

This is the accepted manuscript made available via CHORUS. The article has been published as:

Multiparticle form factors of the principal chiral model at large N

Axel Cortés Cubero

Phys. Rev. D **86**, 025025 — Published 23 July 2012

DOI: [10.1103/PhysRevD.86.025025](https://doi.org/10.1103/PhysRevD.86.025025)

Multiparticle Form Factors of the Principal Chiral Model At Large N

Axel Cortés Cubero*

*Baruch College, The City University of New York,
17 Lexington Avenue, New York, NY 10010, U.S.A. and
The Graduate School and University Center, The City University of New York,
365 Fifth Avenue, New York, NY 10016, U.S.A.*

We study the sigma model with $SU(N) \times SU(N)$ symmetry in $1+1$ dimensions. The two- and four-particle form factors of the Noether current operators are found, by combining the integrable-bootstrap method with the large- N expansion.

PACS numbers: 11.15.Pg, 11.15.Tk, 11.40.-q, 11.55.Ds, 02.30.Ik

I. INTRODUCTION

The quantum principal chiral sigma model is completely integrable in one space and one time dimension [1], [2]. Its action is

$$S = \frac{N}{2g_0^2} \int d^2x \eta^{\mu\nu} \text{Tr} \partial_\mu U(x)^\dagger \partial_\nu U(x), \quad (\text{I.1})$$

where $U(x) \in SU(N)$, $\mu, \nu = 0, 1$, and where $\eta^{\mu\nu}$ is the Minkowski metric, $\eta^{00} = 1, \eta^{11} = -1, \eta^{01} = \eta^{10} = 0$. The action is invariant under the global transformation $U(x) \rightarrow V_L U(x) V_R$, for $V_L, V_R \in SU(N)$. The model is asymptotically free and has a mass gap m . There are two Noether currents,

$$j_\mu^L(x)_a^c = \frac{-iN}{2g_0^2} \partial_\mu U_{ab}(x) U^{*bc}(x), \quad j_\mu^R(x)_b^d = \frac{-iN}{2g_0^2} U^{*da}(x) \partial_\mu U_{ab}(x), \quad (\text{I.2})$$

where where $a, b = 1, \dots, N$, associated with the symmetries $U \rightarrow V_L U$ and $U \rightarrow U V_R$ respectively.

In this paper, we calculate the two- and four-excitation form factors of the current operators using a large- N expansion and the form-factor bootstrap method [3]. This approach has been used in Reference [4], to find the form factors of the renormalized field operator. We also find the two-particle form factor for all $N > 2$.

In the next section, we review the exact S matrix for the chiral model. We calculate the two-particle form factors in the planar limit in Section III, and for general N in Section IV. In Section V we calculate the four-particle form factor, and we discuss our results in the final section.

II. THE EXACT S-MATRIX AND MULTIPARTICLE STATES

The sigma model has elementary particles of mass m , which carry both left and right colors. These elementary particles form bound states that obey a sine formula [5]

$$m_r = m \frac{\sin(\frac{\pi r}{N})}{\sin(\frac{\pi}{N})}, \quad r = 1, \dots, N-1, \quad (\text{II.1})$$

where m_r is the mass of a r -particle bound state. In the large- N limit, the mass of a r -particle bound state is $m_r = mr$, for finite r . This means that there are no bound states of a finite number of elementary particles in the planar limit, since the binding energy vanishes.

We introduce particle and antiparticle creation operators $\mathfrak{A}_P^\dagger(\theta)_{ab}$ and $\mathfrak{A}_A^\dagger(\theta)_{ba}$, respectively, where θ is the particle rapidity, defined in terms of the momentum vector by $p_0 = m \cosh \theta$, $p_1 = m \sinh \theta$, and $a, b = 1, \dots, N$ are left and right color indices, respectively. A product of creation operators acting on the vacuum in order of increasing rapidity, from left to right, gives the multiparticle state

$$|P, \theta_1, a_1, b_1; A, \theta_2, b_2, a_2; \dots\rangle_{\text{in}} = \mathfrak{A}_P^\dagger(\theta_1)_{a_1 b_2} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \dots |0\rangle, \quad \text{where } \theta_1 > \theta_2 > \dots \quad (\text{II.2})$$

*Electronic address: acortes_cubero@gc.cuny.edu

The S matrix of two particles, with incoming rapidities θ_1 and θ_2 , outgoing rapidities θ'_1 and θ'_2 , is

$${}_{\text{out}}\langle P, \theta'_1, c_1, d_1; P, \theta'_2, c_2, d_2 | P, \theta_1, a_1, b_1; P, \theta_2, a_2, b_2 \rangle_{\text{in}} = S_{PP}(\theta)_{a_1 b_1; a_2 b_2}^{c_2 d_2; c_1 d_1} 4\pi\delta(\theta'_1 - \theta_1) 4\pi\delta(\theta'_2 - \theta_2),$$

where $\theta = \theta_1 - \theta_2$. We follow convention and call the function $S_{PP}(\theta)_{a_1 b_1; a_2 b_2}^{c_2 d_2; c_1 d_1}$ the S matrix. It is given by

$$S_{PP}(\theta)_{a_1 b_1; a_2 b_2}^{c_2 d_2; c_1 d_1} = \chi(\theta) S_{\text{CGN}}(\theta)_{a_1; a_2}^{c_2; c_1} S_{\text{CGN}}(\theta)_{b_1; b_2}^{d_2; d_1}, \quad (\text{II.3})$$

where $S_{\text{CGN}}(\theta)$ is the S matrix of two elementary excitations of the $SU(N)$ chiral Gross-Neveu model [6], [7]:

$$S_{\text{CGN}}(\theta)_{a_1; a_2}^{c_2; c_1} = \frac{\Gamma(i\theta/2\pi + 1)\Gamma(-i\theta/2\pi - 1/N)}{\Gamma(i\theta/2\pi + 1 - 1/N)\Gamma(-i\theta/2\pi)} \left(\delta_{a_1}^{c_1} \delta_{a_2}^{c_2} - \frac{2\pi i}{N\theta} \delta_{a_2}^{c_1} \delta_{a_1}^{c_2} \right),$$

and $\chi(\theta)$ is the CDD factor [8]:

$$\chi(\theta) = \frac{\sinh\left(\frac{\theta}{2} - \frac{\pi i}{N}\right)}{\sinh\left(\frac{\theta}{2} + \frac{\pi i}{N}\right)}. \quad (\text{II.4})$$

The particle-antiparticle S matrix is related to the particle-particle S matrix by crossing, *i.e.* $\theta \rightarrow \hat{\theta} = \pi i - \theta$. The S matrix for a particle with incoming rapidity θ_1 and outgoing rapidity θ'_1 , and an antiparticle with incoming rapidity θ_2 and outgoing rapidity θ'_2 is

$$S_{AP}(\theta)_{a_1 b_2; b_2 a_2}^{d_2 c_2; c_1 d_1} = S(\hat{\theta}, N) \times \left[\delta_{b_2}^{d_2} \delta_{a_2}^{c_2} \delta_{a_1}^{c_1} \delta_{b_1}^{d_1} - \frac{2\pi i}{N\hat{\theta}} \left(\delta_{a_1 a_2} \delta^{c_1 c_2} \delta_{b_2}^{d_2} \delta_{b_1}^{d_1} + \delta_{a_2}^{c_2} \delta_{a_1}^{c_1} \delta_{b_1 b_2} \delta^{d_1 d_2} \right) - \frac{4\pi^2}{N^2 \hat{\theta}^2} \delta_{a_1 a_2} \delta^{c_1 c_2} \delta_{b_1 b_2} \delta^{d_1 d_2} \right], \quad (\text{II.5})$$

where

$$S(\theta, N) = \frac{\sinh\left(\frac{\theta}{2} - \frac{\pi i}{N}\right)}{\sinh\left(\frac{\theta}{2} + \frac{\pi i}{N}\right)} \left[\frac{\Gamma(i\theta/2\pi + 1)\Gamma(-i\theta/2\pi - 1/N)}{\Gamma(i\theta/2\pi + 1 - 1/N)\Gamma(-i\theta/2\pi)} \right]^2 = 1 + \mathcal{O}\left(\frac{1}{N^2}\right). \quad (\text{II.6})$$

The creation operators satisfy the Zamolodchikov algebra:

$$\begin{aligned} \mathfrak{A}_P^\dagger(\theta_1)_{a_1 b_1} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} &= S_{PP}(\theta)_{a_1 b_1; a_2 b_2}^{c_2 d_2; c_1 d_1} \mathfrak{A}_P^\dagger(\theta_2)_{c_2 d_2} \mathfrak{A}_P^\dagger(\theta_1)_{c_1 d_1}, \\ \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} &= S_{AA}(\theta)_{b_1 a_1; b_2 a_2}^{d_2 c_2; d_1 c_1} \mathfrak{A}_A^\dagger(\theta_2)_{d_2 c_2} \mathfrak{A}_A^\dagger(\theta_1)_{d_1 c_1}, \\ \mathfrak{A}_P^\dagger(\theta_1)_{a_1 b_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} &= S_{AP}(\theta)_{a_1 b_1; b_2 a_2}^{d_2 c_2; c_1 d_1} \mathfrak{A}_A^\dagger(\theta_2)_{d_2 c_2} \mathfrak{A}_P^\dagger(\theta_1)_{c_1 d_1}. \end{aligned} \quad (\text{II.7})$$

The r -excitation form factor of an operator $\mathfrak{B}(x)$ is defined as

$$\langle 0 | \mathfrak{B}(x) | I_1, \theta_1, C_1; \dots; I_r, \theta_r, C_r \rangle = e^{-i \sum_{k=1}^r x \cdot p_k} \mathcal{F}_{C_1, \dots, C_r}^{\mathfrak{B}}(\theta_1, \dots, \theta_r),$$

where $I_k = P$ if the k^{th} excitation is a particle, and $I_k = A$ if the k^{th} excitation is an antiparticle, C_k is the set of indices a_k, b_k for $I_k = P$ or b_k, a_k for $I_k = A$. The x -dependence of the form factor is trivial, due to Lorentz invariance.

The vacuum expectation value of two operators $\mathfrak{B}(x)$ and $\mathfrak{C}(y)$ can be expressed in terms of form factors, using completeness of in states:

$$\begin{aligned} \langle 0 | \mathfrak{B}(x) \mathfrak{C}(y) | 0 \rangle &= \langle 0 | \mathfrak{B}(x) | 0 \rangle \langle 0 | \mathfrak{C}(y) | 0 \rangle \\ &+ \sum_{r=1}^{\infty} \sum_{t=1}^{\infty} \int \frac{d\theta_1 \dots d\theta_r, d\phi_1 \dots d\phi_t}{(2\pi)^{r+t}(r+t)!} \langle 0 | \mathfrak{B}(x) | P, \theta_1, a_1, b_1; \dots; P, \theta_r, a_r, b_r; A, \phi_1, d_1, c_1; \dots; A, \phi_t, d_t, c_t \rangle \\ &\times \langle P, \theta_1, a_1, b_1; \dots; P, \theta_r, a_r, b_r; A, \phi_1, d_1, c_1; \dots; A, \phi_t, d_t, c_t | \mathfrak{C}(y) | 0 \rangle. \end{aligned} \quad (\text{II.8})$$

III. SMIRNOV'S AXIOMS AND THE TWO-PARTICLE FORM FACTORS

In this section we calculate the first nonvanishing form factor of the current operators at large N . We will discuss only the left-handed current $j_\mu^L(x)_a^c$ in detail, since the same method yields the right-handed-current form factor.

Under a global $SU(N) \times SU(N)$ transformation, the current and the particle and antiparticle creation operators transform as

$$j_\mu^L(x) \rightarrow V_L j_\mu^L(x) V_L^\dagger, \quad \mathfrak{A}_P^\dagger(\theta) \rightarrow V_R^\dagger \mathfrak{A}_P^\dagger(\theta) V_L^\dagger, \quad \mathfrak{A}_A^\dagger(\theta) \rightarrow V_L \mathfrak{A}_A^\dagger(\theta) V_R.$$

Only form factors with equal number of particles and antiparticles are invariant under such global transformations. The first nontrivial form factor is

$$\begin{aligned} \langle 0 | j_\mu^L(x)_{a_0 c_0} | A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2 \rangle &= \langle 0 | j_\mu^L(x)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} | 0 \rangle \\ &= (p_1 - p_2)_\mu e^{-ix \cdot (p_1 + p_2)} F(\theta) \left(\delta_{a_0 a_2} \delta_{b_1 b_2} \delta_{c_0 a_1} - \frac{1}{N} \delta_{a_0 c_0} \delta_{b_1 b_2} \delta_{a_1 a_2} \right), \end{aligned} \quad (\text{III.1})$$

for $\theta_1 > \theta_2$, and

$$\begin{aligned} \langle 0 | j_\mu^L(x)_{a_0 c_0} | P_1, \theta_1, a_1, b_1; A, \theta_2, b_2, a_2 \rangle &= \langle 0 | j_\mu^L(x)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} | 0 \rangle \\ &= (p_1 - p_2)_\mu e^{-ix \cdot (p_1 + p_2)} F'(\theta) \left(\delta_{a_0 a_2} \delta_{b_1 b_2} \delta_{c_0 a_1} - \frac{1}{N} \delta_{a_0 c_0} \delta_{b_1 b_2} \delta_{a_1 a_2} \right), \end{aligned} \quad (\text{III.2})$$

for $\theta_2 > \theta_1$, where, as before, $\theta = \theta_1 - \theta_2$. The $\mathcal{O}(\frac{1}{N})$ term in (III.1) ensures the tracelessness the current operator. Lorentz invariance requires that the function $F(\theta)$ depends only on the rapidity difference θ [3].

We next apply the scattering axiom, also known as Watson's theorem [3]. This axiom follows from the Zamolodchikov algebra (II.7) on the creation operators of the in-state. This gives a relation between $F(\theta)$ and $F'(\theta)$:

$$\langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} | 0 \rangle = S_{AP}(\theta)_{a_2 b_2; b_1 a_1}^{d_1 c_1; c_2 d_2} \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{d_1 c_1} \mathfrak{A}_P^\dagger(\theta_2)_{c_2 d_2} | 0 \rangle. \quad (\text{III.3})$$

or

$$F'(\theta) = S(\hat{\theta}, N) \left(1 - \frac{2\pi i}{\hat{\theta}} \right) F(\theta). \quad (\text{III.4})$$

In obtaining (III.4), some factors of $1/N$ in the S matrix were canceled by summing over group indices in (III.3).

We next consider the Smirnov periodicity axiom [3], which follows from crossing symmetry. For the M -excitation form factor of an operator $\mathfrak{B}(0)$, the periodicity axiom is

$$\begin{aligned} \langle 0 | \mathfrak{B}(0) \mathfrak{A}_{I_1}^\dagger(\theta_1)_{C_1} \mathfrak{A}_{I_1}^\dagger(\theta_2)_{C_2} \dots \mathfrak{A}_{I_M}^\dagger(\theta_M)_{C_M} | 0 \rangle \\ = \langle 0 | \mathfrak{B}(0) \mathfrak{A}_{I_M}^\dagger(\theta_M - 2\pi i)_{C_M} \mathfrak{A}_{I_1}^\dagger(\theta_1)_{C_1} \dots \mathfrak{A}_{I_{M-1}}^\dagger(\theta_{M-1})_{C_{M-1}} | 0 \rangle. \end{aligned} \quad (\text{III.5})$$

For more discussion of this axiom see References [4], [9].

Applying the periodicity axiom to our form factors (III.1), we find the two equivalent conditions:

$$\begin{aligned} \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} | 0 \rangle &= \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1 - 2\pi i)_{b_1 c_1} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} | 0 \rangle \\ \Rightarrow F'(\theta) &= F(\theta - 2\pi i), \end{aligned} \quad (\text{III.6})$$

and

$$\begin{aligned} \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_2)_{a_2 b_2} | 0 \rangle &= \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_2 - 2\pi i)_{a_2 b_2} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} | 0 \rangle \\ \Rightarrow F(\theta) &= F'(\theta + 2\pi i). \end{aligned} \quad (\text{III.7})$$

Combining (III.4) with (III.6) gives

$$F(\theta - 2\pi i) = \hat{S}(\theta, N) \left(\frac{\theta + \pi i}{\theta - \pi i} \right) F(\theta), \quad (\text{III.8})$$

where we have defined the function $\hat{S}(\theta, N) \equiv S(\hat{\theta}, N)$.

For large N , we expand $\hat{S}(\theta, N) = 1 + \mathcal{O}(\frac{1}{N^2})$ and $F(\theta) = F^0(\theta) + \frac{1}{N} F^1(\theta) + \frac{1}{N^2} F^2(\theta) + \dots$, so that

$$F^0(\theta - 2\pi i) = \left(\frac{\theta + \pi i}{\theta - \pi i} \right) F^0(\theta). \quad (\text{III.9})$$

The general solution to (III.9) is

$$F^0(\theta) = \frac{g(\theta)}{\theta + \pi i}, \quad (\text{III.10})$$

where $g(\theta)$ satisfies the periodicity condition $g(\theta - 2\pi i) = g(\theta)$. The minimal choice is to take $g(\theta) = g$, a constant. We do not present a proof that this is the right choice for the function $g(\theta)$, but it is the simplest solution and is thus likely to be the correct physical solution.

Next we determine the value of g . There is a conserved charge $Q_{a_0 c_0}^L$, associated with the current operator. This charge is

$$Q_{a_0 c_0}^L = \int dx^1 j_0^L(x)_{a_0 c_0}.$$

We fix the value of g by requiring that the charge generates the $SU(N)$ Lie algebra:

$$Q_a^L{}^a = 0, \quad [Q_{a_1}^L{}^{c_1}, Q_{a_2}^L{}^{c_2}] = i f_{a_1 a_2 c_3}^{c_1 c_2 a_3} Q_{a_3}^L{}^{c_3}, \quad (\text{III.11})$$

where the structure coefficients are

$$f_{a_1 a_2 c_3}^{c_1 c_2 a_3} = i (\delta_{a_1}^{c_2} \delta_{a_2}^{a_3} \delta_{c_3}^{c_1} - \delta_{a_2}^{c_1} \delta_{a_1}^{a_3} \delta_{c_3}^{c_2}).$$

We cross the incoming particle from Equation (III.1) to an outgoing antiparticle, *via* $\theta_2 \rightarrow \theta_2 - \pi i$, to find

$$\begin{aligned} \langle A, \theta_2, b_2, a_2 | j_0^L(x)_{a_0 c_0} | A, \theta_1, b_1, a_1 \rangle \\ = m(\cosh \theta_1 + \cosh \theta_2) \exp\{-im[x^0(\cosh \theta_1 - \cosh \theta_2) - x^1(\sinh \theta_1 - \sinh \theta_2)]\} \\ \times F_1(\theta + \pi i) \left(\delta_{a_0 a_2} \delta_{b_1 b_2} \delta_{c_0 a_1} - \frac{1}{N} \delta_{a_0 c_0} \delta_{b_1 b_2} \delta_{a_1 a_2} \right). \end{aligned}$$

The integral over x^1 gives the matrix element of the charge operator:

$$\begin{aligned} \langle A, \theta_1, b_2, a_2 | Q_{a_0 c_0}^L | A, \theta_1, b_1, a_1 \rangle \\ = (2\pi)^2 2(p_1)_0 \delta(\theta_1 - \theta_2) \left(\delta_{a_0 a_2} \delta_{b_1 b_2} \delta_{c_0 a_1} - \frac{1}{N} \delta_{a_0 c_0} \delta_{b_1 b_2} \delta_{a_1 a_2} \right) F_1(\pi i). \end{aligned}$$

The matrix element of the commutator of two charges is found by inserting a complete set of one-antiparticle intermediate states:

$$\begin{aligned} \langle A, \theta_2, b_2, a_2 | [Q_{a_0 c_0}^L, Q_{a_4 c_4}^L] | A, \theta_1, b_1, a_1 \rangle \\ = \int \frac{d\theta_3}{4\pi} \langle A, \theta_2, b_2, a_2 | Q_{a_0 c_0}^L | A, \theta_3, b_3, a_3 \rangle \langle A, \theta_3, b_3, a_3 | Q_{a_4 c_4}^L | A, \theta_1, b_1, a_1 \rangle \\ - \int \frac{d\theta_3}{4\pi} \langle A, \theta_2, b_2, a_2 | Q_{a_4 c_4}^L | A, \theta_3, b_3, a_3 \rangle \langle A, \theta_3, b_3, a_3 | Q_{a_0 c_0}^L | A, \theta_1, b_1, a_1 \rangle. \end{aligned} \quad (\text{III.12})$$

With the choice $F(\pi i) = 1$, Equation (III.12) becomes

$$\begin{aligned} \langle A, \theta_2, b_2, a_2 | [Q_{a_0}^L{}^{c_0}, Q_{a_4}^L{}^{c_4}] | A, \theta_1, b_1, a_1 \rangle \\ = i f_{a_0 a_4 c_5}^{c_0 c_4 a_5} \langle A, \theta_2, b_2, a_2 | Q_{a_5}^L{}^{c_5} | A, \theta_1, b_1, a_1 \rangle, \end{aligned}$$

which is equivalent to (III.11). This fixes the constant $g = 2\pi i$.

We have not yet discussed the annihilation-pole axiom [3]. This axiom relates the form factors of M particles to the form factors of $M - 2$ particles. The general multiparticle form factor of the current operator is

$$\begin{aligned} \langle 0 | j_\mu^L(0)_{a_0 c_0} | A, \theta_1, b_1, a_1; \dots; A, \theta_l, b_l, a_l; P, \theta_{l+1}, a_{l+1}, b_{l+1}; \dots; P, \theta_{2l}, a_{2l}, b_{2l}; A, \theta_{n-1}, b_{n-1}, a_{n-1}; P, \theta_n, a_n, b_n \rangle \\ = [p_1 + \dots + p_l - (p_{l+1} + \dots + p_{2l}) + p_{n-1} - p_n]_\mu \mathcal{F}^{\mathcal{O}}(\theta_1, \dots, \theta_n)_{a_0 c_0 a_1 \dots a_n; b_1 \dots b_n}. \end{aligned} \quad (\text{III.13})$$

Here we have factored out the vector-valued pre-factor in square brackets, consisting of a linear combination of the particle momenta, chosen to make $\mathcal{F}^{\mathcal{O}}(\theta_1, \dots, \theta_n)$ a Lorentz scalar. We define a Lorentz-scalar-valued operator $\mathcal{O}_{a_0 c_0}$ by

$$\begin{aligned} \langle 0 | \mathcal{O}_{a_0 c_0} | A, \theta_1, b_1, a_1; \dots; A, \theta_l, b_l, a_l; P, \theta_{l+1}, a_{l+1}, b_{l+1}; \dots; P, \theta_{2l}, a_{2l}, b_{2l}; A, \theta_{n-1}, b_{n-1}, a_{n-1}; P, \theta_n, a_n, b_n \rangle \\ \equiv \mathcal{F}^{\mathcal{O}}(\theta_1, \dots, \theta_n)_{a_0 c_0 a_1 \dots a_n; b_1 \dots b_n}. \end{aligned}$$

The form factor has a pole at $\theta_{n-1n} \equiv \theta_{n-1} - \theta_n = -\pi i$, corresponding to annihilation of the $(n-1)^{\text{st}}$ and n^{th} excitations. We cross the n^{th} particle to an outgoing antiparticle, yielding

$$\begin{aligned} & \langle A, \theta_n, b_n, a_n | j_\mu^L(0)_{a_0 c_0} | A, \theta_1, b_1, a_1; \dots; A, \theta_l, b_l, a_l; P, \theta_{l+1}, a_{l+1}, b_{l+1}; \dots; P, \theta_{2l}, a_{2l}, b_{2l}; A, \theta_{n-1}, b_{n-1}, a_{n-1} \rangle \\ &= [p_1 + \dots + p_l - (p_{l+1} + \dots + p_{2l}) + p_{n-1} + p_n]_\mu \\ &\quad \times \langle A, \theta_n, b_n, a_n | \mathcal{O}_{a_0 c_0} | A, \theta_1, b_1, a_1; \dots; A, \theta_l, b_l, a_l; P, \theta_{l+1}, a_{l+1}, b_{l+1}; \dots; P, \theta_{2l}, a_{2l}, b_{2l}; A, \theta_{n-1}, b_{n-1}, a_{n-1} \rangle. \end{aligned}$$

By the generalized crossing formula [9],

$$\begin{aligned} & \langle A, \theta_n, b_n, a_n | \mathcal{O}_{a_0 c_0} | A, \theta_1, b_1, a_1; \dots; A, \theta_l, b_l, a_l; P, \theta_{l+1}, a_{l+1}, b_{l+1}; \dots; P, \theta_{2l}, a_{2l}, b_{2l}; A, \theta_{n-1}, b_{n-1}, a_{n-1} \rangle \\ &= \langle A, \theta_n, b_n, a_n | A, \theta_1, b_1, a_1 \rangle \mathcal{F}^\mathcal{O}(\theta_2, \dots, \theta_{n-1})_{a_0 c_0 a_2 \dots a_{n-1}; b_2 \dots b_{n-1}} \\ &\quad + \mathcal{F}^\mathcal{O}(\theta_n - i\pi_-, \theta_1, \dots, \theta_{n-1})_{a_0 c_0 a_n a_1 \dots a_{n-1}; b_n b_1 \dots b_{n-1}} \quad \text{for } \theta_n \geq \theta_1 > \dots > \theta_{n-1} \end{aligned}$$

or

$$\begin{aligned} &= \langle A, \theta_n, b_n, a_n | A, \theta_{n-1}, b_{n-1}, a_{n-1} \rangle \mathcal{F}^\mathcal{O}(\theta_1, \dots, \theta_{2l})_{a_0 c_0 a_1 \dots a_{2l}; b_1 \dots b_{2l}} \\ &\quad + \mathcal{F}^\mathcal{O}(\theta_1, \dots, \theta_{n-1}, \theta_n + i\pi_-)_{a_0 c_0 a_1 \dots a_n; b_1 \dots b_n} \quad \text{for } \theta_1 > \dots > \theta_{n-1} \geq \theta_n, \end{aligned} \quad (\text{III.14})$$

where the right-hand side contains the n - and the $n-2$ -particle form factors, and $\pi_- = \pi - \epsilon$. Near the annihilation pole at $\theta_{n-1n} = -\pi i$ the form factors are of the form:

$$\begin{aligned} \mathcal{F}^\mathcal{O}(\theta_n - i\pi_-, \theta_1, \dots, \theta_{n-1})_{a_0 c_0 a_1 \dots a_n; b_1 \dots b_n} &= \frac{1}{\theta_{n-1} - \theta_n + i\epsilon} h(\theta_1, \dots, \theta_n)_{a_0 c_0 a_1 \dots a_n; b_1 \dots b_n}, \\ \text{and} \\ \mathcal{F}^\mathcal{O}(\theta_1, \dots, \theta_{n-1}, \theta_n + i\pi_-)_{a_0 c_0 a_1 \dots a_n; b_1 \dots b_n} &= \frac{1}{\theta_{n-1} - \theta_n - i\epsilon} h(\theta_1, \dots, \theta_n)_{a_0 c_0 a_1 \dots a_n; b_1 \dots b_n}, \end{aligned}$$

where $h(\theta_1, \dots, \theta_n)_{a_0 c_0 a_1 \dots a_n; b_1 \dots b_n}$ is an analytic function in θ_{n-1n} . We use the identity

$$\frac{1}{\theta_{n-1} - \theta_n \pm i\epsilon} = \mathbf{P} \left\{ \frac{1}{\theta_{n-1} - \theta_n} \right\} \mp i\pi \delta(\theta_{n-1} - \theta_n),$$

where $\mathbf{P} \{f(\theta_{n-1}, \theta_n)\}$ is the principal value of $f(\theta_{n-1}, \theta_n)$. We apply Watson's theorem to Equation (III.14), and find

$$\begin{aligned} & \langle A, \theta_n, b_n, a_n | \mathcal{O}_{a_0 c_0} | A, \theta_1, b_1, a_1; \dots; A, \theta_l, b_l, a_l; P, \theta_{l+1}, a_{l+1}, b_{l+1}; \dots; P, \theta_{2l}, a_{2l}, b_{2l}; A, \theta_{n-1}, b_{n-1}, a_{n-1} \rangle \\ &= \langle A, \theta_n, b_n, a_n | A, \theta_{n-1}, b'_{n-1}, a'_{n-1} \rangle S_{AA}(\theta_{1n-1})_{d_1 c_1; b_1 a_1}^{b'_{n-1} a'_{n-1}; b'_1 a'_1} \times \dots \times S_{AA}(\theta_{ln-1})_{d_l c_l; b_l a_l}^{d_{l-1} c_{l-1}; b'_l a'_l} \\ &\quad \times S_{AP}(\theta_{n-1l+1})_{c_{l+1} d_{l+1}; a_{l+1} b_{l+1}}^{d_l c_l; a'_{l+1} b'_{l+1}} \times \dots \times S_{AP}(\theta_{n-12l})_{c_{2l} d_{2l}; a_{2l} b_{2l}}^{c_{2l-1} d_{2l-1}; a'_{2l} b'_{2l}} \times \mathcal{F}^\mathcal{O}(\theta_1, \dots, \theta_{2l})_{a_0 c_0 a'_1 \dots a'_{2l}; b'_1 \dots b'_{2l}} \\ &\quad + \left(\mathbf{P} \left\{ \frac{1}{\theta_{n-1} - \theta_n} \right\} - i\pi \delta(\theta_{n-1} - \theta_n) \right) h(\theta_1, \dots, \theta_n)_{a_0 c_0 a_1 \dots a_n; b_1 \dots b_n} \\ &= \langle A, \theta_n, b_n, a_n | A, \theta_{n-1}, b_{n-1}, a_{n-1} \rangle \mathcal{F}^\mathcal{O}(\theta_1, \dots, \theta_{2l})_{a_0 c_0 a_1 \dots a_{2l}; b_1 \dots b_{2l}} \\ &\quad + \left(\mathbf{P} \left\{ \frac{1}{\theta_{n-1} - \theta_n} \right\} + i\pi \delta(\theta_{n-1} - \theta_n) \right) h(\theta_1, \dots, \theta_n)_{a_0 c_0 a_1 \dots a_n; b_1 \dots b_n}. \end{aligned} \quad (\text{III.15})$$

We will use the normalization $\langle A, \theta_n, b_n, a_n | A, \theta_{n-1}, b_{n-1}, a_{n-1} \rangle = 4\pi \delta_{a_{n-1} a_n} \delta_{b_{n-1} b_n} \delta(\theta_{n-1} - \theta_n)$. Comparing the terms proportional to $\delta(\theta_{n-1} - \theta_n)$ in (III.15), we recover the annihilation pole axiom [9]:

$$\begin{aligned} & h(\theta_1, \dots, \theta_{n-1}, \theta_{n-1})_{a_0 c_0 a_1 \dots a_n; b_1 \dots b_n} \\ &= \text{Res}_{|\theta_{n-1n} = -\pi i} \mathcal{F}^\mathcal{O}(\theta_1, \dots, \theta_{2l}, \theta_{n-1}, \theta_n)_{a_0 c_0 a_1 \dots a_{2l} a_{n-1} a_n; b_1 \dots b_{2l} b_{n-1} b_n} \\ &= 2i \mathcal{F}^\mathcal{O}(\theta_1, \dots, \theta_{2l})_{a_0 c_0 a'_1 \dots a'_{2l}; b'_1 \dots b'_{2l}} \delta_{a'_{n-1} a_n} \delta_{b'_{n-1} b_n} \\ &\quad \times \left(\delta_{a'_1 a_1} \dots \delta_{a'_{n-1} a_{n-1}} \delta_{b'_1 b_1} \dots \delta_{b'_{n-1} b_{n-1}} - S_{AA}(\theta_{1n-1})_{d_1 c_1; b_1 a_1}^{b'_{n-1} a'_{n-1}; b'_1 a'_1} \times \dots \times S_{AA}(\theta_{ln-1})_{d_l c_l; b_l a_l}^{d_{l-1} c_{l-1}; b'_l a'_l} \right. \\ &\quad \left. \times S_{AP}(\theta_{n-1l+1})_{c_{l+1} d_{l+1}; a_{l+1} b_{l+1}}^{d_l c_l; a'_{l+1} b'_{l+1}} \times \dots \times S_{AP}(\theta_{n-12l})_{c_{2l} d_{2l}; a_{2l} b_{2l}}^{c_{2l-1} d_{2l-1}; a'_{2l} b'_{2l}} \right). \end{aligned} \quad (\text{III.16})$$

IV. TWO-PARTICLE FORM FACTORS AT FINITE N

In this section, we find the exact two-particle form factor of the current operator, for arbitrary $N \geq 2$. For $N = 2$, the principal chiral model is equivalent to an $O(4)$ -symmetric vector model. The form factors of currents of the $O(4)$ model were found in Reference [10].

Our result for the two-particle form factor, for general N , is

$$\begin{aligned} \langle 0 | j_\mu^L(0)_{a_0 c_0} | A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2 \rangle \\ = (p_1 - p_2)_\mu F(\theta) \left(\delta_{a_0 a_2} \delta_{b_1 b_2} \delta_{c_0 a_1} - \frac{1}{N} \delta_{a_0 c_0} \delta_{b_1 b_2} \delta_{a_1 a_2} \right), \end{aligned}$$

where $F_1(\theta)$ satisfies equation (III.8). We insert

$$F(\theta) = \frac{g(\theta)}{\theta + \pi i},$$

into (III.8), finding

$$g(\theta - 2\pi i) = \hat{S}(\theta, N) g(\theta). \quad (\text{IV.1})$$

We solve Equation (IV.1) by a contour-integration method first used in Reference [10]. We define a contour C to be that from $-\infty$ to ∞ and from $\infty + 2\pi i$ to $-\infty + 2\pi i$, bounding the strip in which the form factor is holomorphic. Then

$$\ln g(\theta) = \int_C \frac{dz}{4\pi i} \coth \frac{z - \theta}{2} \ln g(z) = \int_{-\infty}^{\infty} \frac{dz}{4\pi i} \coth \frac{z - \theta}{2} \ln \frac{g(z)}{g(z + 2\pi i)}.$$

We differentiate both sides with respect to θ , and use (IV.1) to write

$$\frac{d}{d\theta} [\ln g(\theta)] = \frac{1}{8\pi i} \int_{-\infty}^{\infty} \frac{dz}{\sinh^2 \frac{1}{2}(z - \theta)} \ln \hat{S}(z, N). \quad (\text{IV.2})$$

The solution to (IV.2) is

$$g(\theta) = g \exp \int_0^\infty dx A(x, N) \frac{\sin^2[x(\pi i - \theta)/2\pi]}{\sinh x}, \quad (\text{IV.3})$$

where the function $A(x, N)$ is defined by

$$\hat{S}(\theta, N) = \exp \int_0^\infty dx A(x, N) \sinh \left(\frac{x\theta}{\pi i} \right), \quad (\text{IV.4})$$

and g is a constant. Note that expanding the S matrix in powers of $1/N$ yields $A(x, N) = \frac{1}{N^2} B(x) + \mathcal{O}(\frac{1}{N^3})$.

To express the function $\hat{S}(\theta, N)$, presented in (II.6), in the form (IV.4), we use the integral formula of the gamma function [11], [12],

$$\Gamma(z) = \exp \int_0^\infty \frac{dx}{x} \left[\frac{e^{-xz} - e^{-x}}{1 - e^{-x}} + (z - 1)e^{-x} \right], \quad \text{for } \text{Re } z > 0.$$

Then

$$\left[\frac{\Gamma\left(\frac{i\hat{\theta}}{2\pi} + 1\right) \Gamma\left(\frac{-i\hat{\theta}}{2\pi} - \frac{1}{N}\right)}{\Gamma\left(\frac{i\hat{\theta}}{2\pi} + 1 - \frac{1}{N}\right) \Gamma\left(\frac{-i\hat{\theta}}{2\pi}\right)} \right]^2 = \exp \int_0^\infty \frac{dx}{x} \frac{4e^{-x} (e^{2x/N} - 1)}{1 - e^{-2x}} \sinh \left(\frac{x\theta}{\pi i} \right), \quad (\text{IV.5})$$

for $N > 2$. We use the formula [9]

$$\frac{\sin \frac{\pi}{2}(z + a)}{\sin \frac{\pi}{2}(z - a)} = \exp 2 \int_0^\infty \frac{dx}{x} \frac{\sinh x(1 - z)}{\sinh x} \sinh(xa), \quad \text{for } 0 < z < 1,$$

to write the CDD factor as

$$\frac{\sinh\left(\frac{\hat{\theta}}{2} - \frac{\pi i}{N}\right)}{\sinh\left(\frac{\hat{\theta}}{2} + \frac{\pi i}{N}\right)} = \frac{\sin\frac{\pi}{2}\left(\left(1 - \frac{2}{N}\right) - \frac{\theta}{\pi i}\right)}{\sin\frac{\pi}{2}\left(\left(1 - \frac{2}{N}\right) + \frac{\theta}{\pi i}\right)} = \exp\int_0^\infty \frac{dx}{x} \frac{-2\sinh(2x/N)}{\sinh x} \sinh\left(\frac{x\theta}{\pi i}\right), \quad (\text{IV.6})$$

for $N > 2$. Combining (IV.5) and (IV.6) gives

$$\hat{S}(\theta, N) = \exp\int_0^\infty \frac{dx}{x} \left[\frac{-2\sinh(2x/N)}{\sinh x} + \frac{4e^{-x}(e^{2x/N} - 1)}{1 - 2^{-2x}} \right] \sinh\left(\frac{x\theta}{\pi i}\right). \quad (\text{IV.7})$$

From (IV.1) and (IV.3), the form factor is

$$F_1(\theta) = \frac{g}{(\theta + \pi i)} \exp\int_0^\infty \frac{dx}{x} \left[\frac{-2\sinh\left(\frac{2x}{N}\right)}{\sinh x} + \frac{4e^{-x}(e^{2x/N} - 1)}{1 - e^{-2x}} \right] \frac{\sin^2[x(\pi i - \theta)/2\pi]}{\sinh x}. \quad (\text{IV.8})$$

The condition $F_1(\pi i) = 1$ implies $g = 2\pi i$.

V. FOUR-PARTICLE FORM FACTORS

Next we find the four-excitation form factor of the current operator, in the large- N limit. Only the form factor with two particles and two antiparticles is non-zero, because of the global symmetry. The most general Lorentz- and $SU(N) \times SU(N)$ -invariant four-particle form factor, respecting the tracelessness of the current operator is

$$\begin{aligned} & \langle 0 | j_\mu^L(0)_{a_0 c_0} | A, \theta_1, b_1, a_1; A, \theta_2, b_2, a_2; P, \theta_3, a_3, b_3; P, \theta_4, a_4, b_4 \rangle \\ &= \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} | 0 \rangle \\ &= \frac{1}{N} [p_1 + p_2 - p_3 - p_4]_\mu \vec{F}(\theta_1, \theta_2, \theta_3, \theta_4) \cdot \vec{D}_{a_0 c_0 a_1 a_2 a_3 a_4; b_1 b_2 b_3 b_4}, \end{aligned} \quad (\text{V.1})$$

for $\theta_1 > \theta_2 > \theta_3 > \theta_4$,

$$\begin{aligned} & \langle 0 | j_\mu^L(0)_{a_0 c_0} | A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2; A, \theta_3, b_3, a_3; P, \theta_4, a_4, b_4 \rangle \\ &= \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} | 0 \rangle \\ &= \frac{1}{N} [p_1 + p_2 - p_3 - p_4]_\mu \vec{G}(\theta_1, \theta_2, \theta_3, \theta_4) \cdot \vec{D}_{a_0 c_0 a_1 a_2 a_3 a_4; b_1 b_2 b_3 b_4}, \end{aligned} \quad (\text{V.2})$$

for $\theta_1 > \theta_3 > \theta_2 > \theta_4$,

$$\begin{aligned} & \langle 0 | j_\mu^L(0)_{a_0 c_0} | A, \theta_1, b_1, a_1; P, \theta_2, a_2, b_2; P, \theta_3, a_3, b_3; A, \theta_4, b_4, a_4 \rangle \\ &= \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} | 0 \rangle \\ &= \frac{1}{N} [p_1 + p_2 - p_3 - p_4]_\mu \vec{H}(\theta_1, \theta_2, \theta_3, \theta_4) \cdot \vec{D}_{a_0 c_0 a_1 a_2 a_3 a_4; b_1 b_2 b_3 b_4}, \end{aligned} \quad (\text{V.3})$$

for $\theta_1 > \theta_3 > \theta_4 > \theta_2$,

$$\begin{aligned} & \langle 0 | j_\mu^L(0)_{a_0 c_0} | P, \theta_1, a_1, b_1; A, \theta_2, b_2, a_2; P, \theta_3, a_3, b_3; A, \theta_4, b_4, a_4 \rangle \\ &= \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} | 0 \rangle \\ &= \frac{1}{N} [p_1 + p_2 - p_3 - p_4]_\mu \vec{K}(\theta_1, \theta_2, \theta_3, \theta_4) \cdot \vec{D}_{a_0 c_0 a_1 a_2 a_3 a_4; b_1 b_2 b_3 b_4}, \end{aligned} \quad (\text{V.4})$$

for $\theta_3 > \theta_1 > \theta_4 > \theta_2$,

$$\begin{aligned} & \langle 0 | j_\mu^L(0)_{a_0 c_0} | P, \theta_1, a_1, b_1; P, \theta_2, a_2, b_2; A, \theta_3, b_3, a_3; A, \theta_4, b_4, a_4 \rangle \\ &= \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} | 0 \rangle \\ &= \frac{1}{N} [p_1 + p_2 - p_3 - p_4]_\mu \vec{L}(\theta_1, \theta_2, \theta_3, \theta_4) \cdot \vec{D}_{a_0 c_0 a_1 a_2 a_3 a_4; b_1 b_2 b_3 b_4}, \end{aligned} \quad (\text{V.5})$$

for $\theta_3 > \theta_4 > \theta_1 > \theta_2$,

$$\begin{aligned}
& \langle 0 | j_\mu^L(0)_{a_0 c_0} | P, \theta_1, a_1, b_1; A, \theta_2, b_2, a_2; A, \theta_3, b_3, a_3; P, \theta_4, a_4, b_4 \rangle \\
&= \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 a_3} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} | 0 \rangle \\
&= \frac{1}{N} [p_1 + p_2 - p_3 - p_4]_\mu \vec{Q}(\theta_1, \theta_2, \theta_3, \theta_4) \cdot \vec{D}_{a_0 c_0 a_1 a_2 a_3 a_4; b_1 b_2 b_3 b_4},
\end{aligned} \tag{V.6}$$

for $\theta_3 > \theta_1 > \theta_2 > \theta_4$,

$$\begin{aligned}
& \langle 0 | j_\mu^L(0)_{a_0 c_0} | A, \theta_2, b_2, a_2; A, \theta_1, b_1, a_1; P, \theta_3, a_3, b_3; P, \theta_4, a_4, b_4 \rangle \\
&= \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} | 0 \rangle \\
&= \frac{1}{N} [p_1 + p_2 - p_3 - p_4]_\mu \vec{F}(\theta_2, \theta_1, \theta_3, \theta_4) \cdot \vec{D}_{a_0 c_0 a_1 a_2 a_3 a_4; b_1 b_2 b_3 b_4},
\end{aligned} \tag{V.7}$$

for $\theta_2 > \theta_1 > \theta_3 > \theta_4$, and

$$\begin{aligned}
& \langle 0 | j_\mu^L(0)_{a_0 c_0} | A, \theta_1, b_1, a_1; A, \theta_2, b_2, a_2; P, \theta_4, a_4, b_4; P, \theta_3, a_3, b_3 \rangle \\
&= \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} | 0 \rangle \\
&= \frac{1}{N} [p_1 + p_2 - p_3 - p_4]_\mu \vec{F}(\theta_1, \theta_2, \theta_4, \theta_3) \cdot \vec{D}_{a_0 c_0 a_1 a_2 a_3 a_4; b_1 b_2 b_3 b_4},
\end{aligned} \tag{V.8}$$

for $\theta_1 > \theta_2 > \theta_4 > \theta_3$, where we define the eight-component vectors

$$\begin{aligned}
[\vec{D}_{a_0 c_0 a_1 a_2 a_3 a_4; b_1 b_2 b_3 b_4}] &= \begin{pmatrix} \delta_{a_0 a_3} \delta_{a_1 c_0} \delta_{a_2 a_4} \delta_{b_1 b_3} \delta_{b_2 b_4} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_1 a_3} \delta_{a_2 a_4} \delta_{b_1 b_3} \delta_{b_2 b_4} \\ \delta_{a_0 a_3} \delta_{a_1 c_0} \delta_{a_2 a_4} \delta_{b_1 b_4} \delta_{b_2 b_3} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_1 a_3} \delta_{a_2 a_4} \delta_{b_1 b_4} \delta_{b_2 b_3} \\ \delta_{a_0 a_4} \delta_{a_1 c_0} \delta_{a_2 a_3} \delta_{b_1 b_3} \delta_{b_2 b_4} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_1 a_4} \delta_{a_2 a_3} \delta_{b_1 b_3} \delta_{b_2 b_4} \\ \delta_{a_0 a_4} \delta_{a_1 c_0} \delta_{a_2 a_3} \delta_{b_1 b_4} \delta_{b_2 b_3} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_1 a_4} \delta_{a_2 a_3} \delta_{b_1 b_4} \delta_{b_2 b_3} \\ \delta_{a_0 a_3} \delta_{a_1 a_4} \delta_{a_2 c_0} \delta_{b_1 b_4} \delta_{b_2 b_3} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_2 a_3} \delta_{a_1 a_4} \delta_{b_1 b_4} \delta_{b_2 b_3} \\ \delta_{a_0 a_3} \delta_{a_1 a_4} \delta_{a_2 c_0} \delta_{b_1 b_3} \delta_{b_2 b_4} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_2 a_3} \delta_{a_1 a_4} \delta_{b_1 b_3} \delta_{b_2 b_4} \\ \delta_{a_0 a_4} \delta_{a_1 a_3} \delta_{a_2 c_0} \delta_{b_1 b_3} \delta_{b_2 b_4} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_2 a_4} \delta_{a_1 a_3} \delta_{b_1 b_3} \delta_{b_2 b_4} \\ \delta_{a_0 a_4} \delta_{a_1 a_3} \delta_{a_2 a_0} \delta_{b_1 b_4} \delta_{b_2 b_3} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_2 a_4} \delta_{a_1 a_3} \delta_{b_1 b_4} \delta_{b_2 b_3} \end{pmatrix}, \tag{V.9} \\
[\vec{F}(\theta_1, \theta_2, \theta_3, \theta_4)] &= \begin{pmatrix} F_1(\theta_1, \theta_2, \theta_3, \theta_4) \\ F_2(\theta_1, \theta_2, \theta_3, \theta_4) \\ F_3(\theta_1, \theta_2, \theta_3, \theta_4) \\ F_4(\theta_1, \theta_2, \theta_3, \theta_4) \\ F_5(\theta_1, \theta_2, \theta_3, \theta_4) \\ F_6(\theta_1, \theta_2, \theta_3, \theta_4) \\ F_7(\theta_1, \theta_2, \theta_3, \theta_4) \\ F_8(\theta_1, \theta_2, \theta_3, \theta_4) \end{pmatrix},
\end{aligned}$$

and similarly for \vec{G} , \vec{H} , \vec{K} , \vec{L} and \vec{Q} .

Watson's theorem relates the form factors with different ordering of rapidities, yielding

$$\begin{aligned}
& \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} | 0 \rangle \\
&= S_{AP}(\theta_{23})_{a_3 b_3; b_2 a_2}^{d_2 c_2; c_3 d_3} \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{d_2 c_2} \mathfrak{A}_P^\dagger(\theta_3)_{c_3 d_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} | 0 \rangle, \\
& \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} | 0 \rangle \\
&= S_{AP}(\theta_{24})_{a_4 b_4; b_2 a_2}^{d_2 c_2; c_4 d_4} \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_A^\dagger(\theta_2)_{d_2 c_2} \mathfrak{A}_P^\dagger(\theta_4)_{c_4 d_4} | 0 \rangle, \\
& \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} | 0 \rangle \\
&= S_{AP}(\theta_{13})_{a_3 b_3; b_1 a_1}^{d_1 c_1; c_3 d_3} \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{d_1 c_1} \mathfrak{A}_P^\dagger(\theta_3)_{c_3 d_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} | 0 \rangle, \\
& \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} | 0 \rangle \\
&= S_{AP}(\theta_{14})_{a_4 b_4; b_1 a_1}^{d_1 c_1; c_4 d_4} \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_A^\dagger(\theta_1)_{d_1 c_1} \mathfrak{A}_P^\dagger(\theta_4)_{c_4 d_4} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} | 0 \rangle, \\
& \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} | 0 \rangle \\
&= S_{AP}(\theta_{13})_{a_3 b_3; b_1 a_1}^{d_1 c_1; c_3 d_3} \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{d_1 c_1} \mathfrak{A}_P^\dagger(\theta_3)_{c_3 d_3} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} | 0 \rangle, \\
& \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} | 0 \rangle \\
&= S_{AA}(\theta_{12})_{b_1 a_1; b_2 a_2}^{d_2 c_2; d_1 c_1} \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_2)_{d_2 c_2} \mathfrak{A}_A^\dagger(\theta_1)_{d_1 c_1} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} | 0 \rangle, \\
& \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} | 0 \rangle \\
&= S_{PP}(\theta_{34})_{a_3 b_3; a_4 b_4}^{c_4 d_4; c_3 d_3} \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_4)_{c_4 d_4} \mathfrak{A}_P^\dagger(\theta_3)_{c_3 d_3} | 0 \rangle.
\end{aligned}$$

These imply, respectively,

$$\begin{aligned}
& \vec{G}(\theta_1, \theta_2, \theta_3, \theta_4) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-2\pi i}{N\theta_{23}} & \left(1 - \frac{2\pi i}{\theta_{23}}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-2\pi i}{N\theta_{23}} & 0 & \left(1 - \frac{2\pi i}{\theta_{23}}\right) & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{23}} & 0 \\ 0 & \frac{-1}{N} \left(\frac{2\pi i}{\theta_{23}} + \frac{4\pi^2}{\theta_{23}^2}\right) & \frac{-1}{N} \left(\frac{2\pi i}{\theta_{23}} + \frac{4\pi^2}{\theta_{23}^2}\right) & \left(1 - \frac{4\pi i}{\theta_{23}} - \frac{4\pi^2}{\theta_{23}^2}\right) & 0 & 0 & 0 & \frac{-1}{N} \left(\frac{2\pi i}{\theta_{23}} + \frac{4\pi^2}{\theta_{23}^2}\right) \\ 0 & 0 & 0 & 0 & \left(1 - \frac{2\pi i}{\theta_{23}}\right) & \frac{-2\pi i}{N\theta_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{23}} & \left(1 - \frac{2\pi i}{\theta_{23}}\right) \end{pmatrix} \\
&\times \vec{F}(\theta_1, \theta_2, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right) \\
&\equiv \overleftarrow{M}_1(\theta_2, \theta_3) \vec{F}(\theta_1, \theta_2, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{V.10}
\end{aligned}$$

$$\begin{aligned}
& \vec{H}(\theta_1, \theta_2, \theta_3, \theta_4) \\
&= \begin{pmatrix} \left(1 - \frac{4\pi i}{\theta_{24}} - \frac{4\pi^2}{\theta_{24}^2}\right) & \frac{-1}{N} \left(\frac{2\pi i}{\theta_{24}} + \frac{4\pi^2}{\theta_{24}^2}\right) & \frac{-1}{N} \left(\frac{2\pi i}{\theta_{24}} + \frac{4\pi^2}{\theta_{24}^2}\right) & 0 & 0 & \frac{-1}{N} \left(\frac{2\pi i}{\theta_{24}} - \frac{4\pi^2}{\theta_{24}^2}\right) & 0 & 0 \\ 0 & \left(1 - \frac{2\pi i}{\theta_{24}}\right) & 0 & \frac{-2\pi i}{N\theta_{24}} & \frac{-2\pi i}{N\theta_{24}^2} & 0 & 0 & 0 \\ 0 & 0 & \left(1 - \frac{2\pi i}{\theta_{24}}\right) & \frac{-2\pi i}{N\theta_{24}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{24}} & \left(1 - \frac{2\pi i}{\theta_{24}}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \left(1 - \frac{2\pi i}{\theta_{24}}\right) & \frac{-2\pi i}{N\theta_{24}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&\quad \times \vec{G}(\theta_1, \theta_2, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right) \\
&\equiv \overleftrightarrow{M}_2(\theta_2, \theta_4) \vec{G}(\theta_1, \theta_2, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{V.11}
\end{aligned}$$

$$\begin{aligned}
& \vec{K}(\theta_1, \theta_2, \theta_3, \theta_4) \\
&= \begin{pmatrix} \left(1 - \frac{2\pi i}{\theta_{13}}\right) & \frac{-2\pi i}{N\theta_{13}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \left(1 - \frac{2\pi i}{\theta_{13}}\right) & \frac{-2\pi i}{N\theta_{13}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{13}} & \left(1 - \frac{2\pi i}{\theta_{13}}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{N} \left(\frac{2\pi i}{\theta_{13}} + \frac{4\pi^2}{\theta_{13}^2}\right) & \left(1 - \frac{4\pi i}{\theta_{13}} - \frac{4\pi^2}{\theta_{13}^2}\right) & \frac{-1}{N} \left(\frac{2\pi i}{\theta_{13}} + \frac{4\pi^2}{\theta_{13}^2}\right) \\ 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{13}} & \frac{-2\pi i}{N\theta_{13}} & 0 & 0 & \left(1 - \frac{2\pi i}{\theta_{13}}\right) \end{pmatrix} \\
&\quad \times \vec{H}(\theta_1, \theta_2, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right) \\
&\equiv \overleftrightarrow{M}_3(\theta_1, \theta_3) \vec{H}(\theta_1, \theta_2, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{V.12}
\end{aligned}$$

$$\begin{aligned}
& \vec{L}(\theta_1, \theta_2, \theta_3, \theta_4) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-2\pi i}{N\theta_{14}} & \left(1 - \frac{2\pi i}{\theta_{14}}\right) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-2\pi i}{N\theta_{14}} & \left(1 - \frac{2\pi i}{\theta_{14}}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(1 - \frac{4\pi i}{\theta_{14}} - \frac{4\pi^2}{\theta_{14}^2}\right) & \frac{-1}{N} \left(\frac{2\pi i}{\theta_{14}} + \frac{4\pi^2}{\theta_{14}^2}\right) & 0 & \frac{-1}{N} \left(\frac{2\pi i}{\theta_{14}} + \frac{4\pi^2}{\theta_{14}^2}\right) \\ \frac{-2\pi i}{N\theta_{14}} & 0 & 0 & 0 & 0 & \left(1 - \frac{2\pi i}{\theta_{14}}\right) & \frac{-2\pi i}{N\theta_{14}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{14}} & \left(1 - \frac{2\pi i}{\theta_{14}}\right) \end{pmatrix} \\
&\quad \times \vec{K}(\theta_1, \theta_2, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right) \\
&\equiv \overleftrightarrow{M}_4(\theta_1, \theta_4) \vec{K}(\theta_1, \theta_2, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{V.13}
\end{aligned}$$

$$\vec{Q}(\theta_1, \theta_2, \theta_3, \theta_4) = \overleftrightarrow{M}_3(\theta_1, \theta_3) \vec{G}(\theta_1, \theta_2, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{V.14}$$

$$\begin{aligned}
\vec{F}(\theta_1, \theta_2, \theta_3, \theta_4) &= \begin{pmatrix} 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 1 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 \\ \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 1 \\ 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 0 & 1 & \frac{-2\pi i}{N\theta_{12}} \\ 1 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 \\ \frac{-2\pi i}{N\theta_{12}} & 1 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & 0 \\ 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 1 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} \\ 0 & 0 & 1 & \frac{-2\pi i}{N\theta_{12}} & 0 & 0 & \frac{-2\pi i}{N\theta_{12}} & 0 \end{pmatrix} \vec{F}(\theta_2, \theta_1, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right) \\
&\equiv \overleftarrow{T}_1(\theta_1, \theta_2) \vec{F}(\theta_2, \theta_1, \theta_3, \theta_4) + \mathcal{O}\left(\frac{1}{N^2}\right), \tag{V.15}
\end{aligned}$$

$$\begin{aligned}
\vec{F}(\theta_1, \theta_2, \theta_3, \theta_4) &= \begin{pmatrix} 0 & \frac{-2\pi i}{N\theta_{34}} & \frac{-2\pi i}{N\theta_{34}} & 1 & 0 & 0 & 0 & 0 \\ \frac{-2\pi i}{N\theta_{34}} & 0 & 1 & \frac{-2\pi i}{N\theta_{34}} & 0 & 0 & 0 & 0 \\ \frac{-2\pi i}{N\theta_{34}} & 1 & 0 & \frac{-2\pi i}{N\theta_{34}} & 0 & 0 & 0 & 0 \\ 1 & \frac{-2\pi i}{N\theta_{34}} & \frac{-2\pi i}{N\theta_{34}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{34}} & 1 & \frac{-2\pi i}{N\theta_{34}} \\ 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{34}} & 0 & \frac{-2\pi i}{N\theta_{34}} & 1 \\ 0 & 0 & 0 & 0 & 1 & \frac{-2\pi i}{N\theta_{34}} & 0 & \frac{-2\pi i}{N\theta_{34}} \\ 0 & 0 & 0 & 0 & \frac{-2\pi i}{N\theta_{34}} & 1 & \frac{-2\pi i}{N\theta_{34}} & 0 \end{pmatrix} \vec{F}(\theta_1, \theta_2, \theta_4, \theta_3) + \mathcal{O}\left(\frac{1}{N^2}\right) \\
&\equiv \overleftarrow{T}_2(\theta_3, \theta_4) \vec{F}(\theta_1, \theta_2, \theta_4, \theta_3) + \mathcal{O}\left(\frac{1}{N^2}\right). \tag{V.16}
\end{aligned}$$

Next we apply the Smirnov periodicity axiom (III.5):

$$\begin{aligned}
&\langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1 - 2\pi i)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} | 0 \rangle \\
&= \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} | 0 \rangle, \\
&\langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_2 - 2\pi i)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} | 0 \rangle \\
&= \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} | 0 \rangle, \\
&\langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_3 - 2\pi i)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} | 0 \rangle \\
&= \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} | 0 \rangle, \\
&\langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_P^\dagger(\theta_4 - 2\pi i)_{a_4 b_4} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} | 0 \rangle \\
&= \langle 0 | j_\mu^L(0)_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} | 0 \rangle,
\end{aligned}$$

which imply, respectively,

$$\vec{F}(\theta_1 - 2\pi i, \theta_2, \theta_3, \theta_4) = \vec{H}(\theta_2, \theta_1, \theta_3, \theta_4), \tag{V.17}$$

$$\vec{H}(\theta_2 - 2\pi i, \theta_1, \theta_3, \theta_4) = \vec{L}(\theta_1, \theta_2, \theta_3, \theta_4), \tag{V.18}$$

$$\vec{L}(\theta_1, \theta_2, \theta_3 - 2\pi i, \theta_4) = \vec{Q}(\theta_1, \theta_2, \theta_4, \theta_3), \tag{V.19}$$

$$\vec{Q}(\theta_1, \theta_2, \theta_4 - 2\pi i, \theta_3) = \vec{F}(\theta_1, \theta_2, \theta_3, \theta_4). \tag{V.20}$$

We combine Watson's theorem with the periodicity axiom, to express Equations (V.17), (V.18), (V.19) and (V.20) in terms of only $\vec{F}(\theta_1, \theta_2, \theta_3, \theta_4)$. We combine (V.17) with (V.13), (V.12) and (V.15), and find

$$\vec{F}(\theta_1 - 2\pi i, \theta_2, \theta_3, \theta_4) = \overleftarrow{M}_4(\theta_1, \theta_4) \overleftarrow{M}_3(\theta_1, \theta_3) \left[\overleftarrow{T}_1(\theta_1, \theta_2) \right]^{-1} \vec{F}(\theta_1, \theta_2, \theta_3, \theta_4). \tag{V.21}$$

Combining (V.18) with (V.11), (V.10) and (V.15) gives

$$\left[\overleftarrow{T}_1(\theta_1, \theta_2 - 2\pi i) \right]^{-1} \vec{F}(\theta_1, \theta_2 - 2\pi i, \theta_3, \theta_4) = \overleftarrow{M}_2(\theta_2, \theta_4) \overleftarrow{M}_1(\theta_2, \theta_4) \vec{F}(\theta_1, \theta_2, \theta_3, \theta_4). \tag{V.22}$$

Combining (V.19) with (V.12), (V.10) and (V.16) gives

$$\overleftrightarrow{M}_3(\theta_1, \theta_3 - 2\pi i) \overleftrightarrow{M}_1(\theta_2, \theta_3 - 2\pi i) \vec{F}(\theta_1, \theta_2, \theta_3, \theta_4) = \left[\overleftrightarrow{T}_2(\theta_3, \theta_4) \right]^{-1} \vec{F}(\theta_1, \theta_2, \theta_3, \theta_4). \quad (\text{V.23})$$

Finally, we combine (V.20) with (V.13), (V.11) and (V.16) to find

$$\overleftrightarrow{M}_4(\theta_1, \theta_4 - 2\pi i) \overleftrightarrow{M}_2(\theta_2, \theta_4 - 2\pi i) \left[\overleftrightarrow{T}_2(\theta_3, \theta_4 - 2\pi i) \right]^{-1} \vec{F}(\theta_1, \theta_2, \theta_3, \theta_4 - 2\pi i) = \vec{F}(\theta_1, \theta_2, \theta_3, \theta_4). \quad (\text{V.24})$$

The set of equations (V.21), (V.22), (V.23) and (V.24) are difficult to solve, for finite N . In the large- N limit, the matrices $\overleftrightarrow{M}_{1,2,3,4}$ become diagonal and mutually commute, and the matrices $\overleftrightarrow{T}_{1,2}$ become their own inverses. This greatly simplifies the problem, allowing us to find the form factors. We expand the form factors in powers of $1/N$ as $\vec{F}(\theta_1, \theta_2, \theta_3, \theta_4) = \vec{F}^0(\theta_1, \theta_2, \theta_3, \theta_4) + \frac{1}{N} \vec{F}^1(\theta_1, \theta_2, \theta_3, \theta_4) + \dots$, simplifying the periodicity conditions for $\vec{F}^0(\theta_1, \theta_2, \theta_3, \theta_4)$. We combine (V.21) and (V.22) to get

$$\vec{F}^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) = \overleftrightarrow{M}_4(\theta_1, \theta_4) \overleftrightarrow{M}_3(\theta_1, \theta_3) \overleftrightarrow{M}_2(\theta_2, \theta_4) \overleftrightarrow{M}_1(\theta_2, \theta_3) \vec{F}^0(\theta_1, \theta_2, \theta_3, \theta_4), \quad (\text{V.25})$$

or explicitly, in terms of the components of $\vec{F}^0(\theta_1, \theta_2, \theta_3, \theta_4)$,

$$\begin{aligned} F_1^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{13} + \pi i}{\theta_{13} - \pi i} \right) \left(\frac{\theta_{24} + \pi i}{\theta_{24} - \pi i} \right)^2 F_1^0(\theta_1, \theta_2, \theta_3, \theta_4), \\ F_2^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{14} + \pi i}{\theta_{14} - \pi i} \right) \left(\frac{\theta_{23} + \pi i}{\theta_{23} - \pi i} \right) \left(\frac{\theta_{24} + \pi i}{\theta_{24} - \pi i} \right) F_2^0(\theta_1, \theta_2, \theta_3, \theta_4), \\ F_3^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{13} + \pi i}{\theta_{13} - \pi i} \right) \left(\frac{\theta_{23} + \pi i}{\theta_{23} - \pi i} \right) \left(\frac{\theta_{24} + \pi i}{\theta_{24} - \pi i} \right) F_3^0(\theta_1, \theta_2, \theta_3, \theta_4), \\ F_4^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{14} + \pi i}{\theta_{14} - \pi i} \right) \left(\frac{\theta_{23} + \pi i}{\theta_{23} - \pi i} \right)^2 F_4^0(\theta_1, \theta_2, \theta_3, \theta_4), \\ F_5^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{14} + \pi i}{\theta_{14} - \pi i} \right)^2 \left(\frac{\theta_{23} + \pi i}{\theta_{23} - \pi i} \right) F_5^0(\theta_1, \theta_2, \theta_3, \theta_4), \\ F_6^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{14} + \pi i}{\theta_{14} - \pi i} \right) \left(\frac{\theta_{13} + \pi i}{\theta_{13} - \pi i} \right) \left(\frac{\theta_{24} + \pi i}{\theta_{24} - \pi i} \right) F_6^0(\theta_1, \theta_2, \theta_3, \theta_4), \\ F_7^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{13} + \pi i}{\theta_{13} - \pi i} \right)^2 \left(\frac{\theta_{24} + \pi i}{\theta_{24} - \pi i} \right) F_7^0(\theta_1, \theta_2, \theta_3, \theta_4), \\ F_8^0(\theta_1 - 2\pi i, \theta_2 - 2\pi i, \theta_3, \theta_4) &= \left(\frac{\theta_{14} + \pi i}{\theta_{14} - \pi i} \right) \left(\frac{\theta_{13} + \pi i}{\theta_{13} - \pi i} \right) \left(\frac{\theta_{23} + \pi i}{\theta_{23} - \pi i} \right) F_8^0(\theta_1, \theta_2, \theta_3, \theta_4). \end{aligned}$$

The solution that satisfies (V.25), (V.15) and (V.16) is

$$\begin{aligned} F_1^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_1(\theta_1, \theta_2, \theta_3, \theta_4)}{(\theta_{13} + \pi i)(\theta_{24} + \pi i)^2}, \\ F_2^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_2(\theta_1, \theta_2, \theta_3, \theta_4)}{(\theta_{14} + \pi i)(\theta_{23} + \pi i)(\theta_{24} + \pi i)}, \\ F_3^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_2(\theta_1, \theta_2, \theta_4, \theta_3)}{(\theta_{13} + \pi i)(\theta_{23} + \pi i)(\theta_{24} + \pi i)}, \\ F_4^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_1(\theta_1, \theta_2, \theta_4, \theta_3)}{(\theta_{14} + \pi i)(\theta_{23} + \pi i)^2}, \\ F_5^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_1(\theta_2, \theta_1, \theta_3, \theta_4)}{(\theta_{14} + \pi i)^2(\theta_{23} + \pi i)}, \\ F_6^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_2(\theta_2, \theta_1, \theta_3, \theta_4)}{(\theta_{14} + \pi i)(\theta_{13} + \pi i)(\theta_{24} + \pi i)}, \\ F_7^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_1(\theta_2, \theta_1, \theta_4, \theta_3)}{(\theta_{13} + \pi i)^2(\theta_{24} + \pi i)}, \\ F_8^0(\theta_1, \theta_2, \theta_3, \theta_4) &= \frac{g_2(\theta_2, \theta_1, \theta_4, \theta_3)}{(\theta_{14} + \pi i)(\theta_{13} + \pi i)(\theta_{23} + \pi i)}, \end{aligned} \quad (\text{V.26})$$

where the functions $g_1(\theta_1, \theta_2, \theta_3, \theta_4)$ and $g_2(\theta_1, \theta_2, \theta_3, \theta_4)$ are periodic under $\theta_{1,2} \rightarrow \theta_{1,2} - 2\pi i$.

Instead of the analysis of the previous paragraph, we could have combined (V.23) and (V.24) to obtain

$$\begin{aligned} & \overleftrightarrow{M}_4(\theta_1, \theta_4 - 2\pi i) \overleftrightarrow{M}_3(\theta_1, \theta_3 - 2\pi i) \overleftrightarrow{M}_2(\theta_2, \theta_4 - 2\pi i) \overleftrightarrow{M}_1(\theta_2, \theta_3 - 2\pi i) \vec{F}^0(\theta_1, \theta_2, \theta_3 - 2\pi i, \theta_4 - 2\pi i) \\ &= \vec{F}^0(\theta_1, \theta_2, \theta_3, \theta_4). \end{aligned} \quad (\text{V.27})$$

The condition (V.27) is equivalent to (V.25). The solution of (V.27) is (V.26)

The minimal choice for the functions $g_{1,2}(\theta_1, \theta_2, \theta_3, \theta_4)$ is to set them equal to constants, $g_1(\theta_1, \theta_2, \theta_3, \theta_4) = g_1$, $g_2(\theta_1, \theta_2, \theta_3, \theta_4) = g_2$. These constants are fixed using the annihilation-pole axiom. There is an annihilation pole at $\theta_{24} = -\pi i$. The annihilation-pole axiom (Equation (III.16)) implies

$$\begin{aligned} & \text{Res}|_{\theta_{24}=-\pi i} \langle 0 | \mathcal{O}_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} \mathfrak{A}_A^\dagger(\theta_2)_{b_2 a_2} \mathfrak{A}_P^\dagger(\theta_4)_{a_4 b_4} | 0 \rangle \\ &= 2i \left\{ \langle 0 | \mathcal{O}_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} | 0 \rangle \delta_{a_2 a_4} \delta_{b_2 b_4} \right. \\ & \quad \left. - \langle 0 | \mathcal{O}_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b'_1 a'_1} \mathfrak{A}_P^\dagger(\theta_3)_{a'_3 b'_3} | 0 \rangle \delta_{a'_2 a_4} \delta_{b'_2 b_4} S_{AA}(\theta_{12})_{d_1 c_1; b'_1 a'_1}^{b'_2 a'_2; b'_1 a'_1} S_{AP}(\theta_{23})_{a_3 b_3; b_2 a_2}^{d_1 c_1; a'_3 b'_3} \right\}. \end{aligned} \quad (\text{V.28})$$

We substitute (III.10) into the right-hand side of (V.28) to find

$$\begin{aligned} & \langle 0 | \mathcal{O}_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b_1 a_1} \mathfrak{A}_P^\dagger(\theta_3)_{a_3 b_3} | 0 \rangle \delta_{a_2 a_4} \delta_{b_2 b_4} \\ & - \langle 0 | \mathcal{O}_{a_0 c_0} \mathfrak{A}_A^\dagger(\theta_1)_{b'_1 a'_1} \mathfrak{A}_P^\dagger(\theta_3)_{a'_3 b'_3} | 0 \rangle \delta_{a'_2 a_4} \delta_{b'_2 b_4} S_{AA}(\theta_{12})_{d_1 c_1; b'_1 a'_1}^{b'_2 a'_2; b'_1 a'_1} S_{AP}(\theta_{23})_{a_3 b_3; b_2 a_2}^{d_1 c_1; a'_3 b'_3} \\ &= \frac{2\pi}{(\theta_{13} + \pi i)} \left\{ \frac{2\pi i}{N \hat{\theta}_{23}} \left(\delta_{a_0 a_4} \delta_{a_2 a_3} \delta_{c_0 a_1} \delta_{b_1 b_3} \delta_{b_2 b_4} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_1 a_4} \delta_{a_2 a_3} \delta_{b_1 b_3} \delta_{b_2 b_4} \right) \right. \\ & \quad + \frac{1}{N} \left(\frac{-2\pi i}{\hat{\theta}_{23}} + \frac{2\pi i}{\theta_{12}} - \frac{4\pi^2}{\theta_{12} \hat{\theta}_{23}} \right) \left(\delta_{a_0 a_3} \delta_{a_2 a_4} \delta_{a_1 c_0} \delta_{b_2 b_3} \delta_{b_1 b_4} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_1 a_3} \delta_{a_2 a_4} \delta_{b_2 b_3} \delta_{b_1 b_4} \right) \\ & \quad \left. - \frac{2\pi i}{N \theta_{12}} \left(\delta_{a_0 a_3} \delta_{a_1 a_4} \delta_{a_2 c_0} \delta_{b_1 b_3} \delta_{b_2 b_4} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_2 a_3} \delta_{a_1 a_4} \delta_{b_1 b_3} \delta_{b_2 b_4} \right) \right\}. \end{aligned}$$

Equation (V.28) yields for the constants $g_2 = 8\pi^2 i$, $g_1 = 0$. It is worth mentioning that the constants g_1 and g_2 are determined in terms of the normalization constant g of the two-particle form factor. If we chose $g(\theta)$ from Equation (III.10) to be a more general periodic function, and not a constant, then it would be inconsistent with the annihilation pole axiom to make $g_{1,2}(\theta_1, \theta_2, \theta_3, \theta_4)$ constants. In this way, the choice of the functions $g_{1,2}(\theta_1, \theta_2, \theta_3, \theta_4)$ (and probably the arbitrary periodic functions that emerge from form factors with more particles) is at least partially fixed by the choice of solution for the two-particle form factor. We notice that the double poles present in (V.26) vanish, because $g_1 = 0$. The first term on the right-hand side of (V.28) is of order $1/N$. This is the reason we introduced a factor of $1/N$ in Equations (V.1) through (V.8).

The minimal four-particle form factor satisfying all of Smirnov's axioms for large N is

$$\begin{aligned} & \langle 0 | j_\mu^L(0)_{a_0 c_0} | A, \theta_1, b_1, a_1; A, \theta_2, b_2, a_2; P, \theta_3, a_3, b_3; P, \theta_4, a_4, b_4 \rangle \\ &= [p_1 + p_2 - p_3 - p_4]_\mu \frac{8\pi^2 i}{N} \\ & \quad \times \left\{ \frac{1}{(\theta_{14} + \pi i)(\theta_{23} + \pi i)(\theta_{24} + \pi i)} \left(\delta_{a_0 a_3} \delta_{a_1 c_0} \delta_{a_2 a_4} \delta_{b_1 b_4} \delta_{b_2 b_3} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_1 a_3} \delta_{a_2 a_4} \delta_{b_1 b_4} \delta_{b_2 b_3} \right) \right. \\ & \quad + \frac{1}{(\theta_{13} + \pi i)(\theta_{23} + \pi i)(\theta_{24} + \pi i)} \left(\delta_{a_0 a_4} \delta_{a_1 c_0} \delta_{a_2 a_3} \delta_{b_1 b_3} \delta_{b_2 b_4} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_1 a_4} \delta_{a_2 a_3} \delta_{b_1 b_3} \delta_{b_2 b_4} \right) \\ & \quad + \frac{1}{(\theta_{14} + \pi i)(\theta_{13} + \pi i)(\theta_{24} + \pi i)} \left(\delta_{a_0 a_3} \delta_{a_1 a_4} \delta_{a_2 c_0} \delta_{b_1 b_3} \delta_{b_2 b_4} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_2 a_3} \delta_{a_1 a_4} \delta_{b_1 b_3} \delta_{b_2 b_4} \right) \\ & \quad \left. + \frac{1}{(\theta_{14} + \pi i)(\theta_{13} + \pi i)(\theta_{23} + \pi i)} \left(\delta_{a_0 a_4} \delta_{a_1 a_3} \delta_{a_2 c_0} \delta_{b_1 b_4} \delta_{b_2 b_3} - \frac{1}{N} \delta_{a_0 c_0} \delta_{a_2 a_4} \delta_{a_1 a_3} \delta_{b_1 b_4} \delta_{b_2 b_3} \right) \right\}, \end{aligned} \quad (\text{V.29})$$

which is the main result of this section.

VI. CONCLUSIONS

We found the two-particle form factor of the principal-chiral-model current operator, for general N . We were only able to find the four-particle form factor for large N , because the S matrix is much simpler in this limit.

Form factors of more excitations can be calculated at large N , using this method. As we add particles, the number of functions to determine grows very fast. This will be tedious, but perhaps not impossible. We hope it is possible to calculate all the form factors in the planar limit. With knowledge of all the form factors, we can write down an expression for the Wightman function that should be valid at all energy scales. Since the theory is asymptotically free, the high energy (short distances) limit of this Wightman function should correspond to the perturbative, weakly coupled regime. In principle, we should find Wightman and Green functions that contain the results from perturbation theory in the high energy limit. This problem is under investigation.

We are interested in applying the form factors found here to (2+1)-dimensional anisotropic Yang-Mills theory. This is a theory where the coupling constants are weak, but different in different directions. The form factors of the $O(4)$ -symmetric sigma model were used to calculate the string tension [13], and the glueball masses [14] of the $SU(2)$ gauge theory. We can apply our results to extend this treatment beyond the $SU(2)$ gauge group.

Acknowledgments

I wish to thank my advisor Peter Orland for all his suggestions and helpful discussions. This project was supported in part by the National Science Foundation, under Grant No. PHY0855387.

-
- [1] P.B. Wiegmann, Phys. Lett. **142B** (1984) 173.
 - [2] E. Abadalla, M.C.B. Abadalla and M. Lima-Santos, Phys. Lett. **140B** (1984) 71.
 - [3] F.A. Smirnov, **Form Factors in Completely Integrable Models of Quantum Field Theory**, Adv. Series in Math. Phys. **14**, World Scientific (1992).
 - [4] P. Orland, Phys. Rev. **D84** (2011) 105005; P. Orland, arXiv:1205.1763v1 [hep-th] (2012).
 - [5] B. Schroer, T.T. Truong and P. Weisz, Phys. Lett. **63B** (1976) 422.
 - [6] B. Berg, M. Karowski and P. Weisz, Nucl. Phys. **B134** (1978) 125.
 - [7] V. Kurak and J.A. Swieca, Phys. Lett **82B** (1979) 289.
 - [8] A.B. Zamolodchikov and Al. B. Zamolodchikov, Ann. Phys. **120** (1979) 253.
 - [9] H. Babujian, A. Fring, M. Karowski and A. Zapletal, Nucl. Phys. **B538** (1999) 535; H. Babujian and M. Karowski, Nucl. Phys. **B620** (2002) 407.
 - [10] M. Karowski and P. Weisz, Nucl. Phys. **B139** (1978) 455.
 - [11] E.T. Whittaker and G.N. Watson, **A Course of Modern Analysis**, Cambridge University Press, Chapter 12, pg. 249.
 - [12] P. Weisz, Phys. Lett. **B67** (1977) 179.
 - [13] P. Orland, Phys. Rev. **D74** (2006) 085001; Phys. Rev. **D77**, (2008) 025035.
 - [14] P. Orland, Phys. Rev. **D75** (2007) 101702.