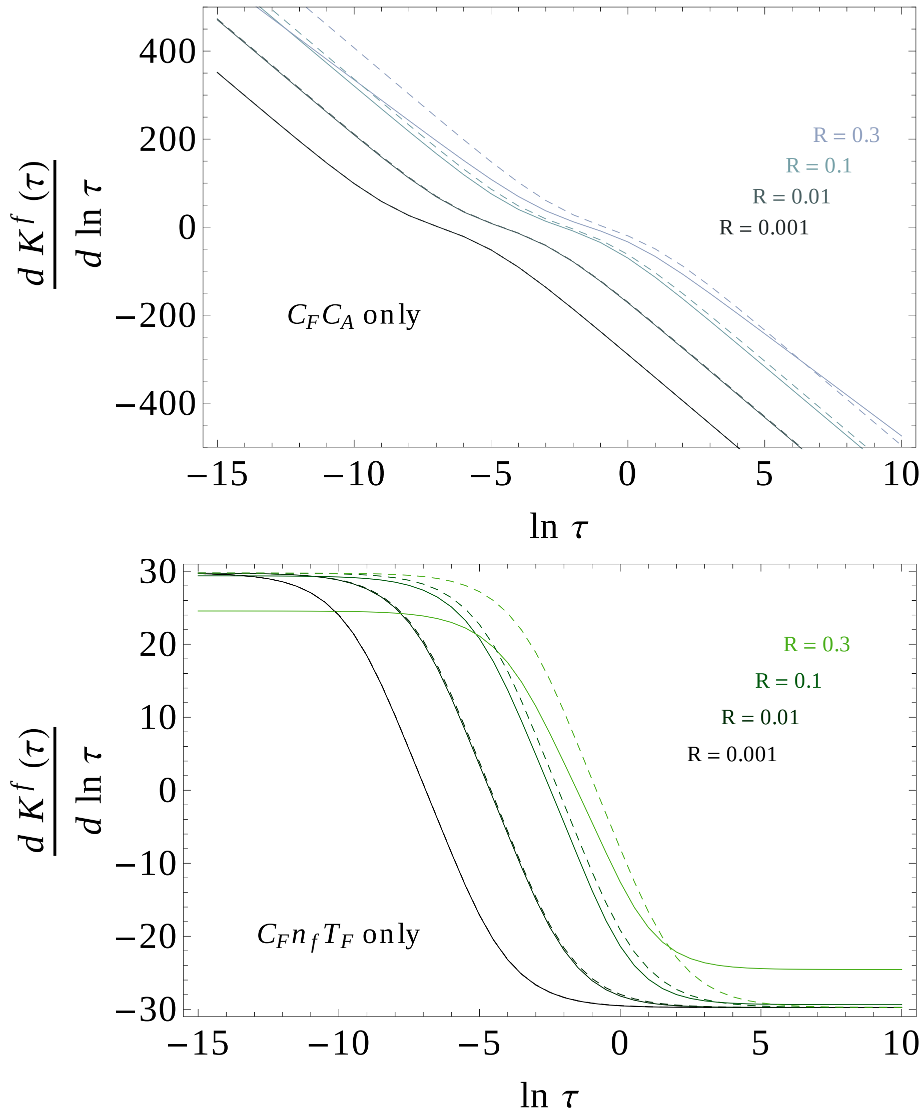

Figure 4: Contributions to the differential jet thrust distribution coming from the non-global terms in the soft function. We show $\frac{\mathrm{d}}{\mathrm{d}\ln(\tau)}K_R^f(\tau)$ (solid) and its small R limit, $\frac{\mathrm{d}}{\mathrm{d}\ln\tau}K_{R\to 0}^f(\tau)$ (dashed), plotted as a function of $\tau = \frac{\tau_\omega Q}{2\omega}$ for various R.

of the singular R dependence present in the contributions to the integrated τ_ω distribution not determined by the refactorization ansatz. After multiplying the above expression by $1/2$ and making the substitution $\ln\left(\frac{\tau_\omega Q}{2R\omega}\right) \to \ln(z)$, we find that Eq. (5.17) exactly reproduces the hemisphere NGLs of Eq. (54) in Ref. [28]. This shows that, as hoped, the next-to-leading NGLs in our integrated jet thrust distribution are simply related to corresponding next-to-leading NGLs in the integrated hemisphere soft function. In fact, if it turns out that, as seems likely, the anti-k_T NGLs coincide with ours in the small R limit, we have shown that there is no essential difference between the small R NGLs that arise in the integrated anti-k_T jet thrust distribution and the NGLs that arise in the hemisphere mass distribution.

Clearly, the simplicity of the NGLs derived above suggests that a simple factorized picture emerges in the small R limit. It's true at least at $\mathcal{O}(\alpha_s^2)$ and probably at higher orders that, for sufficiently small R, the integrated τ_ω soft function refactorizes in perturbation theory as follows:

$$\int_0^{M_L} dk_L \int_0^{M_R} dk_R \int_0^{\omega} d\lambda\, S_R(k_L,k_R,\lambda,\mu) = \mathcal{R}_\mu\left(\frac{M_L}{\sqrt{R}\mu}\right)\mathcal{R}_\mu\left(\frac{M_R}{\sqrt{R}\mu}\right)\mathcal{R}_{\rm out}\left(\frac{2\sqrt{R}\omega}{\mu},R\right) \times\mathcal{R}_f\left(\frac{M_L}{2R\omega}\right)\mathcal{R}_f\left(\frac{M_R}{2R\omega}\right)+\mathcal{O}(R). \tag{5.18}$$

Here, the functions $\mathcal{R}_\mu(z)$ and $\mathcal{R}_f(z)$ denote respectively the μ-dependent and μ-independent parts of the hemisphere soft function [12, 28, 40]. Writing the soft function in this refactorized form makes the relationship between the small R exclusive jet mass soft function and the hemisphere soft function completely explicit. It is worth pointing out that $\mathcal{R}_\mu(z)$ and $\mathcal{R}_f(z)$ have no explicit R dependence; their R dependence is simply encoded in their arguments, $\frac{M}{\sqrt{R}\mu}$ and $\frac{M}{2R\omega}$. In fact, the only part of (5.18) which is not a simple rescaling of part of the hemisphere soft function is $\mathcal{R}_{\rm out}\left(\frac{2\sqrt{R}\omega}{\mu},R\right)$. Moreover, the dependence on the first argument of $\mathcal{R}_{\rm out}\left(\frac{2\sqrt{R}\omega}{\mu},R\right)$ is fixed by RG invariance. Actually, it is possible to fix the entire singular R dependence of $\mathcal{R}_{\rm out}(z,R)$ with arguments similar to the ones made in Sections 5.1 and 5.2. This analysis, however, is beyond the scope of the current paper.

When we expand Eq. (5.18) to $\mathcal{O}\left(\alpha_s^2\right)$ and further convolute M_L and M_R into $\tau_\omega Q$ using

$$K_R(\tau_\omega,\omega,\mu) = \int dk_L dk_R \int_0^{\omega} d\lambda\, S_R(k_L,k_R,\lambda,\mu)\delta\left(\tau_\omega - \frac{k_L+k_R}{Q}\right), \tag{5.19}$$

we find that

$$K_R^f(\tau_\omega,\omega) = 2\mathcal{R}_f\left(\frac{\tau_\omega Q}{2R\omega}\right)+\mathcal{O}(R)\,. \tag{5.20}$$

This is completely consistent with the function $K^f_{R\to 0}(\tau_\omega,\omega)$, Eq. (5.14), coming from the expansion of the exact result, up to a term that is independent of τ_ω and ω, $-2\mathcal{R}_f(1/R)$. Recall, such a term was present to ensure the normalization condition $K_R^f(2\omega/Q,\omega)=0$. We point out that while the normalization of $K_R^f(\tau_\omega,\omega)$ is arbitrary since any τ_ω- and ω-independent constant can be assigned differently between $K_R^f(\tau_\omega,\omega)$ and the two-loop constant term of $\mathcal{R}_{\rm out}(2\sqrt{R}\,\omega/\mu,R)$ [5], we find that the choice of normalization in Eq. (5.20) is more natural when considering the rescaling arguments of this section. Furthermore, since we have confirmed this refactorization at two loops, including non-global logs, the refactorization and $\Gamma_{\rm cusp}$ ansatz of Ref. [27] are also confirmed at two loops – Eq. (5.18) reproduces them when $\mathcal{R}_f(z)=1$.

[5] We remind the reader that the two-loop constant term $\mathcal{R}_\mu\left(\frac{M_{L,R}}{\sqrt{R}\mu}\right)$ is completely fixed by rescaling.

6. Conclusions

Over the last several years it has become increasingly apparent that jet substructure will play an important role in collider physics, and that theoretical calculations of jet substructure will require new organizational tools. For example, the simplest substructure observable, jet mass, when examined in an exclusive context, such as for the hardest or two hardest jets, is associated with poorly understood dependence on the multiple relevant scales (jet masses and veto scales). In this paper, by studying in detail the non-global structure of the soft, scale-dependent contributions to the integrated cone jet thrust distribution at two-loops, we took an important step towards understanding the interplay of these scales. We presented compelling evidence that the non-global logarithms that arise in a realistic class of jet algorithms when expanded around small jet size R are in fact no more complicated than the non-global logarithms that arise when one uses the hemisphere jet algorithm.

It seems that at each order in perturbation theory, it is possible to take the limit of small jet cone size in a consistent and useful way at the level of the integrands and that taking this limit effectively reduces all subsequent integrations to the corresponding hemisphere ones. This vastly simplifies the non-global parts of the distribution. A comparison of the non-global contribution to the integrated hemisphere mass distribution given in Section 5.3 and the analogous integrated jet thrust distribution given in 4.2 illustrates the point – the hemisphere results of Ref. [28] reproduced in Eq. (5.15) are far simpler than the finite R results in Eq. (4.5), derived in this paper. Naïvely, one might think that this procedure, explained in more detail in Sections 5.1 and 5.2, only gives an accurate approximation if one works with unrealistically small jet resolution parameters. However, as illustrated in Figure 4, this is not the case at all; we found that approximating our result for the finite R differential distribution using this procedure works great even for $R \sim 0.1$.

One point worth emphasizing is that the refactorization ansatz of Ref. [27] played an important role in our analysis. In Figure 1 we compared the $\mathcal{O}\left(\alpha_s^2\right)$ prediction of this ansatz to full QCD at the same order using EVENT2 and showed that the refactorization ansatz reproduces the singular terms in the QCD differential distribution very nicely, up to subleading jet algorithm and NGL effects. Indeed, in Section 4, we defined the non-global contribution to the integrated jet thrust distribution by subtracting the refactorization ansatz in integrated form from our complete result for the scale-dependent part of the integrated distribution. If we had used a different subtraction procedure it is not clear that we would have been successful in making contact with the integrated hemisphere function. This remark is related to the fact that, at the outset of this work, it was not clear what appearance, if any, the jet cone size R would make in the non-global contribution that remains once one subtracts off the integrated refactorization ansatz.

Our results suggest that the non-global contribution to the integrated jet thrust distribution at small R is simply

$$K_{R\to 0}^{f,\,\text{exact}}(\tau_\omega,\omega) = 2\,\mathcal{R}_f\left(\frac{\tau_\omega Q}{2R\omega}\right)\,, \tag{6.1}$$

where $\mathcal{R}_f(z)$ is the non-global contribution to the integrated hemisphere mass distribution. We have only shown this up to a τ_ω- and ω-independent function of R, due to the fact that we consistently dropped all contributions that depended on no scales at all or on R but not on a ratio of dimensionful scales. This small R limit depends on the precise definition of $K_R^f(\tau_\omega,\omega)$. In our definition, $K_R^f(\tau_\omega,\omega)$ is everything not given by the refactorization from [27]. With this definition, Eq. (6.1) implies that all of the remaining singular R dependence is inextricably linked to the ratio $\frac{\tau_\omega Q}{2\omega}$.

The assertion of the last paragraph is very sensible because we showed in Section 5.4 that the leading and subleading non-global logarithms are

$$K_{R\to 0}^{f,\,\mathrm{NGL}}(\tau_\omega,\omega) = C_F C_A\left[-\frac{8\pi^2}{3}\ln^2\left(\frac{\tau_\omega Q}{2R\omega}\right)+\left(-\frac{8}{3}+\frac{88\pi^2}{9}-16\zeta_3\right)\left|\ln\left(\frac{\tau_\omega Q}{2R\omega}\right)\right|\right]$$
$$+C_F n_f T_F\left(\frac{16}{3}-\frac{32\pi^2}{9}\right)\left|\ln\left(\frac{\tau_\omega Q}{2R\omega}\right)\right|+\cdots. \tag{6.2}$$

In the appropriate limit, Eq. (6.1) reduces to Eq. (6.2) up to terms not constrained by our calculation. The form of Eq. (6.2) also shows that the NGLs that arise in the integrated cone jet thrust distribution are naturally R dependent. The great strength of our approach is that we abandoned the leading-logarithm approximation, preferring instead to first perform exact calculations and only then attempt to determine an appropriate approximation scheme.

The results recapitulated above motivated us to modify the refactorization formula of Ref. [27] to include the non-global contributions as well. In Section 5.4, we discussed the possibility of a refined refactorization formula for the integrated soft function

$$\int_0^{M_L} dk_L \int_0^{M_R} dk_R \int_0^{\omega} d\lambda\, S_R(k_L,k_R,\lambda,\mu) = \mathcal{R}_\mu\left(\frac{M_L}{\sqrt{R}\mu}\right)\mathcal{R}_\mu\left(\frac{M_R}{\sqrt{R}\mu}\right)\mathcal{R}_{\mathrm{out}}\left(\frac{2\sqrt{R}\omega}{\mu},R\right)$$
$$\times\mathcal{R}_f\left(\frac{M_L}{2R\omega}\right)\mathcal{R}_f\left(\frac{M_R}{2R\omega}\right)+\mathcal{O}(R). \tag{6.3}$$

where the functions $\mathcal{R}_\mu(z)$ and $\mathcal{R}_f(z)$ denote respectively the μ-dependent and μ-independent parts of the integrated hemisphere soft function [12, 40]. Eq. (6.3) is consistent with all of the observations made in this paper and, if true, would make precise the statement that resumming the NGLs in the cone jet thrust distribution is equivalent to resumming the NGLs in the hemisphere mass distribution.

In future work, it would be useful to understand to what extent refactorizations like Eq. (6.3) hold beyond two loops. If this is understood, similar arguments should be appropriate for precision predictions of jet substructure observables at hadron colliders. In any case, it is of great interest to try and generalize Eq. (6.3), both to multi-jet processes at e^+e^- colliders and, if possible, to di- and multi-jet processes at hadron colliders.

In order to understand non-global structure in more detail, it may be necessary to explore jet algorithms distinct from the thrust cone algorithm used in this paper. Most prominent among the alternatives is the anti-k_T jet algorithm which, as noted in Section 5, is very similar to the thrust cone one and has the advantage that it is widely used by experimentalists. It would also be interesting to have all of the R-dependent constant terms that were dropped in writing down the non-global contribution to the integrated jet thrust distribution, since they would significantly clarify the structure of the $\ln(R)$ terms that remain in the small R limit.

Finally, even if Eq. (6.3) turns out to be a useful approximation, one still has to deal with the hemisphere NGLs themselves. Resummation of these NGLs seems difficult [20, 22, 28, 32, 33]. Perhaps one can choose scales so that the non-global structure has a numerically small effect. Nevertheless, it would be nice to understand NGLs in more detail, beyond the leading NGL. One way forward would be to perform an explicit three-loop calculation of the integrated hemisphere soft function. While it goes without saying that this will be no easy task, we expect that the integrals which arise at three loops will turn out to be solvable using modern multi-loop computational techniques.

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A. Calculation of the Integrated Jet Thrust Distribution

In this appendix, we calculate the six terms in Eq. (4.1) in the order in which we discussed the corresponding contributions to the NLO τ_ω soft function in Section 3. We begin with the charge renormalization contributions. The result is

$$
\begin{aligned}
K_R^{\rm Ren}(\tau_\omega,\omega,\mu) = C_F C_A\Bigg[&\frac{176}{9}\ln^3\left(\frac{\mu}{\tau_\omega Q}\right)+\frac{88}{3}\ln(r)\ln^2\left(\frac{\mu}{\tau_\omega Q}\right)+\left(\frac{44\ln^2(r)}{3}\right.\\
&\left.-\frac{22\pi^2}{9}\right)\ln\left(\frac{\mu}{\tau_\omega Q}\right)+\ln\left(\frac{\mu}{2\omega}\right)\left(\frac{176{\rm Li}_2(-r)}{3}+\frac{44\ln^2(r)}{3}+\frac{44\pi^2}{9}\right)-\frac{88}{3}\ln(r)\ln^2\left(\frac{\mu}{2\omega}\right)\Bigg]\\
&+C_F n_f T_F\Bigg[-\frac{64}{9}\ln^3\left(\frac{\mu}{\tau_\omega Q}\right)-\frac{32}{3}\ln(r)\ln^2\left(\frac{\mu}{\tau_\omega Q}\right)+\left(\frac{8\pi^2}{9}-\frac{16\ln^2(r)}{3}\right)\ln\left(\frac{\mu}{\tau_\omega Q}\right)\\
&+\ln\left(\frac{\mu}{2\omega}\right)\left(-\frac{64{\rm Li}_2(-r)}{3}-\frac{16}{3}\ln^2(r)-\frac{16\pi^2}{9}\right)+\frac{32}{3}\ln(r)\ln^2\left(\frac{\mu}{2\omega}\right)\Bigg].
\end{aligned}
\tag{A.1}
$$

For the real-virtual interference contributions, the result is

$$
K_R^{\rm R-V}(\tau_\omega,\omega,\mu) = C_F C_A\Bigg[\frac{64}{3}\ln^4\left(\frac{\mu}{\tau_\omega Q}\right)+\frac{128}{3}\ln(r)\ln^3\left(\frac{\mu}{\tau_\omega Q}\right)+\big(32\ln^2(r)
$$

$$
-8\pi^2)\ln^2\left(\frac{\mu}{\tau_\omega Q}\right)+\ln\left(\frac{\mu}{\tau_\omega Q}\right)\left(\frac{32\ln^3(r)}{3}-8\pi^2\ln(r)-\frac{64\zeta(3)}{3}\right)
$$

$$
+\ln^2\left(\frac{\mu}{2\omega}\right)\left(128\mathrm{Li}_2(-r)+32\ln^2(r)+\frac{32\pi^2}{3}\right)+\ln\left(\frac{\mu}{2\omega}\right)\left(128\mathrm{Li}_3\left(\frac{1}{r+1}\right)\right.
$$

$$
-128\mathrm{Li}_3\left(\frac{r}{r+1}\right)-128\mathrm{Li}_2(-r)\ln(r)+256\mathrm{Li}_2(-r)\ln(r+1)-\frac{32}{3}\ln^3(r)
$$

$$
\left.+64\ln^2(r+1)\ln(r)+8\pi^2\ln(r)+\frac{64}{3}\pi^2\ln(r+1)\right)-\frac{128}{3}\ln(r)\ln^3\left(\frac{\mu}{2\omega}\right)\Bigg] \tag{A.2}
$$

and for the same-side in-in contribution, we have

$$
K_R^{r_1}(\tau_\omega,\omega,\mu)=C_FC_A\Bigg[-\frac{64}{3}\ln^4\left(\frac{\mu}{\tau_\omega Q}\right)+\left(-\frac{128\ln(r)}{3}-\frac{352}{9}\right)\ln^3\left(\frac{\mu}{\tau_\omega Q}\right)
$$

$$
+\left(-32\ln^2(r)-\frac{176\ln(r)}{3}+\frac{16\pi^2}{3}-\frac{536}{9}\right)\ln^2\left(\frac{\mu}{\tau_\omega Q}\right)+\ln\left(\frac{\mu}{\tau_\omega Q}\right)\left(-\frac{32}{3}\ln^3(r)\right.
$$

$$
\left.-\frac{88\ln^2(r)}{3}+\frac{16}{3}\pi^2\ln(r)-\frac{536\ln(r)}{9}+\frac{232\zeta(3)}{3}-\frac{22\pi^2}{9}-\frac{1544}{27}\right)\Bigg]
$$

$$
+C_Fn_fT_F\left[\frac{128}{9}\ln^3\left(\frac{\mu}{\tau_\omega Q}\right)+\left(\frac{64\ln(r)}{3}+\frac{160}{9}\right)\ln^2\left(\frac{\mu}{\tau_\omega Q}\right)+\left(\frac{32\ln^2(r)}{3}+\frac{160\ln(r)}{9}\right.\right.
$$

$$
\left.\left.+\frac{8\pi^2}{9}+\frac{304}{27}\right)\ln\left(\frac{\mu}{\tau_\omega Q}\right)\right]. \tag{A.3}
$$

The calculation of the opposite-side in-in contributions is somewhat less trivial since the integral over $S_R^{r_2}(k_L,k_R,\lambda,\mu)$ requires sector decomposition. Carrying out this analysis leads to

$$
C_FC_A\Bigg[-2\ln\left(\frac{\mu}{\tau_\omega Q}\right)\int_0^1\frac{\mathrm{d}z}{z}\left(\frac{2f_{C_A}^{(0)}\left(\frac{z}{2-z},r\right)}{2-z}-f_{C_A}^{(0)}(0,r)\right)
$$

$$
+2\ln(2)f_{C_A}^{(0)}(0,r)\ln\left(\frac{\mu}{\tau_\omega Q}\right)+f_{C_A}^{(1)}(0,r)\ln\left(\frac{\mu}{\tau_\omega Q}\right)+2f_{C_A}^{(0)}(0,r)\ln^2\left(\frac{\mu}{\tau_\omega Q}\right)\Bigg]
$$

$$
+C_Fn_fT_F\Bigg[-2\ln\left(\frac{\mu}{\tau_\omega Q}\right)\int_0^1\frac{\mathrm{d}z}{z}\left(\frac{2f_{n_f}^{(0)}\left(\frac{z}{2-z},r\right)}{2-z}-f_{n_f}^{(0)}(0,r)\right)
$$

$$
+2\ln(2)f_{n_f}^{(0)}(0,r)\ln\left(\frac{\mu}{\tau_\omega Q}\right)+f_{n_f}^{(1)}(0,r)\ln\left(\frac{\mu}{\tau_\omega Q}\right)+2f_{n_f}^{(0)}(0,r)\ln^2\left(\frac{\mu}{\tau_\omega Q}\right)\Bigg] \tag{A.4}
$$

for the finite part of the scale-dependent terms. Eq. (A.4) is written in terms of the functions introduced in Section 3. We can easily evaluate the above expression using the moments tabulated in Eqs. (3.19) and the result is

$$K_R^{r_2}(\tau_\omega,\omega,\mu)=C_FC_A\Bigg[\Bigg(\frac{176\mathrm{Li}_2\left(r^2\right)}{3}+32\mathrm{Li}_3\left(\frac{r^2}{r^2-1}\right)-32\mathrm{Li}_2\left(r^2\right)\ln\left(1-r^2\right)$$
$$+24\mathrm{Li}_2\left(r^2\right)\ln\left(r^2\right)+\frac{16r^2}{3-3r^2}-\frac{16}{3}\ln^3\left(1-r^2\right)+\frac{88}{3}\ln\left(r^2\right)\ln\left(1-r^2\right)$$
$$+\frac{32r^2\ln(r)}{3\left(r^2-1\right)^2}\Bigg)\ln\left(\frac{\mu}{\tau_\omega Q}\right)+\left(64\mathrm{Li}_2(-r)+64\mathrm{Li}_2(r)\right)\ln^2\left(\frac{\mu}{\tau_\omega Q}\right)\Bigg]+C_Fn_fT_F\Bigg[\Bigg(-\frac{64\mathrm{Li}_2\left(r^2\right)}{3}$$
$$+\frac{32r^2}{3\left(r^2-1\right)}-\frac{64r^2\ln(r)}{3\left(r^2-1\right)^2}-\frac{64}{3}\ln(r)\ln\left(1-r^2\right)\Bigg)\ln\left(\frac{\mu}{\tau_\omega Q}\right)\Bigg]. \tag{A.5}$$

The calculation of the in-out contributions requires sector decomposition as well. In terms of the functions introduced in Section 3, the scale-dependent terms are given by

$$C_FC_A\Bigg[g_{C_A}^{(1)}(0,r)\ln\left(\frac{\mu}{2\omega}\right)+g_{C_A}^{(0)}(0,r)\ln^2\left(\frac{\mu}{2\omega}\right)+g_{C_A}^{(0)}(0,r)\ln^2\left(\frac{\mu}{\tau_\omega Q}\right)$$
$$-\ln\left(\frac{\mu}{2\omega}\right)\int_0^1\frac{\mathrm{d}z}{z}\left(g_{C_A}^{(0)}(z,r)+g_{C_A}^{(0)}(1/z,r)-2g_{C_A}^{(0)}(0,r)\right)-\frac{1}{2}g_{C_A}^{(1)}(0,r)\ln\left(\frac{\tau_\omega Q}{2\omega}\right)$$
$$-\frac{1}{2}g_{C_A}^{(0)}(0,r)\ln^2\left(\frac{\tau_\omega Q}{2\omega}\right)+\int_1^{\frac{\tau_\omega Q}{2\omega}}\frac{\mathrm{d}z_2}{z_2}\int_0^1\frac{\mathrm{d}z_1}{z_1}\left(g_{C_A}^{(0)}\left(\frac{z_2}{z_1},r\right)-g_{C_A}^{(0)}(0,r)\right)\Bigg]$$
$$+C_Fn_fT_F\Bigg[g_{n_f}^{(1)}(0,r)\ln\left(\frac{\mu}{2\omega}\right)+g_{n_f}^{(0)}(0,r)\ln^2\left(\frac{\mu}{2\omega}\right)+g_{n_f}^{(0)}(0,r)\ln^2\left(\frac{\mu}{\tau_\omega Q}\right)$$
$$-\ln\left(\frac{\mu}{2\omega}\right)\int_0^1\frac{\mathrm{d}z}{z}\left(g_{n_f}^{(0)}(z,r)+g_{n_f}^{(0)}(1/z,r)-2g_{n_f}^{(0)}(0,r)\right)-\frac{1}{2}g_{n_f}^{(1)}(0,r)\ln\left(\frac{\tau_\omega Q}{2\omega}\right)$$
$$-\frac{1}{2}g_{n_f}^{(0)}(0,r)\ln^2\left(\frac{\tau_\omega Q}{2\omega}\right)+\int_1^{\frac{\tau_\omega Q}{2\omega}}\frac{\mathrm{d}z_2}{z_2}\int_0^1\frac{\mathrm{d}z_1}{z_1}\left(g_{n_f}^{(0)}\left(\frac{z_2}{z_1},r\right)-g_{n_f}^{(0)}(0,r)\right)\Bigg]. \tag{A.6}$$

Using the moments tabulated in Eqs. (3.22), we find an explicit expression for Eq. (A.6):

$$K_R^{r_3}(\tau_\omega,\omega,\mu)=C_FC_A\Bigg[\left(-64\mathrm{Li}_2(-r)-64\mathrm{Li}_2(r)+\frac{16\pi^2}{3}\right)\ln^2\left(\frac{\mu}{\tau_\omega Q}\right)$$
$$+\left(-64\mathrm{Li}_2(-r)-64\mathrm{Li}_2(r)+\frac{16\pi^2}{3}\right)\ln^2\left(\frac{\mu}{2\omega}\right)+\ln\left(\frac{\mu}{2\omega}\right)\Bigg(-\frac{352\mathrm{Li}_2\left(r^2\right)}{3}$$
$$+128\mathrm{Li}_3(1-r)+64\mathrm{Li}_3(r)-64\mathrm{Li}_3\left(\frac{1}{r+1}\right)+64\mathrm{Li}_3\left(\frac{r}{r+1}\right)-128\mathrm{Li}_2(-r)\ln(r+1)$$
$$+128\mathrm{Li}_2(r)\ln(1-r)-64\mathrm{Li}_2(r)\ln(r)+\frac{16\left(r^2+1\right)}{3\left(r^2-1\right)}-\frac{64r^2\ln(r)}{3\left(r^2-1\right)^2}-32\ln(r)\ln^2(r+1)$$
$$+64\ln^2(1-r)\ln(r)-\frac{64}{3}\pi^2\ln(1-r)-\frac{352}{3}\ln(1-r)\ln(r)-\frac{352}{3}\ln(r)\ln(r+1)$$
$$-\frac{32}{3}\pi^2\ln(r+1)-64\zeta_3+\frac{176\pi^2}{9}\Bigg)+\ln\left(\frac{\tau_\omega Q}{2\omega}\right)\Bigg(8\mathrm{Li}_3\left(r^2\right)-64\mathrm{Li}_3\left(\frac{r}{r+1}\right)+48\zeta_3$$

$$-64\mathrm{Li}_3(1-r)+64\mathrm{Li}_2(-r)\ln(r+1)-32\mathrm{Li}_2(-r)\ln(r)-64\mathrm{Li}_2(r)\ln(1-r)+\frac{32}{3}\ln^3(r+1)$$
$$+\frac{32}{3}\pi^2\ln(1-r)-32\ln^2(1-r)\ln(r)\Big)+\ln^2\left(\frac{\tau_\omega Q}{2\omega}\right)\left(32\mathrm{Li}_2(-r)-\frac{8\pi^2}{3}+32\mathrm{Li}_2(r)\right)$$
$$+\chi_{C_A}\left(\frac{\tau_\omega Q}{2\omega},r\right)-\chi_{C_A}(1,r)\Bigg]+C_F n_f T_F\left[\ln\left(\frac{\mu}{2\omega}\right)\left(\frac{128\mathrm{Li}_2\left(r^2\right)}{3}-\frac{64\pi^2}{9}-\frac{32\left(r^2+1\right)}{3\left(r^2-1\right)}\right.\right.$$
$$\left.\left.+\frac{128r^2\ln(r)}{3\left(r^2-1\right)^2}+\frac{128}{3}\ln(r)\ln\left(1-r^2\right)\right)+\chi_{n_f}\left(\frac{\tau_\omega Q}{2\omega},r\right)-\chi_{n_f}(1,r)\right], \tag{A.7}$$

where $\chi_{C_A}(x,r)$ and $\chi_{n_f}(x,r)$ are non-trivial functions built out of one- and two-dimensional harmonic polylogarithms. They are given explicitly in Appendix B.

Finally, for the out-out contribution, the result is

$$K_R^{r_4}(\tau_\omega,\omega,\mu)=C_F C_A\Bigg[\ln\left(\frac{\mu}{2\omega}\right)\Bigg(-32\mathrm{Li}_3\left(\frac{r^2}{r^2-1}\right)+\frac{352\mathrm{Li}_2(r)}{3}$$
$$-64\mathrm{Li}_3(r)-64\mathrm{Li}_3\left(\frac{1}{r+1}\right)+64\mathrm{Li}_3\left(\frac{r}{r+1}\right)+64\mathrm{Li}_2(-r)\ln(1-r)$$
$$+32\mathrm{Li}_2(-r)\ln(r)-64\mathrm{Li}_2(-r)\ln(r+1)-32\mathrm{Li}_2(r)\ln(r)+64\mathrm{Li}_2(r)\ln(r+1)$$
$$+\frac{32r^2\ln(r)}{3\left(r^2-1\right)^2}+\frac{16}{3}\ln^3(1-r)+\frac{32\ln^3(r)}{3}+\frac{16}{3}\ln^3(r+1)-64\ln(r)\ln^2(1-r)$$
$$+16\ln(r+1)\ln^2(1-r)+16\ln^2(r+1)\ln(1-r)-\frac{88\ln^2(r)}{3}-32\ln(r)\ln^2(r+1)$$
$$+\frac{176}{3}\ln(r)\ln(1-r)+\frac{64}{3}\pi^2\ln(1-r)-\frac{16}{3}\pi^2\ln(r)+\frac{536\ln(r)}{9}-\frac{32}{3}\pi^2+64\zeta_3$$
$$-\frac{176\pi^2}{9}+\frac{176}{3}\ln(r)\ln(r+1)\ln(r+1)-128\mathrm{Li}_3(1-r)+\frac{8r^2+8}{3-3r^2}-64\mathrm{Li}_2(r)\ln(1-r)\Bigg)$$
$$+\ln^2\left(\frac{\mu}{2\omega}\right)\left(-64\mathrm{Li}_2(-r)+64\mathrm{Li}_2(r)-32\ln^2(r)+\frac{176\ln(r)}{3}-16\pi^2\right)$$
$$+\frac{128}{3}\ln(r)\ln^3\left(\frac{\mu}{2\omega}\right)\Bigg]$$
$$+C_F n_f T_F\Bigg[\ln\left(\frac{\mu}{2\omega}\right)\Bigg(-\frac{128\mathrm{Li}_2(r)}{3}-\frac{16\left(r^2+1\right)}{3\left(1-r^2\right)}-\frac{64r^2\ln(r)}{3\left(1-r^2\right)^2}+\frac{32\ln^2(r)}{3}-\frac{160\ln(r)}{9}$$
$$-\frac{64}{3}\ln(r)\ln\left(\frac{1-r}{r+1}\right)-\frac{128}{3}\ln(r)\ln(r+1)+\frac{64\pi^2}{9}\Bigg)-\frac{64}{3}\ln(r)\ln^2\left(\frac{\mu}{2\omega}\right)\Bigg]. \tag{A.8}$$

Now that all the pieces are in place, we can combine them together and study the result.

B. Analytic Expressions For Non-Trivial Two-Parameter Integrals

The non-trivial two-parameter integrals

$$\int_1^x\frac{\mathrm{d}z_2}{z_2}\int_0^1\frac{\mathrm{d}z_1}{z_1}\left(g_{C_A}^{(0)}\left(\frac{z_2}{z_1},r\right)-g_{C_A}^{(0)}(0,r)\right)$$

$$\int_1^x \frac{\mathrm{d}z_2}{z_2} \int_0^1 \frac{\mathrm{d}z_1}{z_1} \left(g^{(0)}_{n_f}\left(\frac{z_2}{z_1}, r\right) - g^{(0)}_{n_f}(0,r) \right) , \tag{B.1}$$

first encountered in Section 3, were parametrized there in terms of two functions, $\chi_{C_A}(x,r)$ and $\chi_{n_f}(x,r)$, such that:

$$\begin{aligned}
\int_1^x \frac{\mathrm{d}z_2}{z_2} \int_0^1 \frac{\mathrm{d}z_1}{z_1} \left(g^{(0)}_{C_A}\left(\frac{z_2}{z_1}, r\right) - g^{(0)}_{C_A}(0,r) \right) &= \chi_{C_A}(x,r) - \chi_{C_A}(1,r) \\
\int_1^x \frac{\mathrm{d}z_2}{z_2} \int_0^1 \frac{\mathrm{d}z_1}{z_1} \left(g^{(0)}_{n_f}\left(\frac{z_2}{z_1}, r\right) - g^{(0)}_{n_f}(0,r) \right) &= \chi_{n_f}(x,r) - \chi_{n_f}(1,r) \, .
\end{aligned} \tag{B.2}$$

As explained in Section 3, $g^{(0)}_{C_A}(x,r)$ and $g^{(0)}_{n_f}(x,r)$ are simply the leading order terms in the epsilon expansion of the in-out contributions to the τ_ω soft function. In this appendix we provide analytical expressions for both $\chi_{C_A}(x,r)$ and $\chi_{n_f}(x,r)$. In what follows the H functions are one- or two-dimensional harmonic polylogarithms (introduced in Refs. [50] and [51] respectively).

For the $C_F C_A$ color structure we have

$$\begin{aligned}
\chi_{C_A}(x,r) &= -\frac{32r^2(x-1)}{3(2r+1)(xr+r+x)} + \frac{32\left(2H(0,r)+H\left(\frac{r^2}{r+1},x\right)\right)x^3}{3(x+1)(xr+r+x)} \\
&- \frac{4\left(H(-1,r)-5H(0,r)+H\left(0,1-r^2\right)+4H(0,x)-4H\left(\frac{r}{r+1},x\right)\right)x}{3(x+1)} \\
&+ \frac{16(1-2x)H\left(\frac{1}{r+1},x\right)x}{3(xr+r+x)} + \frac{16\left(2H(0,r)+H\left(\frac{r^2}{r+1},x\right)\right)x^3}{3(x+1)^2(xr+r+x)} \\
&- \frac{16(r+1)x}{3(x+1)(xr+r+x)}\Big(H(-1,r)-H(0,r)+H\left(0,1-r^2\right)+H(0,x)\Big) \\
&+ \frac{32x}{3(r+1)}\left(-2H(0,r)+H\left(\frac{1}{r+1},x\right)-H\left(\frac{r^2}{r+1},x\right)\right) - \frac{32\left(H(0,x)-H\left(\frac{1}{r+1},x\right)\right)}{3\left(r^2-1\right)} \\
&+ \frac{4\left(3H(-1,r)+5H(0,r)+3H\left(0,1-r^2\right)+8H\left(\frac{r}{r+1},x\right)\right)}{3(x+1)} + \frac{16}{3(r+1)^2}\Bigg(-2H(0,r) \\
&+ H\left(\frac{1}{r+1},x\right) - H\left(\frac{r^2}{r+1},x\right)\Bigg) - \frac{16}{3(x+1)(rx+x+1)}\Bigg(H(-1,r)-2H(0,r) \\
&+ H\left(0,1-r^2\right)+H(0,x)-H\left(\frac{r^2}{r+1},x\right)\Bigg) - \frac{32}{3(r+1)}\Bigg(-H(0,r)+H\left(\frac{r}{r+1},x\right) \\
&- H\left(\frac{r^2}{r+1},x\right)\Bigg) - \frac{16(x+2)\left(2H(0,r)+H\left(\frac{r^2}{r+1},x\right)\right)}{3(x+1)^2} - \frac{8r}{3(r+1)^2}\Bigg(2H(0,r)\Bigg(H(0,x) \\
&- 4H\left(\frac{r}{r+1},x\right)\Bigg) - H\left(0,\frac{1}{r+1},x\right) + H\left(0,\frac{r^2}{r+1},x\right) + 4H\left(\frac{r}{r+1},\frac{1}{r+1},x\right) \\
&- 4H\left(\frac{r}{r+1},\frac{r^2}{r+1},x\right)\Bigg) + \frac{16x}{3(x+1)^2}\Bigg(2H(0,r)H(0,x) - \Bigg(H\left(\frac{1}{r+1},x\right)
\end{aligned} \tag{B.3}$$

$$
\begin{aligned}
&-H\left(\frac{r}{r+1},x\right)\Bigg)\,H(0,x)-\Bigg(H(-1,r)-2H(0,r)+H\left(0,1-r^2\right)\Bigg)H\left(\frac{1}{r+1},x\right)\\
&+\Bigg(H(-1,r)-3H(0,r)+H\left(0,1-r^2\right)\Bigg)H\left(\frac{r}{r+1},x\right)+2H(0,-1,r)-5H(0,0,r)\\
&-H\left(0,1,r^2\right)+H\left(0,\frac{1}{r+1},x\right)-2H\left(0,\frac{r}{r+1},x\right)+H\left(0,\frac{r^2}{r+1},x\right)\\
&+H\left(\frac{1}{r+1},\frac{r^2}{r+1},x\right)-H\left(\frac{r}{r+1},\frac{r^2}{r+1},x\right)\Bigg)+\frac{8r}{3(r-1)^2}\Bigg(2H(0,r)H(0,x)\\
&-H\left(0,\frac{1}{r+1},x\right)+H\left(0,\frac{r^2}{r+1},x\right)\Bigg)+\frac{176}{3}H(-1,x)\Bigg(2H(0,-1,r)-5H(0,0,r)\\
&-H\left(0,1,r^2\right)\Bigg)+\frac{176}{3}H(0,x)\Bigg(2H(0,r)H\left(0,1-r^2\right)+2H\left(0,1,r^2\right)-\frac{\pi^2}{3}\Bigg)\\
&+\frac{176}{3}\Bigg(-H(-1,x)H(0,x)H\left(\frac{1}{r+1},x\right)+\Bigg(H(-1,x)+H(0,x)\Bigg)H\left(0,\frac{1}{r+1},x\right)\\
&-\Bigg(2H(-1,x)+H(0,x)\Bigg)H\left(0,\frac{r}{r+1},x\right)+H(0,x)\Bigg(H\left(-1,\frac{r}{r+1},x\right)\\
&+H\left(\frac{1}{r+1},-1,x\right)\Bigg)+H\left(-1,0,\frac{r^2}{r+1},x\right)+H\left(-1,\frac{1}{r+1},\frac{r^2}{r+1},x\right)\\
&-H\left(-1,\frac{r}{r+1},\frac{r^2}{r+1},x\right)+H\left(0,-1,\frac{r}{r+1},x\right)-3H\left(0,0,\frac{1}{r+1},x\right)\\
&+3H\left(0,0,\frac{r}{r+1},x\right)-H\left(0,\frac{1}{r+1},-1,x\right)-H\left(0,\frac{1}{r+1},\frac{r^2}{r+1},x\right)\\
&+2H\left(0,\frac{r}{r+1},-1,x\right)+H\left(0,\frac{r}{r+1},\frac{1}{r+1},x\right)\Bigg)+\frac{176}{3}\Bigg(H(-1,r)\\
&+H\left(0,1-r^2\right)\Bigg)\Bigg(-H\left(-1,\frac{1}{r+1},x\right)+H\left(-1,\frac{r}{r+1},x\right)+H\left(0,\frac{1}{r+1},x\right)\\
&-H\left(0,\frac{r}{r+1},x\right)\Bigg)+\frac{176}{3}H(0,r)\Bigg(2H(-1,0,x)+2H\left(-1,\frac{1}{r+1},x\right)\\
&-3H\left(-1,\frac{r}{r+1},x\right)-2H\left(0,\frac{1}{r+1},x\right)+H\left(0,\frac{r}{r+1},x\right)\Bigg)+8H(0,-1,x)\Bigg(\frac{\pi^2}{3}\\
&-4H(0,-1,r)+10H(0,0,r)\Bigg)+16H(-1,r)\Bigg(H\left(0,-1,\frac{1}{r+1},x\right)\\
&-H\left(0,-1,\frac{r}{r+1},x\right)-H\left(0,0,\frac{1}{r+1},x\right)+H\left(0,0,\frac{r}{r+1},x\right)\Bigg)\\
&-16H(0,r)\Bigg(2H(0,-1,0,x)+2H\left(0,-1,\frac{1}{r+1},x\right)-3H\left(0,-1,\frac{r}{r+1},x\right)\\
&-2H\left(0,0,-1,\frac{x}{r}\right)-2H\left(0,0,\frac{1}{r+1},x\right)+3H\left(0,0,\frac{r}{r+1},x\right)\Bigg)\\
&+16\Bigg(H(0,x)\Bigg(H\left(0,-1,\frac{1}{r+1},x\right)-H\left(0,0,\frac{1}{r+1},x\right)+H\left(0,0,\frac{r}{r+1},x\right)\Bigg)\\
&-H\left(0,-1,0,\frac{r^2}{r+1},x\right)-H\left(0,-1,\frac{1}{r+1},\frac{r^2}{r+1},x\right)-H\left(0,-1,\frac{r}{r+1},0,x\right)
\end{aligned}
$$

$$+H\left(0,-1,\frac{r}{r+1},\frac{r}{r+1},x\right)-H\left(0,-1,\frac{r}{r+1},\frac{1}{r+1},x\right)+H\left(0,-1,\frac{r}{r+1},\frac{r^2}{r+1},x\right)$$
$$-2H\left(0,0,-1,\frac{1}{r+1},x\right)-H\left(0,0,-1,\frac{1}{r+1},\frac{x}{r}\right)+H\left(0,0,-1,\frac{r}{r+1},\frac{x}{r}\right)$$
$$+2H\left(0,0,0,\frac{1}{r+1},x\right)-2H\left(0,0,0,\frac{r}{r+1},x\right)+H\left(0,0,\frac{1}{r+1},\frac{r^2}{r+1},x\right)$$
$$-H\left(0,0,\frac{r}{r+1},\frac{r}{r+1},x\right)+H\left(0,0,\frac{r}{r+1},\frac{1}{r+1},x\right)-H\left(0,0,\frac{r}{r+1},\frac{r^2}{r+1},x\right)\Bigg)$$
$$+32H(0,x)\Bigg(H(0,r)H\left(0,1,r^2\right)-H\left(0,0,1,r^2\right)+\zeta_3\Bigg)$$

and for the $C_F n_f T_F$ color structure we have

$$\chi_{n_f}(x,r)=-\frac{64(1-x)r^2}{3(2r+1)(xr+r+x)}+\frac{32(r+1)r}{3(xr+r+x)}\left(-H(-1,r)+H(0,r)\right. \tag{B.4}$$
$$\left.-H\left(0,1-r^2\right)-H(0,x)\right)+\frac{32H\left(\frac{1}{r+1},x\right)r}{(r+1)(xr+r+x)}-\frac{64H\left(\frac{1}{r+1},x\right)r}{3(r+1)^2(xr+r+x)}$$
$$+\frac{32\left(2H(0,r)-H\left(\frac{1}{r+1},x\right)+H\left(\frac{r^2}{r+1},x\right)\right)r}{3(r+1)^2}-\frac{32(r-1)\left(H(0,r)+H\left(\frac{r}{r+1},x\right)\right)}{3(r+1)}$$
$$+\frac{64H\left(\frac{1}{r+1},x\right)r}{3\left(1-r^2\right)}-\frac{32\left(-H(0,r)+H\left(\frac{r}{r+1},x\right)-H\left(\frac{r^2}{r+1},x\right)\right)}{3(x+1)}-\frac{32H(0,r)r}{3(x+1)}$$
$$+\frac{32\left(r^2+1\right)H(0,x)}{3\left(r^2-1\right)}-\frac{64\left(2H(0,r)+H\left(\frac{r^2}{r+1},x\right)\right)r^4}{3(r+1)^2(xr+r+x)}$$
$$+\frac{32\left(r^2-1\right)\left(H(-1,r)-2H(0,r)+H\left(0,1-r^2\right)+H(0,x)-H\left(\frac{r^2}{r+1},x\right)\right)}{3(x+1)r}$$
$$+\frac{32(r+1)\left(H(-1,r)-2H(0,r)+H\left(0,1-r^2\right)+H(0,x)-H\left(\frac{r^2}{r+1},x\right)\right)}{3(rx+x+1)r}$$
$$+\frac{32\left(2H(0,r)+H\left(\frac{r^2}{r+1},x\right)\right)r^3}{(r+1)(xr+r+x)}-\frac{16(H(-1,r)H(0,x)+H(0,0,x))r}{3(r-1)^2}$$
$$-\frac{64\left((2H(0,r)-H(-1,r))H(0,x)-H(0,0,x)-H\left(0,\frac{1}{r+1},x\right)+H\left(0,\frac{r^2}{r+1},x\right)\right)r^2}{3\left(r^2-1\right)^2}$$
$$-\frac{32x}{3(x+1)^2}\Bigg(H(0,x)\Bigg(H\left(\frac{r}{r+1},x\right)-H\left(\frac{1}{r+1},x\right)\Bigg)+H(0,r)\Bigg(2H(0,x)$$
$$-H\left(\frac{r}{r+1},x\right)\Bigg)+\left(H(-1,r)-2H(0,r)+H\left(0,1-r^2\right)\right)\Bigg(H\left(\frac{r}{r+1},x\right)$$
$$-H\left(\frac{1}{r+1},x\right)\Bigg)-5H(0,0,r)+2H(0,1,r)-2H\left(0,1,r^2\right)+H\left(0,\frac{1}{r+1},x\right)$$
$$-2H\left(0,\frac{r}{r+1},x\right)+H\left(0,\frac{r^2}{r+1},x\right)+H\left(\frac{1}{r+1},\frac{r^2}{r+1},x\right)-H\left(\frac{r}{r+1},\frac{r^2}{r+1},x\right)\Bigg)$$

$$
\begin{aligned}
&+\frac{16r}{3(r+1)^2}\Bigg(H(-1,r)H(0,x)-8H(0,r)H\left(\frac{r}{r+1},x\right)+H(0,0,x)\\
&+4\Bigg(H\left(\frac{r}{r+1},\frac{1}{r+1},x\right)-H\left(\frac{r}{r+1},\frac{r^2}{r+1},x\right)\Bigg)\Bigg)+\frac{128}{3}H(0,x)\left(-H\left(0,1,r^2\right)\right.\\
&\left.+\frac{\pi^2}{6}-H(0,r)H\left(0,1-r^2\right)\right)+\frac{64}{3}H(-1,x)\left(-2H(0,-1,r)+5H(0,0,r)\right.\\
&\left.+H\left(0,1,r^2\right)\right)+\frac{64}{3}H(0,r)\Bigg(-H\left(-1,\frac{1}{r+1},x\right)+2\Bigg(H\left(-1,\frac{r}{r+1},x\right)\\
&-H(-1,0,x)\Bigg)+H\left(0,\frac{1}{r+1},x\right)\Bigg)+\frac{64}{3}H(0,x)\Bigg(H\left(-1,\frac{1}{r+1},x\right)\\
&-H\left(-1,\frac{r}{r+1},x\right)-H\left(0,\frac{1}{r+1},x\right)+H\left(0,\frac{r}{r+1},x\right)\Bigg)+\frac{64}{3}\Bigg(H(-1,r)\\
&-H(0,r)+H\left(0,1-r^2\right)\Bigg)\Bigg(H\left(-1,\frac{1}{r+1},x\right)-H\left(-1,\frac{r}{r+1},x\right)-H\left(0,\frac{1}{r+1},x\right)\\
&+H\left(0,\frac{r}{r+1},x\right)\Bigg)+\frac{64}{3}H(-1,x)\Bigg(2H\left(0,\frac{r}{r+1},x\right)-H\left(0,\frac{1}{r+1},x\right)\Bigg)\\
&-\frac{64}{3}\Bigg(H\left(-1,0,\frac{r^2}{r+1},x\right)+H\left(-1,\frac{1}{r+1},\frac{r^2}{r+1},x\right)-H\left(-1,\frac{r}{r+1},\frac{r^2}{r+1},x\right)\\
&+H\left(0,-1,\frac{r}{r+1},x\right)-3H\left(0,0,\frac{1}{r+1},x\right)+3H\left(0,0,\frac{r}{r+1},x\right)-H\left(0,\frac{1}{r+1},-1,x\right)\\
&-H\left(0,\frac{1}{r+1},\frac{r^2}{r+1},x\right)+2H\left(0,\frac{r}{r+1},-1,x\right)+H\left(0,\frac{r}{r+1},\frac{1}{r+1},x\right)\Bigg)\ .
\end{aligned}
$$

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