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r-Duality and "Instead-of-Confinement" Mechanism in $\mathcal{N}=1$ Supersymmetric QCD

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Abstract

We consider $\mathcal{N}=2$ SQCD with the U(N) gauge group and N_f flavors ($N_f > N$) perturbed by an $\mathcal{N}=2$ breaking deformation – a small mass term μ for the adjoint matter. We study r-vacua, with the constraint $\frac{2}{3}N_f < r \leq N$. At large values of the parameter $\xi \sim \mu m$ (m is a typical value of the quark masses) r quark flavors condense, by construction. The effective low-energy theory with the gauge group U(r)×U(1)$^{N-r}$ is at weak coupling. Upon reducing ξ the original theory undergoes a crossover transition from weak to strong coupling.

As the original theory becomes strongly coupled, at low energies it is described by a weakly coupled infrared-free *dual* theory with the gauge group U($N_f - r$)×U(1)$^{N-N_f+r}$ and N_f light dyon flavors. These dyons condense triggering formation of non-Abelian strings which still confine monopoles, rather than quarks, contrary to naive duality arguments. "Instead-of-confinement" mechanism for quarks and gauge bosons of the original theory takes place: screened quarks and gauge bosons of the original theory decay, on curves of the marginal stability (CMS), into confined monopole-antimonopole pairs that form stringy mesons.

Next, we increase the deformation parameter μ thus decoupling the adjoint fields. Then our theory flows to $\mathcal{N}=1$ SQCD. The gauge group of the dual theory becomes U($N_f - r$). We show that the dual theory is weakly coupled if we are sufficiently close to the Argyres–Douglas point. The "instead-of-confinement" mechanism for quarks and gauge bosons survives in the limit of large μ. It determines low-energy non-Abelian dynamics in the r-vacua of $\mathcal{N}=1$ SQCD.

1 Introduction

The mechanism of confinement based on the monopole condensation [1] was shown to work [2, 3] in the monopole vacua of $\mathcal{N} = 2$ supersymmetric QCD. This confinement *per se* is essentially Abelian [4, 5, 6, 7]. Non-Abelian gauge group is broken down to an Abelian subgroup by condensation of the adjoint scalars at a high scale, with the subsequent monopole condensation at a much lower scale, in a low-energy Abelian theory. Simultaneously, formation of confining flux tubes (strings) occurs.

In $\mathcal{N} = 1$ supersymmetric QCD there are no adjoint scalars. One may hope that, starting from $\mathcal{N} = 2$ QCD and decoupling the adjoint scalars, one can arrive at a regime with non-Abelian confinement.

Motivated by this idea we recently found [8, 9] a novel non-Abelian duality in the quark vacuum of $\mathcal{N} = 2$ supersymmetric QCD with the U(N) gauge group and N_f flavors of fundamental matter (quarks), where $N < N_f < \frac{3}{2}N$. The theory was perturbed [9] by a mass term μ for the adjoint matter. At small μ the deformation superpotential reduces to the Fayet–Iliopoulos (FI) [10] F-term with the effective FI parameter ξ determined by $\xi \sim \mu m$, where m presents a typical scale of the quark masses. In [8, 9] we focused exclusively on the so-called $r = N$ vacuum in which $r = N$ quarks condense, thus completely Higgsing the U(N) gauge group. A global color-flavor locked symmetry survives in the limit of equal quark mass terms.

At large ξ this theory is at weak coupling and supports non-Abelian flux tubes (strings) [11, 12, 13, 14] (for reviews see also [15, 16, 17, 18]). It is the formation of these strings that ensures confinement of monopoles. Monopoles manifest themselves as junctions of two different strings. If $\xi \gg \Lambda^2_{\mathcal{N}=2}$, the problem can be treated quasiclassically (here $\Lambda_{\mathcal{N}=2}$ is the scale of $\mathcal{N} = 2$ SQCD).

Now, what happens if the value of ξ decreases? Upon reducing the ξ parameter, the theory undergoes a crossover transition [8, 19, 20] in a strongly coupled regime. Needless to say, quasiclassical description in terms of the original theory fails.

At small ξ, dynamics can be described in terms of a weakly coupled *dual* $\mathcal{N} = 2$ SQCD, with the $\mathrm{U}(N_f - N)\times\mathrm{U}(1)^{2N-N_f}$ gauge group and N_f flavors of light *dyons*.[1] This structure is similar to Seiberg's duality in $\mathcal{N} = 1$ theories [22, 23] where emergence of the dual $\mathrm{SU}(N_f - N)$ group was first observed.

The dual theory supports non-Abelian strings due to condensation of light dyons in much the same way as the string formation in the original theory is due to condensation of squarks. Importantly, the strings of the dual theory confine monopoles, rather than quarks [8]. This is due to the fact that the light dyons that condense in the dual theory carry weight-like chromoelectric charges (in addition to chromomagnetic charges). In other words, they carry the quark charges. The strings formed through condensation of these dyons can confine only the states with the root-like magnetic charges, i.e. the monopoles [8]. Thus,

[1] This is in perfect agreement with the results obtained in [21] where the $\mathrm{SU}(N_f - N)$ dual gauge group was identified at the root of the baryonic Higgs branch in the SU(N) gauge theory with massless (s)quarks.

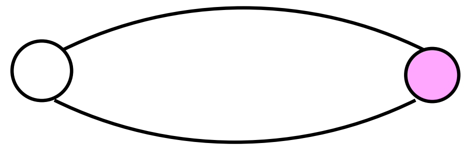

Figure 1: Meson formed by a monopole-antimonopole pair connected by two strings. Open and closed circles denote the monopole and antimonopole, respectively.

our $\mathcal{N}=2$ non-Abelian duality is *not* electromagnetic.

Then, there is no confinement of the chromoelectric charges; on the contrary, they are *Higgs-screened*.

At strong coupling where the dual description is applicable, the quarks and gauge bosons of the original theory are in what we call "instead-of-confinement" phase. Namely, the quarks and gauge bosons decay into monopole-antimonopole pairs on the curves of marginal stability (CMS) [8, 19]. The (anti)monopoles forming the pair are confined. In other words, the original quarks and gauge bosons evolve in the strong coupling domain of small ξ to become stringy mesons with two constituents being connected by two strings as shown in Fig. 1. These mesons are expected to lie on Regge trajectories.

Moreover, deep in the non-Abelian quantum regime the confined monopoles were demonstrated [19] to belong to the *fundamental representation* of the global (color-flavor locked) group. Therefore, the monopole-antimonopole mesons can be both, in the adjoint and singlet representation of this group. This pattern seems to be similar to what we have in actuality. The role of the "constituent quarks" inside the mesons is played by the monopoles.

The above referred to small values of the deformation parameter $|\mu|$. Next, we increased its value, thus decoupling the adjoint fields and sending the original theory to the limit of $\mathcal{N}=1$ SQCD. At large μ the dual theory was demonstrated [9] to be weakly coupled and infrared (IR) free, with the $\mathrm{U}(N_f-N)$ gauge group and N_f light dyons D^{lA}, (here $l=1,...,N_f-N$ is the color index in the dual gauge group, while $A=1,...,N_f$ is the flavor index). Our proof is valid provided that the dyon condensate $\sim\xi\sim\mu m$ is small enough which, in turn, requires the quark masses to be small in the large μ limit. Non-Abelian strings (albeit this time non-BPS saturated) still confine monopoles while the quark and gauge bosons of original $\mathcal{N}=1$ SQCD are presented by stringy mesons built from the monopole-antimonopoles pairs connected by two non-Abelian strings, see Fig. 1.

"Instead-of-confinement" mechanism is still at work.

In this paper we make a next step by exploring other vacua of the μ-deformed $\mathcal{N}=2$ theory, with the number of condensed quarks r smaller than N. Namely, we focus on the interval

$$\frac{2}{3}N_f < r \leq N\,. \tag{1.1}$$

The difference between these r vacua from that with $r=N$ is that for $r<N$ a U(1) factor of the $\mathrm{U}(N)$ gauge group always remains unbroken [24] and therefore residual long-range forces are present. The theory is not fully Higgsed. Still we will show that the low-energy

physics is rather similar to that of the $r = N$ case.

Strategically we follow the route similar to the analysis of [9]. First we study non-Abelian duality at small $|\mu|$, not far from the $\mathcal{N} = 2$ limit, and then increase $|\mu|$ sending the theory to $\mathcal{N} = 1$ SQCD. At large ξ the low-energy physics is determined by a weakly coupled U(r)×U(1)$^{N-r}$ gauge theory broken by the condensation of squarks down to U(1).

Upon reducing ξ the theory goes through a crossover transition to strong coupling. At small ξ the low-energy physics can be described by a dual weakly coupled IR free theory. The gauge group of the dual theory is

$$U(\nu) \times U(1)^{N-\nu}, \qquad \nu = \begin{cases} r, & r \leq \frac{N_f}{2} \\ N_f - r, & r > \frac{N_f}{2} \end{cases} . \tag{1.2}$$

Given the constraint (1.1) we focus on the case $\nu = N_f - r$. We will refer to this non-Abelian duality as "r-duality." Only if $r = N$ our r-duality reduces to Seiberg-like duality which we had studied in [8, 9].

Note, that the presence of the SU(ν)×U(1)$^{N_f-\nu}$ gauge groups at the roots of the non-baryonic branches in massless ($\xi = 0$) $\mathcal{N} = 2$ SU(N) SQCD was first recognized in [21]. Also, the relation between r and ν given in Eq. (1.2) was noted in [25, 26], where it was interpreted as a correspondence between "classical and quantum r-vacua." Our interpretation is different: we interpret it as a dual description emerging upon reducing ξ below the crossover transition line.

Light matter of the dual theory is represented by N_f flavors of dyons charged with respect to the gauge group (1.2). We calculate their electric and magnetic charges and show that they are, in fact, quark-like states with weight-like electric and root-like magnetic charges. Upon condensation of these dyons non-Abelian string are formed. We show explicitly that these strings confine monopoles, rather than quarks, in much the same way as in the $r = N$ vacuum.

The distinction between the $r < N$ and $r = N$ vacua is that one Z_N string (let us say, the N-th, there are N Z_N strings altogether) is always absent in the $r < N$ vacua. The associated flux of the unbroken U(1) gauge factor is not confined. Instead, it is spread in accordance with the Coulomb law. As a result, non-Abelian strings become metastable in the $r < N$ vacua: they can be broken by monopole-antimonopole pair creation, with monopoles being junctions of one of the first r Z_N strings and the would-be N-th string (which is in fact absent). At large quark masses these monopoles are heavy and strings are almost stable.

Next, we will increase μ thus decoupling the adjoint matter, together with the U(1) factors of the dual gauge group (1.2) and singlet dyons. The dual theory then reduces to a gauge theory with the gauge group

$$U(\nu) \times U(1)^{\rm unbr} \tag{1.3}$$

and N_f non-Abelian quark-like dyons. Here U(1)$^{\rm unbr}$ denotes the unbroken U(1) gauge factor. Dyons are neutral with respect to U(1)$^{\rm unbr}$. We integrate out heavy fields and present a

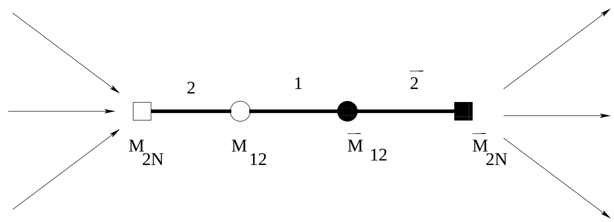

Figure 2: Example of the dipole meson formed as result of breaking of 2-nd string by pair creation of monopole M_{2N} (shown by boxes) interpolating between 2-nd string and would-be N-th string, which is absent. Arrows denote unconfined flux. Circles denote monopoles $M_{KK'}$, $K, K' = 1, ..., \nu$. Open and closed circles/boxes denote monopoles and antimonopoles, respectively.

superpotential for the light dyons. We show that this theory stays at week coupling as we increase $|\mu|$ provided that we stay close enough to the Argyres–Douglas (AD) point [27] in the quark mass parameter space. Formation of the non-Abelian strings and monopole confinement ensue.

Our main results can be summarized as follows.

We found that strongly coupled low-energy dynamics of $\mathcal{N} = 1$ supersymmetric SQCD in the r-vacua in the range (1.1) are *not* what one might naively expect from electromagnetic duality. The dual gauge group is U(ν) (where $\nu = N_f - r$) with N_f flavors of light quark-like dyons. Their condensation leads to formation of non-Abelian strings which still confine monopoles, rather than quarks. The quarks and gauge bosons of the original theory are in the "instead-of-confinement" phase: upon crossing CMS from weak to strong coupling they decay into confined monopole-antimonopole pairs that form stringy mesons. For $r < N$ the strings in the stringy mesons depicted in Fig. 1 can be broken by a pair creation of particular monopoles which interpolate between the K-th string ($K = 1, ..., \nu$) and the would-be N-th string, which is in fact absent. An example of the meson emerging in this way is shown in Fig. 2.

The endpoints emit fluxes of the unbroken U(1) gauge field. This makes this meson a dipole-like configuration. Note that the non-Abelian fluxes of the SU(ν) gauge group are always trapped and squeezed in the non-Abelian strings. Long-range forces are associated only with the unbroken U(1) gauge factor. Monopoles inside the dipole meson cannot annihilate if the overall flavor representation of the meson is nontrivial, say, the meson is in adjoint.

In a forthcoming publication [28] we will compare the r-duality with *Seiberg's* duality [22, 23].

To this end we will consider a generalization [29] of Seiberg's duality to r vacua (originally Seiberg's duality was formulated for the monopole $r = 0$ vacua). In the $r = N$ vacuum our dual gauge group U($\nu = N_f - r$) coincides with Seiberg's dual group U($N_f - N$). Moreover, in this case Seiberg's dual superpotential has a classical vacuum.

We will show that, upon integrating out heavy mesonic M-fields, this superpotential coincides with our dual superpotential obtained in [9], while Seiberg's "dual quarks" in fact reduce to our quark-like dyons D^{lA}.

At the same time, in the window $\frac{2}{3}N_f < r < N$ vacua our r duality does not match Seiberg's duality. Our dual theory has the U(ν) gauge group instead of U($\tilde{N}$) and a different superpotential for light matter. Our dual theory does have a supersymmetric classical vacuum and, in a certain regime (with small ξ), stays at weak coupling. Thus, it is appropriate to speak of triality.

For the r vacua in the range $\frac{2}{3}N_f < r < N$ Seiberg's dual superpotential has no supersymmetric classical vacua if the quark mass terms are nonvanishing. Integrating out Seiberg's "dual quarks" one obtains a continuation of the Affleck–Dine–Seiberg superpotential [30] to $N_f > N$. This superpotential correctly reproduces the quark and gaugino condensates and gives the correct number of the quantum vacua [29, 32].

We interpret this as follows [28]. In the r vacua in the range $\frac{2}{3}N_f < r < N$ the generalized Seiberg dual theory is in fact in the strong coupling regime and therefore is not useful in describing low-energy physics in its entirety. However, it does describe the chiral sector in the sense of the Veneziano–Yankielowicz effective superpotential [31] (which is not a genuine low-energy superpotential). Namely, chiral condensates are correctly reproduced. The spectrum of excitations is not.

Low-energy physics in the r vacua is described (in the range $\frac{2}{3}N_f < r < N$) by weakly coupled r-dual theory with the dual gauge group U($\nu = N_f - r$) rather than U($N_f - N$).

We also show in [28] that classical supersymmetric vacua of Seiberg's dual theory detected in [29, 32] correspond to smaller r, namely to $r < (N_f - N)$. In this range Seiberg's dual theory is at weak coupling and hence describes low-energy physics in full. This range, however, is beyond the scope of the present paper.

In this paper we only consider the r-vacua in the range (1.1). The detailed study of the r-vacua with $r \leq \frac{2}{3}N_f$ is left for future work. Still, we make a few qualitative comments about these vacua. Our picture suggests that we have a conformal window in the r-vacua in the range

$$\frac{1}{3}N_f \leq r \leq \frac{2}{3}N_f \,. \tag{1.4}$$

This means that even if we take $\mathcal{N} = 1$ SQCD with $N < N_f < \frac{3}{2}N$, the r-vacua in the range (1.4) are described by a conformal theory in the IR.

If $r < \frac{1}{3}N_f$ then Eq. (1.2) gives $\nu = r$; therefore, there is no crossover transition upon reducing ξ. The dual theory has the same gauge group U(r) as the original one. This suggests that in the dual theory we have a regular Higgs phase for quarks, and "instead-of-confinement" mechanism does not work. Quarks and gauge bosons at strong coupling are just Higgs-screened, rather than transformed into stringy mesons of the type shown in Fig. 1 or Fig. 2.

A problem for future studies is extrapolating our construction of r duality to $r \leq \frac{2}{3}N_f$ and comparing it in this range with Seiberg's duality, in particular, of importance is the range $r < (N_f - N)$ where the Seiberg's dual theory is at weak coupling.

The paper is organized as follows. In Sec. 2 we describe our basic theory, μ-deformed

$\mathcal{N}=2$ SQCD.[2] In Sec. 3, as a preparation for original explorations, we summarize what is known about the non-Abelian duality and "instead-of-confinement" mechanism in the $r=N$ vacuum. Then, in Sec. 4, we proceed to the r-duality. We consider the Seiberg–Witten curve and derive Eq. (1.2). Section 5 is devoted to a thorough study of the $r=N-1$ vacuum. In this particular example we describe in detail the low-energy theory at large ξ and in the small-μ limit. The passage to still smaller r becomes qualitatively clear. In Sec. 6 we reduce ξ and calculate the light dyon charges in the dual theory. Monopole confinement is demonstrated. We present the action of the dual theory and use exact Seiberg–Witten curves to calculate the vacuum expectation values (VEVs) of the dyon fields. In Sec. 7 we increase the value of the deformation parameter μ, decouple the adjoint matter and derive effective superpotential for light non-Abelian dyons. Section 8 summarizes our conclusions. In Appendices A–D we present calculational details of our analysis.

2 Basic Model: μ-Deformed $\mathcal{N}=2$ SQCD

The gauge symmetry of our basic model is U(N)=SU(N)×U(1). In the absence of deformation the model under consideration is $\mathcal{N}=2$ SQCD with N_f massive quark hypermultiplets. We assume that $N_f > N$ but $N_f < \frac{3}{2}N$. The latter inequality ensures that the dual theory can be infrared free.

In addition, we will introduce the mass term μ for the adjoint matter breaking $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$.

The field content is as follows. The $\mathcal{N}=2$ vector multiplet consists of the U(1) gauge field A_μ and the SU(N) gauge field A^a_μ, where $a=1,...,N^2-1$, and their Weyl fermion superpartners plus complex scalar fields a, and a^a and their Weyl superpartners, respectively. The N_f quark multiplets of the U(N) theory consist of the complex scalar fields q^{kA} and $\tilde{q}_{Ak}$ (squarks) and their fermion superpartners — all in the fundamental representation of the SU(N) gauge group. Here $k=1,...,N$ is the color index while A is the flavor index, $A=1,...,N_f$. We will treat q^{kA} and $\tilde{q}_{Ak}$ as rectangular matrices with N rows and N_f columns.

Let us first discuss the undeformed $\mathcal{N}=2$ theory. The superpotential has the form

$$\mathcal{W}_{\mathcal{N}=2}=\sqrt{2}\sum_{A=1}^{N_f}\left(\frac{1}{2}\,\tilde{q}_A\mathcal{A}q^A+\tilde{q}_A\mathcal{A}^a\,T^a q^A+m_A\,\tilde{q}_A q^A\right)\,, \tag{2.1}$$

where $\mathcal{A}$ and $\mathcal{A}^a$ are chiral superfields, the $\mathcal{N}=2$ superpartners of the gauge bosons of U(1) and SU(N), respectively.

Next, we add a mass term for the adjoint fields which breaks $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$,

$$\mathcal{W}_{\rm br}=\sqrt{\frac{N}{2}}\,\frac{\mu_0}{2}\mathcal{A}^2+\frac{\mu}{2}(\mathcal{A}^a)^2, \tag{2.2}$$

[2] For a detailed review of this model see [17].

where μ_0 and μ is are mass parameters for the chiral superfields in $\mathcal{N}=2$ gauge supermultiplets, U(1) and SU(N), respectively.[3] In this paper we will consider the single-trace perturbation which amounts to choosing μ_0 such, that the parameter

$$\gamma = 1 - \sqrt{\frac{2}{N}}\frac{\mu_0}{\mu} = 0. \tag{2.3}$$

Clearly, the mass term (2.2) splits the $\mathcal{N}=2$ supermultiplets, breaking $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$.

The bosonic part of the action of our basic theory has the form (for details see [17])

$$\begin{aligned} S &= \int d^4x \left[\frac{1}{4g_2^2}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{4g_1^2}\left(F_{\mu\nu}\right)^2 + \frac{1}{g_2^2}\left|D_\mu a^a\right|^2 + \frac{1}{g_1^2}\left|\partial_\mu a\right|^2 \right. \\ &+ \left. \left|\nabla_\mu q^A\right|^2 + \left|\nabla_\mu \bar{\tilde{q}}^A\right|^2 + V(q^A, \tilde{q}_A, a^a, a)\right] . \end{aligned} \tag{2.4}$$

Here D_μ is the covariant derivative in the adjoint representation of SU(N), while

$$\nabla_\mu = \partial_\mu - \frac{i}{2}\,A_\mu - iA^a_\mu\,T^a \tag{2.5}$$

acts in the fundamental representation. We suppress the color SU(N) indices of the matter fields. The normalization of the SU(N) generators T^a is as follows

$$\mathrm{Tr}\,(T^aT^b) = \tfrac{1}{2}\,\delta^{ab}\,.$$

The coupling constants g_1 and g_2 correspond to the U(1) and SU(N) sectors, respectively. With our conventions, the U(1) charges of the fundamental matter fields are $\pm 1/2$, see Eq. (2.5).

The scalar potential $V(q^A, \tilde{q}_A, a^a, a)$ in the action (2.4) is the sum of D and F terms,

$$\begin{aligned} V(q^A, \tilde{q}_A, a^a, a) &= \frac{g_2^2}{2}\left(\frac{1}{g_2^2}\,f^{abc}\bar{a}^b a^c + \bar{q}_A\,T^a q^A - \tilde{q}_A T^a\,\bar{\tilde{q}}^A\right)^2 \\ &+ \frac{g_1^2}{8}\left(\bar{q}_A q^A - \tilde{q}_A\bar{\tilde{q}}^A\right)^2 \\ &+ 2g_2^2\left|\tilde{q}_A T^a q^A + \frac{1}{\sqrt{2}}\,\frac{\partial\mathcal{W}_{\rm br}}{\partial a^a}\right|^2 + \frac{g_1^2}{2}\left|\tilde{q}_A q^A + \sqrt{2}\,\frac{\partial\mathcal{W}_{\rm br}}{\partial a}\right|^2 \\ &+ \frac{1}{2}\sum_{A=1}^{N_f}\left\{\left|(a + \sqrt{2}m_A + 2T^a a^a)q^A\right|^2 \right. \\ &+ \left. \left|(a + \sqrt{2}m_A + 2T^a a^a)\bar{\tilde{q}}^A\right|^2\right\} . \end{aligned} \tag{2.6}$$

[3]Without loss of generality one can assume them to be real.

Here f^{abc} denote the structure constants of the SU(N) group, m_A is the mass term for the A-th flavor, and the sum over the repeated flavor indices A is implied.

The vacua of the theory (2.4) are determined by the zeros of the potential (2.6). In general, the theory has a number of the so called r-vacua, in which (quasiclassically) r squarks condense. Later we will show that this quasiclassical analysis is valid if we require the parameter $\xi \sim \mu m$ to be large, with m being a typical scale of the quark masses. The overall range of variation of r is $r = 0, ..., N$. Say, the $r = 0$ vacua (there are N such vacua) are always at strong coupling. These are in fact the monopole vacua of [2, 3].

3 Duality in the $r = N$ vacuum

In this section we will briefly review non-Abelin duality in the $r = N$ vacua established in [8, 19, 9]. These vacua have the maximal possible number of condensed quarks, $r = N$. Moreover, the gauge group U(N) is completely Higgsed in these vacua, and, as a result, they support non-Abelian strings [11, 12, 13, 14]. The occurrence of these strings ensures confinement of the monopoles in these vacua.

First, we will assume that μ is small, much smaller than the quark masses

$$\mu \ll |m_A|, \qquad A = 1, ..., N_f\,. \tag{3.1}$$

3.1 Vacuum structure at large ξ

Now we assume that our theory is at weak coupling, so that we can analyze it quasiclassically. With generic values of the quark masses we have

$$C_{N_f}^{N} = \frac{N_f!}{N!(N_f - N)!} \tag{3.2}$$

isolated r-vacua in which $r = N$ quarks (out of N_f) develop vacuum expectation values (VEVs). Consider, say, the vacuum in which the first N flavors develop VEVs, to be denoted as (1, 2 ..., N). In this vacuum the adjoint fields develop VEVs too, namely,

$$\langle\Phi\rangle = -\frac{1}{\sqrt{2}}\begin{pmatrix} m_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & m_N \end{pmatrix}, \tag{3.3}$$

where

$$\Phi \equiv \frac{1}{2}\,a + T^a\,a^a\,. \tag{3.4}$$

For generic values of the quark masses, the SU(N) subgroup of the gauge group is broken down to U(1)$^{N-1}$. However, in the *special limit* of equal masses,

$$m_1 = m_2 = ... = m_{N_f}\,, \tag{3.5}$$

the adjoint field VEVs do not break the SU(N)×U(1) gauge group. In this limit the theory acquires a global flavor SU(N_f) symmetry.

With all quark masses equal (and limiting ourselves to the leading order in μ), the mass term for the adjoint matter (2.2) reduces to the Fayet–Iliopoulos F-term of the U(1) factor of the SU(N)×U(1) gauge group, which does *not* break $\mathcal{N}=2$ supersymmetry [5, 7]. Higher orders in the parameter μ break $\mathcal{N}=2$ supersymmetry by splitting all $\mathcal{N}=2$ multiplets.

If the quark masses are unequal the U(N) gauge group is broken down to U(1)^N by the adjoint field VEVs (3.3).

Using (2.2) and (3.3) it is not difficult to obtain the quark field VEVs from Eq. (2.6). Up to a gauge rotation they can be written as [33]

$$\begin{aligned}\langle q^{kA}\rangle &= \langle \bar{\tilde{q}}^{kA}\rangle = \frac{1}{\sqrt{2}}\left(\begin{array}{cccccc}\sqrt{\xi_1} & \ldots & 0 & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & \ldots & \sqrt{\xi_N} & 0 & \ldots & 0\end{array}\right),\\ k &= 1,...,N\,, \qquad A=1,...,N_f\,,\end{aligned} \tag{3.6}$$

where we present the quark fields as matrices in the color (k) and flavor (A) indices. The Fayet–Iliopoulos F-term parameters for each U(1) gauge factor are given (in the quasiclassical approximation) by the following expressions:

$$\xi_P \approx 2\,\mu m_P, \qquad P=1,...,N. \tag{3.7}$$

While the adjoint VEVs do not break the SU(N)×U(1) gauge group in the limit (3.5), the quark condensate (3.6) does result in the spontaneous breaking of both gauge and flavor symmetries. A diagonal global SU(N) combining the gauge SU(N) and an SU(N) subgroup of the flavor SU(N_f) group survives, provided that the quark masses are equal. This is color-flavor locking. Below we will refer to this diagonal global symmetry as to SU$(N)_{C+F}$.

Thus, the pattern of the color and flavor symmetry breaking is as follows:

$$\mathrm{U}(N)_{\rm gauge}\times \mathrm{SU}(N_f)_{\rm flavor}\to \mathrm{SU}(N)_{C+F}\times \mathrm{SU}(N_f-N)_F\times \mathrm{U}(1)\,. \tag{3.8}$$

Here SU$(N)_{C+F}$ is a global unbroken color-flavor rotation, which involves the first N flavors, while the SU$(N_f-N)_F$ factor stands for the flavor rotation of the (N_f-N) quarks. As we will see shortly, the global symmetry of the dual theory is, of course, the same, albeit the physical origin is different. The presence of the global SU$(N)_{C+F}$ group is instrumental for formation of the non-Abelian strings [11, 12, 13, 14, 33]. Tensions of N elementary strings are determined [33] by the parameters ξ_P, see (3.7),

$$T_P = 2\pi\xi_P\,. \tag{3.9}$$

Since the global (flavor) SU(N_f) group is broken by the quark VEVs anyway, it will be helpful for our purposes to consider the following mass splitting:

$$m_P = m_{P'}, \qquad m_K = m_{K'}, \qquad m_P - m_K = \Delta m \tag{3.10}$$

where

$$P, P' = 1, ..., N \quad \text{and} \quad K, K' = N+1, ..., N_f \,. \tag{3.11}$$

This mass splitting respects the global group (3.8) in the $(1, 2, ..., N)$ vacuum. Moreover, this vacuum becomes isolated. No Higgs branches develop. We will often focus on this limit below in this section.

Now, let us briefly discuss the perturbative excitation spectrum. Since both U(1) and SU(N) gauge groups are broken by the squark condensation, all gauge bosons become massive.

To the leading order in μ, $\mathcal{N}=2$ supersymmetry is not broken. In fact, with nonvanishing ξ_P's (see Eq. (3.7)), both the quarks and adjoint scalars combine with the gauge bosons to form long $\mathcal{N}=2$ supermultiplets [7], for a review see [17]. In the limit (3.10) $\xi_P \equiv \xi$, and all states come in representations of the unbroken global group (3.8), namely, in the singlet and adjoint representations of SU$(N)_{C+F}$,

$$(1,\, 1), \quad (N^2-1,\, 1), \tag{3.12}$$

and in the bifundamental representations

$$(\bar{N},\, N_f-N), \quad (N,\, \bar{N}_f-\bar{N})\,. \tag{3.13}$$

We mark representations in (3.12) and (3.13) with respect to two non-Abelian factors in (3.8). The singlet and adjoint fields are (i) the gauge bosons, and (ii) the first N flavors of the squarks q^{kP} $(P = 1, ..., N)$, together with their fermion superpartners. The bifundamental fields are the quarks q^{kK} with $K = N+1, ..., N_f$. These quarks transform in the two-index representations of the global group (3.8) due to the color-flavor locking. Singlet and adjoint fields have masses of order $g\sqrt{\xi}$, while masses of bifundamental fields are equal to Δm.

The above quasiclassical analysis is valid if the theory is at weak coupling. This is the case if the quark VEVs are sufficiently large so that the gauge coupling constant is frozen at a large scale. From (3.6) we see that the quark condensates are of the order of $\sqrt{\mu m}$ (see also [2, 3, 21, 29]). The weak coupling condition reduces to

$$\sqrt{\mu m} \gg \Lambda_{\mathcal{N}=2}\,, \tag{3.14}$$

where $\Lambda_{\mathcal{N}=2}$ is the scale of the $\mathcal{N}=2$ theory, and we assume that all quark masses are of the same order $m_A \sim m$. In particular, the condition (3.14), combined with the condition (3.1) of smallness of μ, implies that the average quark mass m is very large.

3.2 Dual theory

Now we will relax the condition (3.14) and pass to the strong coupling domain at

$$|\sqrt{\xi_P}| \ll \Lambda_{\mathcal{N}=2}\,, \qquad |m_A| \ll \Lambda_{\mathcal{N}=2}\,, \tag{3.15}$$

still keeping μ small.

In [8, 9] it was shown that the theory (2.4) in the $r = N$ vacuum undergoes a crossover transition as the value of ξ decreases. The domain (3.15) can be described in terms of weakly coupled (infrared free) dual theory with with the *gauge group*

$$\mathrm{U}(N_f - N) \times \mathrm{U}(1)^{2N-N_f}\,, \tag{3.16}$$

and N_f flavors of light *dyons*.[4]

Light dyons D^{lA} ($l = 1, ..., (N_f - N)$ and $A = 1, ..., N_f$) are in the fundamental representation of the gauge group $\mathrm{SU}(N_f - N)$ and are charged under the Abelian factors indicated in Eq. (3.16). In addition, there are $(2N - N_f)$ light dyons D^J ($J = (N_f - N + 1), ..., N$), neutral under the $\mathrm{SU}(N_f - N)$ group, but charged under the U(1) factors.

The dyon condensates are as follows:

$$\langle D^{lA}\rangle = \langle\bar{\tilde{D}}^{lA}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \ldots & 0 & \sqrt{\xi_1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & \ldots & \sqrt{\xi_{(N_f-N)}} \end{pmatrix}, \tag{3.17}$$

$$\langle D^J\rangle = \langle\bar{\tilde{D}}^J\rangle = \sqrt{\frac{\xi_J}{2}}, \qquad J = (N_f - N + 1), ..., N\,. \tag{3.18}$$

The most important feature apparent in (3.17), as compared to the squark VEVs in the original theory (3.6), is a "vacuum leap" [8],

$$(1, ..., N)_{\sqrt{\xi}\gg\Lambda_{\mathcal{N}=2}} \to (N+1, ..., N_f,\ (N_f - N + 1), ..., N)_{\sqrt{\xi}\ll\Lambda_{\mathcal{N}=2}}\,. \tag{3.19}$$

In other words, if we pick up the vacuum with nonvanishing VEVs of the first N quark flavors in the original theory at large ξ, Eq. (2.4), and then reduce ξ below $\Lambda_{\mathcal{N}=2}$, the system goes through a crossover transition and ends up in the vacuum of the *dual* theory with the nonvanishing VEVs of $(N_f - N)$ last dyons (plus VEVs of $(2N - N_f)$ dyons that are $\mathrm{SU}(N_f - N)$ singlets).

The Fayet–Iliopoulos parameters ξ_P in (3.17), (3.18) are determined by the quantum version of the classical expressions (3.7) [33]. Defining

$$u_k = \left\langle \mathrm{Tr}\left(\frac{1}{2}\,a + T^a\,a^a\right)^k\right\rangle, \qquad k = 1, ..., N\,, \tag{3.20}$$

we perform a quantum generalization in the two relevant terms in the third line of the potential in (2.6),

$$\frac{\partial\mathcal{W}_{\rm br}}{\partial a^a} \to \mu\,\frac{\partial u_2}{\partial a^a}\,, \qquad \frac{\partial\mathcal{W}_{\rm br}}{\partial a} \to \mu\,\frac{\partial u_2}{\partial a}\,. \tag{3.21}$$

[4]Previously the $\mathrm{SU}(N_f - N)$ gauge group was identified [21] as dual on the Coulomb branch at the root of the baryonic Higgs branch in the $\mathcal{N} = 2$ supersymmetric $\mathrm{SU}(N)$ Yang–Mills theory with massless quarks.

From this we obtain [33]

$$\xi_P = -2\sqrt{2}\,\mu\,E_P\,, \tag{3.22}$$

where E_P ($P = 1, ..., N$) are the diagonal elements of the $N \times N$ matrix

$$E = \frac{1}{N}\,\frac{\partial u_2}{\partial a} + T^{\tilde a}\,\frac{\partial u_2}{\partial a^{\tilde a}}\,. \tag{3.23}$$

Here $T^{\tilde a}$ are the Cartan generators of the SU(N) gauge group (the subscript $\tilde a$ runs over $\tilde a = 1, ..., (N-1)$).

The parameters E_P are expressible in terms of the roots of the Seiberg–Witten curve. Namely, in the given $r = N$ vacuum they are [33]

$$E_P = e_P, \qquad P = 1, ..., N\,, \tag{3.24}$$

where e_P are the double roots of the Seiberg–Witten curve [21],

$$y^2 = \prod_{P=1}^{N}(x-\phi_P)^2 - 4\left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}}\right)^{2N-N_f}\prod_{A=1}^{N_f}\left(x+\frac{m_A}{\sqrt{2}}\right), \tag{3.25}$$

while ϕ_P are gauge invariant parameters on the Coulomb branch.

In the $r = N$ vacuum the curve (3.25) has N double roots and reduces to

$$y^2 = \prod_{P=1}^{N}(x-e_P)^2, \tag{3.26}$$

where quasiclassically (at large masses) e_P's are given by the mass parameters, $\sqrt{2}e_P \approx -m_P$ ($P = 1, ..., N$).

Thus, the dyon condensates at small ξ in the $r = N$ vacuum are determined by

$$\xi_P = -2\sqrt{2}\,\mu\,e_P\,. \tag{3.27}$$

We will see below that the expressions (3.6), (3.17) and (3.22) are quite general and valid also for the $r < N$ vacua, while the relation (3.24) gets modified in the $r < N$ vacua.

As long as we keep ξ_P and masses small enough (i.e. in the domain (3.15)) the coupling constants of the infrared-free dual theory (frozen at the scale of the dyon VEVs) are small: the dual theory is at weak coupling.

At small masses, in the region (3.15), the double roots of the Seiberg–Witten curve are

$$\sqrt{2}e_I = -m_{I+N}, \qquad \sqrt{2}e_J = \Lambda_{\mathcal{N}=2}\,\exp\left(\frac{2\pi i}{2N-N_f}J\right) \tag{3.28}$$

for $2N - N_f > 1$, where

$$I = 1, ..., (N_f - N) \;\;\text{and}\;\; J = (N_f - N + 1), ..., N\,. \tag{3.29}$$

In particular, the $(N_f - N)$ first roots are determined by the masses of the last $(N_f - N)$ quarks — a reflection of the fact that the non-Abelian sector of the dual theory is not asymptotically free and is at weak coupling in the domain (3.15).

3.3 "Instead-of-confinement" mechanism

Now, let us consider either the equal quark masses or the special choice (3.10). Both, the gauge group and the global flavor $\mathrm{SU}(N_f)$ group, are broken in the vacuum. In the case of (3.10) the flavor $\mathrm{SU}(N_f)$ group is explicitly broken down to $\mathrm{SU}(N)\times\mathrm{SU}(N_f-N)$ by masses. However, the color-flavor locked form apparent in (3.17) under the given mass choice guarantees that the diagonal global $\mathrm{SU}(N_f-N)_{C+F}$ symmetry survives. More exactly, the unbroken *global* group of the dual theory is

$$\mathrm{SU}(N)_F \times \mathrm{SU}(N_f-N)_{C+F} \times \mathrm{U}(1)\,. \tag{3.30}$$

The $\mathrm{SU}(N_f-N)_{C+F}$ factor in (3.30) is a global unbroken color-flavor rotation, which involves the last (N_f-N) flavors, while the $\mathrm{SU}(N)_F$ factor stands for the flavor rotation of the first N dyons.

Thus, color-flavor locking takes place in the dual theory too. In much the same way as in the original theory, the presence of the global $\mathrm{SU}(N_f-N)_{C+F}$ symmetry is the reason behind formation of the non-Abelian strings. Their tensions are still given by Eq. (3.9), where the parameters ξ_P are determined by (3.27) [33, 9]. For generic quark masses the global symmetry (3.8) is broken down to $\mathrm{U}(1)^{N_f-1}$.

In the equal mass limit, or given the special choice (3.10), the global unbroken symmetry (3.30) of the dual theory at small ξ coincides with the global group (3.8) which manifests itself in the $r=N$ vacuum of the original theory at large ξ.

Note, however, that this global symmetry is realized in two very distinct ways in the dual pair at hand. As was already mentioned, the quarks and $\mathrm{U}(N)$ gauge bosons of the original theory at large ξ come in the following representations of the global group (3.8):

$$(1,1),\ (N^2-1,1),\ (\bar{N},(N_f-N)),\ \text{and}\ (N,(\bar{N}_f-\bar{N}))\,.$$

At the same time, the dyons and $\mathrm{U}(N_f-N)$ gauge bosons of the dual theory form

$$(1,1),\ (1,(N_f-N)^2-1),\ (N,(\bar{N}_f-\bar{N})),\ \text{and}\ (\bar{N},(N_f-N))$$

representations of (3.30). We see that the adjoint representations of the $(C+F)$ subgroup are different in two theories. How can this happen?

The quarks and gauge bosons which form the adjoint (N^2-1) representation of $\mathrm{SU}(N)$ at large ξ and the dyons and gauge bosons which form the adjoint $((N_f-N)^2-1)$ representation of $\mathrm{SU}(N_f-N)$ at small ξ are, in fact, *distinct* states. The (N^2-1) adjoints of $\mathrm{SU}(N)$ become heavy and decouple as we pass from large to small ξ along the line $\xi \sim \Lambda_{\mathcal{N}=2}$. Moreover, some composite $((N_f-N)^2-1)$ adjoints of $\mathrm{SU}(N_f-N)$, which are heavy and invisible in the low-energy description at large ξ become light at small ξ and form the D^{lK} dyons $(K=N+1,...,N_f)$ and gauge bosons of $\mathrm{U}(N_f-N)$. The phenomenon of the level crossing takes place. Although this crossover is smooth in the full theory, from the standpoint of the low-energy description the passage from large to small ξ means a dramatic change: the low-energy theories in these domains are completely different; in particular, the degrees of freedom in these theories are different.

This logic leads us to the following conclusion [8]. In addition to light dyons and gauge bosons included in the low-energy theory at small ξ we must have heavy fields which form the adjoint representation $(N^2-1, 1)$ of the global symmetry (3.30). These are screened quarks and gauge bosons from the large-ξ domain.

As has been already noted in Sec. 1, at small ξ they decay into the monopole-antimonopole pairs on the curves of marginal stability (CMS).[5] This is in accordance with the results obtained in [2, 3, 34] for $\mathcal{N}=2$ SU(2) gauge theories, on the Coulomb branch at vanishing ξ. For the theory at hand this picture was established in [19]. The general rule is that the only states that exist at strong coupling inside CMS are those which can become massless on the Coulomb branch [2, 3, 34]. For our theory these are light dyons shown in Eq. (3.17), gauge bosons of the dual gauge group and monopoles.

At small nonvanishing values of ξ the monopoles and antimonopoles produced in the decay process of the adjoint $(N^2-1, 1)$ states cannot escape from each other and fly off to asymptotically large separations because they are confined. Therefore, the (screened) quarks or gauge bosons evolve into stringy mesons in the strong coupling domain of small ξ – the monopole-antimonopole pairs connected by two strings [8, 9], as shown in Fig. 1. This is what we call "instead-of-confinement" mechanism for quarks and gauge bosons.

3.4 $r = N$ Duality at large μ

From Eqs. (3.17), (3.22) and (3.28) we see that the VEVs of the non-Abelian dyons D^{lA} are determined by $\sqrt{\mu m}$ and are much smaller than the VEVs of the Abelian dyons D^J in the domain (3.15). The latter are of the order of $\sqrt{\mu \Lambda_{\mathcal{N}=2}}$. This circumstance is most crucial for us. It allows us to increase μ and decouple the adjoint fields without spoiling the weak coupling condition in the dual theory [9].

Now we assume that

$$|\mu| \gg |m_A|, \qquad A = 1, ..., N_f\,. \tag{3.31}$$

The VEVs of the Abelian dyons become large at large μ. This makes U(1) gauge fields of the dual group (3.16) heavy. Decoupling these gauge factors, together with the adjoint matter and the Abelian dyons themselves, we obtain the low-energy theory with the

$$U(N_f - N) \tag{3.32}$$

gauge fields and the non-Abelian dyons D^{lA} $(l = 1, ..., N_f - N,\ A = 1, ..., N_f)$. For the

[5] An explanatory remark regarding our terminology is in order. Strictly speaking, such pairs can be formed by monopole-antidyons and dyon-antidyons as well, the dyons carrying root-like electric charges. In this paper we refer to all such states collectively as to "monopoles." This is to avoid confusion with dyons which appear in Eq. (3.17). The latter dyons carry weight-like electric charges and, roughly speaking, behave as quarks, see [8] for further details.

single-trace perturbation (2.2) with $\gamma = 0$ the superpotential for D^{lA} has the form [9]

$$\mathcal{W} = -\frac{1}{2\mu}\,(\tilde{D}_A D^B)(\tilde{D}_B D^A) + m_A\,(\tilde{D}_A D^A)\,, \tag{3.33}$$

where the color indices are contracted inside each parentheses.

The minimization of this superpotential leads to the dyon VEVs,

$$\langle D^{lA}\rangle = \langle \bar{\tilde{D}}^{lA}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \dots & 0 & \sqrt{\xi_1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & \sqrt{\xi_{(N_f-N)}} \end{pmatrix}, \tag{3.34}$$

where those ξ's that enter Eq. (3.34) (cf. Eq. (3.17)) are of the order of μm, see (3.28). Other ξ's (see Eq. (3.18)) become irrelevant, since all U(1) gauge fields become heavy at large μ and decouple.

Below the scale μ our theory becomes dual to $\mathcal{N} = 1$ SQCD with the scale

$$\tilde{\Lambda}^{3N-2N_f} = \frac{\Lambda_{\mathcal{N}=2}^{2N-N_f}}{\mu^{N_f-N}}\,. \tag{3.35}$$

The only condition we impose to keep this infrared-free theory in the weak coupling regime is

$$|\sqrt{\mu m}| \ll \tilde{\Lambda}\,. \tag{3.36}$$

This means that at large μ we must keep the quark masses sufficiently small.

We would like to stress that if VEV's of dyons were all of order of $\sqrt{\mu\Lambda_{\mathcal{N}=2}}$, it would not be possible to decouple the adjoint matter keeping the dual theory at weak coupling. Once we increased μ above the scale $\sqrt{\mu\Lambda_{\mathcal{N}=2}}$, we would get that these VEVs are much larger than $\tilde{\Lambda}$, which breaks the weak coupling condition in the dual theory. Thus, the non-Abelian structure present in the dual theory is the most important element of the continuation to large μ.

To summarize, at large μ and small ξ the original $\mathcal{N} = 1$ SQCD in the $r = N$ vacuum goes through a crossover transition at strong coupling. In the domain (3.36) it is described by the weakly coupled infrared-free dual theory, U($N_f - N$) SQCD, with N_f light dyon flavors. Condensation of the light dyons D^{lA} in this theory triggers formation of the non-Abelian strings and confinement of monopoles. For quarks and gauge bosons of the original $\mathcal{N} = 1$ SQCD we have an "instead-of-confinement" phase: they decay into the monopole-antimonopole pairs on CMS and form stringy mesons shown in Fig. 1.

4 r-Duality

Now we are finally ready to turn to the main topic of this paper – the study of the $r < N$ vacua. First we consider the small-μ domain in which the theory is close to the $\mathcal{N} = 2$ limit.

Our task is to analyze the transition from large to small ξ. In much the same way as in [8] we will do this in two steps. First, we will assume the quark mass differences to be large. In this domain the theory stays at weak coupling, and we can safely decrease the value of the parameter ξ. Next, we will use the exact Seiberg–Witten solution of the theory on the Coulomb branch [2, 3] (i.e. at $\xi \to 0$) to perform the passage from the domain of the large quark mass differences to the domain of the small quark mass differences.

With large mass differences, the quark sector of the theory in the r-vacuum is at weak coupling and can be analyzed semiclassically. The number of the r-vacua with $r < N$ in our theory is [29]

$$(N-r)\, C^r_{N_f} = (N-r)\,\frac{N_f!}{r!(N_f-r)!}\,, \tag{4.1}$$

i.e. is equal to the number of choices one can pick up r quarks which develop VEVs (out of N_f quarks) times the Witten index (the number of vacua) in the classically unbroken $\mathrm{SU}(N-r)$ pure gauge theory.

Below we consider a particular vacuum where the *first* r quarks develop VEVs (cf. Sec. 3), to be labeled by $(1, ..., r)$. Quasiclassically at large mass differences the VEVs of the adjoint scalars are given by

$$\left\langle \mathrm{diag}\left(\frac{1}{2}\,a + T^a\,a^a\right)\right\rangle \approx -\frac{1}{\sqrt{2}}\,[m_1, ..., m_r, 0, ..., 0]\,, \tag{4.2}$$

where the first r diagonal elements are proportional to the quark masses, while the last $(N-r)$ entries classically vanish. In quantum theory they become of order of $\Lambda_{\mathcal{N}=2}$.

Now we have to identify this vacuum in terms of the Seiberg–Witten curve. In our theory (2.4) it has the form [21]

$$y^2 = \prod_{k=1}^{N}(x-\phi_k)^2 - 4\left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}}\right)^{2N-N_f}\,\prod_{A=1}^{N_f}\left(x+\frac{m_A}{\sqrt{2}}\right), \tag{4.3}$$

where ϕ_k are gauge invariant parameters on the Coulomb branch. Semiclassically,

$$\mathrm{diag}\left(\frac{1}{2}\,a + T^a\,a^a\right) \approx [\phi_1, ..., \phi_N]\,. \tag{4.4}$$

Therefore, in the $(1, ..., r)$ quark vacuum we have

$$\phi_P \approx -\frac{m_P}{\sqrt{2}}, \quad P = 1, ..., r\,, \qquad \phi_P \sim \Lambda_{\mathcal{N}=2}, \quad P = r+1, ..., N \tag{4.5}$$

in the large m_A limit, see (4.2).

To identify the $r < N$ vacuum in terms of the curve (4.3) it is necessary to find such values of ϕ_P which would ensure the curve to have $N-1$ double roots. r parameters ϕ_P's are determined by the quark masses in the semiclassical limit, see (4.5). $N-1$ double roots are associated with r condensed quarks and $N-r-1$ condensed monopoles. Altogether, $N-1$ condensed states.

In contrast, in the $r=N$ vacuum we have the maximal possible number of the condensed states (quarks), namely, N in U(N) theory. This difference is related to the presence of the unbroken U(1) gauge group in the $r<N$ vacua [24]. In the classically unbroken (after quark condensation) U($N-r$) gauge group $N-r-1$ monopoles condense at the quantum level, thus breaking the non-Abelian SU($N-r$) subgroup. One U(1) factor remains unbroken because the monopoles do not interact with it.

Now we pass to the limit of the equal quark masses (3.5) and address the following question. What is the maximal number of ϕ's which are determined by the quark masses exactly, without $\Lambda_{\mathcal{N}=2}$ corrections? Let us denote this number by ν. Let us rewrite the curve (4.3) as

$$y^2 = \left(x+\frac{m}{\sqrt{2}}\right)^{2\nu} \times \left\{\prod_{k=\nu+1}^{N}(x-\phi_k)^2 - 4\left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}}\right)^{2N-N_f}\left(x+\frac{m}{\sqrt{2}}\right)^{N_f-2\nu}\right\}, \tag{4.6}$$

where the first ν ϕ's are given by

$$\phi_P = -\frac{m}{\sqrt{2}}, \qquad P=1,...,\nu\,. \tag{4.7}$$

This curve has ν double roots located at

$$e_P = -\frac{m}{\sqrt{2}}, \qquad P=1,...,\,\nu\,. \tag{4.8}$$

Now, the reduced curve in the curly brackets has $(N-\nu)$ colors and $(N_f-2\nu)$ flavors. If the maximal number of quarks (all of them) condense in this reduced theory, the rank of the classically unbroken gauge group would be $(N-\nu)-(N_f-2\nu)$. This number should be equal to the rank of the classically unbroken group in the r-vacuum of the full theory. This gives

$$(N-\nu)-(N_f-2\nu) = N-r\,, \tag{4.9}$$

which entails

$$\nu = N_f - r. \tag{4.10}$$

Note, that the number of flavors in the reduced curve should be, of course, non-negative. This gives $N_f-2\nu \geq 0$ or

$$r \geq N_f/2\,.$$

For smaller r it is obvious that $\nu=r$. Thus, we arrive at

$$\nu = \begin{cases} r, & r\leq \frac{N_f}{2} \\ N_f-r, & r>\frac{N_f}{2} \end{cases}\,. \tag{4.11}$$

The main feature of the solution (4.7) is the absence of $O(\Lambda_{\mathcal{N}=2})$ corrections to the first ν ϕ's. This means that in the equal mass limit these ν ϕ's become equal. This is a signal of restoration of the non-Abelian SU(ν) gauge group, i.e. the gauge group of the dual theory at small ξ.

Namely, the dual gauge group in the equal mass limit becomes

$$\mathrm{U}(\nu) \times \mathrm{U}(1)^{N-\nu}. \tag{4.12}$$

This is in perfect agreement with the results obtained in [21, 29] where non-Abelian gauge groups were identified at the roots of the nonbaryonic Higgs branches in the SU(N) gauge theory with the massless quarks.

The novel element of our analysis presented in this section is that we started from the non-Abelian r-vacuum at large ξ and demonstrated that, as we reduce ξ, the theory in this vacuum undergoes crossover to a different non-Abelian regime, with the dual low-energy gauge group (4.12). As was already mentioned, the physical reason for the emergence of the non-Abelian gauge group is that the low-energy effective theory with the dual gauge group (4.12) is infrared-free in the equal mass limit and stays at weak coupling. Therefore, the classical analysis showing that the non-Abelian gauge group is restored in the equal mass limit remains intact in quantum theory.

As was already mentioned, we interpret (4.11) as a crossover transition with respect to the parameter ξ. If $r > N_f/2$ the rank of the dual non-Abelian gauge subgroup SU(ν) at small ξ is different from the rank of the original non-Abelian subgroup SU(r). This difference imply a "vacuum leap" (see Secs. 3.2 and 6.2) and occurrence of "instead-of-confinement" mechanism.

For $r < N_f/2$ there is no crossover.

5 $r = N - 1$ vacuum at large ξ

Our main example of the r vacuum in this paper is

$$r = N - 1\,, \tag{5.1}$$

in the theory (2.4). We will use the same strategy as for the study of the $r = N$ vacuum: first assume that μ is small and the theory is close to the $\mathcal{N} = 2$ limit, so we can use the exact Seiberg–Witten solution valid near the Coulomb branch. We will study the crossover from the large-ξ domain where the low-energy gauge group is

$$\mathrm{U}(r = N - 1) \times \mathrm{U}(1)^{\mathrm{unbr}} \tag{5.2}$$

to the small-ξ domain where the dual theory has the gauge group

$$\mathrm{U}(\nu = N_f - N - 1) \times \mathrm{U}(1)^{N-\nu-1} \times \mathrm{U}(1)^{\mathrm{unbr}}. \tag{5.3}$$

At the last stage we will increase μ thus decoupling the adjoint matter.

Although in this paper we mostly consider the $r = (N-1)$ vacuum as a particular example of $r < N$ vacua in the theory (2.4), we believe that our results are general and can be applied to all r vacua.

We also note, that while we keep μ small to ensure the proximity of the theory at hand to the $\mathcal{N} = 2$ limit, we need a weaker condition to have a crossover into strong coupling, namely $r > N_f/2$, see (4.11). At the last stage, in Sec. 7, we make μ large and assume that $r > \frac{2}{3} N_f$ in order to keep the dual $\mathcal{N} = 1$ theory infrared free.

5.1 Low-energy theory

The low-energy theory in the $r = N-1$ vacuum at large ξ is presented in Appendix A. It includes non-Abelian gauge fields A^n_μ $(n = 1, ..., r^2 - 1)$ as well as Abelian fields A_μ and $A^{N^2-1}_\mu$. The last one is associated with the last Cartan generator of the SU(N) group. These fields have scalar $\mathcal{N} = 2$ superpartners a^n, a and a^{N^2-1}. Light matter consists of quarks q^{kA} $(k = 1, ..., r)$. Note, that all non-Abelian gauge fields from the sector SU(N)/SU(r) are heavy and decouple in the large mass limit due to the structure of the adjoint VEV's (4.2). Also q^{NA} quarks are heavy and not included in the low-energy theory.

The potential (A.3) determines the vacuum structure in the $r = N-1$ vacuum. The adjoint VEV's have the form

$$\langle \mathrm{diag}\,(\Phi) \rangle \approx -\frac{1}{\sqrt{2}}\,[m_1, ..., m_{N-1}, 0]\,, \tag{5.4}$$

while the (s)quark VEV's are

$$\begin{aligned} \langle q^{kA} \rangle &= \langle \bar{\tilde{q}}^{kA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1} & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \sqrt{\xi_{(N-1)}} & 0 & \ldots & 0 \end{pmatrix}, \\ k &= 1, ..., (N-1)\,, \qquad A = 1, ..., N_f\,, \end{aligned} \tag{5.5}$$

where now the first $(N-1)$ parameters ξ are given quasiclassically by (3.7) while

$$\xi_N = 0\,. \tag{5.6}$$

The last condition reflects the fact that the N-th quark is heavy and develops no VEV.

To see that this is the case we can use the general formula (3.22) for ξ's where the quasiclassical expression for the matrix E reduces to

$$\mathrm{diag}\,(E) \approx \langle \mathrm{diag}\,(\Phi) \rangle \approx -\frac{1}{\sqrt{2}}\,[m_1, ..., m_{N-1}, 0]\,. \tag{5.7}$$

As is seen from Eq. (A.2), the quarks interact with a particular linear combination of the U(1) gauge fields A_μ and $A_\mu^{N^2-1}$, namely,

$$A_\mu + \sqrt{\frac{2}{N(N-1)}}\, A_\mu^{N^2-1}. \tag{5.8}$$

The quark VEVs make this combination massive. The orthogonal combination

$$\sqrt{\frac{2}{N(N-1)}}\, A_\mu - A_\mu^{N^2-1}. \tag{5.9}$$

remains massless and corresponds to the unbroken $\mathrm{U}(1)^{\mathrm{unbr}}$ gauge group.

In the equal mass limit the global flavor symmetry $\mathrm{SU}(N_f)$ is broken in the r vacuum down to

$$\mathrm{SU}(r)_{C+F} \times \mathrm{SU}(\nu = N_f - r)_F \times \mathrm{U}(1)\,. \tag{5.10}$$

Now $\mathrm{SU}(r)_{C+F}$ is a global unbroken color-flavor rotation, which involves only the first r flavors, while the $\mathrm{SU}(\nu = N_f - r)_F$ factor stands for the flavor rotation of the remainder of the quark sector.

Since the global (flavor) $\mathrm{SU}(N_f)$ group is broken by the quark VEVs anyway, it is useful to consider the following mass splitting:

$$m_P = m_{P'}, \qquad m_K = m_{K'}, \qquad m_P - m_K = \Delta m,$$
$$P, P' = 1, ..., r \;\text{ and }\; K, K' = r+1, ..., N_f\,. \tag{5.11}$$

This mass splitting respects the global group (5.10) in the $(1, 2, ..., r)$ vacuum. This vacuum becomes isolated.

In much the same way as in the $r = N$ vacuum, in the $r = N - 1$ vacuum all states in the limit (5.11) come in representations of the unbroken global group (5.10), namely, in the singlet and adjoint representations of $\mathrm{SU}(r)_{C+F}$,

$$(1,\, 1), \quad (r^2 - 1,\, 1), \tag{5.12}$$

and in the bifundamental representations

$$(\bar{r},\, \nu), \quad (r,\, \bar{\nu})\,. \tag{5.13}$$

We mark representations in (3.12) and (3.13) with respect to two non-Abelian factors in (5.10). The singlet and adjoint fields are the gauge bosons, and the first r flavors of the quarks q^{kP} $(P = 1, ..., r)$. The bifundamental fields are the quarks q^{kK} with $K = r+1, ..., N_f$. Singlet and adjoint fields have masses of order $g\sqrt{\xi}$, where ξ is the common value of the first r parameters ξ in the limit (5.11), while the bifundamental field masses are equal to Δm.

The above quasiclassical analysis applies provided that the theory is at weak coupling. The weak coupling condition is

$$|\sqrt{\xi}| \sim |\sqrt{\mu m}| \gg \Lambda^{\rm LE}_{\mathcal{N}=2}\,, \tag{5.14}$$

where $\Lambda^{\rm LE}_{\mathcal{N}=2}$ is the scale of the low-energy theory (A.1) determined by

$$\Lambda^{2N-N_f}_{\mathcal{N}=2} = m^2\,(\Lambda^{\rm LE}_{\mathcal{N}=2})^{2(N-1)-N_f}\,. \tag{5.15}$$

5.2 Strings and confinement of monopoles at large ξ

As quarks develop VEVs in the $r = N-1$ vacuum the monopoles should be confined, in much the same way as they are in the $r = N$ vacuum. As was already mentioned, the distinction is that a single U(1) factor of the gauge group remains unbroken; therefore the associated magnetic flux should be unconfined. In this section we will determine the elementary string fluxes in the classical limit at large ξ to show that the elementary monopole fluxes can be absorbed by two strings. Hence, the monopoles are indeed represented by the junctions of two different strings. The exceptions are the monopoles M_{PN} ($P = 1, ..., r$) interpolating between an P-th elementary string and the N-th would-be string (which is in fact absent).

To make our discussion simpler we will consider here (and, often, below) the theory with U($N = 4$) gauge group and $N_f = 5$ as an example,

$$N = 4\,, \qquad N_f = 5\,, \qquad r = 3, \qquad \nu = 2\,. \tag{5.16}$$

In this case the low-energy theory (A.1) has the gauge group U(3)$\times$ U(1)$_{15}$, where U(1)$_{15}$ describes the gauge field $A^{N^2-1}_\mu$ with $N = 4$.

If the quark masses are unequal, the U(3) gauge group is broken down to U(1)^3 and the non-Abelian strings become $Z_{N=4}$ Abelian strings, see [17] for more details.[6] Let us calculate their fluxes. Charges of three quarks q^{kA}, $k = 1, 2, 3$ in (A.1) can be written as

$$\vec{n}_{q^1} = \left(\frac{1}{2}, 0;\ \frac{1}{2}, 0;\ \frac{1}{2\sqrt{3}}, 0;\ \frac{1}{2\sqrt{6}}, 0\right),$$

$$\vec{n}_{q^2} = \left(\frac{1}{2}, 0;\ -\frac{1}{2}, 0;\ \frac{1}{2\sqrt{3}}, 0;\ \frac{1}{2\sqrt{6}}, 0\right),$$

$$\vec{n}_{q^3} = \left(\frac{1}{2}, 0;\ 0, 0;\ -\frac{1}{\sqrt{3}}, 0;\ \frac{1}{2\sqrt{6}}, 0\right), \tag{5.17}$$

respectively, where we use the notation

$$\vec{n} = \left(n_e, n_m;\ n^3_e, n^3_m;\ n^8_e, n^8_m;\ n^{15}_e, n^{15}_m\right), \tag{5.18}$$

[6] One of these strings is absent in the $r = 3$ vacuum.

and n_e and n_m denote electric and magnetic charges of a given state with respect to the U(1) gauge group, while n_e^3, n_m^3, n_e^8, n_e^8 and n_e^{15}, n_e^{15} stand for the electric and magnetic charges with respect to the Cartan generators of the SU(4) gauge group (broken down to $U(1)^3$ by quark mass differences). In Appendix B for convenience we present weights and roots of the SU(4) algebra. Quark charges correspond to the weights of this algebra. Note, that the 4-th quark is heavy and does not enter in the low-energy theory (A.1).

Consider one of the Z_4 strings which is formed due to the winding of the q^{11} quark at $r \to \infty$ (see [17, 8] for a more detailed discussion of the construction of the non-Abelian strings),

$$q^{11} \sim \sqrt{\frac{\xi_1}{2}}\, e^{i\alpha}, \qquad q^{22} \sim \sqrt{\frac{\xi_2}{2}}, \qquad q^{33} \sim \sqrt{\frac{\xi_3}{2}}, \tag{5.19}$$

see (5.5). Here r and α are the polar coordinates in the plane $i = 1, 2$ orthogonal to the string axis. Note that in the $r = N = 4$ vacuum there is one extra condition associated with the fourth quark [8]. In the $r = 3$ vacuum this condition is absent. Equations (5.19) imply the following behavior of the gauge potentials at $r \to \infty$:

$$\begin{aligned}
&\frac{1}{2}A_i + \frac{1}{2}A_i^3 + \frac{1}{2\sqrt{3}}A_i^8 + \frac{1}{2\sqrt{6}}A_i^{15} \sim \partial_i\alpha\,,\\
&\frac{1}{2}A_i - \frac{1}{2}A_i^3 + \frac{1}{2\sqrt{3}}A_i^8 + \frac{1}{2\sqrt{6}}A_i^{15} \sim 0\,,\\
&\frac{1}{2}A_i - \frac{1}{\sqrt{3}}A_i^8 + \frac{1}{2\sqrt{6}}A_i^{15} \sim 0\,,
\end{aligned} \tag{5.20}$$

see the quark charges in (5.17). In the $r = 3$ vacuum we have to supplement these conditions with one extra condition which ensures that the combination (5.9) of the gauge potentials A_μ and A_μ^{15}, which has no interaction with quarks, is not excited, namely,

$$\frac{1}{\sqrt{6}}\,A_i - A_i^{15} \sim 0\,. \tag{5.21}$$

The solution to equations (5.20) is

$$\begin{aligned}
&A_i \sim \frac{4}{7}\,\partial_i\alpha\,, \qquad A_i^3 \sim \partial_i\alpha\,,\\
&A_i^8 \sim \frac{1}{\sqrt{3}}\,\partial_i\alpha\,, \qquad A_i^{15} \sim \frac{4}{7\sqrt{6}}\,\partial_i\alpha\,.
\end{aligned} \tag{5.22}$$

It determines the string gauge fluxes $\int dx_i A_i$, $\int dx_i A_i^3$, $\int dx_i A_i^8$ and $\int dx_i A_i^{15}$, respectively. The integration above is performed over a large circle in the $(1, 2)$ plane. Let us call this string S_1.

Next, we define the string charges [8] as

$$\begin{aligned}
&\int dx_i (A_i^D,\, A_i;\, A_i^{3D},\, A_i^3;\, A_i^{8D},\, A_i^8;\, A_i^{15D},\, A_i^{15})\\
&= 4\pi\,(-n_e,\, n_m;\, -n_e^3,\, n_m^3;\, -n_e^8,\, n_m^8;\, -n_e^{15},\, n_m^{15})\,.
\end{aligned} \tag{5.23}$$

This definition ensures that the string has the same charge as a trial monopole which can be attached to the string endpoint. In other words, the flux of the given string is the flux of a trial monopole[7] sitting on string's end, with the charge defined by (5.23).

In particular, according to this definition, the charge of the string with the fluxes (5.22) is

$$\vec{n}_{S_1} = \left(0,\ \frac{2}{7};\ 0,\ \frac{1}{2};\ 0,\ \frac{1}{2\sqrt{3}};\ 0,\ \frac{2}{7\sqrt{6}}\right). \tag{5.24}$$

Since this string is formed through the quark condensation, it is magnetic.

There are two other elementary strings S_2 and S_3 which arise due to winding of q^{22} and q^{33} quarks, respectively. Repeating the above procedure for these strings we get their charges,

$$\vec{n}_{S_2} = \left(0,\ \frac{2}{7};\ 0,\ -\frac{1}{2};\ 0,\ \frac{1}{2\sqrt{3}};\ 0,\ \frac{2}{7\sqrt{6}}\right),$$

$$\vec{n}_{S_3} = \left(0,\ \frac{2}{7};\ 0,\ 0;\ 0,\ -\frac{1}{\sqrt{3}};\ 0,\ \frac{2}{7\sqrt{6}}\right). \tag{5.25}$$

Note, that the fourth string S_4 of the U(4) gauge group is absent in the $r = 3$ vacuum since the fourth quark is heavy, have no VEV and, therefore, can have no winding.

It is easy to check that each of the three elementary SU(4) monopoles associated with first three roots of the SU(4) algebra (see Appendix B) is confined by two elementary strings. Consider, say, two elementary monopoles from the SU($r = 3$) subgroup with the charges $\vec{n}_{M_{12}} = (0,\ 0;\ 0,\ 1;\ 0,\ 0;\ 0,\ 0)$ and $\vec{n}_{M_{23}} = (0,\ 0;\ 0,\ -\frac{1}{2};\ 0,\ \frac{\sqrt{3}}{2};\ 0,\ 0)$. These charges can be written as a difference of the charges of two elementary strings, namely,

$$\vec{n}_{M_{12}} = (0,\ 0;\ 0,\ 1;\ 0,\ 0;\ 0,\ 0) = \vec{n}_{S_1} - \vec{n}_{S_2}\,,$$

$$\vec{n}_{M_{23}} = (0,\ 0;\ 0,\ -\frac{1}{2};\ 0,\ \frac{\sqrt{3}}{2};\ 0,\ 0) = \vec{n}_{S_2} - \vec{n}_{S_3}\,. \tag{5.26}$$

This means that each of these monopoles (at large ξ) is in fact a junction of two strings, with one string having the outgoing flux while the other incoming. The third M_{13} monopole from the SU($r = 3$) subgroup can be considered as a bound state of two elementary ones in (5.26).

So far the monopole confinement in the $r = N - 1$ vacuum looks quite similar to that in the $r = N$ vacuum [8]. The distinction becomes apparent once we consider the SU($N = 4$) monopole which does not belong to the SU($r = 3$) subgroup. Let us consider the M_{34} monopole with charges

$$\vec{n}_{M_{34}} = \left(0,\ 0;\ 0,\ 0;\ 0,\ -\frac{1}{\sqrt{3}};\ 0,\ \sqrt{\frac{2}{3}}\right). \tag{5.27}$$

[7]This trial monopole does not necessarily exist in our theory. In the U(N) theories the SU(N) monopoles are rather string junctions, so they are attached to *two* strings, [13, 8].

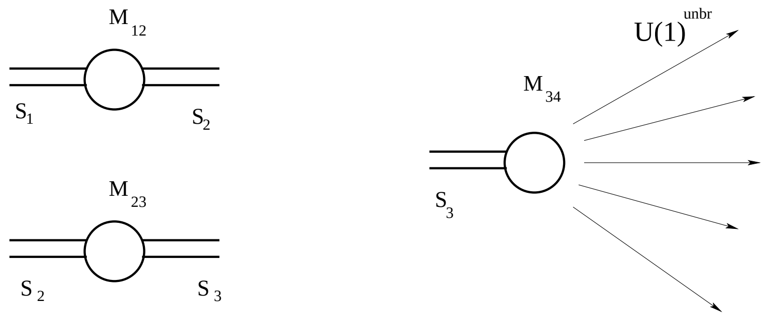

Figure 3: The monopole confinement in the $r = 3$ vacuum. The thick double lines denote strings, while the circles denote monopoles. Unconfined U(1) flux is shown by arrows.

In the $r = 4$ vacuum this monopole is a junction of two strings S_3 and S_4. In the $r = 3$ vacuum the S_4 string is absent. Let us calculate the unconfined flux of the S_3 string with the monopole M_{34} attached to its end. To this end consider the difference

$$\vec{n}_{\rm unconf} = \vec{n}_{S_3} - \vec{n}_{M_{34}} = \frac{2\sqrt{6}}{7}\left(0,\, \frac{1}{\sqrt{6}};\, 0,\, 0;\, 0,\, 0;\, 0,\, -1\right). \tag{5.28}$$

We see that the n_m^8 charge is cancelled, and the resulting charge is a source of the U(1) gauge magnetic field corresponding to the following combination:

$$\frac{1}{\sqrt{6}}\, A_\mu - A_\mu^{15}, \tag{5.29}$$

This is exactly the field of the unbroken $\mathrm{U}(1)^{\rm unbr}$ gauge group, see (5.9).

Thus, the S_3 string can terminate on the M_{34} monopole producing a magnetic source for the unbroken $\mathrm{U}(1)^{\rm unbr}$ gauge field. All other monopole fluxes, in particular, all non-Abelian fluxes from the SU(3) subgroup, are absorbed and squeezed in the confining strings S_1, S_2 and S_3.

The picture of the monopole confinement in the $r = 3$ vacuum is shown in Fig. 3.

To conclude this section let us determine the tensions of three elementary strings in the $r = 3$ vacuum. To the leading order in μ, close to the $\mathcal{N} = 2$ limit, these strings are BPS saturated. The Bogomol'nyi representation for non-Abelian strings stabilized by the Fayet–Iliopoulos F-term is considered in [33]. The boundary terms in this representation determine the string tensions,

$$T = \mathrm{Tr}\left\{ \left(\begin{array}{ccc} \xi_1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \xi_N \end{array} \right) \int dx_i \left(\frac{1}{2} A_i + T^a A_i^a \right) \right\}. \tag{5.30}$$

The first diagonal matrix here is associated with quark condensates determined by ξ's, while the second matrix linear in A's represent the string flux. This formula is quite general and applies to any vacuum. Say, in the $r = N$ vacuum the fluxes of the elementary Z_N strings are [17, 33]

$$\int dx_i \,\mathrm{diag}\left(\frac{1}{2}A_i + T^a A_i^a\right)_{S_P} = 2\pi \ (0, ..., 1, 0, ..., 0) \tag{5.31}$$

with the only nonvanishing element located at the P-th position, $P = 1, ..., N$. This implies the result [33] for the tension of the P-th string quoted in (3.9).

In the $r = 3$ vacuum at hand the string fluxes are determined by Eqs. (5.24), (5.25). Thus, we have

$$\int dx_i \,\mathrm{diag}\left(\frac{1}{2}A_i + T^a A_i^a\right)_{S_{1,2,3}} = 2\pi \,\mathrm{diag}\left\{\left(1, 0, 0, \frac{1}{7}\right),\right.$$
$$\left.\left(0, 1, 0, \frac{1}{7}\right), \quad \left(0, 0, 1, \frac{1}{7}\right)\right\}. \tag{5.32}$$

This gives the tensions for three elementary strings

$$T_{S_{1,2,3}} = 2\pi \,\xi_{1,2,3}. \tag{5.33}$$

Note, that the last (nonvanishing) element in (5.32) (i.e. 1/7) does not contribute because of the condition $\xi_{N=4} = 0$.

We see that the string tensions in the $r = N - 1$ vacuum are still determined by nonvanishing ξ's, in much the same way as in the $r = N$ vacuum. In fact, we can fine-tune the quark masses in such a way that the $r = 3$ vacuum coalesces with the $r = 4$ vacuum (this amounts to taking $\xi_4 \to 0$). Then Eqs. (3.9) and (5.33) show continuity of string tensions.

6 Dual theory in the $r = N - 1$ vacuum

Now we will decrease the parameter ξ passing in the domain of small ξ. Then the original theory (A.1) finds itself in the strong coupling regime. As we already explained in Sec. 4 ,(see also [8]) in order to study the transition from large to small ξ we first assume the quark mass differences $\Delta m_{AB} = m_A - m_B$ to be large,

$$|\Delta m_{AB}| \gg \Lambda_{\mathcal{N}=2} \,.$$

In this domain the theory stays at weak coupling, and we can safely decrease the value of ξ.

Next, we use the exact Seiberg–Witten solution of the theory on the Coulomb branch [2, 3] to pass from the domain of the large quark mass differences to that with small quark mass differences,

$$|\Delta m_{AB}| \ll \Lambda_{\mathcal{N}=2} \,.$$

In doing so we keep the quark masses themselves large,

$$|m_A| \gg \Lambda_{\mathcal{N}=2} \,.$$

In this limit the non-Abelian subgroup of the low-energy gauge group is U($r = N-1$) at large ξ (see Sec. 5) and, therefore, the crossover to strong coupling as well as duality in the r-vacuum look very similar to the those in the $r = N$ vacuum in the U(N) theory studied in [8].

Summarizing, in this section we will assume the following conditions for the dual theory:

$$|\Delta m_{AB}| \ll \Lambda_{\mathcal{N}=2}, \qquad |m_A| \gg \Lambda_{\mathcal{N}=2}, \qquad |\xi_P| \ll \Lambda^2_{\mathcal{N}=2}, \qquad |\mu| \ll \Lambda_{\mathcal{N}=2}. \tag{6.1}$$

To be more precise, the Seiberg–Witten curve factorizes in the $r = N-1$ vacuum in the folowing way [35]:

$$\begin{aligned} y^2 &= \prod_{k=1}^{N}(x-\phi_k)^2 - 4\left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}}\right)^{2N-N_f} \prod_{A=1}^{N_f}\left(x+\frac{m_A}{\sqrt{2}}\right) \\ &= \prod_{P=1}^{N-1}(x-e_P)^2\,(x-e_N^+)(x-e_N^-)\,. \end{aligned} \tag{6.2}$$

It has $r = (N-1)$ double roots associated with the quark condensation, so that for the large mass differences e_P's are given by the mass parameters, $\sqrt{2}e_P \approx -m_P$ ($P = 1, ..., N-1$). The last two roots (and ϕ_N) are of order of $\Lambda_{\mathcal{N}=2}$. For single-trace deformation superpotential (2.2), with $\gamma = 0$, (see (2.3)) their sum vanishes [35],

$$e_N^+ + e_N^- = 0\,. \tag{6.3}$$

This condition is equivalent to the following physical condition:

$$\xi_N = -2\sqrt{2}\mu\, E_N = 0, \tag{6.4}$$

which is valid because the N-th quark is heavy; therefore, it develops no VEV. We already obtained this condition in the classical limit, (see (5.6)). Below we will see that it is satisfied also in the quantum theory. The root e_N^+ determines the value of the gaugino condensate [24].

Once $\Delta m_{AB} \ll \Lambda_{\mathcal{N}=2}$ (while $m_A \approx m \gg \Lambda_{\mathcal{N}=2}$) x is close to $-m/\sqrt{2}$, if we are interested in double roots of the curve. Then the curve can be approximately written as

$$\begin{aligned} y^2 &\approx \left(\frac{m}{\sqrt{2}}\right)^2 \left\{\prod_{k=1}^{r}(x-\phi_k)^2 - 4\left(\frac{\Lambda^{\rm LE}_{\mathcal{N}=2}}{\sqrt{2}}\right)^{2r-N_f} \prod_{A=1}^{N_f}\left(x+\frac{m_A}{\sqrt{2}}\right)\right\} \\ &\approx \left(\frac{m}{\sqrt{2}}\right)^2 \prod_{P=1}^{r}(x-e_P)^2\,, \end{aligned} \tag{6.5}$$

where the parameter $\Lambda^{\rm LE}_{\mathcal{N}=2}$ is given in (5.15).

We see that the curve reduces to the curve for the r-vacuum in the U(r) theory. Now we use the results obtained in [8] where the transition to the strong coupling (small Δm_{AB}) was studied in this case.

To conclude this subsection we present, as an illustration, the ϕ values and roots of the curve (6.2) for the particular theory (5.16), in the limit of large masses (6.1). In this limit ϕ's are

$$\phi_{1,2} = -\frac{m_{1,2}}{\sqrt{2}}, \qquad \phi_3 \approx -\frac{1}{\sqrt{2}}(m_3 + \Lambda^{\rm LE}_{\mathcal{N}=2}), \qquad \phi_4 \approx 0, \tag{6.6}$$

while the roots have the form

$$e_{1,2} = -\frac{m_{1,2}}{\sqrt{2}}, \quad e_3 \approx -\frac{1}{\sqrt{2}}(m_3 - \Lambda^{\rm LE}_{\mathcal{N}=2}), \quad e_4^{\pm} \approx \pm\sqrt{2m_3\Lambda^{\rm LE}_{\mathcal{N}=2}}\,. \tag{6.7}$$

Here we assume for simplicity that $m_4 = m_1$ and $m_5 = m_2$, cf. [8]. We see that $e_{1,2}$ are exactly given by the masses (see Sec. 4), while e_4^{+} is much smaller than the double roots.

6.1 Monodromies

In this section we will study how quantum numbers of the massless quarks $q^{11}, ..., q^{rr}$ in the $(1, ..., r)$ vacuum change as we reduce Δm_{AB} to pass from weak coupling to the strong coupling domain along the Coulomb branch at $\xi = 0$.

To simplify our discussion we will consider a particular case (5.16) so that the dual group has the smallest nontrivial rank $\nu = 2$. We will consider the $(1, 2, 3)$ vacuum. The monodromies upon reducing the quark mass differences for the $(1, 2, 3)$ vacuum in the U(3) theory was studied in [8]. As was explained above, we can use these results for our $r = 3$ vacuum in the U(4) theory if we keep $m \gg \Lambda_{\mathcal{N}=2}$.

The quark quantum numbers change due to monodromies with respect to Δm_{AB}. The complex planes of Δm_{AB} have cuts, and when we cross these cuts, the a and a_D fields acquire monodromies; the quantum numbers of the corresponding states change accordingly. The method used in [8] to calculate the quark monodromies was the study of the Seiberg–Witten curve of the theory in the proximity of the Argyres–Douglas points [27] in Δm_{AB} variables. In these AD points our $(1, 2, 3)$ vacuum collides with the monopole singularities. There are two relevant AD points for the theory at hand [8]. The first one occurs at

$$\Delta m_{31} = \Lambda^{\rm LE}_{\mathcal{N}=2}, \qquad e_1 = e_3 = -\frac{m_1}{\sqrt{2}}\,, \tag{6.8}$$

where two double roots of the Seiberg–Witten curve (6.5) coincide, while the second is at

$$\Delta m_{32} = \Lambda^{\rm LE}_{\mathcal{N}=2}, \qquad e_2 = e_3 = -\frac{m_2}{\sqrt{2}}\,, \tag{6.9}$$

where the other two double roots coincide. In these AD points the monopoles M_{13} and M_{23}, *respectively*, become massless. In [8] it was shown that passing through these AD points the quarks pick up magnetic charges of the corresponding monopoles, while the monopoles do not change their charges. As a result, below the AD points the charges of the massless dyons are

$$\vec{n}_{D^1} = \left(\frac{1}{2}, 0;\ \frac{1}{2}, \frac{1}{2};\ \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2};\ \frac{1}{2\sqrt{6}}, 0\right),$$

$$\vec{n}_{D^2} = \left(\frac{1}{2}, 0;\ -\frac{1}{2}, -\frac{1}{2};\ \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2};\ \frac{1}{2\sqrt{6}}, 0\right),$$

$$\vec{n}_{D^3} = \left(\frac{1}{2}, 0;\ 0, 0;\ -\frac{1}{\sqrt{3}}, -\sqrt{3};\ \frac{1}{2\sqrt{6}}, 0\right), \qquad (6.10)$$

see (5.17) and (B.4). Here we adjust results of [8] taking into account the presence of the extra charge along T^{15} in the U(4) theory. This amounts to just adding the quark charges with respect to this Cartan generator in (6.10), since the M_{13} and M_{23} monopoles have no n_m^{15} charges, see (5.17) and (B.4).

Note, that as we decrease Δm_{AB} we do not encounter other AD points in which the M_{P4} monopoles ($P = 1, 2, 3$) become massless. To approach these points one has to reduce m_A (see (6.7)), but we keep $|m_A|$ large at the moment.

Two remarks are in order here. First, it is crucially important to note that the massless dyons D^1 and D^2 have both electric and magnetic charges 1/2 with respect to the T^3 generator of the dual U($\nu = 2$) gauge group. This means that they can fill the fundamental representation of this group. Moreover, all dyons D^{lA} ($l = 1, ..., \nu = 2$) can form color doublets. This is another confirmation of the conclusion made in Sect. 4, that the non-Abelian factor SU($\nu = 2$) of the dual gauge group gets restored in the equal mass limit.

A general reason ensuring that the D^{lA} ($l = 1, ..., \nu$) dyons fill the fundamental representation of the U(ν) group is as follows: due to monodromies the D^{lA} dyons pick up magnetic charges of particular monopoles of SU(r). The magnetic charges of these particular monopoles are represented by weights rather than roots of the U(ν) subgroup ($\pm 1/2$ for U($\nu = 2$), see (B.4)). This is related to the absence of the AD points associated with collisions of the first ν double roots, see (4.8). In other words, the dual SU(ν) theory is infrared-free and no monopole singularities occur in this subsector.

The second comment is that the dyon charges with respect to each U(1) generator are proportional to each other. This guarantees that these dyons are mutually local.

6.2 "Vacuum leap"

In this section we will present the low-energy dual theory for the $r = N - 1$ vacuum at small ξ. The gauge group of the theory is indicated in (4.12). One of the U(1) factors of this group corresponds to the unbroken U(1)$^{\text{unbr}}$. The light matter sector consists of dyons which carry weight-like electric charges as well as root-like magnetic charges. Non-Abelian dyons

D^{lA} ($l = 1, ..., \nu$, $A = 1, ..., N_f$) are in the fundamental representation of the SU(ν) dual gauge group. There are also dyon singlets D^J ($J = (\nu + 1), ..., r$) charged with respect to the U(1) factors of the dual gauge group. In the particular example (5.16), the dyon charges were calculated in Sec. 6.1. In this example we have a doublet of the non-Abelian dyons D^{lA} ($l = 1, 2$) plus one singlet dyon D^3. The action of the dual theory for this case is presented in Appendix C.

The potential of this theory determines the dyons VEV's. In the generic $r = N - 1$ vacuum we have

$$\langle D^{lA} \rangle = \langle \bar{\tilde{D}}^{lA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \dots & 0 & \sqrt{\xi_1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & \sqrt{\xi_\nu} \end{pmatrix},$$

$$\langle D^J \rangle = \langle \bar{\tilde{D}}^J \rangle = \sqrt{\frac{\xi_J}{2}}, \qquad J = (\nu + 1), ..., r\,. \tag{6.11}$$

In much the same way as in the $r = N$ vacuum the most important feature in (3.17) is a "vacuum leap" [8],

$$(1, ..., r)_{\sqrt{\xi} \gg \Lambda_{\mathcal{N}=2}} \to (r + 1, ..., N_f,\ (\nu + 1), ..., r)_{\sqrt{\xi} \ll \Lambda_{\mathcal{N}=2}}\,. \tag{6.12}$$

In other words, if we pick up the vacuum with nonvanishing VEVs of the first r quark flavors in the original theory at large ξ, and then reduce ξ below $\Lambda_{\mathcal{N}=2}$, the system will go through a crossover transition and end up in the vacuum of the dual theory with the nonvanishing VEVs of ν last dyons (plus VEVs of $(r - \nu)$ SU(ν) singlets).

The occurrence of this "vacuum leap" was demonstrated previously in [8] in a particular example of the $r = 3$ vacuum in the U(3) gauge theory with $N_f = 5$ flavors. This was done as follows. The curve (6.5) was studied with small mass differences Δm_{14} and Δm_{25}. It was shown that if at large $(m_3 - m_{1,2})$ $\phi_{1,2}$ and $e_{1,2}$ were approximately given by $-m_{1,2}/\sqrt{2}$, respectively, then at small $(m_3 - m_{1,2})$ they approach $-m_{4,5}/\sqrt{2}$.

The ξ_P parameters in (6.11) can be calculated from the potential (C.4), see also (C.5). It turns out that they are still determined by Eq. (3.22), in much the same way as in the $r = N$ vacuum, where the matrix E is given by (3.23). However, for the $r = N - 1$ vacuum the relation (3.24) between E_P and the roots of the Seiberg–Witten curve modifies. In Appendix D we consider the simplest example of the $r = 1$ vacuum in the U(2) gauge theory to find this relation. An obvious generalization of the result (D.10) is

$$E_P = \sqrt{(e_P - e_N^+)(e_P - e_N^-)}, \quad P = 1, ..., (N - 1), \quad E_N = 0\,, \tag{6.13}$$

which leads to our final expressions for the dyon VEVs in terms of the roots of the Seiberg–Witten curve,

$$\xi_P = -2\sqrt{2}\,\mu \sqrt{(e_P - e_N^+)(e_P - e_N^-)}, \quad P = 1, ..., (N - 1), \quad \xi_N = 0\,. \tag{6.14}$$

Note that, at small Δm_{AB}, in the domain (6.1), the first ν roots are determined by the masses of the last ν quarks,

$$\sqrt{2}e_I = -m_{I+r}, \qquad I = 1, ..., \nu \tag{6.15}$$

(up to small corrections of order of $\Delta m^2/\Lambda_{\mathcal{N}=2}$). This is because the non-Abelian sector of the dual theory is infrared-free and is at weak coupling in the domain (6.1). As long as we keep ξ_P small the dual theory is at weak coupling. For large masses (see (6.1)) this amounts to making μ sufficiently small.

6.3 "Instead-of-confinement" mechanism in the $r = (N-1)$ vacuum

The phenomenon of the "vacuum leap" ensures that we have "instead-of-confinement" mechanism for the quarks and gauge bosons in the $r = (N-1)$-vacuum, in much the same way as in the $r = N$ vacuum.

Indeed, consider the mass choice (5.11). Both, the gauge group and the global flavor $\mathrm{SU}(N_f)$ group, are broken in the vacuum. However, the color-flavor locked form of (6.11) shows that the unbroken global group of the dual theory is

$$\mathrm{SU}(r)_F \times \mathrm{SU}(\nu)_{C+F} \times \mathrm{U}(1)\,. \tag{6.16}$$

The $\mathrm{SU}(\nu)_{C+F}$ factor in (6.16) is a global unbroken color-flavor rotation, which involves the last ν flavors, while the $\mathrm{SU}(r)_F$ factor stands for the flavor rotation of the first r dyons.

In the equal mass limit, or given the mass choice (5.11), the global unbroken symmetry (6.16) of the dual theory at small ξ coincides with the global group (5.10) in the the original theory at large ξ. However, again this global symmetry is realized in two different ways in the dual pair at hand. The quarks and gauge bosons of the original theory at large ξ come in the $(1,1)$, $(r^2-1,1)$, $(\bar{r},\nu)$, and $(r,\bar{\nu})$ representations (see (5.12), (5.13)), while the dyons and $\mathrm{U}(\nu)$ gauge bosons form

$$(1,1), \qquad (1, \nu^2-1) \tag{6.17}$$

and

$$(r,\bar{\nu}), \qquad (\bar{r},\nu) \tag{6.18}$$

representations of (6.16). We see again that the adjoint representations of the $(C+F)$ subgroup are different in two theories.

This means that quarks and gauge bosons which form the adjoint (r^2-1) representation of $\mathrm{SU}(r)$ at large ξ and the dyons and gauge bosons which form the adjoint (ν^2-1) representation of $\mathrm{SU}(\nu)$ at small ξ are different states. What happens to quarks and gauge bosons at small ξ?

In much the same way as in the $r = N$ vacuum, the screened quarks and gauge bosons in the $r = (N-1)$ vacuum from the large-ξ domain decay in the monopole-antimonopole pairs on the CMS. As we will show in Sect. (6.4), at small nonvanishing ξ the monopoles and antimonopoles produced in the decay process of the adjoint $(r^2-1,1)$ states are confined. Therefore, the (screened) quarks or gauge bosons evolve into stringy mesons in the strong coupling domain of small ξ – the monopole-antimonopole pairs connected by two strings, as shown in Fig. 1. The difference with "instead-of-confinement" phase in the $r=N$ vacuum is that in the $r=(N-1)$ vacuum the strings can be broken by M_{PN}-monopole-antimonopole pairs (see the next subsection); here $P=1,...,r$. As a result, dipole stringy states emitting unbroken $\mathrm{U}(1)^{\mathrm{unbr}}$ magnetic gauge fields are formed, see Fig. 2. Non-Abelian SU(ν) fluxes are confined in these stringy dipoles.

Note, that in the large mass limit (6.1) the M_{PN} monopoles are very heavy, with masses of order of m/g_2^2; therefore, stringy mesons in Fig. 1 are almost stable.

6.4 Strings and monopole confinement in the dual theory

Now we will use the light dyon charges (6.10) to obtain the fluxes of the Z_4 strings in the dual theory and show that these strings still confine monopoles.

Consider the $\tilde{S}_1$ string arising due to winding of the D^{14} dyon. At $r\to\infty$ we have

$$
\begin{aligned}
D^{14}(r\to\infty) &\sim \sqrt{\frac{\xi_1}{2}}\,e^{i\alpha}, \qquad D^{25}(r\to\infty)\sim\sqrt{\frac{\xi_2}{2}},\\
D^{3}(r\to\infty) &\sim \sqrt{\frac{\xi_3}{2}},
\end{aligned}
\tag{6.19}
$$

see (6.11). Note again that the condition associated with the fourth dyon is absent in the $r=3$ vacuum. Taking into account the dyon charges in Eq. (6.10) we obtain the behavior of the gauge potentials at infinity,

$$
\begin{aligned}
&\frac{1}{2}A_i+\frac{1}{2}A_i^3+\frac{1}{2}A_i^{3D}+\frac{1}{2\sqrt{3}}A_i^8+\frac{\sqrt{3}}{2}A_i^{8D}+\frac{1}{2\sqrt{6}}A_i^{15}\sim\partial_i\alpha\,,\\
&\frac{1}{2}A_i-\frac{1}{2}A_i^3-\frac{1}{2}A_i^{3D}+\frac{1}{2\sqrt{3}}A_i^8+\frac{\sqrt{3}}{2}A_i^{8D}+\frac{1}{2\sqrt{6}}A_i^{15}\sim 0\,,\\
&\frac{1}{2}A_i-\frac{1}{\sqrt{3}}A_i^8-\sqrt{3}A_i^{8D}+\frac{1}{2\sqrt{6}}A_i^{15}\sim 0\,,
\end{aligned}
\tag{6.20}
$$

which, in turn, implies

$$
\begin{aligned}
&A_i+\frac{1}{\sqrt{6}}A_i^{15}\sim\frac{2}{3}\,\partial_i\alpha\,,\\
&\frac{1}{2}A_i^3+\frac{1}{2}A_i^{3D}\sim\frac{1}{2}\,\partial_i\alpha\,,\\
&\frac{1}{2\sqrt{3}}A_i^8+\frac{\sqrt{3}}{2}A_i^{8D}\sim\frac{1}{6}\,\partial_i\alpha\,.
\end{aligned}
\tag{6.21}
$$

The combinations orthogonal to those which appear in (6.21) are required to tend to zero at infinity, namely, $A_i^3 - A_i^{3D} \sim 0$, $A_i^{8D} - 3A_i^8 \sim 0$ and $A_i^{15D} \sim 0$. Also taking into account (5.21) which stays intact in the dual theory we get

$$
\begin{aligned}
& A_i \sim \frac{4}{7}\,\partial_i\alpha\,, \qquad A_i^D \sim 0\,, \\
& A_i^3 \sim \frac{1}{2}\,\partial_i\alpha\,, \qquad A_i^{3D} \sim \frac{1}{2}\,\partial_i\alpha\,, \\
& A_i^8 \sim \frac{1}{10\sqrt{3}}\,\partial_i\alpha\,, \qquad A_i^{8D} \sim \frac{\sqrt{3}}{10}\,\partial_i\alpha \\
& A_i^{15} \sim \frac{4}{7\sqrt{6}}\,\partial_i\alpha\,, \qquad A_i^{15D} \sim 0\,.
\end{aligned}
\tag{6.22}
$$

These expressions determine the charges of the $\tilde{S}_1$ string,

$$
\vec{n}_{\tilde{S}_1} = \left(0,\, \frac{2}{7};\, -\frac{1}{4},\, \frac{1}{4};\, -\frac{\sqrt{3}}{20},\, \frac{1}{20\sqrt{3}};\, 0,\, \frac{2}{7\sqrt{6}}\right). \tag{6.23}
$$

Paralleling the above analysis we determine the charges of the other two Z_4 strings which are due to windings of the fields D^{25} and D^3, respectively. We get

$$
\begin{aligned}
& \vec{n}_{\tilde{S}_2} = \left(0,\, \frac{1}{3};\, \frac{1}{4},\, -\frac{1}{4};\, -\frac{\sqrt{3}}{20},\, \frac{1}{20\sqrt{3}};\, 0,\, \frac{2}{7\sqrt{6}}\right), \\
& \vec{n}_{\tilde{S}_3} = \left(0,\, \frac{1}{3};\, 0,\, 0;\, \frac{\sqrt{3}}{10},\, -\frac{1}{10\sqrt{3}};\, 0,\, \frac{2}{7\sqrt{6}}\right).
\end{aligned}
\tag{6.24}
$$

Now we can check that each of three monopoles from the SU(3) subgroup of SU(4) can be confined by two strings. For the M_{12} and M_{23} monopoles we have

$$
\begin{aligned}
& \vec{n}_{M_{12}} = (\vec{n}_{\tilde{S}_1} - \vec{n}_{\tilde{S}_2}) + \frac{1}{2}\,(\vec{n}_{D^{14}} - \vec{n}_{D^{25}})\,, \\
& \vec{n}_{M_{23}} = (\vec{n}_{\tilde{S}_1} - \vec{n}_{\tilde{S}_2}) + \frac{3}{10}\,(\vec{n}_{D^{25}} - \vec{n}_{D^{3}}) - \frac{1}{10}\,(\vec{n}_{D^{14}} - \vec{n}_{D^{25}})\,,
\end{aligned}
\tag{6.25}
$$

where $\vec{n}_{D^{14}}$, $\vec{n}_{D^{25}}$ and $\vec{n}_{D^3}$ are charges of the condensed dyons given in (6.10). Only a part of the monopole flux is confined inside the strings. The remainder of its flux is screened by the condensate of the D^{14}, D^{25} and D^3 dyons.

We see that, although the quark charges change as we pass from the large-ξ domain to small-ξ, and the quarks turn into dyons, this does *not* happen with the monopoles. The monopole states do not change their charges. They are confined in both, strong and weak coupling domains, being represented by the junctions of two different elementary strings. In the strong coupling domain in the dual theory there is a peculiarity: not the entire monopole flux is carried by two attached strings; a part of it is screened by the dyon condensate.

Consider now the M_{P4} monopoles ($P = 1, 2, 3$). In much the same way as in the original theory (see Sec. 5.2), their fluxes in the dual theory are not completely confined in the $r = 3$ vacuum. Consider, say, the M_{34} monopole (see (5.27)) attached to the string $\tilde{S}_3$. In the $r = 3$ vacuum the $\tilde{S}_4$ string is absent due to the fact that $\xi_4 = 0$, and the flux of the above configuration is unconfined.

Let us calculate this unconfined flux. It is easy to check that

$$\begin{aligned} \vec{n}_{\rm unconf} &= \vec{n}_{\tilde{S}_3} - \vec{n}_{M_{34}} + \frac{1}{10}\,(2\vec{n}_{D^3} - \vec{n}_{D^{14}} - \vec{n}_{D^{25}}) \\ &= \frac{2\sqrt{6}}{7}\left(0,\, \frac{1}{\sqrt{6}};\, 0,\, 0;\, 0,\, 0;\, 0,\, -1\right). \end{aligned} \tag{6.26}$$

Here we add in the right-hand side a linear combination of the charges the of D^{14}, D^{25} and D^3 dyons. This linear combination is screened by their condensates. In much the same way as in the original theory, we see that the n_m^8 charge is canceled and the resulting charge is, in fact, a source for the U(1) gauge magnetic field exactly corresponding to the field of the unbroken $\mathrm{U}(1)^{\rm unbr}$ gauge group, see (5.29).

Thus, the $\tilde{S}_3$ string can terminate on the monopole M_{34} producing a magnetic source of the unbroken $\mathrm{U}(1)^{\rm unbr}$ gauge field. All other monopole fluxes are absorbed by confining the $\tilde{S}_1$, $\tilde{S}_2$ and $\tilde{S}_3$ strings. The picture of the monopole confinement in the $r = 3$ vacuum of the dual theory is qualitatively the same as that in the original theory, see Fig. 3. Basically, the only difference is the fact that now confined non-Abelian fluxes are associated with the dual gauge group SU($\nu = 2$), rather than with the original SU($r = 3$) group.

Note, that at large quark masses (see (6.1)) the M_{P4} monopole masses ($P = 1, 2, 3$) are very large; therefore, the $\tilde{S}_P$ strings are almost stable in this limit.

Note also, that, in much the same way as in the original theory, the tensions of $\tilde{S}_P$ strings are still given by Eq. (5.33), where the ξ_P parameters are determined by (6.14).

7 r-Duality at large μ

Now we are ready to increase μ and decouple the adjoint matter. Our theory (2.4) will flow to $\mathcal{N} = 1$ SQCD.

7.1 Moving to the Argyres–Douglas point

In order to keep our dual theory at weak coupling we need to keep the ξ parameters (at least ν of them) sufficiently small. At large μ this creates a problem. In the $r = N$ vacuum this problem was overcame in [9] by assuming the quark masses to be small. The ξ parameters in the $r = N$ vacuum are given by (3.27), while the first $(N_f - N)$ roots of the Seiberg–Witten curve are determined by the quark masses, with no $\Lambda_{\mathcal{N}=2}$-corrections, see (3.28). This allows us to increase μ thus decoupling the adjoint matter as well as the U(1) factors, while keeping the low-energy U$(N_f - N)$ gauge theory at week coupling.

Inspecting Eq. (6.14) we immediately see that this strategy does not work in the $r = N-1$ vacuum.

Although the first ν roots of the Seiberg–Witten curve are determined by the quark masses (see (6.15)), the last two undouble roots $e_N^{\pm}$ are of order of $\Lambda_{\mathcal{N}=2}$ at small masses. Therefore, at large μ the ξ parameters become large at small masses,

$$\sim \mu \Lambda_{\mathcal{N}=2} \,,$$

destroying the weak coupling condition.

Thus, in the $r < N$ vacua we need a different, novel strategy. Equation (6.14) shows that if we keep the mass differences very small and force the average value of the ν double roots (determined by the quark masses, that are almost equal) to lie in the proximity of one of the roots $e_N^{\pm}$, we make ν parameters ξ small. Say, we fine-tune the quark masses to ensure the limit

$$e_P \to e_N^{+}, \quad \Delta m_{KK'} \ll \Lambda_{\mathcal{N}=2}, \quad P = 1, ..., \nu, \quad K, K' = (r+1), ..., N_f. \tag{7.1}$$

Note, that it is possible to place all ν double roots close to e_N^{+} because it is the quark masses rather than $\Lambda_{\mathcal{N}=2}$ that determine the "non-Abelian" roots of the Seiberg–Witten curve and the VEVs of the non-Abelian dyons, see (6.15).

This limit means moving to the AD points. To see that this is indeed the case observe that masses of ν monopoles M_{PN} $(P = 1, ..., \nu)$ on the Coulomb branch are determined by the differences $(e_P - e_N^{+}) \to 0$, the corresponding β-cycles shrink.

Thus, besides the light dyons D^{lA} and D^J which are always present in our r vacuum, we get extra light monopoles that are mutually nonlocal with the dyons. If we were on the Coulomb branch (at $\xi_P = 0$) this would definitely mean moving to strong coupling. In fact, the running coupling constant of our dual theory is determined by the light dyon loops. If the monopoles simultaneously become light, their loops give logarithmic contributions to the inverse coupling, making the overall coupling constant of order of unity.

However, at small but nonvanishing ξ we are *not* on the Coulomb branch. In fact, the monopoles are confined. In particular, ν monopoles M_{PN} $(P = 1, ..., \nu)$ in question form stringy dipole states shown in Fig. 2. Although the masses of the M_{PN} monopoles become very small in the limit (7.1), the mass of the stringy dipole state formed by one of these monopoles (and an antimonopole) is determined by the string tension and, therefore, is much larger. It is of order of $\sqrt{\xi_P}$. The masses of the D^{lA} dyons are of order of $\tilde{g}\sqrt{\xi}$. Starting from weak coupling in the dual theory and calculating the renormalization of the coupling constant $\tilde{g}$ we see that the monopole-antimonopole states are heavier, and their loops are suppressed. In the theory (C.1) the coupling constant renormalization is determined by the dyon loops. This ensures that the renormalized coupling constant is small, provided that we keep ξ's small enough.

In other words, away from the Coulomb branch (at $\mu \neq 0$) the dual theory has no nontrivial conformal AD-regime, which appears on the Coulomb branch in the limit (7.1) [27]. It stays infrared-free. Note, however, that the effective two-dimensional sigma model on the non-Abelian string goes into a nontrivial conformal regime at the AD-point [36]. This is because condensates of the scalar fields tend to zero inside the string core, and on the string we are essentially back to the Coulomb branch of the four-dimensional bulk theory.

Let us stress, that this is the most important observation which allows us to extend our r-duality from $\mathcal{N}=2$ SQCD to $\mathcal{N}=1$.

The fact that the light matter VEVs tend to zero in the AD point was first recognized in [41] in the Abelian case.

7.2 Decoupling the U(1) factors

Now we can continue following the same road as in [9], where the large-μ limit was studied in the $r=N$ vacuum. First we will take the limit (7.1) still keeping μ small.

The VEVs of the non-Abelian dyons D^{lA} become much smaller than the VEVs of the Abelian dyons D^J, see (6.11), (6.14), and (6.15). In particular, the VEVs of the D^J dyons are determined by the differences $(e_J - e_N^+)$ for $J=(\nu+1),...,r$ which are not small and stay of order of $\Lambda_{\mathcal{N}=2}$ in the limit (7.1).

As a result, $(N-\nu-1)$ U(1) gauge fields of the dual gauge group (4.12) as well as the D^J dyons themselves acquire large masses, $\sim \sqrt{\mu\Lambda_{\mathcal{N}=2}}$, and decouple. At low energies we are left with the

$$\mathrm{U}(\nu) \times \mathrm{U}(1)^{\mathrm{unbr}} \tag{7.2}$$

gauge theory of non-Abelian D^{lA} dyons ($l=1,...,\nu$, $A=1,...,N_f$). The gauge field corresponding to $\mathrm{U}(1)^{\mathrm{unbr}}$ does not interact with the dyons and remains massless. The VEVs of the non-Abelian dyons are given by

$$\langle D^{lA}\rangle = \langle \bar{\tilde{D}}^{lA}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \ldots & 0 & \sqrt{\xi_1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & \ldots & \sqrt{\xi_\nu} \end{pmatrix}, \tag{7.3}$$

see (6.11), where the first ν parameters ξ_P are small in the limit (7.1).

The superpotential of this theory can be written as

$$\mathcal{W} = \sqrt{2} \sum_{A=1}^{N_f} \left(\frac{1}{2}\, \tilde{D}_A b_{U(1)} D^A + \tilde{D}_A b^p\, T^p D^A \right.$$
$$\left. + m_A\, \tilde{D}_A D^A \right) + \mu\, u_2(b_{U(1)}, b^p, a^{\mathrm{unbr}}). \tag{7.4}$$

Here $b_{U(1)}$ is a chiral superfield, the $\mathcal{N}=2$ superpartner of $B_\mu^{U(1)}$, where $B_\mu^{U(1)}$ is a particular linear combination of the dual gauge fields not interacting with the D^J dyons. We normalized

$b_{U(1)}$ so that the charges of the D^{lA} dyons with respect to this field are $\frac{1}{2}$. This amounts to redefining its coupling constant $\tilde{g}^2_{U(1)}$.

Moreover, b^p (with $p = 1, ..., \nu^2-1$) is an SU(ν) adjoint chiral field, the $\mathcal{N} = 2$ superpartner of the dual SU(ν) gauge field, see (C.1). We also use the standard normalization for the non-Abelian charges of D^{lA} absorbing $\sqrt{2}$ present in (C.2) in the definition of the gauge fields. Finally, $a^{\rm unbr}$ is a superpartner of the gauge field of the U(1)$^{\rm unbr}$, see (5.9).

7.3 Decoupling adjoint matter

Now we increase μ and make it

$$|\mu| \gg |\sqrt{\xi_P}|, \qquad P = 1, ..., \nu \tag{7.5}$$

decoupling adjoint matter. In order to keep the dual theory at weak coupling we go to the AD limit (7.1) and require

$$|\sqrt{\xi_P}| \ll \tilde{\Lambda}, \qquad P = 1, ..., \nu\,, \tag{7.6}$$

where

$$\tilde{\Lambda}^{r-2\nu} = \frac{\Lambda^{r-\nu}_{\mathcal{N}=2}}{\mu^{\nu}}\,. \tag{7.7}$$

We also assume that the quark mass differences are very small, even smaller than E_P, namely,

$$\Delta m_{KK'} \ll E_P = \sqrt{(e_P^2 - e_N^2)}, \quad P = 1, ..., \nu, \quad K, K' = (r+1), ..., N_f\,. \tag{7.8}$$

Given the superpotential (7.4) we can explicitly integrate out the adjoint matter. First we find the adjoint scalar VEVs. Say, in the simplest example $\nu = 2$ we have

$$b^3 = -\frac{1}{\sqrt{2}}\left(m_{N_f-1} - m_{N_f}\right), \qquad b_{\rm U(1)} = -\frac{1}{\sqrt{2}}\left(m_{N_f-1} + m_{N_f}\right). \tag{7.9}$$

Next we find $a^{\rm unbr}$ from Eq. (7.4) and expand the resulting function u_2 in powers of b^p and deviations of $b_{\rm U(1)}$ from its VEV in (7.9),

$$\begin{aligned} u_2(b_{\rm U(1)}, b^p) &= c_1\,(b^p)^2 + c_2\,\Delta b_{\rm U(1)} + c_3\,(\Delta b_{\rm U(1)})^2 \\ &+ O\left(\frac{\mu_2\,(b^p)^4}{\Lambda^2_{\mathcal{N}=2}}\right) + O\left(\frac{\mu_2\,(\Delta b_{\rm U(1)})^3}{\Lambda_{\mathcal{N}=2}}\right), \end{aligned} \tag{7.10}$$

Since $\Delta b_{\rm U(1)}$ and b^p are of order of E_P (the VEVs of b^p are also small, of order of $\Delta m_{KK'}$, see (7.9)) we can neglect higher-order terms in the expansion (7.10) and keep only linear and quadratic terms. Higher-order terms are suppressed by powers of $E_P/\Lambda_{\mathcal{N}=2}$.

Now, substituting (7.10) into (7.4) and integrating over $\Delta b_{\rm U(1)}$ and b^p we get the superpotential which depends only on D^{lA}. Minimizing it and requiring the VEVs of D^{lA} to be given by (7.3) (see also (6.14)) we fix the coefficients c_1 and c_2. Say, for $\nu = 2$ we get

$$c_1 = -\frac{1}{2\sqrt{2}}\,\frac{\hat{m}}{\hat{E}}\,, \qquad c_2 = 2\hat{E}\,, \tag{7.11}$$

where

$$\hat{m} = \frac{1}{\nu}\sum_{P=1}^{\nu} m_{r+P}, \qquad \hat{E} = \frac{1}{\nu}\sum_{P=1}^{\nu} E_P\,. \tag{7.12}$$

Note that the constant c_3 cannot be fixed by this procedure. In principle, c_3 can be fixed by studying the behavior of u_2 near the AD points.

After eliminating the adjoint matter the superpotential takes the form

$$\begin{aligned}\mathcal{W} &= \frac{\hat{E}}{\sqrt{2}\,\hat{m}\,\mu}\,(\tilde{D}_A D^B)(\tilde{D}_B D^A) + \left[(m_A - \hat{m}) + \frac{(\sqrt{2}\,\hat{E})^2}{\hat{m}}\right](\tilde{D}_A D^A) \\ &+ c\left[\frac{1}{2\mu}(\tilde{D}_A D^A)^2 + \sqrt{2}\nu\,\hat{E}\,(\tilde{D}_A D^A)\right].\end{aligned} \tag{7.13}$$

This equation presents our final large-μ result for the superpotential of the theory dual to $\mathcal{N}=1$ SQCD in the $(1,\,\dots\,,r)$ vacuum. The constant $c \sim 1$ remains undetermined; it is related to c_3 above.

One can check that minimization of this superpotential leads to correct values of the dyon VEVs, Eq. (7.3). The theory with the superpotential (7.13) possesses many other vacua in which different dyons (and different number of dyons) develop VEVs. We consider only one particular vacuum here. As was explained in Sec. 6.2, if we choose the $(1,\,\dots\,,r)$ vacuum in the original theory above the crossover, then we end up in the $(0,...,0,r+1,...,N_f)$ vacuum in the dual theory below the crossover, see (7.3). Vacua with the number of condensed D's less than the maximum possible one (equal ν) seen in (7.13) are spurious.

7.4 Perturbative mass spectrum

Now we briefly summarize the perturbative mass spectrum of our dual theory with superpotential (7.13) given the quark mass choice (5.11).

The U(ν) gauge group is completely Higgsed, and the masses of the gauge bosons are

$$m_{\text{SU}(\nu)} = \tilde{g}_2\sqrt{\xi} \tag{7.14}$$

for the SU($\tilde{N}$) gauge bosons, and

$$m_{\text{U}(1)} = \tilde{g}_1\sqrt{\frac{\nu}{2}}\,\sqrt{\xi}\,. \tag{7.15}$$

for the U(1) gauge boson. Here $\tilde{g}_1$ and $\tilde{g}_2$ are dual gauge couplings for the U(1) and SU(ν) gauge bosons, respectively, while ξ is a common value of the first ν parameters ξ_P (see Eqs. (6.14) and (6.15)),

$$\xi = -2\,\mu\,\sqrt{\hat{m}^2 - 2e_N^2}\,. \tag{7.16}$$

The dyon masses are determined by the D-term potential

$$V_D^{\rm dual} = \frac{\tilde{g}_2^2}{2}\left(\bar{D}_A T^p D_A - \tilde{D}_A T^p \bar{\tilde{D}}^A\right)^2 + \frac{\tilde{g}_1^2}{8}\left(|D^A|^2 - |\tilde{D}_A|^2\right)^2 \qquad (7.17)$$

and the F-term potential following from the superpotential (7.13). Diagonalizing the quadratic form given by these two potentials we find that, out of $4\nu N_F$ real degrees of freedom of the scalar dyons, ν^2 are eaten up in the Higgs mechanism, $\nu^2 - 1$ real scalar dyons have the same mass as the non-Abelian gauge fields, Eq. (7.14), while one scalar dyon has the mass (7.15). These dyons are scalar superpartners of the SU(ν) and U(1) gauge bosons in $\mathcal{N} = 1$ massive vector supermultiplets, respectively.

Another $2(\nu^2 - 1)$ dyons form a $(1, \nu^2 - 1)$ representation of the global group (6.16). Their mass is as follows:

$$m_{(1,\nu^2-1)} = \frac{\hat{E}^2}{\hat{m}}\,, \qquad (7.18)$$

while two real singlet dyons are heavier, their mass

$$m_{(1,\,1)} \sim \hat{E} \qquad (7.19)$$

is determined by the last term (the one with unknown coefficient) in (7.13). Here

$$\hat{E} = \frac{1}{\sqrt{2}}\sqrt{\hat{m}^2 - 2e_N^2}\,, \qquad (7.20)$$

see (6.13).

The masses of $4N\nu$ bifundamental fields are given by the mass split of r first and ν last quark masses, see (5.11),

$$m_{(\bar{r},\,\nu)} = \Delta m\,. \qquad (7.21)$$

All these dyons are the scalar components of the $\mathcal{N} = 1$ chiral multiplets.

We see that the masses of the gauge multiplets and those of chiral matter get a large split in the limit of large μ and small $\hat{E}$. Chiral matter becomes much lighter than the gauge multiplets cf. [42, 17].

7.5 Summary

To summarize, at large μ, upon reducing ξ, the original $\mathcal{N} = 1$ SQCD in the $r = N - 1$ vacuum undergoes a crossover transition at strong coupling. In the domain (7.6) in the vicinity of the AD points (7.1) it is described by the weakly coupled infrared-free dual theory, U(ν)×U(1)$^{\rm unbr}$ SQCD, with N_f light dyon flavors. Condensation of the light dyons D^{lA} in this theory leads to formation of the non-Abelian strings and confinement of monopoles. Quarks and gauge bosons of the original $\mathcal{N} = 1$ SQCD are in the "instead-of-confinement" phase: they decay into the monopole-antimonopole pairs on CMS and form stringy mesons. In fact, in the AD-regime (7.1) the M_{PN} monopoles ($P = 1, ..., \nu$) become very light and, therefore, strings are unstable. As a result, stringy mesons shown in Fig. 1 decay into stringy dipoles, see Fig 2. Stringy dipoles with non-trivial charges with respect to the SU(r) part of the global group (for example from the adjoint representation) are stable.

8 Conclusions

Our main task was to extend non-Abelian duality, that was observed previously [8] in the $r = N$ vacuum, to vacua with a smaller number of condensed quarks, which we referred to as the r vacua. The second task was exploration of the confinement mechanism both in the original and dual theories, as it reveals itself in the r vacua. As in [8] we start from the $\mathcal{N} = 2$ theory slightly deformed by the adjoint field mass parameter μ and study the transition from large values of the FI parameters ξ to small values. At large ξ it is the original theory that is weakly coupled. As we move to smaller ξ the original theory becomes coupled exceedingly stronger. A dual description becomes more appropriate. We identify the dual gauge group (which, surprisingly, is not the Seiberg dual group if $r < N$), dual matter and dual theory as a whole. Remarkably, the "dual quarks" are *not* monopoles. We identify an "instead-of-confinement" mechanism.

Then we increase the deformation parameter μ and repeat the whole program. At large μ the adjoint fields decouple, and our theory flows to $\mathcal{N} = 1$ SQCD. The gauge group of the dual theory becomes $\mathrm{U}(N_f - r)$. We show that the dual theory is still weakly coupled if we approach the Argyres–Douglas point. The "instead-of-confinement" mechanism for quarks and gauge bosons survives in the limit of large μ. It determines low-energy non-Abelian dynamics in the r-vacua of $\mathcal{N} = 1$ SQCD.

Our main example in this paper is the $r = (N - 1)$ vacuum. Still we expect that our results are quite general and can be applied to all $r > \frac{2}{3}N_f$ vacua. In particular, a generic r vacuum has $(N - r - 1)$ condensed monopoles at large ξ, in addition to r condensed quarks. These monopoles are charged with respect to Abelian U(1) factors of the gauge group. At large μ and small ξ in the dual theory all $\mathrm{SU}(\nu)$ singlets (including these monopoles) become heavy and decouple. They do play no role in the low-energy dynamics of the dual theory at large μ. The light matter charged with respect to the dual gauge group $\mathrm{U}(\nu)$ consists of the D^{lA} dyons which are quark-like states. In particular, condensation of these dyons leads to confinement of monopoles.

A very crucial question is comparison of the r duality we studied here with the Seiberg duality. This will be carried out in a separate publication [28].

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Appendix A: Low-energy action of the U(N) theory in the $r = N-1$ vacuum at large ξ

The low-energy action has the form

$$S = \int d^4x \left[\frac{1}{4g_2^2} \left(F^n_{\mu\nu}\right)^2 + \frac{1}{4g_1^2} \left(F_{\mu\nu}\right)^2 + \frac{1}{4g_2^2} \left(F^{(N^2-1)}_{\mu\nu}\right)^2 + \frac{1}{g_2^2} \left|D_\mu a^n\right|^2 \right.$$

$$\left. + \frac{1}{g_1^2} \left|\partial_\mu a\right|^2 + \frac{1}{g_2^2} \left|\partial_\mu a^{(N^2-1)}\right|^2 + \left|\nabla_\mu q^A\right|^2 + \left|\nabla_\mu \bar{\tilde{q}}^A\right|^2 + V \right], \tag{A.1}$$

where the fundamental and adjoint color indices are $k = 1, ..., r$ and $n = 1, ..., r^2 - 1$, respectively, while the U(1) gauge field $A_\mu^{N^2-1}$ and its scalar superpartner a^{N^2-1} are associated with the last Cartan generator of SU(N). Note that all non-Abelian gauge fields from the SU(N)/SU(r) sector are heavy and decouple in the large mass limit due to the structure of the adjoint VEVs, see (4.2). Also the q^{NA} quarks are heavy and not included in the low-energy theory. The covariant derivative

$$\nabla_\mu = \partial_\mu - \frac{i}{2}\, A_\mu - \frac{i}{\sqrt{2N(N-1)}}\, A_\mu^{N^2-1} - iA^n_\mu T^n \tag{A.2}$$

acts in the fundamental representation.

The scalar potential $V(q^A, \tilde{q}_A, a^n, a, a^{(N^2-1)})$ in the action (A.1) is

$$V(q^A, \tilde{q}_A, a^n, a, a^{(N^2-1)}) = \frac{g_2^2}{2} \left(\frac{1}{g_2^2}\, f^{nms} \bar{a}^m a^s + \bar{q}_A\, T^n q^A - \tilde{q}_A T^n\, \bar{\tilde{q}}^A \right)^2$$

$$+ \frac{g_1^2}{8} \left(\bar{q}_A q^A - \tilde{q}_A \bar{\tilde{q}}^A\right)^2 + \frac{g_2^2}{4N(N-1)} \left(\bar{q}_A q^A - \tilde{q}_A \bar{\tilde{q}}^A\right)^2$$

$$+ 2g_2^2 \left| \tilde{q}_A T^n q^A + \frac{1}{\sqrt{2}}\, \frac{\partial \mathcal{W}_{\rm br}}{\partial a^n} \right|^2 + \frac{g_1^2}{2} \left| \tilde{q}_A q^A + \sqrt{2}\, \frac{\partial \mathcal{W}_{\rm br}}{\partial a} \right|^2$$

$$+ 2g_2^2 \left| \frac{1}{\sqrt{2N(N-1)}}\, \tilde{q}_A q^A + \frac{1}{\sqrt{2}}\, \frac{\partial \mathcal{W}_{\rm br}}{\partial a^{(N^2-1)}} \right|^2$$

$$+ \frac{1}{2} \sum_{A=1}^{N_f} \left\{ \left| \left(a + \sqrt{2} m_A + 2T^n a^n + \sqrt{\frac{2}{N(N-1)}}\, a^{(N^2-1} \right) q^A \right|^2 \right.$$

$$\left. + \left| \left(a + \sqrt{2} m_A + 2T^n a^n + \sqrt{\frac{2}{N(N-1)}}\, a^{(N^2-1} \right) \bar{\tilde{q}}^A \right|^2 \right\}. \tag{A.3}$$

Appendix B:
Weights and roots of the SU(4) algebra

In this Appendix we present, for completeness, weights and roots of the SU(4) algebra which we repeatedly use in the main text. Weights determine quark charges, while roots determine monopole charges. The diagonal (Cartan) generators of SU(N) are defined as

$$\begin{aligned} T_{ij}^{\tilde{a}=(m+1)^2-1} &= \frac{1}{\sqrt{2m(m+1)}}\left(\sum_{k=1}^{m}\delta_{ik}\,\delta_{jk} - m\,\delta_{i,m+1}\,\delta_{j,m+1}\right), \\ m &= 1, ..., N-1. \end{aligned} \tag{B.1}$$

For SU(4) the index values $m = 1, 2, 3$ corerespond to the Cartan generators T^3, T^8 and T^{15}.

In three-dimensional Cartan plane the weights of the SU(4) algebra are

$$\begin{aligned} w_1 &= \left(\frac{1}{2};\ \frac{1}{2\sqrt{3}};\ \frac{1}{2\sqrt{6}}\right), \\ w_2 &= \left(-\frac{1}{2};\ \frac{1}{2\sqrt{3}};\ \frac{1}{2\sqrt{6}}\right), \\ w_3 &= \left(0;\ -\frac{1}{\sqrt{3}};\ \frac{1}{2\sqrt{6}}\right), \\ w_4 &= \left(0;\ 0;\ -\frac{3}{2\sqrt{6}}\right). \end{aligned} \tag{B.2}$$

The roots can be obtained as

$$\alpha_{ij} = w_i - w_j, \qquad i < j. \tag{B.3}$$

This implies

$$\begin{aligned} \alpha_{12} &= (1;\ 0;\ 0)\,, \qquad \alpha_{13} = \left(\frac{1}{2};\ \frac{\sqrt{3}}{2};\ 0\right), \\ \alpha_{23} &= \left(-\frac{1}{2};\ \frac{\sqrt{3}}{2};\ 0\right), \qquad \alpha_{14} = \left(\frac{1}{2};\ \frac{1}{2\sqrt{3}};\ \sqrt{\frac{2}{3}}\right), \\ \alpha_{24} &= \left(-\frac{1}{2};\ \frac{1}{2\sqrt{3}};\ \sqrt{\frac{2}{3}}\right), \qquad \alpha_{34} = \left(0;\ -\frac{1}{\sqrt{3}};\ \sqrt{\frac{2}{3}}\right). \end{aligned} \tag{B.4}$$

For the monopole with charges determined by the root α_{ij} we use the notation M_{ij}. From the expressions above we find charges of all monopoles in SU(4). Say, for the M_{23} monopole we have

$$\vec{n}_{M_{23}} = (0,\, 0;\, 0,\, -\frac{1}{2};\, 0,\, \frac{\sqrt{3}}{2};\, 0,\, 0) \tag{B.5}$$

in notations (5.18).

Appendix C:
Low-energy action of the dual theory in the $r = 3$ vacuum for $N = 4$

The dual theory for the $r = 3$ vacuum in the U(3) gauge theory was found in [8]. To utilize these results in the $r = 3$ vacuum in the U(4) theory at hand we make a minor adjustment which takes into account the presence of an extra U(1) gauge field associated with the T^{15} generator in the U(4) theory. The dual gauge group is $\mathrm{U}(2)\times\mathrm{U}(1)^8\times\mathrm{U}(1)^{15}$. The bosonic part of the action is

$$\begin{aligned} S_{\mathrm{dual}} &= \int d^4x \left[\frac{1}{4\tilde{g}_2^2}\left(F^p_{\mu\nu}\right)^2 + \frac{1}{4g_1^2}\left(F_{\mu\nu}\right)^2 + \frac{1}{4\tilde{g}_8^2}\left(F^8_{\mu\nu}\right)^2 + \frac{1}{4\tilde{g}_{15}^2}\left(F^{15}_{\mu\nu}\right)^2 \right. \\ &+ \frac{1}{\tilde{g}_2^2}\left|\partial_\mu b^p\right|^2 + \frac{1}{g_1^2}\left|\partial_\mu a\right|^2 + \frac{1}{\tilde{g}_8^2}\left|\partial_\mu b^8\right|^2 + \frac{1}{\tilde{g}_{15}^2}\left|\partial_\mu a^{15}\right|^2 \\ &+ \left. \left|\nabla^1_\mu D^A\right|^2 + \left|\nabla^1_\mu \tilde{D}_A\right|^2 \left|\nabla^2_\mu D^3\right|^2 + \left|\nabla^2_\mu \tilde{D}_3\right|^2 + V \right], \end{aligned} \tag{C.1}$$

Here covariant derivatives are defined in accordance with the charges of the D^l ($l = 1, 2$) and D^3 dyons in (6.10). Namely,

$$\begin{aligned} \nabla^1_\mu &= = \partial_\mu - i\left(\frac{1}{2}A_\mu + \sqrt{2}\,B^p_\mu\frac{\tau^p}{2} + \frac{\sqrt{10}}{2\sqrt{3}}\,B^8_\mu + \frac{1}{2\sqrt{6}}A^{15}_\mu\right), \\ \nabla^2_\mu &= = \partial_\mu - i\left(\frac{1}{2}A_\mu - \frac{\sqrt{10}}{\sqrt{3}}\,B^8_\mu + \frac{1}{2\sqrt{6}}A^{15}_\mu\right), \end{aligned} \tag{C.2}$$

where the B^p_μ gauge fields ($p = 1, 2, 3$), B^8_μ, and their scalar superpartners b^p and b^8 are

$$\begin{aligned} B^3_\mu &= \frac{1}{\sqrt{2}}\,(A^3_\mu + A^{3D}_\mu), \qquad b^3 = \frac{1}{\sqrt{2}}\,(a^3 + a^3_D) \quad \text{for} \quad p = 3, \\ B^8_\mu &= \frac{1}{\sqrt{10}}\,(A^8_\mu + 3A^{8D}_\mu), \qquad b^8 = \frac{1}{\sqrt{10}}\,(a^8 + 3a^8_D)\,. \end{aligned} \tag{C.3}$$

The coupling constants g_1, $\tilde{g}_8$, $\tilde{g}_{15}$ and $\tilde{g}_2$ correspond to three U(1)'s and the SU(2) gauge groups, respectively. The scalar potential $V(D,\tilde{D},b^p,b^8,a,a^{15})$ in the action (C.1) is

$$
\begin{aligned}
V &= \frac{\tilde{g}_2^2}{4}\left(\frac{1}{\tilde{g}_2^2}\,f^{nms}\bar{a}^m a^s + \bar{D}_A\tau^p D_A - \tilde{D}_A\tau^p\bar{\tilde{D}}^A\right)^2 \\
&+ \frac{10}{3}\frac{\tilde{g}_8^2}{8}\left(|D^A|^2 - |\tilde{D}_A|^2 - 2|D^3|^2 + 2|\tilde{D}_3|^2\right)^2 \\
&+ \frac{\tilde{g}_1^2}{8}\left(|D^A|^2 - |\tilde{D}_A|^2 + |D^3|^2 - |\tilde{D}_3|^2\right)^2 \\
&+ \frac{\tilde{g}_{15}^2}{48}\left(|D^A|^2 - |\tilde{D}_A|^2 + |D^3|^2 - |\tilde{D}_3|^2\right)^2 \\
&+ \frac{\tilde{g}_2^2}{2}\left|\sqrt{2}\tilde{D}_A\tau^p D_A + \sqrt{2}\,\frac{\partial\mathcal{W}_{\rm br}}{\partial b^p}\right|^2 + \frac{\tilde{g}_1^2}{2}\left|\tilde{D}_A D^A + \tilde{D}_3 D_3 + \sqrt{2}\,\frac{\partial\mathcal{W}_{\rm br}}{\partial a}\right|^2 \\
&+ \frac{\tilde{g}_8^2}{2}\left|\sqrt{\frac{10}{3}}\tilde{D}_A D^A - 2\sqrt{\frac{10}{3}}\tilde{D}_3 D^3 + \sqrt{2}\,\frac{\partial\mathcal{W}_{\rm br}}{\partial b^8}\right|^2 \\
&+ \frac{\tilde{g}_{15}^2}{2}\left|\frac{1}{\sqrt{6}}(\tilde{D}_A D^A + \tilde{D}_3 D_3) + \sqrt{2}\,\frac{\partial\mathcal{W}_{\rm br}}{\partial a^{15}}\right|^2 \\
&+ \frac{1}{2}\left\{\left|(a + \tau^p\sqrt{2}\,b^p + \sqrt{\frac{10}{3}}\,b^8 + \frac{1}{\sqrt{6}}\,a^{15} + \sqrt{2}m_A)D^A\right|^2\right. \\
&+ \left|(a + \tau^p\sqrt{2}\,b^p + \sqrt{\frac{10}{3}}\,b^8 + \frac{1}{\sqrt{6}}\,a^{15} + \sqrt{2}m_A)\bar{\tilde{D}}_A\right|^2 \\
&+ \left.\left|a - 2\sqrt{\frac{10}{3}}\,b^8 + \frac{1}{\sqrt{6}}\,a^{15} + \sqrt{2}m_3\right|^2\left(|D^3|^2 + |\tilde{D}_3|^2\right)\right\}, \qquad \text{(C.4)}
\end{aligned}
$$

(see also [9]).

The derivatives of the superpotential $\mathcal{W}$ in (C.4) can be calculated using (3.21). Next, we use monodromies found in the Sec. 6.1 to relate the derivatives of u_2 with respect to b^3 and b^8 to those with respect to a^3 and a^8, namely,

$$
\frac{1}{\sqrt{2}}\,\frac{\partial u_2}{\partial b^3} = \frac{\partial u_2}{\partial a^3}, \qquad \frac{1}{\sqrt{10}}\,\frac{\partial u_2}{\partial b^8} = \frac{\partial u_2}{\partial a^8}\,, \qquad \text{(C.5)}
$$

see also [19, 9].

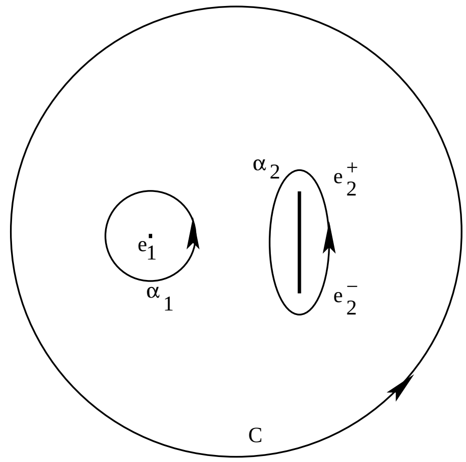

Figure 4: α-contours in x-plane for the U(2) theory. Solid straight line denotes the cut.

Appendix D: The $r = 1$ vacuum in U(2) theory

In this Appendix we find the relation of the matrix E (see (3.23)) determining the quark/dyon VEVs in the original/dual theory with the roots of the Seiberg–Witten curve. We consider the simplest possible example: the $r = 1$ vacuum in the U(2) gauge theory with $1 \leq N_f < 4$.

Let us calculate the diagonal elements of the matrix E given by

$$E = \frac{1}{2}\frac{\partial u_2}{\partial a} + \frac{\tau^3}{2}\frac{\partial u_2}{\partial a^3} \tag{D.1}$$

in this particular case. The Seiberg–Witten curve in this case factorizes as follows:

$$y^2 = (x - e_1)^2\,(x - e_2^+)(x - e_2^-), \tag{D.2}$$

see (6.2). Here the double root at $x = e_1$ corresponds to a single condensed quark in the $r = 1$ vacuum, while two other roots (subject to condition (6.3)) determine the gaugino condensate.

The exact solution of the theory on the Coulomb branch relates the fields a and a^3 to contour integrals running along the contours α_i $(i = 1, 2)$ in x-plane encircling the double root e_1 and the cut which is stretched between the roots $e_2^{\pm}$, see Fig 4.

Using explicit expressions from [37, 38, 39, 40] and their generalization to the U(N) case [33] we can write

$$\frac{\partial \Phi_i}{\partial u_2} = \frac{1}{2}\,\frac{1}{2\pi i}\oint_{\alpha_i}\frac{dx}{y}\,, \qquad \frac{\partial \Phi_i}{\partial u_1} = \frac{1}{2\pi i}\oint_{\alpha_i}\frac{dx}{y}\,[x - (e_1 + e_2)]\,, \tag{D.3}$$

where the variables u_1, u_2 are given by (3.20), and we define

$$(\Phi_1, , ..., \Phi_N) = \text{diag}\left(\frac{1}{2}\,a + T^{\tilde a}\,a^{\tilde a}\right), \tag{D.4}$$

while

$$e_2 = \frac{1}{2}\left(e_2^+ + e_2^-\right). \tag{D.5}$$

In fact, $e_2 = 0$ due to the condition (6.3).

Equation (D.4) gives in the $N = 2$ case

$$a = \Phi_1 + \Phi_2, \qquad a^3 = \Phi_1 - \Phi_2\,. \tag{D.6}$$

For the factorized curve (D.2) the integrals (D.3) can be easily evaluated. The integrals along the α_1 contour are given by their pole contributions. To calculate the integrals along the α_2 contour we write $\alpha_2 = C - \alpha_1$, where C is a large circle at infinity, see Fig 4. This gives us

$$\begin{aligned}
&\frac{\partial \Phi_1}{\partial u_2} = \frac{1}{2}\,\frac{1}{\sqrt{(e_1 - e_2^+)(e_1 - e_2^-)}}, \qquad \frac{\partial \Phi_2}{\partial u_2} = -\frac{1}{2}\,\frac{1}{\sqrt{(e_1 - e_2^+)(e_1 - e_2^-)}},\\
&\frac{\partial \Phi_1}{\partial u_1} = -\frac{e_2}{\sqrt{(e_1 - e_2^+)(e_1 - e_2^-)}},\\
&\frac{\partial \Phi_2}{\partial u_2} = 1 + \frac{e_2}{\sqrt{(e_1 - e_2^+)(e_1 - e_2^-)}}\,.
\end{aligned} \tag{D.7}$$

Using (D.6) we get the derivatives $\partial a/\partial u_1$, $\partial a^3/\partial u_1$, $\partial a/\partial u_2$ and $\partial a^3/\partial u_2$. Inverting this matrix and substituting the result in (D.1) we obtain

$$\operatorname{diag} E = \left(\sqrt{(e_1 - e_2^+)(e_1 - e_2^-)} + e_2,\ e_2\right). \tag{D.8}$$

Now we see that

$$E_{N=2} = e_{N=2} = 0, \tag{D.9}$$

i.e. the two conditions (6.3) and (6.4) are equivalent.

Using these conditions we finally obtain

$$\operatorname{diag} E = \left(\sqrt{(e_1 - e_2^+)(e_1 - e_2^-)},\ 0\right). \tag{D.10}$$

Straightforward generalization of this result to arbitrary N gives Eq. (6.13) that was presented in the main text.

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