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# The Four Dimensional Helicity Scheme Beyond One Loop

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I describe a procedure by which one can transform scattering amplitudes computed in the four dimensional helicity scheme into properly renormalized amplitudes in the 't Hooft-Veltman scheme. I describe a new renormalization program, based upon that of the dimensional reduction scheme and explain how to remove both finite and infrared-singular contributions of the evanescent degrees of freedom to the scattering amplitude.

## I. INTRODUCTION

The Four Dimensional Helicity (FDH) scheme [1, 2] is widely used for computing QCD corrections at next-to-leading order in perturbation theory. It is particularly convenient for use with the helicity method and the techniques of generalized unitarity. Unfortunately, as I have recently shown [3], the FDH is not a unitary regularization scheme. The standard renormalization prescription [2] fails to remove all of the ultraviolet poles, leading to incorrect results at two loops and beyond. Thus the FDH cannot be viewed as a regularization scheme in which one can compute scattering amplitudes. Instead, it should be looked upon as a shortcut for obtaining scattering amplitudes in a unitary regularization scheme. Indeed, this is how the FDH has always been used at one-loop; final results have always been presented in the 't Hooft-Veltman (HV) scheme [4] using the prescription of Kunszt, et al. [5] to transform the FDH scheme result, but it was not clear whether this conversion was necessary or merely expedient, allowing one to match onto standard definitions of the running coupling, etc.

It is now certain that one must convert the results of a calculation in the FDH scheme into results in a properly defined scheme. A first step in this direction was taken by Boughezal, et al. [6], who put forward a prescription for constructing the correct counterterms for renormalization. For inclusive calculations, performed using the optical theorem, like those considered in Refs. [3, 6], such a prescription is sufficient. Experiments, however, measure differential cross sections, and the power of the FDH scheme is that it facilitates the calculation of loop-level amplitudes, giving access to the differential information they contain. To make use of the full amplitude, one must control of both the infrared and ultraviolet structure.

In this paper, I will exploit the close relationship between the FDH and the dimensional reduction (DRED) [7] schemes to develop a prescription for transforming FDH scheme amplitudes, which may be

easier to compute using unitarity methods, into HV scheme amplitudes that can actually be used in calculations. The plan of the paper is: In Section II I will review the regularization schemes that will be used; in Section III I will review the infrared structure of QCD amplitudes; in Section IV I will define the FDH scheme in terms of the DR scheme (a variation on the DRED scheme that I will define below), compute the anomalous dimensions that control the ultraviolet and infrared structure of DR scheme amplitudes through two loops and specify the procedure for transforming FDH scheme results into HV scheme amplitudes.

## II. REGULARIZATION SCHEMES

All of the schemes that I will be working with are variations on dimensional regularization [4], which specifies that loop-momenta are treated as  $D_m = 4 - 2\varepsilon$  dimensional. In dimensional regularization, the singularities (both ultraviolet and infrared) that appear in four-dimensional calculations are transformed into poles in the parameter  $\varepsilon$ . The ultraviolet poles are removed through renormalization, while the infrared poles cancel when one performs “sufficiently inclusive” calculations.

### A. The 't Hooft-Veltman and conventional dimensional regularization schemes

In the original dimensional regularization scheme [4], the HV scheme, observed states are treated as four-dimensional, while internal states (both their momenta and their spin degrees of freedom) are treated as  $D_m$  dimensional. Internal states include states that circulate inside of loop diagrams as well as nominally external states that have infrared overlaps with other nominally external states. It turns out that one can treat internal fermions as having exactly two degrees freedom, just as they have in four dimensions, even though their momenta are  $D_m$  dimensional, but massless internal gauge bosons must have  $(D_m - 2)$  spin degrees of freedom, while massive internal gauge bosons have  $(D_m - 1)$ .

The conventional dimensional regularization (CDR) scheme [8] is closely related to the HV scheme. In the CDR scheme, all states and momenta, both internal and observed, are taken to be  $D_m$  dimensional. This often turns out to be computationally more convenient, especially in infrared sensitive theories like QCD, since one set of rules governs all interactions. Because the HV and CDR schemes handle ultraviolet singularities in the same manner, their behavior under the renormalization group, anomalous dimensions, running coupling, etc., are identical.

In the HV and CDR schemes, internal momenta are taken to be  $D_m = 4 - 2\varepsilon$  dimensional. In general,  $\varepsilon$  is a complex number and it's exact value is unimportant, but taking  $\varepsilon$  to be real and positive (negative) is preferred by ultraviolet (infrared) power-counting arguments. It is important, however, that the  $D_m$ -

dimensional vector space in which momenta take values is *larger* than the standard four-dimensional space-time. This means that the standard four-dimensional metric tensor  $\eta^{\mu\nu}$  spans a smaller space than the  $D_m$  dimensional metric tensor, and the four-dimensional Dirac matrices  $\gamma^{0,1,2,3}$  form a subset of the full  $\gamma^\mu$ . These considerations are of particular importance when considering chiral objects involving  $\gamma_5$  and the Levi-Civita tensor, but cannot be neglected when, as in the HV scheme, one restricts observed states to be strictly four-dimensional.

### B. The dimensional reduction Scheme

The DRED scheme was devised for application to supersymmetric theories. In supersymmetry, it is essential that the number of bosonic degrees of freedom is exactly equal to the number of fermionic degrees of freedom. In the DRED scheme, the continuation to  $D_m$  dimensions is taken as a *compactification* from four dimensions. Thus, while space-time is taken to be four-dimensional and particles have the standard number of degrees of freedom, momenta are regularized dimensionally and span a  $D_m$  dimensional vector space which is *smaller* than four-dimensional space-time.

Because the Ward Identity only applies in the  $D_m$  dimensional vector space in which momenta are defined, the extra  $2\varepsilon$  spin degrees of freedom of gauge bosons are not protected by the Ward Identity and must renormalize differently than the  $2 - 2\varepsilon$  degrees of freedom that are protected. In supersymmetric theories, the supersymmetry provides the missing part of the Ward Identity which demands that the  $2\varepsilon$  spin degrees of freedom be treated as gauge bosons. In non-supersymmetric theories, however, they must be considered to be distinct particles, with distinct couplings and renormalization properties. These extra degrees of freedom are referred to as “ $\varepsilon$ -scalars” or as “evanescent” degrees of freedom.

Since the evanescent degrees of freedom are independent of the gauge bosons, their self-couplings and their coupling to fermions are independent of the gauge coupling and of one another. The quartic self-coupling splits into multiple independent terms; if the gauge theory is  $SU(2)$ , there are two independent quartic self-couplings, in  $SU(3)$ , there are three independent quartic self-couplings, and if the gauge theory is  $SU(N); N \geq 4$ , there are four independent quartic self-couplings [9]. These new couplings run differently from the gauge coupling under the renormalization group and cannot consistently be identified with it.

Notwithstanding its semantic appeal, the insistence on a proper compactification, so that  $D_m \subset 4$  in the DRED scheme, is problematic when dealing with chiral theories [10]. Chirality is a four-dimensional concept and one cannot consistently define chiral operators in a vector space with fewer than four dimensions. One way around this is to adopt a hierarchy of vector spaces  $D_s \supset D_m \supset 4$  (where  $D_m = 4 - 2\varepsilon$  and  $D_s$  is assigned the value  $D_s = 4$ ), as in the FDH scheme (described below). In such a scheme, chiral operators can

be defined in the four-dimensional subspace of  $D_m$ , just as they are in the HV/CDR schemes. Stöckinger and Signer [11, 12] have long advocated that this is the proper definition of the DRED scheme. Aside from the treatment of chiral operators, there are no important computational distinctions between  $D_m \supset 4$  and  $D_m \subset 4$ . In this paper, I will adopt the  $D_m \supset 4$  convention and refer to this variation of dimensional reduction as the DR scheme.

### C. The four dimensional helicity Scheme

In the four-dimensional helicity scheme, one again defines a vector space of dimensionality  $D_m \supset 4$  (again  $D_m = 4 - 2\varepsilon$ ), in which loop momenta take values, and a still larger vector space  $D_s \supset D_m$ , ( $D_s = 4$ ), in which internal spin degrees of freedom take values. Note that the relative numerical values of  $D_s$ ,  $D_m$  and 4 are not important. What is important is that as vector spaces,  $D_s \supset D_m \supset 4$ .

The FDH scheme, like the HV scheme, treats observed states as four-dimensional, except, as in inclusive calculations, where there are infrared overlaps among external states. When infrared overlaps occur, external states are taken to be  $D_s$  dimensional.

As in the DRED scheme, spin degrees of freedom take values in a vector space that is larger than that in which momenta take values. It would seem, therefore, that the same remarks regarding the Ward Identity and the conclusion that the  $D_x = D_s - D_m$  dimensional components of the gauge fields and their couplings must be considered as distinct from the  $D_m$  dimensional gauge fields and couplings would apply.

That is not, however, how the FDH scheme has been used. All field components in the  $D_s$  dimensional space are treated as gauge fields and no distinction is made between the couplings. The reason for doing this is to facilitate the use of helicity amplitudes in conjunction with unitarity methods, the idea being to “sew together” (four dimensional) tree-level helicity amplitudes into loop-level amplitudes. While helicity methods can be used in the CDR scheme [13], they are most transparently and compactly represented using four-dimensional external states. Thus, the FDH scheme demands that the gluons circulating through loop amplitudes have the same number of spin degrees of freedom as the external gluons of helicity amplitudes.

Unfortunately, this framework fails to subtract all of the ultraviolet poles [3] and generates incorrect results. The evanescent couplings and degrees of freedom need to be renormalized separately from their gauge boson counterparts, but there is no mechanism within the FDH for doing so. The errors, however, are only of order  $\mathcal{O}(\varepsilon^1)$  in NLO calculations (which is the level at which the FDH has been used in practical calculations to date) and therefore do not adversely affect those results. At NNLO the errors would be of order  $\mathcal{O}(\varepsilon^0)$  and at N<sup>3</sup>LO and beyond the errors would be singular in  $\varepsilon$ .

### III. THE INFRARED STRUCTURE OF QCD AMPLITUDES

The infrared structure of QCD amplitudes is governed by a set of anomalous dimensions which allow one to predict, for any amplitude, the complete infrared structure [14, 15]. These anomalous dimensions are known completely, in both the massless and massive cases for one and two loop amplitudes, and their properties beyond the two-loop level are being actively studied [16–25].

For a general  $n$  parton scattering process, the set of partons is labeled by  $\mathbf{f} = \{f_i\}_{i=1\dots n}$ . In the formulation of Refs. [15–17], a renormalized amplitude may be factorized into three functions: the jet function  $\mathcal{J}_{\mathbf{f}}$ , which describes the collinear dynamics of the external partons that participate in the collision; the soft function  $\mathbf{S}_{\mathbf{f}}$ , which describes soft exchanges between the external partons; and the hard-scattering function  $|H_{\mathbf{f}}\rangle$ , which describes the short-distance scattering process,

$$\left| \mathcal{M}_{\mathbf{f}} \left( p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon \right) \right\rangle = \mathcal{J}_{\mathbf{f}} \left( \alpha_s(\mu^2), \varepsilon \right) \mathbf{S}_{\mathbf{f}} \left( p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon \right) \left| H_{\mathbf{f}} \left( p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) \right\rangle. \quad (1)$$

The notation indicates that  $|H_{\mathbf{f}}\rangle$  is a vector and  $\mathbf{S}_{\mathbf{f}}$  is a matrix in color space [14, 26, 27]. As with any factorization, there is considerable freedom to move terms about from one function to another. It is convenient [16, 17] to define the jet and soft functions,  $\mathcal{J}_{\mathbf{f}}$  and  $\mathbf{S}_{\mathbf{f}}$ , so that they contain all of the infrared poles but only contain infrared poles, while all infrared finite terms, including those at higher-order in  $\varepsilon$ , are absorbed into  $|H_{\mathbf{f}}\rangle$ .

#### A. The jet function in the HV/CDR schemes

The jet function  $\mathcal{J}_{\mathbf{f}}$  is found to be the product of individual jet functions  $\mathcal{J}_{f_i}$  for each of the external partons,

$$\mathcal{J}_{\mathbf{f}} \left( \alpha_s(\mu^2), \varepsilon \right) = \prod_{i \in \mathbf{f}} \mathcal{J}_i \left( \alpha_s(\mu^2), \varepsilon \right). \quad (2)$$

Each individual jet function is naturally defined in terms of the anomalous dimensions of the Sudakov form factor [15],

$$\begin{aligned} \ln \mathcal{J}_i^{\text{CDR}} \left( \alpha_s(\mu^2), \varepsilon \right) = & - \left( \frac{\alpha_s^{\overline{\text{MS}}}}{\pi} \right) \left[ \frac{1}{8\varepsilon^2} \gamma_{Ki}^{(1)} + \frac{1}{4\varepsilon} \mathcal{G}_i^{(1)}(\varepsilon) \right] \\ & + \left( \frac{\alpha_s^{\overline{\text{MS}}}}{\pi} \right)^2 \left\{ \frac{\beta_0^{\overline{\text{MS}}}}{8} \frac{1}{\varepsilon^2} \left[ \frac{3}{4\varepsilon} \gamma_{Ki}^{(1)} + \mathcal{G}_i^{(1)}(\varepsilon) \right] - \frac{1}{8} \left[ \frac{\gamma_{Ki}^{(2)}}{4\varepsilon^2} + \frac{\mathcal{G}_i^{(2)}(\varepsilon)}{\varepsilon} \right] \right\} + \dots, \end{aligned} \quad (3)$$

where

$$\begin{aligned}
\gamma_{Ki}^{(1)} &= 2C_i, & \gamma_{Ki}^{(2)} &= C_i K = C_i \left[ C_A \left( \frac{67}{18} - \zeta_2 \right) - \frac{10}{9} T_f N_f \right], & C_q &\equiv C_F, & C_g &\equiv C_A, \\
\mathcal{G}_q^{(1)} &= \frac{3}{2} C_F + \frac{\varepsilon}{2} C_F (8 - \zeta_2), & \mathcal{G}_g^{(1)} &= 2\beta_0^{\overline{\text{MS}}} - \frac{\varepsilon}{2} C_A \zeta_2, \\
\mathcal{G}_q^{(2)} &= C_F^2 \left( \frac{3}{16} - \frac{3}{2} \zeta_2 + 3 \zeta_3 \right) + C_F C_A \left( \frac{2545}{432} + \frac{11}{12} \zeta_2 - \frac{13}{4} \zeta_3 \right) - C_F T_f N_f \left( \frac{209}{108} + \frac{1}{3} \zeta_2 \right), \\
\mathcal{G}_g^{(2)} &= 4\beta_1^{\overline{\text{MS}}} + C_A^2 \left( \frac{10}{27} - \frac{11}{12} \zeta_2 - \frac{1}{4} \zeta_3 \right) + C_A T_f N_f \left( \frac{13}{27} + \frac{1}{3} \zeta_2 \right) + \frac{1}{2} C_F T_f N_f.
\end{aligned} \tag{4}$$

Although  $\mathcal{G}_i$  and  $\gamma_{Ki}$  are defined through the Sudakov form factor, they can be extracted from fixed-order calculations [28–34].  $\gamma_{Ki}$  is the cusp anomalous dimension and represents a pure pole term. The  $\mathcal{G}_i$  anomalous dimensions contain terms at higher order in  $\varepsilon$ , but I only keep terms in the expansion that contribute poles to  $\ln(\mathcal{G}_i)$ .  $C_F = (N_c^2 - 1)/(2N_c)$  denotes the Casimir operator of the fundamental representation of  $SU(N_c)$ , while  $C_A = N_c$  denotes the Casimir of the adjoint representation.  $N_f$  is the number of quark flavors and  $T_f = 1/2$  is the normalization of the QCD charge of the fundamental representation.  $\zeta_n = \sum_{k=1}^{\infty} 1/k^n$  represents the Riemann zeta-function of integer argument  $n$ .

## B. The soft function in the HV/CDR schemes

The soft function is determined entirely by the soft anomalous dimension matrix  $\mathbf{\Gamma}_{S_f}$ ,

$$\begin{aligned}
\mathbf{S}_f^{\text{CDR}} \left( p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon \right) &= 1 + \frac{1}{2\varepsilon} \left( \frac{\alpha_s^{\overline{\text{MS}}}}{\pi} \right) \mathbf{\Gamma}_{S_f}^{(1)} + \frac{1}{8\varepsilon^2} \left( \frac{\alpha_s^{\overline{\text{MS}}}}{\pi} \right)^2 \mathbf{\Gamma}_{S_f}^{(1)} \times \mathbf{\Gamma}_{S_f}^{(1)} \\
&\quad - \frac{\beta_0^{\overline{\text{MS}}}}{4\varepsilon^2} \left( \frac{\alpha_s^{\overline{\text{MS}}}}{\pi} \right)^2 \mathbf{\Gamma}_{S_f}^{(1)} + \frac{1}{4\varepsilon} \left( \frac{\alpha_s^{\overline{\text{MS}}}}{\pi} \right)^2 \mathbf{\Gamma}_{S_f}^{(2)} + \dots
\end{aligned} \tag{5}$$

In the color-space notation of Refs. [14, 26, 27], the soft anomalous dimension is given by [16, 17]

$$\mathbf{\Gamma}_{S_f}^{(1)} = \frac{1}{2} \sum_{i \in \mathbf{f}} \sum_{j \neq i} \mathbf{T}_i \cdot \mathbf{T}_j \ln \left( \frac{\mu^2}{-s_{ij}} \right), \quad \mathbf{\Gamma}_{S_f}^{(2)} = \frac{K}{2} \mathbf{\Gamma}_{S_f}^{(1)}, \tag{6}$$

where  $K = C_A (67/18 - \zeta_2) - 10 T_f N_f / 9$  is the same constant that relates the one- and two-loop cusp anomalous dimensions. The  $\mathbf{T}_i$  are the color generators in the representation of parton  $i$  (multiplied by  $(-1)$  for incoming quarks and gluons and outgoing anti-quarks).

## IV. THE FDH SCHEME AT TWO LOOPS

The failure of the FDH scheme as a unitary regularization scheme does not mean that it is of no value in computing higher-order corrections beyond the next-to-leading order. Even at NLO, the FDH scheme has

always been used as a means of obtaining scattering amplitudes in the HV scheme. There is no reason for that to change at two loops. The only difference is that one must recognize that the FDH scheme result is not a physical scattering amplitude, but only an intermediate step toward obtaining one.

In formulating a prescription for converting FDH scheme amplitudes into HV scheme amplitudes, the first problem to address, of course, is that of renormalization. One solution to the renormalization problem, dubbed “dimensional reconstruction,” has been proposed by Boughezal, et al. [6]. The idea behind dimensional reconstruction is that if one knows the the one-loop behavior of an amplitude with arbitrary (integer) numbers of extra spin dimensions (momenta are always  $D_m$  dimensional) then the correct two-loop amplitude can be determined from the renormalization constants at different integer spin dimensions. Note that it is a basic assumption of dimensional reconstruction that when one is computing a two-loop amplitude, the tree-level and one-loop terms that contribute via renormalization are essentially trivial, and that there is no appreciable cost to performing extra one-loop calculations if doing so saves effort on the two-loop piece. The transformations that I will develop will also subscribe to this viewpoint.

While dimensional reconstruction is a completely valid approach to the renormalization problem of the FDH scheme, it does have some drawbacks. One drawback is that it appears that one must determine new renormalization constants for each process at each order of perturbation theory. This is quite different from working within a renormalizable theory, where the renormalization constants can be determined in advance through the study of corrections to one-particle-irreducible Green functions. A more serious drawback is that dimensional reconstruction does not address the infrared structure of amplitudes computed in the FDH scheme.

It is certain that the infrared structure of FDH scheme amplitudes is not equal to that of HV scheme amplitudes. It is also clear from optical theorem calculations [3, 6] that once the renormalization problem is fixed, one could proceed with FDH scheme calculations because the infrared overlaps will sort themselves out. For differential calculations, one needs to know the soft and collinear factorization properties of FDH scheme amplitudes in order to implement a subtraction scheme, but this has already been worked out [35–37]. The problem is that all of the FDH scheme amplitudes, real and virtual, contain errors, though the structure of the errors is such that, after renormalization, they cancel in the inclusive sum. Even if one were willing to live with such circumstances, one would still want to match onto standard definitions of the running coupling and would have to face the fact that parton distribution functions are only available in the CDR scheme. A far better choice is to transform the result to a framework like the HV scheme that is known to be unitary and correct and which can be easily connected to the parton distribution functions.



### A. The connection between the FDH and DR schemes

In order to develop a rigorous set of rules for transforming FDH amplitudes, it is necessary to define the FDH scheme in terms of a renormalizable scheme. One can do this by exploiting the close connection between the FDH and DR schemes. When formulating the QCD Lagrangians in these schemes, one starts with the standard Yang-Mills Lagrangian and then extends the fields into  $D_s$ -dimensions. In the FDH scheme, one proceeds directly to the development of Feynman rules involving the  $D_s$ -dimensional metric tensor and Dirac matrices [1, 2]. In the DR scheme, however, one first splits the gluon field into two independent components, the  $D_m$ -dimensional gauge field and the  $D_x$ -dimensional evanescent field [9, 38, 39]. The metric tensor and Dirac matrices also decompose into orthogonal components. Those new terms in the Lagrangian that do not involve gauge fields are assigned new, independent couplings. The evanescent-quark-antiquark coupling is given the value  $g_e$  ( $g_e^2 = 4\pi\alpha_e$ ) and the quartic evanescent boson couplings are given values  $\eta_{i,i=1,2,3}$ , where  $\eta_1$  represents the quartic interaction that has the same color flow as the quartic gluon coupling, while  $\eta_{2,3}$  represent the non-QCD-like interactions.

Thus, all of the DR scheme interactions are contained in those of the FDH scheme, they are simply not labeled by independent couplings and evanescent Lorentz structures. The only exception to this statement concerns the quartic evanescent boson couplings. Because the evanescent bosons are not protected by gauge symmetry, new quartic interactions, with new color-flows among the evanescent bosons, are generated by higher-order corrections which must be renormalized independently of the QCD-like quartic coupling that appears in the classical Lagrangian. In recognition of the fact that such terms will occur, they are usually assigned independent couplings and added to the effective DR Lagrangian. The FDH scheme doesn't have such couplings, but this does not present a problem. The extra quartic terms introduced to the DR Lagrangian clean up the renormalization procedure, but there is no reason that the couplings assigned to these terms could not be chosen such that they do not contribute to a DR scattering process until radiative corrections to the QCD-like interactions demand that they appear.

### B. The connection between the DR and CDR schemes

From the formulation of the Lagrangians, one can also draw a connection between the structure of the amplitudes in the DR and CDR schemes. In particular, the DR scheme Lagrangian contains all of the interactions that the CDR scheme Lagrangian does, plus a host of interactions involving the evanescent bosons. This means that the amplitudes in the DR scheme can be partitioned into a part that is identical to the CDR scheme amplitude and a part that involves the exchange of one or more evanescent bosons. One

need not consider the case of external evanescent bosons since the DR scheme renormalization program ensures that such terms contribute to the S-matrix at order  $\varepsilon$  [9, 40]. The DR scheme sub-amplitude that involves evanescent exchanges will necessarily include a spin-sum over the evanescent degrees of freedom, with the result that this sub-amplitude will be weighted by a factor of  $D_x = 2\varepsilon$ . The only way that a term from the evanescent sub-amplitude can make a finite (or singular) contribution to the full amplitude is if it is weighted by ultraviolet or infrared poles. Thus, the full evanescent contribution to an amplitude up to order  $\varepsilon^0$  is part of the universal (ultraviolet or infrared) structure of the amplitude, and is controlled by anomalous dimensions. This means that the evanescent contribution to an  $n$ -loop amplitude (that is the part that is different from the CDR amplitude) can be determined entirely in terms of ultraviolet counterterms, jet and soft functions and lower-order (0 to  $(n-1)$ -loop) hard-scattering functions. Thus, with a proper rearrangement of terms (the  $\widehat{\text{DR}}$  scheme defined below), at any order  $n$  the hard-scattering functions in the two schemes are related by

$$\left| H_{\mathbf{f}}^{(n)} \right\rangle_{\widehat{\text{DR}}} = \left| H_{\mathbf{f}}^{(n)} \right\rangle_{\text{HV}} + \mathcal{O}(\varepsilon). \quad (7)$$

### C. A new definition of the FDH scheme

Clearly, if one can draw a close connection between the FDH and DR schemes, one should be able to develop a prescription for the direct transformation of an amplitude computed in the FDH scheme to one that is computed in the HV scheme. From the above considerations, it is quite simple to state the connection.

The four-dimensional helicity scheme *is* the DR scheme with two extra conditions:

1. External states are taken to be four dimensional.
2. The evanescent couplings ( $\alpha_e$  and  $\eta_1$ ) are identified with  $\alpha_s$ .

The first condition asserts the same distinction between the FDH and DR schemes as exists between the HV and CDR schemes. The restriction to four-dimensional external states does not affect the anomalous dimensions of the theory. The ultraviolet counterterms and the jet and soft functions are unchanged. The only changes are to the exact form of the finite hard-scattering matrix elements. The four-dimensional condition also forbids the appearance of external evanescent states. As mentioned before, the renormalization program of the DR scheme ensures that evanescent external states can only contribute to the S-matrix at order  $\varepsilon$  or higher, so this restriction is of no consequence.

The second condition is the one that violates unitarity and renders the FDH non-renormalizable. The evanescent couplings need to be renormalized differently than the QCD coupling, but there is no means of

doing so once the couplings have been identified. Therefore, the FDH can only be used to compute bare (unrenormalized) loop amplitudes.

In the DR scheme, on the other hand, one can determine the correct ultraviolet counterterms, and the infrared counterterms needed to remove the evanescent contribution, leaving the HV scheme result. By computing these counterterms in the DR scheme and then identifying the couplings, one obtains the counterterms needed to shift from the FDH to the HV scheme.

#### D. Ultraviolet counterterms for the FDH

When working within massless QCD, it is only necessary to renormalize the couplings. It is common in dimensional reduction to determine ultraviolet counterterms using modified minimal subtraction (this is known as the  $\overline{\text{DR}}$  scheme), dropping evanescent terms, even if they contain ultraviolet poles, because the factor of  $D_x$  renders them finite. This procedure means that the renormalized coupling in the  $\overline{\text{DR}}$  scheme,  $\alpha_s^{\overline{\text{DR}}}$  differs from the standard coupling  $\alpha_s^{\overline{\text{MS}}}$  that appears in HV/CDR calculations by a finite renormalization. This finite renormalization corresponds precisely to the  $D_x/\varepsilon$  terms that were dropped from the  $\beta$ -function. My goal is to remove all evanescent contributions, so I will include  $(D_x/\varepsilon)^n$  terms in my definitions of the  $\beta$ -functions and anomalous dimensions. To distinguish it from the  $\overline{\text{DR}}$  scheme, I will call this the  $\widehat{\text{DR}}$  scheme.

Because there are so many independent couplings in the DR scheme, and because they mix under renormalization, the simple  $\beta_{0,1,2,\dots}$  labeling of the  $\overline{\text{MS}}$  scheme is insufficient. Instead, I write,

$$\begin{aligned} \beta^{\widehat{\text{DR}}} &= \mu^2 \frac{d}{d\mu^2} \frac{\alpha_s^{\widehat{\text{DR}}}}{\pi} = - \left( \varepsilon \frac{\alpha_s^{\widehat{\text{DR}}}}{\pi} + \frac{\alpha_s^{\widehat{\text{DR}}}}{Z_{\alpha_s^{\widehat{\text{DR}}}}} \frac{\partial Z_{\alpha_s^{\widehat{\text{DR}}}}}{\partial \alpha_e^{\widehat{\text{DR}}}} \beta_e^{\widehat{\text{DR}}} + \frac{\alpha_s^{\widehat{\text{DR}}}}{Z_{\alpha_s^{\widehat{\text{DR}}}}} \frac{\partial Z_{\alpha_s^{\widehat{\text{DR}}}}}{\partial \eta_i^{\widehat{\text{DR}}}} \beta_{\eta_i}^{\widehat{\text{DR}}} \right) \left( 1 + \frac{\alpha_s^{\widehat{\text{DR}}}}{Z_{\alpha_s^{\widehat{\text{DR}}}}} \frac{\partial Z_{\alpha_s^{\widehat{\text{DR}}}}}{\partial \alpha_s^{\widehat{\text{DR}}}} \right)^{-1} \\ &= -\varepsilon \frac{\alpha_s^{\widehat{\text{DR}}}}{\pi} - \sum_{i,j,k,l,m} \beta_{ijklm}^{\widehat{\text{DR}}} \left( \frac{\alpha_s^{\widehat{\text{DR}}}}{\pi} \right)^i \left( \frac{\alpha_e^{\widehat{\text{DR}}}}{\pi} \right)^j \left( \frac{\eta_1^{\widehat{\text{DR}}}}{\pi} \right)^k \left( \frac{\eta_2^{\widehat{\text{DR}}}}{\pi} \right)^l \left( \frac{\eta_3^{\widehat{\text{DR}}}}{\pi} \right)^m. \end{aligned} \quad (8)$$

Similar equations yield

$$\begin{aligned} \beta_e^{\widehat{\text{DR}}} &= \mu^2 \frac{d}{d\mu^2} \frac{\alpha_e^{\widehat{\text{DR}}}}{\pi} = -\varepsilon \frac{\alpha_e^{\widehat{\text{DR}}}}{\pi} - \sum_{i,j,k,l,m} \beta_{e,ijklm}^{\widehat{\text{DR}}} \left( \frac{\alpha_s^{\widehat{\text{DR}}}}{\pi} \right)^i \left( \frac{\alpha_e^{\widehat{\text{DR}}}}{\pi} \right)^j \left( \frac{\eta_1^{\widehat{\text{DR}}}}{\pi} \right)^k \left( \frac{\eta_2^{\widehat{\text{DR}}}}{\pi} \right)^l \left( \frac{\eta_3^{\widehat{\text{DR}}}}{\pi} \right)^m, \\ \beta_{\eta_s}^{\widehat{\text{DR}}} &= \mu^2 \frac{d}{d\mu^2} \frac{\eta_s^{\widehat{\text{DR}}}}{\pi} = -\varepsilon \frac{\eta_s^{\widehat{\text{DR}}}}{\pi} - \sum_{i,j,k,l,m} \beta_{s,ijklm}^{\widehat{\text{DR}}} \left( \frac{\alpha_s^{\widehat{\text{DR}}}}{\pi} \right)^i \left( \frac{\alpha_e^{\widehat{\text{DR}}}}{\pi} \right)^j \left( \frac{\eta_1^{\widehat{\text{DR}}}}{\pi} \right)^k \left( \frac{\eta_2^{\widehat{\text{DR}}}}{\pi} \right)^l \left( \frac{\eta_3^{\widehat{\text{DR}}}}{\pi} \right)^m. \end{aligned} \quad (9)$$

The values of the coefficients through three loops (for  $\beta^{\widehat{\text{DR}}}$  and  $\beta_e^{\widehat{\text{DR}}}$ ) are given in Appendix A. Note that with the rearrangement of the evanescent contributions, the terms in  $\beta^{\widehat{\text{DR}}}$  that are not proportional to  $D_x$  are identical to the coefficients of the  $\beta$ -function in the  $\overline{\text{MS}}$  scheme. This indicates that the renormalized coupling of the  $\widehat{\text{DR}}$  scheme coincides with that of the  $\overline{\text{MS}}$  scheme.

The ultraviolet counterterms for FDH amplitudes are computed as follows. First, one computes the lower loop amplitudes in the DR scheme and then expands the bare couplings in terms of the renormalized couplings using the  $\beta$ -functions of the  $\widehat{\text{DR}}$  scheme. Finally, the evanescent couplings are identified with the QCD coupling and the factors of  $D_x$  are evaluated ( $D_x = 2\epsilon$ ).

$$|\mathcal{M}(\alpha_s)\rangle_{\text{FDH}}^{\text{CT}} = |\mathcal{M}(\alpha_s, \alpha_e, \eta_1)\rangle_{\widehat{\text{DR}}}^{\text{CT}} \Big|_{\substack{\alpha_e, \eta_1 \rightarrow \alpha_s \\ D_x \rightarrow 2\epsilon}} \quad (10)$$

This will remove all of the ultraviolet terms, including the evanescent terms that appear to be finite because of the factor of  $D_x$ .

### E. The infrared structure of the DR scheme

The next step is to remove the unwanted evanescent component of the infrared structure of FDH scheme amplitudes. As with the ultraviolet counterterms, the terms to be removed can be identified by studying the structure of DR scheme amplitudes. The basic form of the infrared structure in the DR scheme is the same as in HV/CDR, but the anomalous dimensions receive evanescent corrections. In addition, there are new  $\mathcal{G}$  anomalous dimensions that depend on the evanescent couplings. Through two-loops, the corrections and new anomalous dimensions depend only on the fermion-evanescent coupling, not the quartic evanescent couplings. Furthermore, because the evanescent couplings are not gauge couplings, there are no new counterparts to the cusp or soft anomalous dimensions, which are associated with the exchange of gauge bosons.

I have determined the values of the infrared anomalous dimensions in the DR scheme by the direct calculation of two-loop amplitudes. I first determine the anomalous dimensions for external quarks from the Drell-Yan amplitude. I then obtain the anomalous dimensions for external gluons from the  $q\bar{q} \rightarrow g\gamma$  amplitude [41–43]. In principle, it would be easier to extract the gluon jet function by calculating the amplitude for  $gg \rightarrow H$ , but the Higgs - gluon coupling is governed by a set of effective operators generated by integrating out the top quark. This system, involving operator mixing and higher-order corrections to the Wilson coefficients, has been studied to high order in the CDR scheme [44, 45], but not in the non-supersymmetric DR scheme.

The calculations of the infrared anomalous dimensions as well as the wave-function and vertex corrections used to extract the  $\beta$ -functions were all calculated within the same framework. The Feynman diagrams were generated with QGRAF [46] and the symbolic algebra program FORM [47] was used to implement the Feynman rules and perform algebraic manipulations to reduce the result to a set of Feynman integrals

and their coefficients. The method of Ref. [48] was used to reduce the calculation of the vertex corrections to propagator integrals. The full set of Feynman integrals was reduced to master integrals using the program REDUZE-2 [49]. REDUZE-2 offers significant improvements over the previous version [50] and was particularly effective at reducing the non-planar double-box integrals that contribute to the  $q\bar{q} \rightarrow g\gamma$  amplitude. All of the master integrals needed for these calculations are known in the literature [51–57].

The jet function in the DR scheme takes the form,

$$\begin{aligned} \ln \widehat{\mathcal{J}}_i^{\text{DR}}(\alpha_s(\mu^2), \alpha_e(\mu^2), \varepsilon) = & - \left( \frac{\alpha_s^{\overline{\text{MS}}}}{\pi} \right) \left[ \frac{1}{8\varepsilon^2} \hat{\gamma}_{Ki}^{(1)} + \frac{1}{4\varepsilon} \hat{\mathcal{G}}_i^{(1)}(\varepsilon) \right] - \left( \frac{\alpha_e^{\widehat{\text{DR}}}}{\pi} \right) \frac{\hat{\mathcal{G}}_{i,e}^{(0,1)}(\varepsilon)}{4\varepsilon} \\ & + \left( \frac{\alpha_s^{\overline{\text{MS}}}}{\pi} \right)^2 \left[ \frac{\beta_{20}^{\widehat{\text{DR}}}}{8} \frac{1}{\varepsilon^2} \left( \frac{3}{4\varepsilon} \hat{\gamma}_{Ki}^{(1)} + \hat{\mathcal{G}}_i^{(1)}(\varepsilon) \right) - \frac{1}{8} \left( \frac{\hat{\gamma}_{Ki}^{(2)}}{4\varepsilon^2} + \frac{\hat{\mathcal{G}}_i^{(2)}(\varepsilon)}{\varepsilon} \right) \right] \\ & + \left( \frac{\alpha_s^{\overline{\text{MS}}}}{\pi} \right) \left( \frac{\alpha_e^{\widehat{\text{DR}}}}{\pi} \right) \frac{1}{8} \left[ \frac{\beta_{e,11}^{\widehat{\text{DR}}} \hat{\mathcal{G}}_{i,e}^{(0,1)}(\varepsilon) - \hat{\mathcal{G}}_{i,e}^{(1,1)}(\varepsilon)}{\varepsilon^2} \right] \\ & + \left( \frac{\alpha_e^{\widehat{\text{DR}}}}{\pi} \right)^2 \frac{1}{8} \left[ \frac{\beta_{e,02}^{\widehat{\text{DR}}} \hat{\mathcal{G}}_{i,e}^{(0,1)}(\varepsilon) - \hat{\mathcal{G}}_{i,e}^{(0,2)}(\varepsilon)}{\varepsilon^2} \right] + \dots, \end{aligned} \quad (11)$$

where the anomalous dimensions in the  $\widehat{\text{DR}}$  scheme are

$$\begin{aligned} \hat{\gamma}_{Ki}^{(1)} &= 2C_i, \quad \hat{\gamma}_{Ki}^{(2)} = C_i \hat{K} = C_i \left[ C_A \left( \frac{67}{18} - \zeta_2 \right) - \frac{10}{9} T_f N_f - \frac{2}{9} D_x C_A \right], \quad C_q \equiv C_F, \quad C_g \equiv C_A, \\ \hat{\mathcal{G}}_q^{(1)} &= \frac{3}{2} C_F + \frac{\varepsilon}{2} C_F (8 - \zeta_2), \quad \hat{\mathcal{G}}_g^{(1)} = 2\beta_{20}^{\widehat{\text{DR}}} - \frac{\varepsilon}{2} C_A \zeta_2, \\ \hat{\mathcal{G}}_{q,e}^{(0,1)} &= -\frac{1}{4} D_x C_F, \quad \hat{\mathcal{G}}_{g,e}^{(0,1)} = 0, \\ \hat{\mathcal{G}}_q^{(2)} &= C_F^2 \left( \frac{3}{16} - \frac{3}{2} \zeta_2 + 3 \zeta_3 \right) + C_A C_F \left( \frac{2545}{432} + \frac{11}{12} \zeta_2 - \frac{13}{4} \zeta_3 \right) - C_F T_f N_f \left( \frac{209}{108} + \frac{1}{3} \zeta_2 \right) \\ &\quad - D_x C_A C_F \left( \frac{311}{864} + \frac{1}{24} \zeta_2 \right), \\ \hat{\mathcal{G}}_g^{(2)} &= 4\beta_{30}^{\widehat{\text{DR}}} + C_A^2 \left( \frac{10}{27} - \frac{11}{12} \zeta_2 - \frac{1}{4} \zeta_3 \right) + C_A T_f N_f \left( \frac{13}{27} + \frac{1}{3} \zeta_2 \right) + \frac{1}{2} C_F T_f N_f \\ &\quad + D_x C_A^2 \left( \frac{7}{54} + \frac{1}{24} \zeta_2 \right), \\ \hat{\mathcal{G}}_{q,e}^{(1,1)} &= D_x \left( -\frac{11}{16} C_A C_F + \frac{1}{4} C_F^2 + \frac{1}{4} C_F^2 \zeta_2 \right), \quad \hat{\mathcal{G}}_{g,e}^{(1,1)} = 2\beta_{21}^{\widehat{\text{DR}}}, \\ \hat{\mathcal{G}}_{q,e}^{(0,2)} &= \frac{3}{16} D_x C_F T_f N_f, \quad \hat{\mathcal{G}}_{g,e}^{(0,2)} = 0, \\ \beta_{20}^{\widehat{\text{DR}}} &= \frac{11}{12} C_A - \frac{1}{6} N_f - \frac{1}{24} D_x C_A, \\ \beta_{30}^{\widehat{\text{DR}}} &= \frac{17}{24} C_A^2 - \frac{5}{24} C_A N_f - \frac{1}{8} C_F N_f - \frac{7}{48} D_x C_A^2, \quad \beta_{21}^{\widehat{\text{DR}}} = \frac{1}{16} D_x C_F N_f, \\ \beta_{e,02}^{\widehat{\text{DR}}} &= \frac{1}{2} C_A - C_F - \frac{1}{4} N_f - \frac{1}{4} D_x (C_A - C_F), \quad \beta_{e,11}^{\widehat{\text{DR}}} = \frac{3}{2} C_F. \end{aligned} \quad (12)$$

Note that the QCD coupling is  $\alpha_s^{\overline{\text{MS}}}$ , the same coupling used in HV/CDR calculations. Since I extract the

anomalous dimensions from amplitude calculations, I cannot separate the order  $\varepsilon$  part of the one-loop  $\widehat{\mathcal{G}}$  anomalous dimensions, which contributes at two-loops when multiplied by a  $\beta$ -function coefficient, from the pure two-loop  $\widehat{\mathcal{G}}$  anomalous dimensions. This merely constitutes a rearrangement of terms and does not affect the prediction of the infrared structure.

The soft function changes very little in going to the DR scheme. This is because evanescent exchanges do not add new soft anomalous dimensions, they only add corrections to the existing terms.

$$\begin{aligned} \widehat{\mathbf{S}}_{\mathbf{f}}^{\text{DR}}\left(p_i, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \varepsilon\right) = & 1 + \frac{1}{2\varepsilon} \left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right) \widehat{\Gamma}_{S_f}^{(1)} + \frac{1}{8\varepsilon^2} \left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^2 \widehat{\Gamma}_{S_f}^{(1)} \times \widehat{\Gamma}_{S_f}^{(1)} \\ & - \frac{\beta_{20}^{\widehat{\text{DR}}}}{4\varepsilon^2} \left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^2 \widehat{\Gamma}_{S_f}^{(1)} + \frac{1}{4\varepsilon} \left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^2 \widehat{\Gamma}_{S_f}^{(2)}, \end{aligned} \quad (13)$$

$$\widehat{\Gamma}_{S_f}^{(1)} = \frac{1}{2} \sum_{i \in \mathbf{f}} \sum_{j \neq i} \mathbf{T}_i \cdot \mathbf{T}_j \ln\left(\frac{\mu^2}{-s_{ij}}\right), \quad \widehat{\Gamma}_{S_f}^{(2)} = \frac{\hat{K}}{2} \widehat{\Gamma}_{S_f}^{(1)}, \quad (14)$$

where  $\hat{K} = C_A(67/18 - \zeta_2) - 10/9 T_f N_f - 2/9 D_x C_A$  is again the same constant that relates the one- and two-loop cusp anomalous dimensions, this time in the  $\widehat{\text{DR}}$  scheme.

### F. Transforming FDH amplitudes into HV amplitudes

I have now assembled all of the pieces needed to convert bare amplitudes computed in the FDH scheme into renormalized amplitudes in the HV scheme. To obtain an  $n$ -loop amplitude in the HV scheme, one needs

1. The bare  $n$ -loop amplitude in the FDH scheme.
2. The renormalized  $m$ -loop amplitudes ( $m \in \{0, \dots, n-1\}$ ) to order  $\varepsilon^{2(n-m)}$  in the HV scheme.
3. The jet and soft functions to order  $n$  in the HV scheme.
4. The renormalized  $m$ -loop amplitudes ( $m \in \{0, \dots, n-1\}$ ) to order  $\varepsilon^{2(n-m)}$  in the  $\widehat{\text{DR}}$  scheme.
5. The jet and soft functions to order  $n$  in the  $\widehat{\text{DR}}$  scheme.

Note that computing the  $n$ -loop squared amplitude to order  $\varepsilon^0$  already required the higher-order in  $\varepsilon$  contributions to the lower-loop amplitudes in the HV scheme. The conversion procedure requires them in the  $\widehat{\text{DR}}$  scheme as well.

The first step is to expand Eq. (1) by orders of  $\alpha_s$ ,

$$\begin{aligned} \left| \mathcal{M}^{(n)} \right\rangle_{\text{HV}} &= \sum_{i=0}^n [\mathcal{J} \otimes \mathbf{S}]^{(i)} \left| \mathcal{H}^{(n-i)} \right\rangle_{\text{HV}} \\ \left| \mathcal{M}^{(n)} \right\rangle_{\widehat{\text{DR}}} &= \sum_{i=0}^n \left[ \widehat{\mathcal{J}} \otimes \widehat{\mathbf{S}} \right]^{(i)} \left| \mathcal{H}^{(n-i)} \right\rangle_{\widehat{\text{DR}}} \end{aligned} \quad (15)$$

I now define the ‘‘renormalized’’ FDH scheme amplitude as

$$\left| \mathcal{M}^{(n)} \right\rangle_{\text{FDH}} = \left| \mathcal{M}^{(n)} \right\rangle_{\text{FDH}}^{\text{Bare}} + \left| \mathcal{M}^{(n)} \right\rangle_{\text{FDH}}^{\text{CT}} = \left| \mathcal{M}^{(n)} \right\rangle_{\widehat{\text{DR}}} \Big|_{\substack{\alpha_e, \eta_1 \rightarrow \alpha_s \\ D_x \rightarrow 2\varepsilon}}. \quad (16)$$

From this I find that

$$\left| \mathcal{H}^{(n)} \right\rangle_{\widehat{\text{DR}}} \Big|_{\substack{\alpha_e, \eta_1 \rightarrow \alpha_s \\ D_x \rightarrow 2\varepsilon}} = \left| \mathcal{M}^{(n)} \right\rangle_{\text{FDH}} - \sum_{i=1}^n \left[ \widehat{\mathcal{J}} \otimes \widehat{\mathbf{S}} \right]^{(i)} \left| \mathcal{H}^{(n-i)} \right\rangle_{\widehat{\text{DR}}} \Big|_{\substack{\alpha_e, \eta_1 \rightarrow \alpha_s \\ D_x \rightarrow 2\varepsilon}}. \quad (17)$$

Finally, using Eq. (7), I obtain

$$\left| \mathcal{H}^{(n)} \right\rangle_{\text{HV}} = \left| \mathcal{M}^{(n)} \right\rangle_{\text{FDH}}^{\text{Bare}} + \left| \mathcal{M}^{(n)} \right\rangle_{\text{FDH}}^{\text{CT}} - \sum_{i=1}^n \left[ \widehat{\mathcal{J}} \otimes \widehat{\mathbf{S}} \right]^{(i)} \left| \mathcal{H}^{(n-i)} \right\rangle_{\widehat{\text{DR}}} \Big|_{\substack{\alpha_e, \eta_1 \rightarrow \alpha_s \\ D_x \rightarrow 2\varepsilon}} + \mathcal{O}(\varepsilon). \quad (18)$$

The infrared structure of the HV scheme amplitude can be extracted from  $\left| \mathcal{M}^{(n)} \right\rangle_{\text{FDH}}^{\text{Bare}}$  in a similar way or constructed directly in terms of the lower order hard scattering matrix elements and the jet and soft functions.

Let me now write out explicitly the transformation of a one-loop bare amplitude in the FDH scheme, involving  $n_q$  quarks and anti-quarks and  $n_g$  gluons, into a renormalized one-loop amplitude in the HV scheme. Starting with

$$\left| \mathcal{H}^{(1)} \right\rangle_{\text{HV}} = \left| \mathcal{M}^{(1)} \right\rangle_{\text{FDH}}^{\text{Bare}} + \left| \mathcal{M}^{(1)} \right\rangle_{\text{FDH}}^{\text{CT}} - \left[ \widehat{\mathcal{J}} + \widehat{\mathbf{S}} \right]^{(1)} \left| \mathcal{H}^{(0)} \right\rangle_{\widehat{\text{DR}}} \Big|_{\substack{\alpha_e, \eta_1 \rightarrow \alpha_s \\ D_x \rightarrow 2\varepsilon}} + \mathcal{O}(\varepsilon), \quad (19)$$

I add in the infrared parts of the HV amplitude (note that the one-loop soft functions of the HV and  $\widehat{\text{DR}}$  scheme are identical) to obtain

$$\begin{aligned} \left| \mathcal{M}^{(1)} \right\rangle_{\text{HV}} &= \left| \mathcal{M}^{(1)} \right\rangle_{\text{FDH}}^{\text{Bare}} - \left( \frac{\alpha_s^{\overline{\text{MS}}}}{\pi} \right) \left( \frac{n_q + n_g - 2}{2\varepsilon} \beta_{20}^{\widehat{\text{DR}}} \right) \left| \mathcal{H}^{(0)} \right\rangle_{\text{HV}} \\ &\quad + \left( \mathcal{J}^{(1)} - \widehat{\mathcal{J}}^{(1)} \right) \Big|_{\substack{\alpha_e, \eta_1 \rightarrow \alpha_s \\ D_x \rightarrow 2\varepsilon}} \left| \mathcal{H}^{(0)} \right\rangle_{\text{HV}} + \mathcal{O}(\varepsilon) \\ &= \left| \mathcal{M}^{(1)} \right\rangle_{\text{FDH}}^{\text{Bare}} - \left( \frac{\alpha_s^{\overline{\text{MS}}}}{\pi} \right) \frac{n_q + n_g - 2}{2\varepsilon} \beta_0^{\overline{\text{MS}}} \left| \mathcal{H}^{(0)} \right\rangle_{\text{HV}} \\ &\quad + \left( \frac{\alpha_s^{\overline{\text{MS}}}}{\pi} \right) \left( \frac{n_q + n_g - 2}{24} C_A - \frac{n_q}{8} C_F - \frac{n_g}{24} C_A \right) \left| \mathcal{H}^{(0)} \right\rangle_{\text{HV}} + \mathcal{O}(\varepsilon) \end{aligned} \quad (20)$$

The first line is just the bare one-loop amplitude with standard  $\overline{\text{MS}}$  ultraviolet counterterm, while the second line is the finite shift, broken into ultraviolet, infrared  $n_q$  and infrared  $n_g$  pieces, identified by Kunszt, et al. [5]. Beyond one loop, the transformations are not so simple and involve the structure of the amplitudes in addition to the identities of the external states.

## V. CONCLUSION

In this paper, I have described a procedure for transforming bare loop amplitudes computed in the four dimensional helicity scheme into renormalized amplitudes in the 't Hooft-Veltman scheme. One of the simplifying features of the FDH, the treatment of the evanescent states as if they were gluons, renders the scheme non-renormalizable. Nevertheless, the FDH can be defined in terms of a renormalizable scheme, a variant of the dimensional reduction scheme. Through this connection to the DR scheme, I have shown that the differences between amplitudes calculated in the FDH scheme and the HV scheme (up to order  $\varepsilon^-$ ) are either ultraviolet or infrared in origin and are therefore part of the universal structure of the amplitude which is controlled by anomalous dimensions. By computing these anomalous dimensions in the  $\widehat{\text{DR}}$  scheme, defined above, through two loops, I provide concrete formulæ for the transformation of the amplitudes.

The utility of such transformations lies in the close connection between the FDH scheme and the techniques of generalized unitarity and the helicity method. These techniques are a natural fit for the FDH scheme, but the results need to be transformed into a renormalizable scheme so that they can be used in practical calculations. With the procedures described in this paper, such transformations can be performed.

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### Appendix A: $\widehat{\text{DR}}$ Scheme $\beta$ -functions

In this appendix, I present the QCD and evanescent  $\beta$ -functions in the  $\widehat{\text{DR}}$  subtraction scheme. In the DR regularization scheme, momenta are dimensionally regularized and taken to be  $D_m = 4 - 2\varepsilon$  dimensional. Spin degrees of freedom are defined in a  $D_s = 4$  dimensional vector space. Despite the numerical values assigned to the dimensionalities, as vector spaces  $D_s \supset D_m \supset 4$ . The symbol  $D_x = 2\varepsilon$  is used to denote the difference between the  $D_s$  and  $D_m$  dimensional vector spaces.

The  $\beta$ -functions are defined as

$$\begin{aligned}
 \beta^{\widehat{\text{DR}}} &= \mu^2 \frac{d}{d\mu^2} \frac{\alpha_s^{\widehat{\text{DR}}}}{\pi} = -\varepsilon \frac{\alpha_s^{\widehat{\text{DR}}}}{\pi} - \sum_{i,j,k,l,m} \beta_{ijklm}^{\widehat{\text{DR}}} \left( \frac{\alpha_s^{\widehat{\text{DR}}}}{\pi} \right)^i \left( \frac{\alpha_e^{\widehat{\text{DR}}}}{\pi} \right)^j \left( \frac{\eta_1^{\widehat{\text{DR}}}}{\pi} \right)^k \left( \frac{\eta_2^{\widehat{\text{DR}}}}{\pi} \right)^l \left( \frac{\eta_3^{\widehat{\text{DR}}}}{\pi} \right)^m, \\
 \beta_e^{\widehat{\text{DR}}} &= \mu^2 \frac{d}{d\mu^2} \frac{\alpha_e^{\widehat{\text{DR}}}}{\pi} = -\varepsilon \frac{\alpha_e^{\widehat{\text{DR}}}}{\pi} - \sum_{i,j,k,l,m} \beta_{e,ijklm}^{\widehat{\text{DR}}} \left( \frac{\alpha_s^{\widehat{\text{DR}}}}{\pi} \right)^i \left( \frac{\alpha_e^{\widehat{\text{DR}}}}{\pi} \right)^j \left( \frac{\eta_1^{\widehat{\text{DR}}}}{\pi} \right)^k \left( \frac{\eta_2^{\widehat{\text{DR}}}}{\pi} \right)^l \left( \frac{\eta_3^{\widehat{\text{DR}}}}{\pi} \right)^m, \quad (\text{A1}) \\
 \beta_{\eta_s}^{\widehat{\text{DR}}} &= \mu^2 \frac{d}{d\mu^2} \frac{\eta_s^{\widehat{\text{DR}}}}{\pi} = -\varepsilon \frac{\eta_s^{\widehat{\text{DR}}}}{\pi} - \sum_{i,j,k,l,m} \beta_{s,ijklm}^{\widehat{\text{DR}}} \left( \frac{\alpha_s^{\widehat{\text{DR}}}}{\pi} \right)^i \left( \frac{\alpha_e^{\widehat{\text{DR}}}}{\pi} \right)^j \left( \frac{\eta_1^{\widehat{\text{DR}}}}{\pi} \right)^k \left( \frac{\eta_2^{\widehat{\text{DR}}}}{\pi} \right)^l \left( \frac{\eta_3^{\widehat{\text{DR}}}}{\pi} \right)^m,
 \end{aligned}$$



where  $\beta^{\widehat{\text{DR}}}$ ,  $\beta_e^{\widehat{\text{DR}}}$  and  $\beta_{\eta_s}^{\widehat{\text{DR}}}$  are the QCD, evanescent-quark-antiquark, and quatic evanescent  $\beta$ -functions, respectively.

The non-vanishing coefficients for  $\beta^{\widehat{\text{DR}}}$  through three loops are:

$$\begin{aligned}
\beta_{20}^{\widehat{\text{DR}}} &= \frac{11}{12}C_A - \frac{1}{6}N_f - \frac{1}{24}D_x C_A, \\
\beta_{30}^{\widehat{\text{DR}}} &= \frac{17}{24}C_A^2 - \frac{5}{24}C_A N_f - \frac{1}{8}C_F N_f - \frac{7}{48}D_x C_A^2, & \beta_{21}^{\widehat{\text{DR}}} &= \frac{1}{16}D_x C_F N_f \\
\beta_{40}^{\widehat{\text{DR}}} &= \frac{2857}{3456}C_A^3 - \frac{1415}{3456}C_A^2 N_f - \frac{205}{1152}C_A C_F N_f + \frac{1}{64}C_F^2 N_f + \frac{79}{3456}C_A N_f^2 + \frac{11}{576}C_F N_f^2 \\
&\quad + D_x \left( -\frac{2749}{6912}C_A^3 + \frac{13}{432}C_A^2 N_f + \frac{23}{2304}C_A C_F N_f \right) + \frac{145}{13824}D_x^2 C_A^3 \\
\beta_{31}^{\widehat{\text{DR}}} &= D_x \left( \frac{5}{256}C_A^2 N_f + \frac{7}{32}C_A C_F N_f + \frac{3}{128}C_F^2 N_f \right) \\
\beta_{22}^{\widehat{\text{DR}}} &= D_x \left( -\frac{1}{64}C_A^2 N_f + \frac{7}{128}C_A C_F N_f - \frac{3}{64}C_F^2 N_f + \frac{1}{256}C_A N_f^2 - \frac{7}{256}C_F N_f^2 \right) \\
&\quad + D_x^2 \left( \frac{1}{256}C_A^2 N_f - \frac{5}{256}C_A C_F N_f \right) \\
\beta_{30100}^{\widehat{\text{DR}}} &= \frac{27}{512}D_x(1-D_x), & \beta_{30010}^{\widehat{\text{DR}}} &= -\frac{45}{126}D_x(2+D_x), & \beta_{30001}^{\widehat{\text{DR}}} &= -\frac{9}{256}D_x(1-D_x) \\
\beta_{20200}^{\widehat{\text{DR}}} &= -\frac{81}{512}D_x(1-D_x), & \beta_{20101}^{\widehat{\text{DR}}} &= \frac{27}{128}D_x(1-D_x), \\
\beta_{20020}^{\widehat{\text{DR}}} &= \frac{45}{64}D_x(2+D_x), & \beta_{20002}^{\widehat{\text{DR}}} &= -\frac{63}{256}D_x(1-D_x),
\end{aligned} \tag{A2}$$

where I omit the last three indices if they all vanish.

The coefficients of  $\beta_e^{\widehat{\text{DR}}}$  through two loops are:

$$\begin{aligned}
\beta_{e,02}^{\widehat{\text{DR}}} &= \frac{1}{2}C_A - C_F - \frac{1}{4}N_f - \frac{1}{4}D_x(C_A - C_F), & \beta_{e,11}^{\widehat{\text{DR}}} &= \frac{3}{2}C_F, \\
\beta_{e,03}^{\widehat{\text{DR}}} &= \frac{3}{8}C_A^2 - \frac{5}{4}C_A C_F + C_F^2 - \frac{3}{16}C_A N_f + \frac{3}{8}C_F N_f + D_x \left( -\frac{1}{2}C_A^2 + \frac{3}{2}C_A C_F - C_F^2 + \frac{3}{32}C_A N_f \right) \\
&\quad + D_x^2 \left( \frac{3}{32}C_A^2 - \frac{1}{4}C_A C_F + \frac{9}{64}C_F^2 \right), \\
\beta_{e,12}^{\widehat{\text{DR}}} &= -\frac{3}{8}C_A^2 + \frac{7}{4}C_A C_F - 2C_F^2 - \frac{5}{16}C_F N_f + D_x \left( -\frac{11}{16}C_A C_F + \frac{1}{2}C_F^2 \right), \\
\beta_{e,21}^{\widehat{\text{DR}}} &= -\frac{7}{64}C_A^2 + \frac{61}{48}C_A C_F + \frac{3}{16}C_F^2 + \frac{1}{16}C_A N_f - \frac{5}{24}C_F N_f + D_x \left( \frac{1}{64}C_A^2 - \frac{11}{96}C_A C_F \right), \\
\beta_{e,02100}^{\widehat{\text{DR}}} &= -\frac{9}{8}(1-D_x), & \beta_{e,02010}^{\widehat{\text{DR}}} &= \frac{5}{8}(2+D_x), & \beta_{e,02001}^{\widehat{\text{DR}}} &= \frac{3}{4}(1-D_x), \\
\beta_{e,01200}^{\widehat{\text{DR}}} &= \frac{27}{64}(1-D_x), & \beta_{e,01020}^{\widehat{\text{DR}}} &= -\frac{15}{8}(2+D_x), & \beta_{e,01002}^{\widehat{\text{DR}}} &= \frac{21}{32}(1-D_x), \\
\beta_{e,01101}^{\widehat{\text{DR}}} &= -\frac{9}{16}(1-D_x),
\end{aligned} \tag{A3}$$

The three-loop coefficients that do not involve the quartic couplings are:

$$\begin{aligned}
\beta_{e,04}^{\widehat{\text{DR}}} &= \frac{9}{16} C_A^3 \zeta_3 - C_A^2 C_F \left( \frac{5}{16} + \frac{69}{16} \zeta_3 \right) + C_A C_F^2 \left( \frac{5}{4} + \frac{15}{2} \zeta_3 \right) - C_F^3 \left( \frac{5}{4} + \frac{9}{4} \zeta_3 \right) \\
&\quad + C_A^2 N_f \left( \frac{3}{128} - \frac{9}{32} \zeta_3 \right) - C_A C_F N_f \left( \frac{15}{32} - \frac{51}{32} \zeta_3 \right) + C_F^2 N_f \left( \frac{27}{32} - \frac{33}{16} \zeta_3 \right) + N_f^2 \left( \frac{1}{256} C_A - \frac{1}{128} C_F \right) \\
&\quad + D_x \left[ -C_A^3 \left( \frac{7}{32} + \frac{3}{8} \zeta_3 \right) + C_A^2 C_F \left( \frac{91}{64} + \frac{135}{32} \zeta_3 \right) - C_A C_F^2 \left( \frac{13}{4} + \frac{249}{32} \zeta_3 \right) + C_F^3 \left( \frac{41}{16} + \frac{27}{16} \zeta_3 \right) \right. \\
&\quad \left. + C_A^2 N_f \left( \frac{21}{128} + \frac{3}{64} \zeta_3 \right) - C_A C_F N_f \left( \frac{37}{256} + \frac{33}{64} \zeta_3 \right) - C_F^2 N_f \left( \frac{47}{128} - \frac{27}{32} \zeta_3 \right) - N_f^2 \left( \frac{1}{512} C_A + \frac{3}{64} C_F \right) \right] \\
&\quad + D_x^2 \left[ + \frac{9}{64} C_A^3 - C_A^2 C_F \left( \frac{35}{64} + \frac{69}{64} \zeta_3 \right) + C_A C_F^2 \left( \frac{461}{512} + \frac{147}{64} \zeta_3 \right) - C_F^3 \left( \frac{189}{256} + \frac{9}{32} \zeta_3 \right) \right. \\
&\quad \left. - C_A^2 N_f \left( \frac{29}{512} - \frac{3}{128} \zeta_3 \right) + C_A C_F N_f \left( \frac{49}{512} - \frac{9}{128} \zeta_3 \right) - C_F^2 N_f \left( \frac{43}{1024} - \frac{3}{64} \zeta_3 \right) \right] \\
&\quad + D_x^3 \left[ -C_A^3 \left( \frac{1}{32} - \frac{3}{128} \zeta_3 \right) + \frac{33}{256} C_A^2 C_F - C_A C_F^2 \left( \frac{189}{1024} + \frac{9}{128} \zeta_3 \right) + C_F^3 \left( \frac{109}{1024} - \frac{3}{64} \zeta_3 \right) \right], \\
\beta_{e,13}^{\widehat{\text{DR}}} &= -C_A^3 \left( \frac{25}{64} - \frac{3}{4} \zeta_3 \right) + C_A^2 C_F \left( \frac{85}{32} - \frac{15}{4} \zeta_3 \right) - C_A C_F^2 \left( \frac{11}{2} - 6 \zeta_3 \right) + C_F^3 \left( \frac{7}{2} - 3 \zeta_3 \right) \\
&\quad + C_A^2 N_f \left( \frac{7}{32} - \frac{3}{8} \zeta_3 \right) - C_A C_F N_f \left( \frac{27}{32} - \frac{9}{8} \zeta_3 \right) + C_F^2 N_f \left( \frac{13}{16} - \frac{3}{4} \zeta_3 \right) + \frac{3}{64} C_A N_f^2 \\
&\quad + D_x \left[ -C_A^3 \left( \frac{13}{32} + \frac{3}{4} \zeta_3 \right) + C_A^2 C_F \left( 1 + \frac{63}{16} \zeta_3 \right) + C_A C_F^2 \left( \frac{5}{64} - \frac{105}{16} \zeta_3 \right) - C_F^3 \left( \frac{29}{32} - \frac{27}{8} \zeta_3 \right) \right. \\
&\quad \left. + C_A^2 N_f \left( \frac{1}{128} + \frac{3}{16} \zeta_3 \right) + C_A C_F N_f \left( \frac{51}{128} - \frac{9}{16} \zeta_3 \right) - C_F^2 N_f \left( \frac{25}{128} - \frac{3}{8} \zeta_3 \right) \right] \\
&\quad + D_x^2 \left[ + C_A^3 \left( \frac{13}{128} + \frac{3}{16} \zeta_3 \right) - C_A^2 C_F \left( \frac{25}{128} + \frac{33}{32} \zeta_3 \right) - C_A C_F^2 \left( \frac{3}{128} - \frac{57}{32} \zeta_3 \right) + C_F^3 \left( \frac{1}{8} - \frac{15}{16} \zeta_3 \right) \right], \\
\beta_{e,22}^{\widehat{\text{DR}}} &= C_A^3 \left( \frac{121}{512} - \frac{45}{16} \zeta_3 \right) - C_A^2 C_F \left( \frac{167}{256} - \frac{207}{16} \zeta_3 \right) + C_A C_F^2 \left( \frac{131}{128} - 18 \zeta_3 \right) - C_F^3 \left( \frac{85}{64} - \frac{27}{4} \zeta_3 \right) \\
&\quad - C_A^2 N_f \left( \frac{899}{1024} - \frac{45}{32} \zeta_3 \right) + C_A C_F N_f \left( \frac{273}{128} - \frac{171}{32} \zeta_3 \right) - C_F^2 N_f \left( \frac{641}{256} - \frac{99}{16} \zeta_3 \right) - N_f^2 \left( \frac{1}{256} C_A - \frac{1}{16} C_F \right) \\
&\quad + D_x \left[ -C_A^3 \left( \frac{4355}{1024} - \frac{45}{32} \zeta_3 \right) + C_A^2 C_F \left( \frac{21071}{1024} - \frac{99}{16} \zeta_3 \right) - C_A C_F^2 \left( \frac{3381}{128} - \frac{261}{32} \zeta_3 \right) \right. \\
&\quad \left. + C_F^3 \left( \frac{13}{256} - \frac{45}{16} \zeta_3 \right) + \frac{1}{1024} C_A^2 N_f + \frac{15}{64} C_A C_F N_f + \frac{1}{16} C_F^2 N_f \right] \\
&\quad + D_x^2 \left[ -\frac{1}{1024} C_A^3 + \frac{83}{1024} C_A^2 C_F - \frac{33}{512} C_A C_F^2 \right], \\
\beta_{e,31}^{\widehat{\text{DR}}} &= -\frac{3025}{4608} C_A^3 + \frac{12601}{3456} C_A^2 C_F - \frac{453}{128} C_A C_F^2 + \frac{129}{64} C_F^3 + \frac{475}{2304} C_A^2 N_f - C_A C_F N_f \left( \frac{151}{1728} + \frac{3}{4} \zeta_3 \right) \\
&\quad - C_F^2 N_f \left( \frac{23}{32} - \frac{3}{4} \zeta_3 \right) - \frac{5}{576} C_A N_f^2 - \frac{35}{864} C_F N_f^2 \\
&\quad + D_x \left[ + \frac{643}{9216} C_A^3 - \frac{883}{1728} C_A^2 C_F - \frac{5}{256} C_A C_F^2 - \frac{1}{144} C_A^2 N_f - \frac{19}{864} C_A C_F N_f \right] \\
&\quad + D_x^2 \left[ -\frac{11}{9216} C_A^3 - \frac{5}{13824} C_A^2 C_F \right],
\end{aligned}$$

(A4)

while the three-loop coefficients that do involve the quartic interactions are:

$$\begin{aligned}
\beta_{e,03100}^{\widehat{\text{DR}}} &= -\frac{9}{64}(1-7D_x+6D_x^2) + \frac{135}{128}N_f(1-D_x), \\
\beta_{e,03010}^{\widehat{\text{DR}}} &= \frac{5}{64}(8-18D_x-11D_x^2) - \frac{75}{128}N_f(2+D_x), \\
\beta_{e,03001}^{\widehat{\text{DR}}} &= \frac{3}{64}(2-19D_x+17D_x^2) - \frac{45}{64}N_f(1-D_x), \\
\beta_{e,12100}^{\widehat{\text{DR}}} &= -\frac{51}{8}(1-D_x), \quad \beta_{e,12010}^{\widehat{\text{DR}}} = \frac{85}{24}(2+D_x), \quad \beta_{e,12001}^{\widehat{\text{DR}}} = \frac{17}{4}(1-D_x), \\
\beta_{e,21100}^{\widehat{\text{DR}}} &= -\frac{801}{1024}(1-D_x), \quad \beta_{e,21010}^{\widehat{\text{DR}}} = \frac{375}{256}(2+D_x), \quad \beta_{e,21001}^{\widehat{\text{DR}}} = \frac{507}{512}(1-D_x), \\
\beta_{e,02200}^{\widehat{\text{DR}}} &= \frac{3}{1024}(422-553D_x+131D_x^2) - \frac{405}{1024}N_f(1-D_x), \\
\beta_{e,02020}^{\widehat{\text{DR}}} &= -\frac{5}{384}(652+136D_x-95D_x^2) + \frac{225}{128}N_f(2+D_x), \\
\beta_{e,02002}^{\widehat{\text{DR}}} &= \frac{1}{1536}(394-731D_x+337D_x^2) - \frac{315}{512}N_f(1-D_x), \\
\beta_{e,02110}^{\widehat{\text{DR}}} &= \frac{55}{32}(2-D_x-D_x^2), \\
\beta_{e,02101}^{\widehat{\text{DR}}} &= -\frac{1}{256}(622-773D_x+151D_x^2) + \frac{135}{256}N_f(1-D_x), \\
\beta_{e,02011}^{\widehat{\text{DR}}} &= -\frac{205}{96}(2-D_x-D_x^2), \\
\beta_{e,11200}^{\widehat{\text{DR}}} &= \frac{405}{128}(1-D_x), \quad \beta_{e,11020}^{\widehat{\text{DR}}} = -\frac{225}{16}(2+D_x), \\
\beta_{e,11002}^{\widehat{\text{DR}}} &= \frac{315}{64}(1-D_x), \quad \beta_{e,11101}^{\widehat{\text{DR}}} = -\frac{135}{32}(1-D_x), \\
\beta_{e,01300}^{\widehat{\text{DR}}} &= -\frac{27}{1024}(11-10D_x-D_x^2), \quad \beta_{e,01210}^{\widehat{\text{DR}}} = -\frac{135}{256}(2-D_x-D_x^2), \\
\beta_{e,01201}^{\widehat{\text{DR}}} &= \frac{27}{512}(11-10D_x-D_x^2), \quad \beta_{e,01120}^{\widehat{\text{DR}}} = -\frac{45}{64}(2-D_x-D_x^2), \\
\beta_{e,01111}^{\widehat{\text{DR}}} &= \frac{45}{32}(2-D_x-D_x^2), \quad \beta_{e,01102}^{\widehat{\text{DR}}} = \frac{9}{256}(14-25D_x+11D_x^2), \\
\beta_{e,01030}^{\widehat{\text{DR}}} &= \frac{5}{4}(16+10D_x+D_x^2), \quad \beta_{e,01021}^{\widehat{\text{DR}}} = \frac{105}{64}(2-D_x-D_x^2), \\
\beta_{e,01012}^{\widehat{\text{DR}}} &= -\frac{105}{64}(2-D_x-D_x^2), \quad \beta_{e,01003}^{\widehat{\text{DR}}} = -\frac{7}{256}(14-25D_x+11D_x^2),
\end{aligned} \tag{A5}$$

A consistent description of  $\beta^{\widehat{\text{DR}}}$  and  $\beta_e^{\widehat{\text{DR}}}$  through three loops only requires knowledge of the  $\beta_{\eta_i}^{\widehat{\text{DR}}}$ 's

through one loop. These coefficients are:

$$\begin{aligned}
\beta_{\eta_1,20000}^{\widehat{\text{DR}}} &= -\frac{3}{8}, & \beta_{\eta_1,02000}^{\widehat{\text{DR}}} &= \frac{1}{3}N_f, & \beta_{\eta_1,10100}^{\widehat{\text{DR}}} &= \frac{9}{2}, & \beta_{\eta_1,01100}^{\widehat{\text{DR}}} &= -\frac{1}{2}N_f, \\
\beta_{\eta_1,00200}^{\widehat{\text{DR}}} &= -\frac{11}{8} - \frac{1}{8}D_x, & \beta_{\eta_1,00110}^{\widehat{\text{DR}}} &= -2 - D_x, & \beta_{\eta_1,00101}^{\widehat{\text{DR}}} &= \frac{7}{2} - \frac{1}{2}D_x, \\
\beta_{\eta_2,20000}^{\widehat{\text{DR}}} &= -\frac{9}{16}, & \beta_{\eta_2,02000}^{\widehat{\text{DR}}} &= \frac{1}{24}N_f, & \beta_{\eta_2,10010}^{\widehat{\text{DR}}} &= \frac{9}{2}, & \beta_{\eta_2,01010}^{\widehat{\text{DR}}} &= -\frac{1}{2}N_f, \\
\beta_{\eta_2,00200}^{\widehat{\text{DR}}} &= \frac{3}{16}(1 - D_x), & \beta_{\eta_2,00110}^{\widehat{\text{DR}}} &= \frac{1}{2}(1 - D_x), & \beta_{\eta_2,00101}^{\widehat{\text{DR}}} &= -\frac{1}{2}(1 - D_x), \\
\beta_{\eta_2,00020}^{\widehat{\text{DR}}} &= -\frac{32}{3} - \frac{4}{3}D_x, & \beta_{\eta_2,00011}^{\widehat{\text{DR}}} &= -\frac{7}{6}(1 - D_x), & \beta_{\eta_2,00002}^{\widehat{\text{DR}}} &= \frac{7}{12}(1 - D_x), \\
\beta_{\eta_3,10001}^{\widehat{\text{DR}}} &= \frac{9}{2}, & \beta_{\eta_3,01001}^{\widehat{\text{DR}}} &= -\frac{1}{2}N_f, & \beta_{\eta_3,00110}^{\widehat{\text{DR}}} &= 2 + D_x, & \beta_{\eta_3,00101}^{\widehat{\text{DR}}} &= \frac{5}{2} - D_x, \\
\beta_{\eta_3,00020}^{\widehat{\text{DR}}} &= \frac{5}{3}(2 + D_x), & \beta_{\eta_3,00011}^{\widehat{\text{DR}}} &= -\frac{10}{3}(2 + D_x), & \beta_{\eta_3,00002}^{\widehat{\text{DR}}} &= -\frac{7}{6} + \frac{11}{12}D_x,
\end{aligned} \tag{A6}$$

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