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J. Ramos-Caro, C. A. Agón, and J. F. Pedraza

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Kinetic Theory of Collisionless Self-Gravitating Gases: II. Relativistic Corrections in Galactic Dynamics

J. Ramos-Caro,[1] C. A. Agón,[2] and J. F. Pedraza[3]

[1] *Departamento de Matemática Aplicada, IMECC, Universidade Estadual de Campinas, Campinas, São Paulo 13083-859, Brazil*[*]

[2] *Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apartado Postal 70-543, México D.F. 04510, México*[†]

[3] *Theory Group, Department of Physics and Texas Cosmology Center University of Texas, 1 University Station C1608, Austin, Texas 78712, USA*[‡]

In this paper we study the kinetic theory of many-particle astrophysical systems imposing axial symmetry and extending our previous analysis in Phys. Rev. D **83**, 123007 (2011). Starting from a Newtonian model describing a collisionless self-gravitating gas, we develop a framework to include systematically the first general relativistic corrections to the matter distribution and gravitational potentials for general stationary systems. Then, we use our method to obtain particular solutions for the case of the Morgan & Morgan disks. The models obtained are fully analytical and correspond to the post-Newtonian generalizations of classical ones. We explore some properties of the models in order to estimate the importance of post-Newtonian corrections and we find that, contrary to the expectations, the main modifications appear far from the galaxy cores. As a by-product of this investigation we derive the corrected version of the tensor virial theorem. For stationary systems we recover the same result as in the Newtonian theory. However, for time dependent backgrounds we find that there is an extra piece that contributes to the variation of the inertia tensor.

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I. INTRODUCTION

The dynamics and evolution of collisionless stellar ensembles is a subject of great interest in astrophysics since they are the primary tool for comparisons of observations and theory in galactic dynamics [1]. Such systems are generally composed by billions of stars, so it is neither practical nor worthwhile to follow the orbit of each particle in the ensemble. Most testable predictions depend only on the distribution function $F(\mathbf{x}, \mathbf{v}, t)$ (DF), a quantity that determines the probability of finding a single star in a given phase-space volume $d^3\mathbf{x}d^3\mathbf{v}$ around the position $\mathbf{x}$ and the velocity $\mathbf{v}$. The dynamics of the DF follows from the appropriate kinetic equation and it in turn determines the statistical evolution of the system.

In the framework of the general theory of relativity (GR) it is assumed that the DF satisfies the general relativistic version of the Fokker-Planck equation [2–4] or the collisionless Boltzmann equation (CBE) [5, 6]. The first one is devoted to systems in which local gravitational encounters dominate in their evolution whereas the latter is useful to study systems sufficiently smooth, so that they may be considered to be collisionless [1]. One can actually consider systems in which a number of particle species can collide and produce different species. This is how the formation of the light elements in the big bang nucleosynthesis is calculated (see [7] for a review). However, in systems such as galaxies and galaxy clusters, physical encounters between the stars are very rare, and the effect of gravitational collisions can be neglected for times far longer than the age of the universe. Thus, for these systems, the CBE provides a very good approximation.

For a typical galaxy, the relaxation time, t_{relax}, is arbitrarily large in comparison with the crossing time, t_{cross}. This means that they can be approximated as a continuum rather than concentrated into nearly point-like stars. Now, it is commonly assumed that the main contribution of the mass in a galaxy is concentrated in an axisymmetric flat distribution [1]. For this reason the obtention of idealized thin disk models has been a problem of great astrophysical relevance. In this case, the most straightforward way to construct a self-consistent model is by means of finding the DF for a system with a known gravitational interactions and matter distribution. Since the mass density of the system is defined by the integration of the DF over the velocity, the problem of finding a DF is that of solving an integral equation (see [8–13] and the references therein). At present we have at disposal a variety of self-consistent galaxy models: [14–25].

Now, even though for most systems under consideration Newtonian gravity is believed to be dominant, general relativistic corrections might play an important role in their evolution. As a matter of fact, in recent years it has been an increasing interest in the incorporation of GR in the description of these systems, and up to date we have a variety of fully relativistic galaxy models: [26–34], among others. Perhaps the principal reason of including GR corrections in galactic dynamics, is the hypothesis that it is possible to overcome the problem of the rota-

[*] Electronic address: javier@ime.unicamp.br
[†] Electronic address: cesar.agon@nucleares.unam.mx
[‡] Electronic address: jpedraza@physics.utexas.edu

tion curves predicted by the Newtonian theory. While, some authors argue that by using GR the inclusion of a dark matter halo is unnecessary at galactic scales (see for instance [35–39]), several publications have pointed out that this is not entirely true [40–44]. In particular, the authors of [45] presented a model in which the percentage of dark matter needed to explain flat rotation curves turns out to be $\sim 30\%$ less than the required by the Newtonian theory. It is important to point out that currently there are alternative approaches to GR which address the problem of rotation curves in spiral galaxies, as for example the so-called $f(R)$ gravity (see [46], for references).

Despite the fact that the relativistic contributions do not solve completely the problem of rotation curves in galaxies, it seems that they do introduce significant corrections. Thus, in order to estimate the effects on the various observables we are interested in, it would be nice to have a framework to include systematically general relativistic corrections to a given Newtonian model. The post-Newtonian approximation is perfectly suited for this purpose. The appropriate scheme that describes the effects of the first corrections beyond the Newtonian theory, was first formulated in [47–49] (see [50] for a textbook analysis) and it is known as the first post-Newtonian (1PN) approximation. This approach holds if the particles in the system are moving non relativistically ($\bar{v} \ll c$) (as in the case of a star moving around a typical galaxy) and gives the corrections up to order $\bar{v}^2/c^2$, where $\bar{v}$ is a typical velocity in the system and c is the speed of light. Currently, higher order PN approximations have been developed because of the increasing interest around kinematics and associated emission of gravitational waves by binary pulsars, neutron stars and black holes, with promising candidates for detectors such as LIGO, VIRGO and GEO600 (see [51, 52] for references).

Based in the above considerations, we recently started a general study of self-gravitating gases in the collisionless regime and, as a first step, we derived a version of the CBE that accounts for the first general relativistic corrections [53]. With this tool in hand, we obtained the 1PN version of the Eddington's polytropes, starting from an ergodic DF proportional to E^n. The purpose of this paper is twofold. First, to implement a similar procedure in the axially symmetric case in order to setup our general framework. And second, to obtain a new set of self-consistent models starting from a Newtonian 'seed' and study the impact of relativistic corrections on the various observables.

The rest of the paper is organized as follows. In section II we present a brief overview about the basics of the 1PN approximation, revisiting the field equations, as well as the kinetic theory for arbitrary self-gravitating systems. In section III we show the fundamental equations defining self-consistent models with post-Newtonian corrections. We start dealing with arbitrary systems but then we focus on discoidal configurations with axial symmetry, in order to prepare the ground to construct 1PN galaxy models in section IV. Finally, we summarize the principal results in section V.

II. GENERAL FRAMEWORK

A. The 1PN approximation

The post-Newtonian approximation has been reviewed carefully in a number of references (see for example [50]). However, we will include here the basic definitions and relations for completeness.

First off, note that in Newtonian mechanics the typical kinetic energy is roughly of the same order of magnitude as the typical potential energy, and thus

$$\bar{v}^2 \sim \phi,$$

where $\bar{v}$ is the mean velocity in the system. The idea is then, to express all physical quantities in terms of a series expansion of the small parameter $\bar{v}/c \ll 1$, and keep the leading order beyond the Newtonian theory. The first quantity to consider is the spacetime itself: any manifold can be considered to be locally flat so, for particles that are moving nonrelativistically, we proceed to express the metric tensor as

$$\begin{aligned} g_{00} &= -1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} + \cdots, \\ g_{ij} &= \delta_{ij} + \overset{2}{g}_{ij} + \overset{4}{g}_{ij} + \cdots, \\ g_{0i} &= \overset{1}{g}_{0i} + \overset{3}{g}_{0i} + \overset{5}{g}_{0i} + \cdots, \end{aligned} \tag{1}$$

where the symbol $\overset{n}{g}_{\mu\nu}$ denotes the term in $g_{\mu\nu}$ of order $(\bar{v}/c)^n$. Odd powers of $\bar{v}/c$ appear in g_{0i} because these components must change sign under time-reversal transformation $t \to -t$.

It is natural to assume a similar expansion for the components of the energy momentum tensor. From their interpretation as the energy density, momentum flux and energy flux, we expect that

$$\begin{aligned} T^{00} &= \overset{0}{T}{}^{00} + \overset{2}{T}{}^{00} + \cdots, \\ T^{ij} &= \overset{2}{T}{}^{ij} + \overset{4}{T}{}^{ij} + \cdots, \\ T^{0i} &= \overset{1}{T}{}^{0i} + \overset{3}{T}{}^{0i} + \cdots. \end{aligned} \tag{2}$$

These expansions lead to a consistent solution of Einstein field equations.

Working in harmonic coordinates (i.e. coordinates such that $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$) and to our order of approximation, the various components of the metric tensor can be expressed in terms of the Newtonian potential ϕ and post-Newtonian potentials ψ and ξ_i as

$$\begin{aligned} \overset{2}{g}_{00} &= -2\phi/c^2, \\ \overset{4}{g}_{00} &= -2(\phi^2+\psi)/c^4, \\ \overset{2}{g}_{ij} &= -2\phi\delta_{ij}/c^2, \\ \overset{1}{g}_{0i} &= 0, \\ \overset{3}{g}_{0i} &= \xi_i/c^3. \end{aligned} \tag{3}$$

Thus, the Einstein equations reduce to

$$\nabla^2\phi \;=\; 4\pi G \overset{0}{T}{}^{00}, \tag{4}$$

$$\nabla^2\psi \;=\; 4\pi G c^2(\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii}) + \frac{\partial^2\phi}{\partial t^2}, \tag{5}$$

$$\nabla^2\xi_i \;=\; 16\pi G c \overset{1}{T}{}^{0i}, \tag{6}$$

along with the coordinate condition

$$4\frac{\partial\phi}{\partial t} + \nabla\cdot\xi = 0. \tag{7}$$

One can also consider the motion of test particles in a given background. For general potentials ϕ, ψ and ξ_i one finds that the free falling particle obeys the equation

$$\begin{aligned} \frac{d\mathbf{v}}{dt} \;=\; -\nabla\phi - \frac{1}{c^2}\Bigg[&\nabla\left(2\phi^2+\psi\right) + \frac{\partial\xi}{\partial t} - \mathbf{v}\times(\nabla\times\xi) \\ &- 3\mathbf{v}\frac{\partial\phi}{\partial t} - 4\mathbf{v}(\mathbf{v}\cdot\nabla\phi) + v^2\nabla\phi\Bigg], \end{aligned} \tag{8}$$

which partially resembles the mathematical structure of the Lorentz force experienced by a charged particle, with velocity $\mathbf{v}$, in the presence of an electromagnetic field. Such law of motion will determine, for instance, the rotation curve corresponding to a given galactic model (see Appendix A for details).

It is instructive to point out that the equations of motion (8) can be derived from the Lagrangian [50]

$$\mathfrak{L} = \frac{v^2}{2} - \phi - \frac{1}{c^2}\left(\frac{\phi^2}{2} + \frac{3\phi v^2}{2} - \frac{v^4}{8} + \psi - \mathbf{v}\cdot\xi\right). \tag{9}$$

For stationary spacetimes, the potentials are independent of time and the associated Hamiltonian $\mathcal{H} = \sum_i \dot{x}_i\partial\mathfrak{L}/\partial v_i - \mathfrak{L}$, is a conserved quantity that can be interpreted as the 1PN generalization of the classical energy:

$$E = \frac{v^2}{2} + \phi + \frac{1}{c^2}\left(\frac{3v^4}{8} - \frac{3v^2\phi}{2} + \frac{\phi^2}{2} + \psi\right). \tag{10}$$

Note that this expression is independent of the vector field ξ.

If the source of gravitation is endowed with axial symmetry, the z-component of the angular momentum is an additional integral of motion. In cylindrical coordinates (R,φ,z) we obtain that, for φ-independent potentials, the quantity

$$L_z = Rv_\varphi + \frac{1}{c^2}\left[Rv_\varphi\left(\frac{v^2}{2} - 3\phi\right) + R\xi_\varphi\right] \tag{11}$$

can be interpreted as the 1PN generalization of the azimuthal angular momentum. In this case $\boldsymbol{\xi}$ plays a role, through its rotational component.

B. 1PN statistical mechanics

From a statistical point of view, the state of the system can be determined by its DF, $F(\mathbf{x},\mathbf{v},t)$, depending on the spatial coordinates, velocity and time. Now, as mentioned in the Introduction, for applications in galactic dynamics it is commonly assumed that the encounters between particles are negligible and hence, the evolution of the stellar system must obey the so-called collisionless Boltzmann equation. In 1PN approximation, such relation can be written as [53]

$$\begin{aligned} &\frac{\partial F}{\partial t} + v^i\frac{\partial F}{\partial x^i} - \frac{\partial\phi}{\partial x^i}\frac{\partial F}{\partial v^i} + \frac{1}{c^2}\left(\frac{v^2}{2} - \phi\right)\left(\frac{\partial F}{\partial t} + v^i\frac{\partial F}{\partial x^i}\right) \\ &+\frac{1}{c^2}\left[4v^iv^j\frac{\partial\phi}{\partial x^j} - \left(\frac{3v^2}{2} + 3\phi\right)\frac{\partial\phi}{\partial x^i} - v^j\left(\frac{\partial\xi_i}{\partial x^j} - \frac{\partial\xi_j}{\partial x^i}\right) + 3v^i\frac{\partial\phi}{\partial t} - \frac{\partial\psi}{\partial x^i} - \frac{\partial\xi_i}{\partial t}\right]\frac{\partial F}{\partial v^i} = 0. \end{aligned} \tag{12}$$

For situations in which encounters play a dominant role, the r.h.s of the above expression can be replaced by a collisional term of the Fokker-Planck type [54].

If the self-gravitating system is in a stationary state, $\partial F/\partial t = 0$ and the DF can be expressed as a function of the energy (a first integral of motion). Moreover, if the system is endowed with additional symmetries we can expect that, as in Newtonian theory, the stationary solutions of (12) are also functions of the remaining integrals of motion. In other words, there is a 1PN version of the Jeans theorem, which simply reflects the fact that equation (12) can be rewriten as $dF/dt = 0$ [53].

The DF describes completely the sate of the system. We can extract from it all the relevant physical quantities, like the mass density, the mean velocity, the velocity-dispersion tensor, etc. For instance, by taking the moments of equation (12), we could in principle obtain some useful results regarding the hydrodynamics of self-gravitating systems at 1PN order. This was first derived by Chandrasekhar [55] using a different approach so we will refrain from writing out these results here, since they are not particularly illuminating.

Another important fact that can be derived with the help of the DF, is the 1PN virial theorem (see appendix B for details). This theorem shows that there exist a linear relation between the variation of the moment of inertia tensor I^{ik}, the kinetic energy tensor K^{ik} and the potential energy tensor W^{ik}, according to:

$$\frac{d^2 I^{ik}}{dt^2} = 2K^{ik} + W^{ik} + \frac{1}{2}\frac{d}{dt}\int d^3x \frac{\overset{0}{\rho}}{c^2}\frac{\partial \phi}{\partial t} x^i x^k \,. \qquad (13)$$

This 1PN virial equation has a similar form as the Newtonian one, except for the last term in the right hand side. The temporal variation of ϕ contributes to the variation of I^{ik}. In particular, for stationary states we have

$$W = -2K,$$

as in the Newtonian case. This is, the total gravitational potential energy (a negative quantity), is two times the total kinetic energy.

III. THE 1PN SELF-GRAVITATION EQUATIONS

The integration of the DF over the velocity space, leads to the energy-momentum tensor. It in turn determines the gravitational field via the Einstein equations, so it should be possible to derive a relation between F and the potentials ϕ, ψ and $\boldsymbol{\xi}$. Such relation is represented by a set of couple equations which we shall call the *1PN self-gravitation equations*. At first we deal with the general case of arbitrary self-gravitating systems and then we shall consider the special case of razor thin disks.

A. Fundamental equations for arbitrary systems

We begin by considering the following expression for the energy-momentum tensor, valid for any self-gravitating system:

$$T^{\mu\nu}(x^i,t) = \frac{1}{c}\int \frac{U^\mu U^\nu}{U^0} F(x^i, U^i, t)\sqrt{-g}\, d^3U. \qquad (14)$$

Here U^ν is the four-velocity, which is related to the three-velocity by $U^i = U^0 v^i/c$ (Greek indices run from 0 to 3 and Latin indices from 1 to 3). From the relation $g_{\mu\nu}U^\mu U^\nu = -c^2$ we can compute the quantities we need to derive the 1PN approximation of the energy-momentum tensor. After some calculations, we obtain

$$\frac{U^0}{c} = 1 + \frac{v^2}{2c^2} - \frac{\phi}{c^2}, \qquad (15)$$

and

$$d^3U = \left(1 + \frac{5v^2}{2c^2} - \frac{3\phi}{c^2}\right) d^3v \qquad (16)$$

Now, taking into account that $\sqrt{-g} = 1 - 2\phi/c^2 + ...$, we find that

$$\begin{aligned} T^{00} &= \int \gamma F d^3v, \\ T^{ij} &= \frac{1}{c^2}\int \gamma\, v^i v^j F d^3v, \qquad (17) \\ T^{0i} &= \frac{1}{c}\int \gamma\, v^i F d^3v, \end{aligned}$$

where

$$\gamma = 1 + \frac{3v^2}{c^2} - \frac{6\phi}{c^2} \qquad (18)$$

is the measure of the integration over velocities.

According to the above relations, we expect that the DF can be expanded in power series of $\bar{v}/c$ as

$$F = \overset{0}{F} + \overset{2}{F} + \cdots, \qquad (19)$$

where $\overset{0}{F}$ is the Newtonian contribution to the DF (which is itself a solution of the classical CBE) and $\overset{2}{F}$ is the first post-Newtonian correction. Plugging (19) into (17) leads to the different components of the energy-momentum tensor at the orders required by the 1PN approximation:

$$\overset{0}{T}{}^{00} = \int \overset{0}{F}\, d^3v, \qquad (20)$$

$$\overset{2}{T}{}^{00} = \frac{3}{c^2}\int (v^2 - 2\phi)\, \overset{0}{F}\, d^3v + \int \overset{2}{F}\, d^3v, \qquad (21)$$

$$\overset{2}{T}{}^{ij} = \frac{1}{c^2}\int v^i v^j\, \overset{0}{F}\, d^3v, \qquad (22)$$

$$\overset{1}{T}{}^{0i} = \frac{1}{c}\int v^i\, \overset{0}{F}\, d^3v, \qquad (23)$$

along with $\overset{0}{T}{}^{ij} = 0$, as expected. With these results we are ready to write the 1PN self-gravitation equations for general stationary systems:

$$\nabla^2\phi = 4\pi G\int \overset{0}{F}\, d^3v, \qquad (24)$$

$$\nabla^2\psi = 8\pi G\int (2v^2 - 3\phi)\ \overset{0}{F}\, d^3v + 4\pi G c^2 \int \overset{2}{F}\, d^3v, \qquad (25)$$

$$\nabla^2\xi_i = 16\pi G\int v^i\ \overset{0}{F}\, d^3v. \qquad (26)$$

To summarize, we can say that a stellar system characterized by an equilibrium DF satisfying (12) is described by a matter distribution given by (20)-(23), and gravitational interactions determined by the field equations (4)-(6). In order to provide a self-consistent description, the relations (24)-(26) must be satisfied. All of these equations are written as power expansions in the small parameter $\bar{v}/c$ and, in consequence, we can clearly distinguish between the Newtonian contribution and the post-Newtonian corrections.

B. The case of stationary razor thin disks

The so-called razor thin disks are of special interest in modeling a number of axisymetric galaxies. In this case, the DF depends on velocities and positions in the form

$$F = f(R, v_R, v_\varphi)\delta(z)\delta(v_z), \qquad (27)$$

where δ is the Dirac delta function and f is a reduced phase-space density describing the stellar population placed on the equatorial plane $z = 0$ (for a finite thin disk of radius a, f must vanish for $R > a$). Equations (24) and (25) can be written as

$$\nabla^2\phi = 4\pi G\delta(z)\int \overset{0}{f}\, d^2v, \qquad (28)$$

$$\nabla^2\psi = 4\pi G\delta(z)\left[\int (4v^2 - 6\phi)\ \overset{0}{f}\, d^2v + c^2\int \overset{2}{f}\, d^2v\right] \qquad (29)$$

where $d^2v = dv_R dv_\varphi$. **Equation (26) requires a little more attention. First start with equation (6) (which is equivalent to (26)). The general solution that vanishes at infinity can be written as [50]**

$$\xi_i(\mathbf{x}) = -4Gc\int \frac{\overset{1}{T^{0i}}(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}, \qquad (30)$$

which, in our case, reduces to

$$-4G\int\int \frac{v^i\ \overset{0}{F}\, d^3x'd^3v}{|\mathbf{x}-\mathbf{x}'|} = -4G\int\int \frac{v^i\ \overset{0}{f}\ \delta(z')d^3x'd^2v}{|\mathbf{x}-\mathbf{x}'|},$$

where we have computed the integral with respect to v_z (we use the notation $v^1 = v_x$, $v^2 = v_y$ and $v^3 = v_z$). This expression can be massaged into a useful form by taking into account two facts: (i) the relation between the cartesian components v_x, v_y and the cylindrical components v_R, v_φ, i.e. $v_x = v_R\cos\varphi - v_\varphi\sin\varphi$ and $v_y = v_R\sin\varphi + v_\varphi\cos\varphi$; (ii) since we are dealing with stationary axisymmetric systems, the DF is an even function of v_R [1] and in consequence $\int v_R f dv_R = 0$. Thus, we can write

$$\int\int \frac{v_x\ \overset{0}{f}\ \delta(z')d^3x'd^2v}{|\mathbf{x}-\mathbf{x}'|} = -\sin\varphi\int\int \frac{v_\varphi\ \overset{0}{f}\ \delta(z')d^3x'd^2v}{|\mathbf{x}-\mathbf{x}'|}$$

and

$$\int\int \frac{v_y\ \overset{0}{f}\ \delta(z')d^3x'd^2v}{|\mathbf{x}-\mathbf{x}'|} = \cos\varphi\int\int \frac{v_\varphi\ \overset{0}{f}\ \delta(z')d^3x'd^2v}{|\mathbf{x}-\mathbf{x}'|}.$$

Now, by introducing the relations $\xi_x = \xi_R\cos\varphi - \xi_\varphi\sin\varphi$ and $\xi_y = \xi_R\sin\varphi + \xi_\varphi\cos\varphi$ in (30), we obtain

$$\xi_R\cos\varphi - \xi_\varphi\sin\varphi = \sin\varphi\int\int 4G\frac{v_\varphi\ \overset{0}{f}\ \delta(z')d^3x'd^2v}{|\mathbf{x}-\mathbf{x}'|}$$

and

$$\xi_R\sin\varphi + \xi_\varphi\cos\varphi = -\cos\varphi\int\int 4G\frac{v_\varphi\ \overset{0}{f}\ \delta(z')d^3x'd^2v}{|\mathbf{x}-\mathbf{x}'|}.$$

Since we assume that $\boldsymbol{\xi}$ is φ-independent, each of these expressions leads us to the conclusion that

$$\xi_R = 0, \qquad \textbf{for finite distributions}, \qquad (31)$$

and that ξ_φ is solution of the following equation:

$$\nabla^2\xi_\varphi = 16\pi G\delta(z)\int v_\varphi\ \overset{0}{f}\, d^2v. \qquad (32)$$

The equation for the component ξ_z can be obtained easily by replacing (27) in (26), and the result is the Laplace equation, $\nabla^2\xi_z = 0$. Its solution can be determined through condition (7). A straightforward calculation leads to

$$\xi_z(R) = \xi_{zo}\ln(R/R_o), \qquad (33)$$

where ξ_{zo} and R_o are constants of integration. In the case of distributions with finite extent, we demand as a boundary condition that $\lim_{R\to\infty}\xi_z = 0$. In consequence, we have to choose $\xi_{zo} = 0$, and hence

$$\xi_z = 0, \qquad \textbf{for finite thin disks}. \qquad (34)$$

On the other hand, we expect that f obeys the collisionless Boltzmann equation in the three-dimensional phase-space (R, v_R, v_φ). In fact, by introducing (27) in (12) and performing an integration on z and v_z, it follows that the distribution f obeys the relation

$$
-\left(1+\frac{v^2}{2c^2}-\frac{\phi}{c^2}-\frac{4R}{c^2}\frac{\partial\phi}{\partial R}-\frac{R}{c^2 v_\varphi}\frac{\partial\xi_\varphi}{\partial R}\right)\frac{v_R v_\varphi}{R}\frac{\partial f}{\partial v_\varphi}+\left(1+\frac{v^2}{2c^2}-\frac{\phi}{c^2}\right)v_R\frac{\partial f}{\partial R}
$$

$$
+\left[\left(1+\frac{v^2}{2c^2}-\frac{\phi}{c^2}\right)\frac{v_\varphi^2}{R}\left(1+\frac{3v_\varphi^2-5v_R^2}{2c^2}+\frac{3\phi}{c^2}\right)\frac{\partial\phi}{\partial R}-\frac{1}{c^2}\frac{\partial\psi}{\partial R}+\frac{v_\varphi}{c^2}\frac{\partial\xi_\varphi}{\partial R}\right]\frac{\partial f}{\partial v_R} \;=\; 0, \tag{35}
$$

where $v^2 = v_R^2 + v_\varphi^2$ and $\partial\phi/\partial R$, $\partial\psi/\partial R$ and $\partial\xi_\varphi/\partial R$ are evaluated at $z = 0$. The above relation is the 1PN version of the Boltzmann equation for an axisymetric two-dimensional shell, located at the equatorial plane, and in a stationary state. Of course, $f(R, v_R, v_\varphi)$ plays the role of the reduced DF describing the diskoidal shell.

It is straightforward to show that E and L_z, given by equations (10) and (11), are solutions of (35). This means that, for axially symmetric systems, any f depending on E and L_z is solution of the CBE; conversely, any solution of the CBE can always be expressed as a function of E and L_z. Thus, any two-integral DF, $f(E, L_z)$, provides a complete statistical description for the (two-degree-of-freedom) stellar system. This fact will be very useful for the formulation of post-newtonian models in the next section.

IV. ANALYTICAL MODELS FOR AXISYMMETRIC GALAXIES

The purpose of this section is to show how to implement the formalism developed above in order to obtain axially symmetric galaxy models. For the applications we want to consider here we have to take into account further considerations. First of all, recall that in [53] we proved that Jeans theorem remains valid at 1PN order. This means that any equilibrium solution of the CBE depends only on the integrals of motion of the system, and that any function of the integrals yields an equilibrium solution of the CBE. Thus, for stationary systems with axial symmetry, we can restrict ourselves to DFs depending on the energy (10) and the angular momentum (11), which are themselves integrals of (12).

The next step would be to implement the previous restrictions starting from a given Newtonian **potential-density pair** with a known DF, as was done in [53] for the spherically symmetric case. As a result, one expects two coupled *self-gravitation equations*, providing a method to determine, from a Newtonian model, its associated post-Newtonian corrections. In practice, the present formalism leads to a two coupled ordinary differential equations in the spherically symmetric case. In the axially symmetric situation however, one ends up with two coupled elliptic partial differential equations (for general volumetric matter distributions).

In this case such equations are much more involved than the ones corresponding to the spherically symmetric case, but the configurations we shall deal here permit us to introduce some additional assumptions to simplify the problem. In the next section, we shall show that a dramatic simplification can be achieved by the consideration of thin discoidal distributions in spheroidal oblate coordinates: instead of getting differential equations, in this case the post-Newtonian corrections can be obtained from simple algebraic equations.

We then present a particular application where the resulting equations can be solved analytically, which means that it is possible to obtain *1PN exact solutions*. The importance of these solutions will be evaluated by a comparison between density profiles and rotation curves described by Newtonian theory and the ones predicted by the 1PN approximation. Although focus on the particular models introduced in [19] (revisited by [24]), our framework can be applied to a wider variety of models. In general, the method can be used for situations in which the potentials are separable functions of the spheroidal oblate coordinates.

A. Hunter's method in the 1PN approximation

One can find in the literature a number of self-consistent stellar models representing razor thin disks; here we will deal with models belonging to the family of Morgan & Morgan disks [19, 24]. In the Newtonian formulation, they can be obtained by a formalism developed by Hunter [56]. Such procedure (known as the Hunter's method) provides the surface density of the disks, the gravitational potential and the circular velocity as series of elementary functions, by superposing solutions of the Laplace equation in oblate spheroidal coordinates. Hunter's method can also be implemented in the context of the 1PN approximation as follows.

To begin with, note that in vacuum, the field equations (4)-(6) reduce to three Laplace equations for ϕ, ψ and ξ_φ (remember that $\xi_z = 0$ for distributions of finite extent). Without loss of generality, we assume that the disk is on the equatorial plane, so we have to impose that the gravitational potentials have symmetry of reflection with respect to the plane $z = 0$, i.e. $\phi(R, z) = \phi(R, -z)$, $\psi(R, z) = \psi(R, -z)$ and $\xi_\varphi(R, z) = \xi_\varphi(R, -z)$. Then, it follows that

$$\frac{\partial\phi}{\partial z}(R,-z) = -\frac{\partial\phi}{\partial z}(R,z), \tag{36}$$

$$\frac{\partial\psi}{\partial z}(R,-z) = -\frac{\partial\psi}{\partial z}(R,z), \tag{37}$$

$$\frac{\partial\xi_\varphi}{\partial z}(R,-z) = -\frac{\partial\xi_\varphi}{\partial z}(R,z), \tag{38}$$

in agreement with the attractive character of gravitation. We also assume that $\partial\phi/\partial z$, $\partial\psi/\partial z$ and $\partial\xi_\varphi/\partial z$ do not vanish in the disk's zone, in order to have the corresponding thin distribution of energy-momentum. Such distribution, restricted to a region $0 \leq R \leq a$ in the plane $z = 0$ (from here on, a will denote the disk radius), will be described by a "shell-like" energy-momentum tensor. If we define

$$\overset{0}{T}{}^{00} = \Sigma(R)\delta(z), \tag{39}$$

$$\overset{2}{T}{}^{00} + \overset{2}{T}{}^{ii} = \frac{1}{c^2}\sigma(R)\delta(z), \tag{40}$$

$$\overset{1}{T}{}^{0\varphi} = \frac{1}{c}\Delta(R)\delta(z), \tag{41}$$

for $0 \leq R \leq a$, it follows from Gauss's Law that

$$\Sigma(R) = \frac{1}{2\pi G}\left(\frac{\partial\phi}{\partial z}\right)_{z=0^+}, \tag{42}$$

$$\sigma(R) = \frac{1}{2\pi G}\left(\frac{\partial\psi}{\partial z}\right)_{z=0^+}, \tag{43}$$

$$\Delta(R) = \frac{1}{8\pi G}\left(\frac{\partial\xi_\varphi}{\partial z}\right)_{z=0^+}. \tag{44}$$

Note that Σ represents the surface mass density of the Newtonian theory (i.e. without relativistic corrections), while Δ plays the role of the surface density of φ-momentum. On the other hand, σ is associated both to the pressure and the relativistic corrections to the mass surface density.

The above relations mean that, in order to have a distribution of matter as the described by (42)-(44), we have to demand that

$$\frac{\partial\phi}{\partial z}(R,0^+) \neq 0, \qquad R \leq a, \tag{45}$$

$$\frac{\partial\phi}{\partial z}(R,0^+) = 0, \qquad R > a, \tag{46}$$

with the same requirement for ψ and ξ_φ. At this point it is convenient to introduce oblate spheroidal coordinates, a system that adapts in a natural way to the geometry of the problem. They are related to the cylindrical ones through

$$R = a\sqrt{(1+\zeta^2)(1-\eta^2)}, \tag{47}$$

$$z = a\zeta\eta, \tag{48}$$

where $0 \leq \zeta < \infty$ and $-1 \leq \eta < 1$. Note that (i) the disk itself has coordinates $\zeta = 0$, $\eta^2 = 1 - R^2/a^2$; (ii) conditions (45)-(46) become

$$\left.\frac{\partial\phi}{\partial\zeta}\right|_{\zeta=0} = H(\eta), \tag{49}$$

$$\left.\frac{\partial\phi}{\partial\eta}\right|_{\eta=0} = 0, \tag{50}$$

where H is an even function of η. The general solution of Laplace's equation satisfying the above conditions can be written as

$$\phi(\zeta,\eta) = -\sum_{n=0}^{\infty} A_{2n} q_{2n}(\zeta) P_{2n}(\eta), \tag{51}$$

where A_{2n} are arbitrary constants, $P_{2n}(\eta)$ and $q_{2n}(\zeta) = i^{2n+1}Q_{2n}(i\zeta)$ are the usual Legendre polynomials and the Legendre functions of second kind, respectively. The post-Newtonian potentials ψ and ξ_φ have the same form,

$$\psi(\zeta,\eta) = -\sum_{n=0}^{\infty} B_{2n} q_{2n}(\zeta) P_{2n}(\eta), \tag{52}$$

$$\xi_\varphi(\zeta,\eta) = -\sum_{n=0}^{\infty} C_{2n} q_{2n}(\zeta) P_{2n}(\eta), \tag{53}$$

but here we have denoted the expansion constants as B_{2n} and C_{2n}. We can derive explicit formulae for Σ, σ and Δ in oblate spheroidal coordinates, by introducing (51)-(53) in (42)-(44):

$$\Sigma = \frac{1}{2\pi a G \eta_*}\sum_{n=0}^{\infty} A_{2n}(2n+1) q_{2n+1}(0) P_{2n}(\eta_*), \tag{54}$$

$$\sigma = \frac{1}{2\pi a G \eta_*}\sum_{n=0}^{\infty} B_{2n}(2n+1) q_{2n+1}(0) P_{2n}(\eta_*), \tag{55}$$

$$\Delta = \frac{1}{8\pi a G \eta_*}\sum_{n=0}^{\infty} C_{2n}(2n+1) q_{2n+1}(0) P_{2n}(\eta_*), \tag{56}$$

where η_* represents the value of coordinate η inside the disk:

$$\eta_* = \sqrt{1 - \frac{R^2}{a^2}} \tag{57}$$

Now that we have stated the fundamental structure of models with 1PN corrections, the next step is to demand that the models obtained are self-consistent, i.e. that they have an analytical equilibrium DF that is related consistently to the surface mass distribution. In other words, we have to formulate the corresponding 1PN self-gravitating equations:

$$\sum_{n=0}^{\infty} \tilde{A}_{2n} \frac{P_{2n}(\eta_*)}{\eta_*} = \int \overset{0}{f} \, d^2 v, \tag{58}$$

$$\sum_{n=0}^{\infty} \tilde{B}_{2n} \frac{P_{2n}(\eta_*)}{\eta_*} = \int (4v^2 - 6\phi) \overset{0}{f} \, d^2 v + c^2 \int \overset{2}{f} \, d^2 v, \tag{59}$$

$$\sum_{n=0}^{\infty} \tilde{C}_{2n} \frac{P_{2n}(\eta_*)}{\eta_*} = \int v_\varphi \overset{0}{f} \, d^2 v, \tag{60}$$

where, for the sake of simplicity, we have defined

$$\tilde{A}_{2n} = \frac{(2n+1) q_{2n+1}(0)}{2\pi a G} A_{2n}, \tag{61}$$

the same for $\tilde{B}_{2n}$, and

$$\tilde{C}_{2n} = \frac{(2n+1) q_{2n+1}(0)}{8\pi a G} C_{2n}. \tag{62}$$

Note that, part of the r.h.s of equations (58)-(60) is characterized by three fundamental quantities in Newtonian dynamics of self-gravitating systems: (i) the mass surface density, $\varSigma = \int \overset{0}{f} \, d^2 v$; (ii) the mean square velocity, $\langle v^2 \rangle = \varSigma^{-1} \int v^2 \overset{0}{f} \, d^2 v$, and (iii) the mean circular velocity, $\langle v_\varphi \rangle = \varSigma^{-1} \int v_\varphi \overset{0}{f} \, d^2 v$. In particular, equation (58) is the Newtonian self-gravitation equation.

There is a variety of cases that can be addressed by the formalism presented above. They correspond to situations in which the gravitational fields (i) have symmetry of reflection with respect to the equatorial plane, (ii) are separable functions of the spheroidal oblate coordinates and (iii) correspond to axially symmetric distributions with finite extent. Note that any function with these features can be expressed in the form (51).

B. The Morgan & Morgan disks with 1PN corrections

In Newtonian gravity, the Generalized Kalnajs Disks [24] are finite thin distributions of matter with surface mass density

$$\varSigma_m(R) = \frac{(2m+1)M}{2\pi a^2} \left(1 - \frac{R^2}{a^2}\right)^{m-1/2}, \qquad m \in \mathcal{N} \tag{63}$$

and gravitational potential ϕ_m, given by (51), with the following expansion constants

$$A_{2n}^{(m)} = \frac{MG}{a} \frac{\sqrt{\pi} 2^{-2m-1} (4n+1)(2m+1)!}{(2n+1)(m-n)! \varGamma(m+n+\frac{3}{2}) q_{2n+1}(0)}. \tag{64}$$

These solutions were first obtained by Morgan & Morgan (MM) [19], by solving a boundary value problem in the context of GR. Even so, we refer to the above solutions as MM disks for historical reasons, but keeping in mind that we are dealing with Newtonian gravity.

All the members of this family have interesting features: a monotonically decreasing mass density and a Keplerian rotation curve. The case $m = 1$, corresponding to the well known Kalnajs disk [20] (see also [1]), is an exception because it describes a self-gravitating disk which rotates as a rigid body.

The superposition of members belonging to the MM family, has been used to obtain new models with more realistic properties. For example, in reference [25], the authors constructed a family of models with approximately flat rotation curves considering a particular combination of MM disks. Another example is the family of models introduced by Letelier in [57], representing flat rings (see also [58] for astrophysical applications). Aditionally, in reference [59] it was shown that it is posible to construct models with realistic rotational curves obeying simple polynomial expressions. In particular, the authors constructed models for a number of galaxies of the Ursa Major cluster, and as an application they estimated their corresponding mass distriutions. This fact suggest that there exist superpositions of MM members leading to galactic models in agreement with the so-called *maximum disk hypothesys* [1]. It would be interesting to have in hand the 1PN version of the MM family, along with all of its features.

Here we illustrate how to obtain the post-Newtonian version of the MM disks. We focus in the $m = 2$ model, which is characterized by a density

$$\varSigma = \frac{5M}{2\pi a^2} \eta_*^3, \tag{65}$$

but in principle, the same proceedure can be applied to any of the remaining members (or linear combinations). From here on, we will drop the subindex of $\varSigma_m$, for simplicity.

It can be shown that the above surface density can be obtained from the DF [60]

$$f(E, L_z) = k(\varOmega L_z - E - 5\, \varOmega^2 a^2/4)^{-1/4}, \tag{66}$$

where

$$k = \frac{2}{\sqrt{3}} \left[\frac{10M}{\pi^{11} a^5 G^3}\right]^{1/4}, \qquad \varOmega = \sqrt{\frac{15\pi G M}{32 a^3}}. \tag{67}$$

Note that the DF defined by (66) is function of the Jacobi's integral, $E_J = E - \varOmega L_z$, which can be interpreted as the energy measured from a frame that rotates with constant angular speed $\varOmega$ (see Appendix C).

In order to obtain a 1PN version of the model, we start from the DF given by (66) but using the post-Newtonian expressions for E and L_z, i.e., equations (10) and (11). Thus, at 1PN order we can write

$$f = \overset{0}{f} + \overset{2}{f}, \tag{68}$$

with

$$\overset{0}{f} = kJ_0^{-1/4} \quad \text{and} \quad \overset{2}{f} = -\frac{k}{4} J_2 J_0^{-5/4}, \tag{69}$$

where

$$J_0 = -\phi - v^2/2 + \Omega R v_\varphi - 5\Omega^2 a^2/4, \tag{70}$$

$$J_2 = -c^{-2}\left[\frac{\phi^2}{2} + \psi - \frac{3v^2\phi}{2} + \frac{3v^4}{8} - \Omega R v_\varphi\left(\frac{v^2}{2} - 3\phi\right) - \Omega R \xi_\varphi\right]. \tag{71}$$

These relations determine the 1PN self-gravitation equations through (58)-(60) and the integrals

$$\int \overset{0}{f}\, d^2v = \frac{5M}{2\pi a^2}\eta_*^3, \tag{72}$$

$$\int v^2 \overset{0}{f}\, d^2v = \frac{75GM^2}{448a^3}\left(7 - 7\eta_*^2 + 6\eta_*^4\right)\eta_*^3 \tag{73}$$

$$\int v_\varphi \overset{0}{f}\, d^2v = \frac{5}{8}\sqrt{\frac{15GM^3}{2\pi a^5}}\sqrt{(1-\eta_*^2)}\eta_*^3 \tag{74}$$

$$\int \overset{2}{f}\, d^2v = \frac{1}{c^2\eta_*}\left[\frac{GM^2}{a^3}\sum_{j=0}^{4} b_{2j}\eta_*^{2j} + \sqrt{\frac{160M}{3\pi^3 a^3 G}}\sqrt{(1-\eta_*^2)}\xi_\varphi + \frac{32\psi}{32aG\pi^2}\right] \tag{75}$$

where

$$b_0 = -\frac{2325}{1024}, \qquad b_2 = \frac{75}{512}, \qquad b_4 = \frac{1125}{128}, \qquad b_6 = \frac{1275}{512}, \qquad b_8 = -\frac{38475}{7168}. \tag{76}$$

By introducing (72)-(75) in (58)-(60), we obtain a system of linear equations for the constants B_{2n} and C_{2n} in terms of A_{2n} (which are known *a priori*), that can be solved analytically. After some computations we find that

$$C_{2n} = \left(\frac{GM}{a}\right)^{3/2} \frac{(45 - 48n - 92n^2 + 16n^3 + 16n^4)}{\Gamma(\frac{1}{2}-n)\Gamma(\frac{7}{2}-n)\Gamma(1+n)\Gamma(4+n)} \times \frac{(4n+1)}{(2n+1)q_{2n+1}(0)}\frac{5\pi^{5/2}\sqrt{15}}{256\sqrt{2}} \tag{77}$$

$$B_{2n} = \left(\frac{GM}{a}\right)^2 \frac{I_{2n} - \sum_{m=0}^{\infty}\Pi_{2n,2m}}{(64/3)q_{2n}(0) + \pi(2n+1)q_{2n+1}(0)}, \tag{78}$$

where I_{2n} and $\Pi_{2n,2m}$ are constants defined in the Appendix D, equations (D9) and (D11), respectively.

With the constants C_{2n} and B_{2n} at hand, the post-Newtonian fields are completely determined along with the remaining physical quantities. In galactic dynamics, the circular velocity and the mass profile are two important measurable quantities used to verify a particular model. In our case, the first one is given by the relation (A5) whereas the second one is given by (65) plus the 1PN corrections coming from $\overset{2}{T}{}^{00}$. The corrected surface density can be written explicitly as

$$\overset{0}{\Sigma} + \overset{2}{\Sigma} = \frac{5M}{2\pi a^2}\left[\eta_*^3 - \frac{15\pi\lambda}{224}\left(7 - 7\eta_*^2 + 64\eta_*^4\right)\eta_*^3 + \frac{\lambda}{5\eta_*}\sum_{n=0}^{\infty}\frac{(2n)!!\hat{B}_{2n}}{(2n-1)!!}P_{2n}(\eta_*)\right], \tag{79}$$

where λ is a dimensionless parameter defined by

$$\lambda \equiv \frac{GM}{ac^2}, \tag{80}$$

and $\hat{B}_{2n}$ are the dimensionless constants

$$\hat{B}_{2n} = \frac{a^2 B_{2n}}{G^2 M^2}. \tag{81}$$

The parameter λ, which is also present in the expression for the circular velocity, is a measure of how large are the 1PN corrections. For example, in a galaxy with 10^{12} solar masses and a radius of 10 Kpc, we have $\lambda \approx 5 \times 10^{-6}$. Here, we consider situations where the relativistic corrections are larger and they can be visualized in the behavior of rotation curves and mass profile. In Figure 1, we plot the circular velocity when $\lambda \sim 10^{-3}$ and $\lambda \sim 10^{-2}$. The 1PN corrections become important in the latter case (last two figures), in particular for values of v_φ near to the disk edge. This is somewhat surprising since one would expect major corrections near the galaxy core, where the mass concentration is maximum. A similar phenomenon occurs with the mass profiles (see Figure 2). Their differences with the Newtonian profile become significant for $\lambda \sim 10^{-2}$ and the magnitude of 1PN corrections increase with the radius.

V. CONCLUDING REMARKS

We continued our study of the post-Newtonian kinetic theory of collisionless self-gravitating gases, extending our previous results [53] to systems with axial symmetry. Before dealing with the applications, we developed the 1PN version of the tensor virial theorem, applicable for arbitrary self-gravitating systems. We found that the 1PN virial theorem differs from the Newtonian one by the fact that the temporal variation of ϕ contributes to the variation of the inertia tensor. However, for stationary systems we recover the same result as in Newtonian theory: the absolute value of the total gravitational potential energy is two times the kinetic energy. The problem about the validity of the virial theorem for collisional systems (which remains valid in the Newtonian theory) is an open question that we leave for future studies.

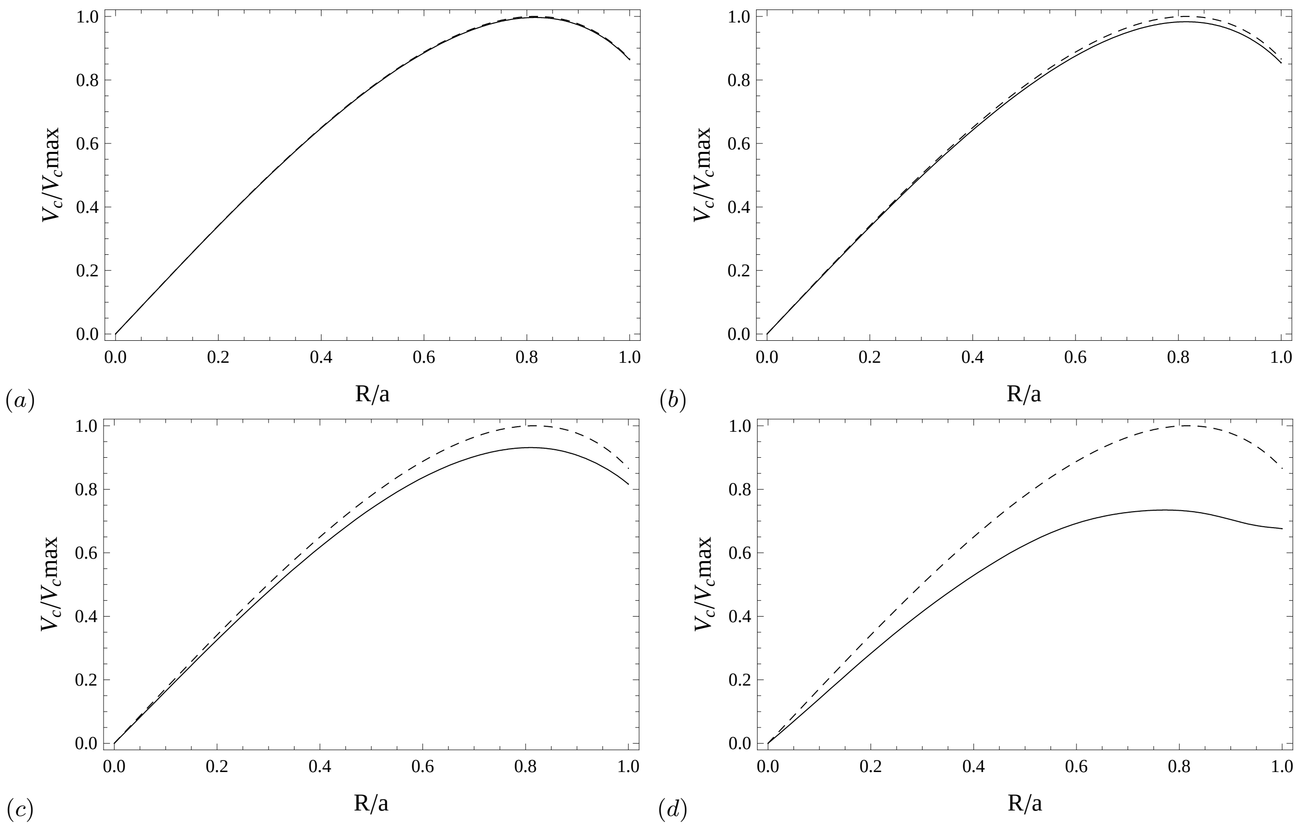

Figure 1: Rotation curves for the 1PN version of the second MM member for different values of the parameter λ: (a) $\lambda = 10^{-3}$, (b) $\lambda = 5 \times 10^{-3}$, (c) $\lambda = 2 \times 10^{-2}$ and (d) $\lambda = 7 \times 10^{-2}$. For $\lambda \sim 10^{-2}$ or more, the differences with the Newtonian model (dashed line) become significant. In this case, the rotation curve tends to flatten as λ grows.

The applications we considered here were devoted specially to address the modeling of systems in galactic dynamics, thus we focused in equilibrium situations with rotational symmetry. We developed a formalism to construct the 1PN version of Newtonian self-consistent models characterized by a stationary DF depending on the Jacobi's integral. The general case is mathematically challenging as it involves a pair of coupled elliptic partial differential equations. However, we restricted our attention to the study of thin models with finite extent, case in which the system reduces to a set of algebraic equations. As an illustrative example of the formalism developed here, we obtained the 1PN version of the second MM disk. The contributions of relativistic corrections found are not significant for $\lambda \sim 10^{-3}$ or less, but they become prominent for larger values of λ (provided that the 1PN approximation is still valid). We noted that the corrections grow with the radial distance, which is surprising since one would expect major corrections near the disk center, where the mass concentration is maximum.

The general formalism starts by implementing the Hunter's method in the 1PN scheme. Then, a stationary DF (solution of the 1PN CBE) is selected and introduced in the self-gravitation equations. Finally, by solving the resulting algebraic equalities, we find the corresponding post-Newtonian fields, ψ and ξ_i. The new 1PN-corrected models obtained are also self-consistent, since this feature is an inherent requirement of the formalism. The method is applicable to self-gravitating thin disks with finite extent provided that: (i) the system has symmetry of reflection with respect to the $z = 0$ plane; (ii) the Newtonian potential is separable in spheroidal oblate coordinates; (iii) The Newtonian DF is only function of E and L_z.

It's worth to point out the interesting work by by Schenk, Shapiro and Teukolsky in 1999 [62]. In this paper the authors focused on the Kalnajs disk (the $m = 1$ member of the MM family) and solved numerically the the Einstein equations coupled to the relativistic CBE, obtaining the first self-consistent model of a rotating relativistic disk. It would be interesting to obtain the 1PN version of the same model by using the formalism developed in this paper in order to establish a comparison of the two approaches and estimate the effects of the higher order relativistic corrections.

As a final remark, it is important to mention that the problem about the stability of these mod-

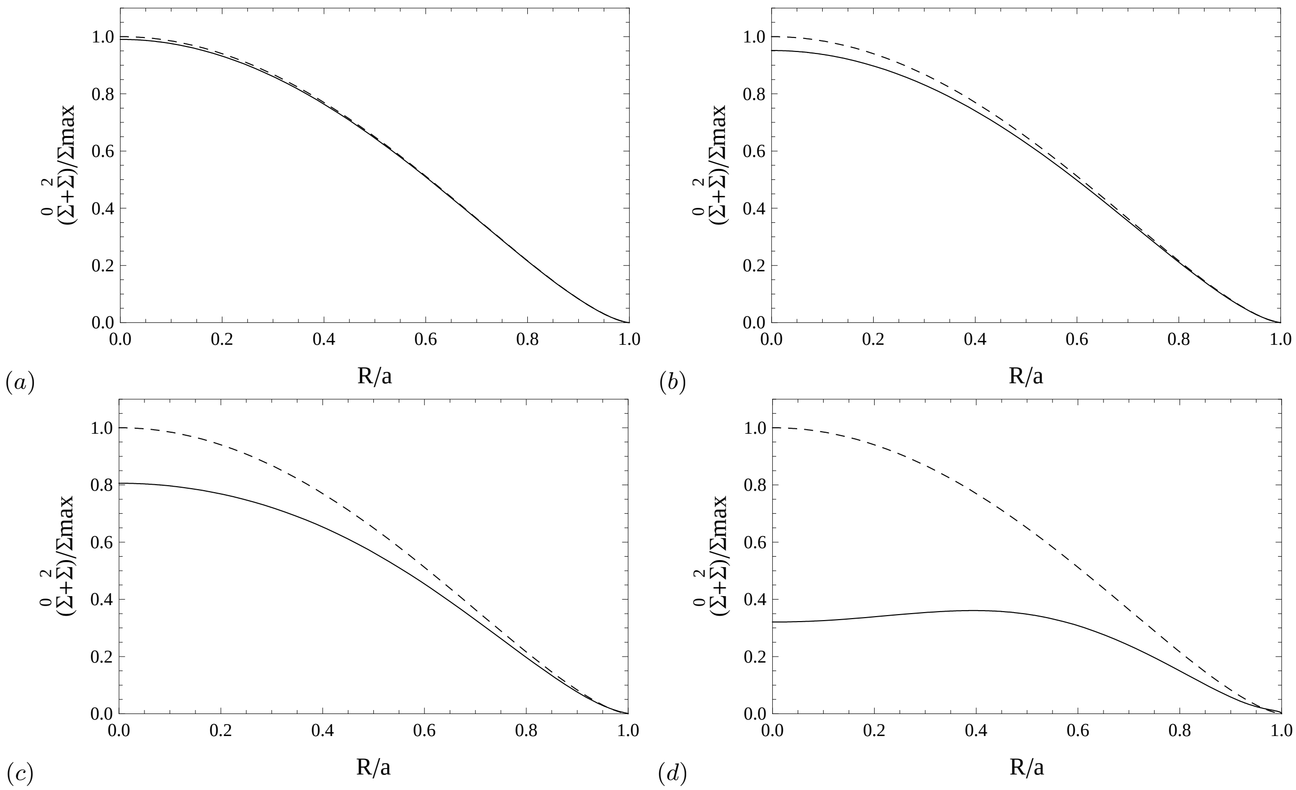

Figure 2: Mass profiles for the 1PN version of the second MM member for the same values of the figure 1. As expected, the differences with the Newtonian model (dashed line) become significant for $\lambda \sim 10^{-2}$ or more. In the case (d), we have a profile which maximum is not at $R = 0$.

els is an important subject that will be addressed in future works. This issue brings several points to be investigated. One fundamental question is to establish whether the discoidal structures considered here are stable by themselves or if they need an additional component (a spheroidal halo, for example) in order to be stable. Another question is to determinate if the marginal stability of Newtonian disks improves or not with the introduction of relativistic corrections. At the present we are performing a preliminary study about the stability of test particles around the solutions obtained, and their response to radial and vertical perturbations in order to compare the results with the Newtonian theory (see for example [25, 63]) and to provide a first test of stability. Then we expect to implement a more conclusive analysis regarding the Jeans-type instabilities by perturbing the DF's by themselves.

Acknowledgments

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Appendix A: Rotation Curves in the 1PN Approximation

The circular velocity is a widely used observable in galactic dynamics, so it is important to have in hand a formula for the rotation curves in the 1PN approximation. In order to do this, we have to study the circular motion of test particles in the equatorial plane. At first, note that according to (8), a star moving around an axisymmetric stationary thin disk with finite extension, obeys the following equations of motion:

$$\begin{aligned}\ddot{R} &= R\dot{\varphi}^2 - \frac{\partial\phi}{\partial R}\left(1 + c^{-2}\left(4\phi - 3\dot{R}^2 + R^2\dot{\varphi}^2 + \dot{z}^2\right)\right) \\ &+c^{-2}\left(4\dot{R}\dot{z}\frac{\partial\phi}{\partial z} - \frac{\partial\psi}{\partial R} + R\dot{\varphi}\frac{\partial\xi_\varphi}{\partial R}\right),\end{aligned} \tag{A1}$$

$$\begin{aligned}\ddot{z} &= -\frac{\partial\phi}{\partial z}\left(1 + c^{-2}\left(4\phi - 3\dot{z}^2 + R^2\dot{\varphi}^2 + \dot{R}^2\right)\right) \\ &+c^{-2}\left(4\dot{z}\dot{R}\frac{\partial\phi}{\partial R} - \frac{\partial\psi}{\partial z} + R\dot{\varphi}\frac{\partial\xi_\varphi}{\partial z}\right),\end{aligned} \tag{A2}$$

$$R\ddot{\varphi} = -2\dot{R}\dot{\varphi} + \frac{4R\dot{\varphi}}{c^2}\left(\dot{R}\frac{\partial\phi}{\partial R} + \dot{z}\frac{\partial\phi}{\partial z}\right) - \frac{\dot{R}}{c^2}\frac{\partial\xi_\varphi}{\partial R} - \frac{\dot{z}}{c^2}\frac{\partial\xi_\varphi}{\partial z}, \tag{A3}$$

where the dot denotes derivation with respect to t. In particular, equatorial circular orbits must satisfy the conditions $\dot{R} = \dot{z} = 0$, $\ddot{R} = \ddot{z} = 0$ and $z = 0$. In this case (A1) reduces to

$$v_\varphi^2\left(1 - \frac{R}{c^2}\frac{\partial\phi}{\partial R}\right) + v_\varphi\frac{R}{c^2}\frac{\partial\xi_\varphi}{\partial R} - R\frac{\partial\phi}{\partial R} - \frac{R}{c^2}\frac{\partial}{\partial R}(2\phi^2 + \psi) = 0, \tag{A4}$$

which can be used to derive an expression for v_φ^2 at 1PN order. In particular, note that we can use the Newtonian expression $v_\varphi^2 = R\partial\phi/\partial R$ each time that v_φ is accompanied by an inverse power of c. After some straightforward algebra we get

$$v_\varphi^2 = R\frac{\partial\phi}{\partial R}\left(1 + \frac{4\phi}{c^2} + \frac{R}{c^2}\frac{\partial\phi}{\partial R}\right) + \frac{R}{c^2}\left(\frac{\partial\psi}{\partial R} - \sqrt{R\frac{\partial\phi}{\partial R}}\frac{\partial\xi_\varphi}{\partial R}\right), \tag{A5}$$

where it is understood that all derivatives are evaluated at $z = 0$. There is a crucial difference between the above relation and the classical formulae for the rotation curves: in the Newtonian case v_φ^2 is linear in $\partial\phi/\partial R$, whereas in the 1PN case, it depends in non linear terms involving ϕ and derivatives of the gravitational potentials. This nonlinear dependence may result significant in some cases.

Appendix B: Virial Theorem in the 1PN Approximation

The virial theorem is an important general result of the kinetic theory relating the contribution of kinetic and potential energy to the temporal change of the inertia tensor associated to the system. In order to obtain the 1PN version of the virial theorem we can start from the conservation laws

$$\frac{\partial T^{\mu\nu}}{\partial x^\mu} = -\Gamma^\nu_{\mu\lambda}T^{\mu\lambda} - \Gamma^\mu_{\mu\lambda}T^{\lambda\nu}. \tag{B1}$$

At first order in $\bar{v}/c$, this expression reproduce the Newtonian mass and momentum conservation laws, which are given by

$$\frac{1}{c}\frac{\partial \overset{0}{T}{}^{00}}{\partial t} + \frac{\partial \overset{1}{T}{}^{i0}}{\partial x^i} = 0\,, \tag{B2}$$

$$\frac{1}{c}\frac{\partial \overset{1}{T}{}^{0i}}{\partial t} + \frac{\partial \overset{2}{T}{}^{ij}}{\partial x^j} = -\frac{1}{c^2}\frac{\partial\phi}{\partial x^i}\overset{0}{T}{}^{00}\,. \tag{B3}$$

The corresponding 1PN corrections of the above laws are obtained by taking into account Christoffel symbols at different orders [50]

$$\frac{1}{c}\frac{\partial \overset{2}{T}{}^{00}}{\partial t}+\frac{\partial \overset{3}{T}{}^{i0}}{\partial x^i} = \frac{1}{c^3}\frac{\partial\phi}{\partial t}\overset{0}{T}{}^{00}\,, \tag{B4}$$

$$\begin{aligned}\frac{1}{c}\frac{\partial \overset{3}{T}{}^{0i}}{\partial t}+\frac{\partial \overset{4}{T}{}^{ij}}{\partial x^j} &= -\overset{4}{\Gamma}{}^i_{00}\overset{0}{T}{}^{00}-\overset{2}{\Gamma}{}^i_{00}\overset{2}{T}{}^{00}-(2\overset{3}{\Gamma}{}^i_{0j}+\delta_{ij}\overset{3}{\Gamma}{}^0_{00}+\delta_{ij}\overset{3}{\Gamma}{}^k_{0k})\overset{1}{T}{}^{0j}\\ &\quad -(\overset{2}{\Gamma}{}^i_{jk}+\delta_{ik}\overset{2}{\Gamma}{}^0_{0j}+\delta_{ik}\overset{2}{\Gamma}{}^l_{lj})\overset{2}{T}{}^{jk}\\ &= -\Big[\frac{\partial}{\partial x^i}\Big(\frac{2\phi^2}{c^4}+\frac{\psi}{c^4}\Big)+\frac{1}{c^4}\frac{\partial\zeta_i}{\partial t}\Big]\overset{0}{T}{}^{00}-\frac{1}{c^2}\frac{\partial\phi}{\partial x^i}\overset{2}{T}{}^{00}\\ &\quad -\frac{1}{c^3}\Big(\frac{\partial\zeta_i}{\partial x^j}-\frac{\partial\zeta_j}{\partial x^i}-4\delta_{ij}\frac{\partial\phi}{\partial t}\Big)\overset{1}{T}{}^{0j}-\frac{1}{c^2}\Big(\delta_{jk}\frac{\partial\phi}{\partial x^i}-4\delta_{ik}\frac{\partial\phi}{\partial x^j}\Big)\overset{2}{T}{}^{jk}\ .\end{aligned} \tag{B5}$$

On the other hand, the energy-momentum tensor, given in (17), can be splitted in two first order contributions:

$$T^{\mu\nu}=\int \overset{0}{F}\,V^\mu V^\nu d^3v+\int\Big[\overset{0}{F}\Big(\frac{3\mathbf{v}^2}{c^2}-\frac{6\phi}{c^2}\Big)+\overset{2}{F}\Big]V^\mu V^\nu d^3v \tag{B6}$$

where $V^\mu=(c,\mathbf{v})$. After some definitions

$$\overset{0}{\rho}\equiv\int \overset{0}{F}\,d^3v,\qquad \overset{2}{\rho}\equiv\int\overset{2}{F}\,d^3v,\quad\text{and}\quad \overset{2}{\tilde\rho}\equiv\overset{2}{\rho}+\overset{0}{\rho}\Big(\frac{3\overline{\mathbf{v}^2}}{c^2}-\frac{6\phi}{c^2}\Big), \tag{B7}$$

and using the different probability densities to compute expectation values,

$$\begin{aligned}\overline{\mathcal{A}} &= (\overset{0}{\rho})^{-1}\int\overset{0}{F}\,\mathcal{A}d^3v\,,\\ \overset{2}{\overline{\mathcal{A}}} &= (\overset{2}{\tilde\rho})^{-1}\int\Big[\overset{0}{F}\Big(\frac{3\mathbf{v}^2}{c^2}-\frac{6\phi}{c^2}\Big)+\overset{2}{F}\Big]\mathcal{A}d^3v\,,\end{aligned} \tag{B8}$$

we get the following simplified expressions for the momentum-energy components at different orders:

$$\overset{0}{T}{}^{00}=c^2\,\overset{0}{\rho}\,,\qquad \overset{2}{T}{}^{00}=c^2\,\overset{2}{\tilde\rho}\,,\qquad \overset{1}{T}{}^{0i}=c\,\overset{0}{\rho}\,\overline{v^i}\,, \tag{B9}$$

$$\overset{3}{T}{}^{0i}=c\,\overset{2}{\rho}\overset{2}{\overline{v^i}}\,,\qquad \overset{2}{T}{}^{ij}=\overset{0}{\rho}\,\overline{v^iv^j}\,,\qquad \overset{4}{T}{}^{ij}=\overset{2}{\tilde\rho}\overset{2}{\overline{v^iv^j}}\ . \tag{B10}$$

Then, equations (B4)-(B5), can be written as

$$\frac{\partial(\overset{2}{\tilde\rho})}{\partial t}+\frac{\partial(\overset{2}{\tilde\rho}\overset{2}{\overline{v^i}})}{\partial x^i} = \frac{\overset{0}{\rho}}{c^2}\frac{\partial\phi}{\partial t}\,, \tag{B11}$$

$$\begin{aligned}\frac{\partial(\overset{2}{\tilde\rho}\overset{2}{\overline{v^i}})}{\partial t}+\frac{\partial(\overset{2}{\tilde\rho}\overset{2}{\overline{v^iv^j}})}{\partial x^j} &= -\overset{0}{\rho}\Big[\frac{\partial}{\partial x^i}\Big(\frac{2\phi^2}{c^2}+\frac{\psi}{c^2}\Big)+\frac{1}{c^2}\frac{\partial\zeta_i}{\partial t}\Big]-\overset{2}{\tilde\rho}\,\frac{\partial\phi}{\partial x^i}\\ &\quad -\overset{0}{\rho}\,\frac{\overline{v^j}}{c^2}\Big(\frac{\partial\zeta_i}{\partial x^j}-\frac{\partial\zeta_j}{\partial x^i}-4\delta_{ij}\frac{\partial\phi}{\partial t}\Big)-\overset{0}{\rho}\,\frac{\overline{v^jv^k}}{c^2}\Big(\delta_{jk}\frac{\partial\phi}{\partial x^i}-4\delta_{ik}\frac{\partial\phi}{\partial x^j}\Big).\end{aligned} \tag{B12}$$

Now we can compute the symmetrized integral of the product $x^k/2$ times the above equation:

$$\frac{1}{2}\frac{d}{dt}\int d^3x\,\overset{2}{\tilde\rho}\,(x^k\,\overset{2}{\overline{v^i}}+x^i\,\overset{2}{\overline{v^k}})=2\,\overset{2}{K}{}^{ik}+\overset{2}{W}{}^{ik} \tag{B13}$$

where

$$\begin{aligned}
\overset{2}{W}{}^{ik} &= -\frac{1}{2}\int d^3x\,\overset{0}{\rho}\,x^k\left[\frac{\partial}{\partial x^i}\Big(\frac{2\phi^2}{c^2}+\frac{\psi}{c^2}\Big)+\frac{1}{c^2}\frac{\partial\zeta_i}{\partial t}+\frac{\overline{v^j}}{c^2}\Big(\frac{\partial\zeta_i}{\partial x^j}-\frac{\partial\zeta_j}{\partial x^i}-4\delta_{ij}\frac{\partial\phi}{\partial t}\Big)+\frac{\overline{v^jv^l}}{c^2}\Big(\delta_{jl}\frac{\partial\phi}{\partial x^i}-4\delta_{il}\frac{\partial\phi}{\partial x^j}\Big)\right]\\
&\quad-\frac{1}{2}\int d^3x\,\overset{0}{\rho}\,x^i\left[\frac{\partial}{\partial x^k}\Big(\frac{2\phi^2}{c^2}+\frac{\psi}{c^2}\Big)+\frac{1}{c^2}\frac{\partial\zeta_k}{\partial t}+\frac{\overline{v^j}}{c^2}\Big(\frac{\partial\zeta_k}{\partial x^j}-\frac{\partial\zeta_j}{\partial x^k}-4\delta_{kj}\frac{\partial\phi}{\partial t}\Big)+\frac{\overline{v^jv^l}}{c^2}\Big(\delta_{jl}\frac{\partial\phi}{\partial x^k}-4\delta_{kl}\frac{\partial\phi}{\partial x^j}\Big)\right]\\
&\quad-\frac{1}{2}\int d^3x\,\overset{2}{\tilde{\rho}}\left(x^k\frac{\partial\phi}{\partial x^i}+x^i\frac{\partial\phi}{\partial x^k}\right), \qquad \text{(B14)}\\
\overset{2}{K}{}^{ik} &= \frac{1}{2}\int d^3x\,\overset{2}{\tilde{\rho}}\,\overline{v^jv^k}\,. \qquad \text{(B15)}
\end{aligned}$$

By defining

$$\overset{2}{I}{}^{ik}=\int d^3x\,\overset{2}{\tilde{\rho}}\,x^ix^k, \qquad \text{(B16)}$$

we can write the time derivative of the equation (B4) as

$$\frac{d\,\overset{2}{I}{}^{ik}}{dt}=\int d^3x\frac{\partial\,\overset{2}{\tilde{\rho}}}{\partial t}x^ix^k=\int d^3x\,\overset{2}{\tilde{\rho}}\,(x^k\,\overset{2}{v^i}+x^i\,\overset{2}{v^k})+\int d^3x\frac{\overset{0}{\rho}}{c^2}\frac{\partial\phi}{\partial t}x^ix^k \qquad \text{(B17)}$$

and its second time derivative

$$\frac{d^2\,\overset{2}{I}{}^{ik}}{dt^2}=\frac{d}{dt}\int d^3x\,\overset{2}{\tilde{\rho}}\,(x^k\,\overset{2}{v^i}+x^i\,\overset{2}{v^k})+\frac{d}{dt}\int d^3x\frac{\overset{0}{\rho}}{c^2}\frac{\partial\phi}{\partial t}x^ix^k\,. \qquad \text{(B18)}$$

Putting all together we obtain the tensor post-Newtonian virial theorem

$$\frac{d^2\,\overset{2}{I}{}^{ik}}{dt^2}=2\,\overset{2}{K}{}^{ik}+\overset{2}{W}{}^{ik}+\frac{1}{2}\frac{d}{dt}\int d^3x\frac{\overset{0}{\rho}}{c^2}\frac{\partial\phi}{\partial t}x^ix^k\,, \qquad \text{(B19)}$$

and taking the trace, we obtain the scalar post-Newtonian virial theorem

$$\frac{d^2\,\overset{2}{I}}{dt^2}=2\,\overset{2}{K}+\overset{2}{W}+\frac{1}{2}\frac{d}{dt}\int d^3x\frac{\overset{0}{\rho}}{c^2}\frac{\partial\phi}{\partial t}\mathbf{x}^2\,, \qquad \text{(B20)}$$

where

$$\overset{2}{O}=\mathrm{trace}(\overset{2}{O}{}^{ik})\,. \qquad \text{(B21)}$$

In summary, we can collect all the above results and to state that if we define a moment of inertia tensor,

$$I^{ik}=\int\int\left(1+\frac{3\overline{v^2}}{c^2}-\frac{6\phi}{c^2}\right)x^ix^kFd^3vd^3x, \qquad \text{(B22)}$$

a kinetic energy tensor,

$$K^{ik}=\frac{1}{2}\int\int\left(1+\frac{3\overline{v^2}}{c^2}-\frac{6\phi}{c^2}\right)\overline{v^iv^k}Fd^3vd^3x, \qquad \text{(B23)}$$

and a potential energy tensor

$$W^{ik}=\overset{0}{W}{}^{ik}+\overset{2}{W}{}^{ik}, \qquad \text{(B24)}$$

where $\overset{0}{W}{}^{ik}$ is the Newtonian potential energy tensor and $\overset{2}{W}{}^{ik}$ is given by (B15), then they satisfy the following relation

$$\frac{d^2 I^{ik}}{dt^2} = 2K^{ik} + W^{ik} + \frac{1}{2}\frac{d}{dt}\int d^3x \frac{\overset{0}{\rho}}{c^2}\frac{\partial \phi}{\partial t} x^i x^k \,, \tag{B25}$$

which can be enunciated as the 1PN tensor virial theorem. The scalar virial equation is obtained by taking the trace of the above relation:

$$\frac{d^2 I}{dt^2} = 2K + W + \frac{1}{2}\frac{d}{dt}\int d^3x \frac{\overset{0}{\rho}}{c^2}\frac{\partial \phi}{\partial t}\mathbf{x}^2.$$

Appendix C: Jacobi's Integral in the 1PN Approximation

For the purposes of the present paper it is important to have in hand an expression of the Jacobi's integral at 1PN order, in order to verify that it has the same form as in Newtonian theory, i.e. $E_J = E - \Omega L_z$.

In a rotating reference frame with angular velocity $\mathbf{\Omega}$, velocities are related through

$$\tilde{\mathbf{v}} = \mathbf{v} - \mathbf{\Omega} \times \mathbf{x}, \tag{C1}$$

where $\tilde{\mathbf{v}}$ and $\mathbf{v}$ are the velocity measured from the rotating and inertial frame, respectively. From this relation, and the 1PN equations of motion it is possible to derive a 1PN corrected version for E_J [61]:

$$\begin{aligned} E_J \;=\; & \frac{\tilde{v}^2}{2} + \phi - \frac{1}{2}(\mathbf{\Omega}\times\mathbf{x})^2 + \frac{1}{c^2}\left[\frac{3\tilde{v}^4}{8} + \frac{\phi^2}{2} + \psi - \frac{3\phi\tilde{v}^2}{2} - (\mathbf{\Omega}\times\mathbf{x})\cdot\boldsymbol{\xi} + \frac{3\phi}{2}(\mathbf{\Omega}\times\mathbf{x})^2 - \frac{1}{8}(\mathbf{\Omega}\times\mathbf{x})^4 \right.\\ & \left. + \frac{\tilde{v}^2}{4}(\mathbf{\Omega}\times\mathbf{x})^2 + \tilde{\mathbf{v}}\cdot(\mathbf{\Omega}\times\mathbf{x})\tilde{v}^2 + \frac{1}{2}(\tilde{\mathbf{v}}\cdot(\mathbf{\Omega}\times\mathbf{x}))^2\right]. \end{aligned} \tag{C2}$$

In particular, if $\mathbf{\Omega} = \Omega\hat{\mathbf{e}}_{\mathbf{z}}$ and $\mathbf{x} = R\hat{\mathbf{e}}_{\mathbf{R}}$, the above formula reduces to

$$\begin{aligned} E_J = & \frac{\tilde{v}^2}{2} + \phi - \frac{\Omega^2 R^2}{2} + \frac{1}{2c^2}\left[\frac{3\tilde{v}^4}{4} + \phi^2 + 2\psi - 3\phi\tilde{v}^2 - 2\Omega R\xi_\varphi + 3\phi\Omega^2R^2 \right.\\ & \left. - \frac{\Omega^4R^4}{4} + \frac{\tilde{v}^2\Omega^2R^2}{2} + 2\Omega R\tilde{v}_\varphi\tilde{v}^2 + \tilde{v}_\varphi^2\Omega^2R^2\right]. \end{aligned} \tag{C3}$$

Now, by replacing $\tilde{v}_R = v_R$, $\tilde{v}_z = v_z$ and $\tilde{v}_\varphi = v_\varphi - \Omega R$ in (C3), we can write

$$E_J = E - \Omega L_z, \tag{C4}$$

where E and L_z are given by (10) and (11) respectively. Thus we see that the total energy associated to a test particle, measured from a rotating frame, has the same form as in the Newtonian theory.

Appendix D: Derivation of the Constants B_{2n} and C_{2n}

For the DF given in (67) and for any function $S(R, v_R, v_\varphi)$, we can write

$$\int S f d^2v = 2^{1/4}k \int_{\Omega R-\mu}^{\Omega R+\mu} dv_\varphi \int_{-\nu}^{\nu} \frac{S(R, v_R, v_\varphi)dv_R}{(\nu^2 - v_R^2)^{1/4}} \tag{D1}$$

where

$$\mu = \sqrt{\frac{45\pi GM}{64a}}\eta_*^2, \qquad \nu = \sqrt{\mu^2 - (v_\varphi - \Omega R)^2}. \tag{D2}$$

In particular, by setting $S = v_\varphi$ in (D1) and introducing the result in (60) we obtain

$$\sum_{n=0}^{\infty} \tilde{C}_{2n} P_{2n}(\eta_*) = \frac{5}{8}\sqrt{\frac{15GM^3}{2\pi a^5}}\sqrt{1-\eta_*^2}\eta_*^4,$$

from which $\tilde{C}_{2n}$ can be found by using the orthogonalization condition $\int_{-1}^{1} P_m(x)P_l(x)dx = 2\delta_{lm}/(2l+1)$. We find

$$\tilde{C}_{2n} = (4n+1)\frac{5}{8}\sqrt{\frac{15GM^3}{2\pi a^5}}\int_0^1 \sqrt{1-x^2}x^4 P_{2n}(x)dx. \tag{D3}$$

The above integral can be solved by setting $x = \cos\theta$ and using the following identity:

$$\int_0^{\pi/2} (\sin\theta)^m P_{2n}(\cos\theta)d\theta = \frac{\pi\left[\Gamma(\frac{m+1}{2})\right]^2}{2\Gamma(1+n+\frac{m}{2})\Gamma(\frac{1}{2}+\frac{m}{2}-n)\Gamma(1+n)\Gamma(\frac{1}{2}-n)}. \tag{D4}$$

After some calculations, we get

$$\int_0^1 \sqrt{1-x^2}x^4 P_{2n}(x)dx = \frac{\pi^2(45-48n-92n^2+16n^3+16n^4)}{128\Gamma(\frac{1}{2}-n)\Gamma(\frac{7}{2}-n)\Gamma(1+n)\Gamma(4+n)}. \tag{D5}$$

Plugging the above result in (D3) we obtain the relation (77).

Now, performing the same procedure on equation (59) and obtain

$$\pi^2 Ga\sum_{n=0}^{\infty}\tilde{B}_{2n}P_{2n}(\eta_*) = \frac{75\pi^2}{7168}\left(\frac{GM}{a}\right)^2 H(\eta_*) + \frac{32\psi}{3} + \sqrt{\frac{160\pi GM}{3a}(1-\eta_*^2)}\xi_\varphi$$

where

$$H(\eta_*) = 375\eta_*^8 - 154\eta_*^6 + 392\eta_*^4 + 14\eta_*^2 - 217. \tag{D6}$$

Again, we use the orthogonality of Legendre polynomials to provide an explicit relation for $\tilde{B}_{2n}$. In the calculations we also find useful (i) the formula

$$\int_{-1}^1 x^{2m}P_{2n}(x)dx = \frac{\sqrt{\pi}2^{-2m}\Gamma(1+2m)}{\Gamma(1+m-n)\Gamma(\frac{3}{2}+m+n)},$$

and (ii) the expression

$$\begin{aligned}\int_{-1}^1 \sqrt{1-x^2}P_{2n}(x)P_{2m}(x)dx = &\sum_{k=0}^{m}\sum_{j=0}^{m-k}\frac{\pi(4m-2k)!(m-k)!}{2^{2m}k!(2m-k)!j!(m-k-j)!(2m-2k)!}\\ &\times\frac{(-1)^{k+j}\left[\Gamma(j+\frac{3}{2})\right]^2}{\Gamma(2+j+n)\Gamma(j-n+\frac{3}{2})\Gamma(1+n)\Gamma(\frac{1}{2}-n)},\end{aligned} \tag{D7}$$

which we derived from (D4) and the series representation of the Legendre polynomials. After some calculations we get

$$\frac{64}{3}B_{2n}q_{2n}(0) + 2\pi^2 aG\tilde{B}_{2n} = \left(\frac{GM}{a}\right)^2\left[I_{2n} - \sum_{m=0}^{\infty}\Pi_{2n,2m}\right], \tag{D8}$$

where

$$I_{2n} = -\frac{75\pi^{5/2}}{16384}\frac{(4n+1)\sum_{k=0}^{8}a_k n^k}{\Gamma(5-n)\Gamma(\frac{11}{2}+n)}, \tag{D9}$$

with

$$\begin{array}{lll} a_0 = 370224 & a_1 = -337854 & a_2 = -628841\\ a_3 = 185300 & a_4 = 174491 & a_5 = -25768\\ a_6 = 16600 & a_7 = 992 & a_8 = 496 \end{array} \tag{D10}$$

and

$$\Pi_{2n,2m} = \frac{25\pi^3}{64}\int_{-1}^1 \sqrt{1-x^2}P_{2n}(x)P_{2m}(x)dx. \tag{D11}$$

From (D8), it is straightforward to derive (78).

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