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Hadronic Light-by-Light and the Pion Polarizability

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We compute the charged pion loop contribution to the light-by-light scattering amplitude for off-shell photons in chiral perturbation theory through next-to-leading order (NLO). We show that for small photon virtualities ($k^2 \ll m_\pi^2$) the NLO contributions are relatively more important due to an accidental numerical supression of the leading-order (LO) terms. This behavior is consistent with previous calculations of the hadronic light-by-light (HLBL) contribution to the muon anomalous magnetic moment, $a_\mu^{\rm HLBL}$, whose leading order value receives $\mathcal{O}(1)$ corrections from models incorporating some of the NLO physics. We also show that models employed thus far for the charged pion loop contribution to $a_\mu^{\rm HLBL}$ are not fully consistent with low-momentum behavior implied by quantum chromodynamics, having omitted potentially significant contributions from the pion polarizability.

In this note, we report on the first computation of the charged pion contribution to the light-by-light (LBL) scattering amplitude for off-shell photons to next-to-leading order in chiral perturbation theory. The LBL amplitude constitutes an important input to the Standard Model (SM) prediction for the anomalous magnetic moment of the muon, $a_\mu = (g_\mu - 2)/2$, an observable that continues be of considerable interest in particle and nuclear physics. To the extent that the SM prediction $a_\mu^{\rm SM}$ is sufficiently reliable, a deviation for the experimental value $a_\mu^{\rm exp}$ could indicate the presence of contributions from physics beyond the SM. Thus, it is important to scrutinize the ingredients in the SM prediction, particularly those associated with hadronic dynamics. The following work represents an effort in this direction.

To set the context, we first review experimental and theoretical situation that motivates this work. The present experimental value, $a_\mu^{\rm exp} = 116592089(63) \times 10^{-11}$ obtained by the E821 Collaboration[1–3] differs from theoretical expectations by 3.6σ assuming the SM of particle physics and state-of-the-art computations of hadronic contributions, including those obtained using data on $\sigma(e^+e^- \to \text{hadrons})$ and dispersion relation methods: $a_\mu^{\rm SM} = 116591802(49) \times 10^{-11}$ (for recent reviews, see Ref. [4, 5] as well as references therein). A deviation of this magnitude can be naturally explained in a number of scenarios for physics beyond the Standard Model (BSM), including (but not limited to) supersymmetry, extra dimensions, or additional neutral gauge bosons [6–8] . A next generation experiment planned for Fermilab would reduce the experimental uncertainty by a factor of four[9]. If a corresponding reduction in the theoretical, SM uncertainty were achieved, the muon anomalous moment could provide an even more powerful indirect probe of BSM physics.

The most significant pieces of the error quoted above for $a_\mu^{\rm SM}$ are associated with the leading order hadronic vacuum polarization (HVP) and the HLBL contributions: $\delta a_\mu^{\rm HVP}({\rm LO}) = \pm 42 \times 10^{-11}$ and $\delta a_\mu^{\rm HLBL} = \pm 26 \times 10^{-11}$ [10] (other authors give somewhat different error estimates for the latter [14–22] , but we will refer to these numbers as points of reference; see [23] for a review). In recent years, considerable scrutiny has been applied to the determination of $a_\mu^{\rm HVP}({\rm LO})$ from data on $\sigma(e^+e^- \to \text{hadrons})$ and hadronic τ decays. Use of the latter indicates a somewhat smaller discrepancy between the SM and experimental values for a_μ than quoted above. Clearly, a significant improvement in this determination will be needed if the levels of theoretical and future experimental precision are to be commensurate.

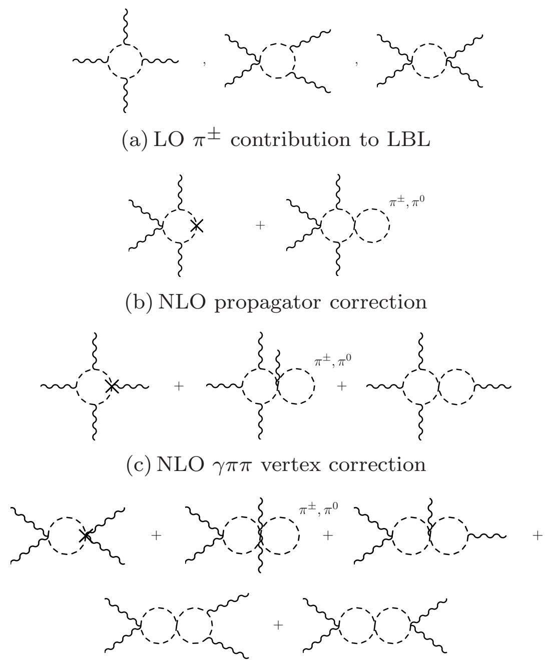
Here, we concentrate on the $a_\mu^{\rm HLBL}$, focusing in particular on the contributions from charged pion loops. Subsequent to the first results from the E821 Collaboration, the theoretical community devoted substantial effort to refining the predictions for pseudoscalar "pole" contributions, which appear at leading order in the expansion of the number of colors N_C and which are numerically dominant. However, the error quoted for the charged pion loop contributions, which enter at subleading order in N_C, is now comparable to the uncertainty associated with the pseudoscalar pole terms. Thus, we are motivated to revisit the former as part of the effort to improve the level of confidence in the theoretical SM prediction for $a_\mu^{\rm HLBL}$.

As a first step in that direction, we have computed the HLBL scattering amplitude for off-shell photons to NLO in Chiral Perturbation Theory (χPT). χPT is an effective field theory for low-energy interactions of hadrons and photons that incorporates the approximate chiral symmetry of quantum chromodynamics (QCD) for light quarks. Long-distance hadronic effects can be computed order-by-order in an expansion of p/Λ_χ, where p is a typical energy scale (such as the pion mass m_π or momentum) and $\Lambda_\chi = 4\pi F_\pi \sim 1$ GeV is the hadronic scale with $F_\pi = 93.4$ MeV being the pion decay constant. At each order in the expansion, presently incalculable strong interaction effects associated with energy scales of order Λ_χ are parameterized by a set of effective operators whose coefficients – "low energy constants" (LECs) – are fit to experimental results and then used to predict other low-energy observables.

χPT has been applied with considerable success to the analysis of a variety of hadronic and electromagnetic processes (for a recent review, see *e.g.* [24]), making it an in principle appropriate and model-independent framework for investigating hadronic contributions to a_μ, another low-energy observable. In the χPT analysis of the pseudoscalar pole contributions to $a_\mu^{\rm HLBL}$, however, one encounters a new LEC that cannot be determined independent of the a_μ measurement itself. Consequently, hadronic modeling is presently unavoidable if one wishes to predict the anomalous moment. Nevertheless, the calculable terms in χPT can be used to test or constrain model input, as any credible model for the LBL amplitude must reproduce behavior in the low-energy regime that is dictated by QCD. Indeed, the χPT computation of the leading $\ln^2$ term in the pion pole contribution revealed a critical sign error in earlier numerical computations of the pion pole contribution[17, 18]. The sub-leading ln term can be obtained from a combination of analytic computation[22] and a determination of the relevant LEC from a determination of the $\pi^0 \to e^+e^-$ branching ratio[25], and it can be used to further constrain the model input.

In this spirit, we have analyzed the charged pion loop contribution to the LBL amplitude to NLO and have compared with corresponding predictions implied by models used in the computation of $a_\mu^{\rm HLBL}$. The leading order (in chiral counting) contribution is finite, contains no LECs, and depends only on m_π and e. As we show below, this contribution is fortuitously suppressed. As a result, higher order contributions are likely to be relatively more important than one might expect on general grounds, rendering this quantity more susceptible to model-dependent uncertainties. Thus, it is arguably all the more important that any model used for the charged pion contribution to $a_\mu^{\rm HLBL}$ respect the requirements of QCD at NLO in the low-momentum regime.

In this respect, we find that models utilized to date have omitted a potentially significant contribution associated with the pion polarizability. Consistently embedding the polarizability in models that can be used to predict the full charged pion loop contribution to $a_\mu^{\rm HLBL}$ will be the subject of a future publication. Although the three-loop point-like pion contribution to $a_\mu^{\rm HLBL}$ is finite, a four-loop pure χPT computation requires an overall counter term, as in the case of pseudoscalar pole contribution. Since the finite part of the counter term (the LEC) cannot be obtained except from the measurement of a_μ itself, obtaining an *a priori* prediction requires appropriate modeling the higher-momentum behavior of the HLBL amplitude. Doing so in a manner that incorporates the polarizability and analyzing the corresponding model-dependent theoretical uncertainty goes beyond the scope of the present study, where we focus on the unambiguous requirements of chiral symmetry for the low-momentum regime.



(a) LO $\pi^\pm$ contribution to LBL

(b) NLO propagator correction

(c) NLO $\gamma\pi\pi$ vertex correction

(d) NLO $\gamma\gamma\pi\pi$ vertex correction

FIG. 1: Representative diagrams for charged pion loop contributions to the LO and NLO to LBL amplitude.

We compute the charged pion contributions to the LBL vertex function $\Pi^{\mu\nu\alpha\beta}$ through NLO from the diagrams in Figure 1, expanding the result as a power series in the external (photon) momentum and pion mass. The LO amplitude that corresponds to a pure scalar QED calculation for point-like charged pions follows from Fig. 1(a) and yields a finite result that is free from any LECs. The result contains two $\mathcal{O}(p^4)$ structures that can be expressed in terms of two dimension eight ($d = 8$) operators, $32\,\mathcal{O}_1^{(8)} \equiv (F^2)^2 \equiv (F_{\mu\nu}F^{\mu\nu})^2$ and $8\,\mathcal{O}_2^{(8)} \equiv F^4 = F_{\alpha\beta}F^{\beta\gamma}F_{\gamma\lambda}F^{\lambda\alpha}$, whose coefficients are given in Table I (the operators are defined to absorb symmetry factors). Naively, one would expect the magnitude of the coefficients to be set by $1/(4\pi)^2 \times 1/m_\pi^4$. However, we find that each operator contains an additional suppression factor of 1/9 and 1/45, respectively. Thus, we anticipate that the NLO contributions from the graphs of Fig. 1(b-d) will be relatively more important.

The graphs in Figures 1(b-d) correspond respectively to the propagator, vertex, and polarizability corrections. The first two classes are divergent and require the introduction of counterterms from the $\mathcal{O}(p^4)$ chiral Lagrangian. We carry out the calculation using dimensional regularization in $d = 4 - 2\epsilon$ dimensions and define the counterterms to remove the contributions proportional to $1/\epsilon - \gamma + \ln 4\pi + 1$ as is the standard convention for χPT[24]. We find that the explicit dependence on the counterterms needed for renormalization of the pion propagator is cancelled by charge and mass renormalization, leaving only a dependence on the $\mathcal{O}(p^4)$ operator associated with the charge radius of the pion:

$$\mathcal{L}_9 = ie\alpha_9 \; F_{\mu\nu} \, \mathrm{Tr} \; \left(Q \left[D^\mu \Sigma, D^\nu \Sigma^\dagger \right] \right) \quad , \tag{1}$$

where $Q = \mathrm{diag}(2/3, -1/3)$ is the electric charge matrix and $\Sigma = \exp(i\tau^a \; \pi^a/F_\pi)$ with $a = 1, 2, 3$ giving the non-linear realization of the spontaneously broken chiral symmetry. After renormalization, one has for the square of the pion charge radius

$$r_\pi^2 = \frac{12}{F_\pi^2} \alpha_9^r(\mu) + \frac{1}{\Lambda_\chi^2} \left[\ln \left(\frac{\mu^2}{m_\pi^2} \right) - 1 \right] \tag{2}$$

where the superscript "r" indicates the finite component after the subtraction of $1/\epsilon - \gamma + \ln 4\pi + 1$ term is performed. Choosing $\mu = m_\rho$ and taking the experimental value for r_π^2 gives $\alpha_9^r(m_\rho) = (7.0 \pm 0.2) \times 10^{-3}$ for two-flavor χPT at $\mathcal{O}(p^4)$. Within error bars, this result is the same as obtained in Ref. [26] for the three-flavor case.

The $\pi\pi\gamma\gamma$ vertex correction shown in Fig. 1(d) is finite, but the polarizability amplitude nevertheless receives an additional finite contribution from $\mathcal{L}_9$ and

$$\mathcal{L}_{10} = e^2 \alpha_{10} \; F^2 \; \mathrm{Tr} \left(Q \Sigma Q \Sigma^\dagger \right) \quad . \tag{3}$$

The corresponding combination entering the LBL amplitude is $\alpha_9^r + \alpha_{10}^r$. As the sum of the one-loop polarizability sub-graphs is finite, this combination of LECs is independent of the renormalization scale. An experimental value $(\alpha_9^r + \alpha_{10}^r)_{\mathrm{exp}} = (1.32 \pm 0.14) \times 10^{-3}$ has been obtained from radiative pion decay [27]. As a cross check on the extraction of these LECs we also consider the determination of α_{10}^r from semileptonic τ-decays given in Ref. [28]. Converting from three- to two-flavor χPT we obtain $\alpha_{10}^r(m_\rho) = -(5.19 \pm 0.06) \times 10^{-3}$, in reasonable agreement with the determination of $\alpha_9^r(m_\rho)$ from the pion form factor and $(\alpha_9^r + \alpha_{10}^r)$ from pion radiative decay. The resulting prediction for the pion polarizability[29], which we confirm by taking the on-shell photon limit of our off-shell $\pi^+\pi^-\gamma\gamma$ computation, disagrees with the latest experimental determination[30] by a factor of two[32].

The final NLO results for the LBL amplitude are summarized in Table I. To lowest order in external momenta, the only change from LO are polarizability corrections which modify the $\mathcal{O}_1^{(8)}$ coefficient. To see the full impact of the (higher momentum) NLO terms, we expand our result to $\mathcal{O}(p^6)$, introducing a complete basis of seven $d = 10$ four-photon operators:

$$\begin{aligned}
16\, \mathcal{O}_1^{(10)} &= \partial_\rho F_{\mu\nu} \partial^\rho F^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \\
8\, \mathcal{O}_2^{(10)} &= \partial_\rho F_{\mu\nu} F^{\mu\nu} \partial^\rho F_{\alpha\beta} F^{\alpha\beta} \\
2\, \mathcal{O}_3^{(10)} &= \partial_\rho F_{\alpha\beta} \partial^\rho F^{\beta\gamma} F_{\gamma\delta} F^{\delta\alpha} \\
4\, \mathcal{O}_4^{(10)} &= \partial_\rho F_{\alpha\beta} F^{\beta\gamma} \partial^\rho F_{\gamma\delta} F^{\delta\alpha} \\
4\, \mathcal{O}_5^{(10)} &= \partial^\mu F_{\mu\nu} F^{\alpha\nu} \partial_\alpha F_{\beta\gamma} F^{\beta\gamma} \\
4\, \mathcal{O}_6^{(10)} &= F_{\mu\nu} F^{\alpha\nu} \partial^\mu F_{\beta\gamma} \partial_\alpha F^{\beta\gamma} \\
2\, \mathcal{O}_7^{(10)} &= F_{\mu\nu} \partial^\mu F_{\alpha\beta} \partial^\nu F^{\beta\gamma} F_{\gamma\alpha}
\end{aligned}$$

The coefficients of these operators are given in Table II. At this order, both vertex and polarizability corrections modify the LO result.

To obtain a sense of the numerical impact of the two-loop corrections, including those involving $\alpha_9^r + \alpha_{10}^r$, we utilize the values of the LECs discussed above. In the case of $\mathcal{O}_1^{(8)}$, the NLO (two-loop) contribution represents a $\sim 20\%$ correction to the LO term, substantially larger than the $\sim m_\pi^2/\Lambda_\chi^2 \sim 0.01$ magnitude one might expect from power counting arguments. In the case of the $d = 10$ operators, the NLO corrections range from a few to $\sim 30\%$. The largest impact of the charge radius corrections is on $\mathcal{O}_1^{(10)}$ ($\sim 30\%$) while the most important effect of the polarizability is on $\mathcal{O}_2^{(10)}$ ($\sim 10\%$). These results, while illustrating the relative importance of the NLO terms due to the LO suppression, may not be fully indicative of their impact on a_μ^{HLBL}, as they cover only a small portion of the kinematic regime relevant to the a_μ^{HLBL} calculation. As we discuss below, pion form factors have been included in previous works [12, 13, 15, 16] which reproduce the charge radius terms at low energies. These model based attempts to include higher order physics all result in significant, $\mathcal{O}(1)$ shifts to the leading order a_μ^{HLBL} value. The impact on a_μ^{HLBL} of extending these models to correctly include the polarization contribution remains to be seen.

TABLE I: Coefficients of lowest dimension ($d = 8$) operators contributing to the HLBL amplitude, scaled by $e^{-4}(4\pi)^2 m_\pi^4$. Second and third columns give LO and NLO contributions in χPT, while final column indicates the VMD result [13].

Operator	1 loop χPT	2 loop	VMD
$\mathcal{O}_1^{(8)}$	1/9	$\frac{m_\pi^2}{F_\pi^2} \frac{16}{3} (\alpha_9^r + \alpha_{10}^r)$	0
$\mathcal{O}_2^{(8)}$	1/45	0	0

We now compare the explicit NLO results in χPT with the corresponding expectations for the operators in Tables I and II derived from models used to compute the charged pion loop contribution to a_μ^{HLBL}. For concreteness, we focus on the extended Nambu-Jona-Lasinio (ENJL) model adopted in Ref. [13]. In that work, the point-like contributions to the LBL vertex

TABLE II: Coefficients of $d = 10$ operators $\mathcal{O}_n^{(10)}$ contributing to the HLBL amplitude, scaled by $e^{-4}(4\pi)^2 m_\pi^6$. First column denotes operator index n. Second and third columns give LO and NLO contributions in χPT, while final column indicates VMD result. Identifying $r_\pi^2 = 6/M_V^2$ (see text) implies agreement between the two-loop χPT and VMD predictions for the charge radius contribution.

n	1 loop	2 loop	VMD
1	$\frac{1}{45}$	$\frac{1}{3}\{\frac{1}{9}(m_\pi r_\pi)^2 + \frac{4}{5}(\frac{m_\pi}{F_\pi})^2(\alpha_9^r + \alpha_{10}^r)\}$	$\frac{2}{9}\frac{m_\pi^2}{M_V^2}$
2	$\frac{2}{45}$	$\frac{1}{9}\{\frac{1}{3}(m_\pi r_\pi)^2 + \frac{1}{2}\frac{m_\pi^2}{\Lambda_\chi^2} + \frac{44}{5}(\frac{m_\pi}{F_\pi})^2(\alpha_9^r + \alpha_{10}^r)\}$	$\frac{2}{9}\frac{m_\pi^2}{M_V^2}$
3	$\frac{2}{315}$	$\frac{1}{135}(m_\pi r_\pi)^2$	$\frac{2}{45}\frac{m_\pi^2}{M_V^2}$
4	$\frac{1}{189}$	$\frac{1}{135}(m_\pi r_\pi)^2$	$\frac{2}{45}\frac{m_\pi^2}{M_V^2}$
5	$\frac{1}{135}$	$\frac{4}{45}(\frac{m_\pi}{F_\pi})^2(\alpha_9^r + \alpha_{10}^r)$	0
6	$\frac{1}{315}$	0	0
7	$\frac{1}{945}$	0	0

function $\Pi^{\mu\nu\alpha\beta}$ are modified by the inclusion of vector meson dominance (VMD) type propagator functions $V_{\mu\lambda}(k^2) = (g_{\mu\lambda}M_V^2 - p_\mu p_\lambda)/(M_V^2 - p^2)$ as

$$\Pi^{\mu\nu\alpha\beta} \to V_{\mu\lambda}(p_1)V_{\nu\sigma}(p_2)V_{\alpha\rho}(p_3)V_{\beta\eta}(p_4)\ \Pi^{\lambda\sigma\rho\eta} \quad , \quad (4)$$

with the "vector meson mass" M_V in general a function of the photon momentum p_j^2. The Ward identities imply that the $p_\mu p_\lambda$ terms do not contribute to the overall LBL vertex function; hence, the replacement of Eq. (4) is equivalent introducing a VMD form factor for each photon when M_V is taken to be a constant. The corresponding prediction for the charge radius is $(r_\pi^2)_{\rm VMD} = 6/M_V^2$. For $M_V = m_\rho$, one obtains a value for r_π^2 in good agreement with experiment. An analogous treatment using a Hidden Local Symmetry approach [15, 16] agrees with the ENJL prescription to $\mathcal{O}(p^6)$.

Expanding the right hand side of Eq. (4) to first order in p^2/M_V^2 we obtain the VMD model prediction for the NLO operator coefficients given in the last column of Tables I and II. Since $\Pi^{\mu\nu\alpha\beta}$ is already $\mathcal{O}(p^4)$ the leading order expansion of Eq. (4) is $\mathcal{O}(p^6)$. Hence, corrections to Wilson coefficients from VMD start at $d = 10$ operators in Table II. Identifying $6/M_V^2$ with the corresponding quantity that gives the pion charge radius , we observe that the VMD model reproduces some but not all of the physics that one expects at NLO for the LBL amplitude. In particular, the polarizability contributions to $\mathcal{O}_1^{(8)}$ as well as $\mathcal{O}_{1,2,5}^{(10)}$ are absent from the VMD prescription. As a point of principle, the results of this comparison imply that the VMD-type models employed for $a_\mu^{\rm HLBL}$ are not fully consistent with the strictures of QCD for the low-momentum behavior of $\Pi^{\mu\nu\alpha\beta}$ and that use of a more consistent model prescription is warranted.

On a practical level, given the relative magnitudes of the $\alpha_9^r + \alpha_{10}^r$ and α_9^r, one has reason to suspect that the omission of the polarizability contribution could have numerically significant implications for $a_\mu^{\rm HLBL}$. As discussed earlier, a comparison of the low-momentum LO and NLO contributions to the low-momentum HLBL amplitude indicates that the both the charge radius and polarizability contributions that appear at NLO can generate substantially larger corrections than one might expect based on power counting, due to the fortuitous numerical suppression of the LO terms. Moreover, the charge radius and polarizability contributions can have comparable magnitudes in the case of some operators, while for others, one or the other dominates.

At this point, one may only speculate as to the effect on $a_\mu^{\rm HLBL}$ of the previously neglected polarizability contribution. Nevertheless, it is instructive to refer to existing model computations that introduce a pion form factor at the $\pi^+\pi^-\gamma$ vertices. In the original computation of Ref. [31], inclusion of the form factor via a VMD prescription reduced the magnitude of the charged pion loop contribution to $a_\mu^{\rm HLBL}$ by a factor of three from the scalar QED/point-like pion result. The subsequent computation using the HLS procedure yielded an even stronger suppression (a factor of ten)[15, 16]. The ENJL calculation of Ref. [13] leads to a result that is about four times larger than the HLS computation, but still strongly suppressed compared to the point-like pion/scalar QED limit. In all cases, the use of a VMD type procedure that matches onto the r_π^2 terms for the HLBL amplitude at low-momentum has a much more significant numerical impact on $a_\mu^{\rm HLBL}$ than the low-momentum comparisons would suggest. Given that the latter already indicate a substantial contribution from the pion polarizability, one has ample motivation to include the corresponding physics in modeling the charged pion contribution to $a_\mu^{\rm HLBL}$. An effort to do so will be reported in forthcoming work.

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