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In AdS/CFT, the holographic Weyl anomaly computation relates the a-anomaly coefficient to the properties of the bulk action at the UV fixed point. This universal behavior suggests the possibility of a holographic c-theorem for the a-anomaly under flows to the IR. We prove such a c-theorem for higher curvature Lovelock gravity, where the bulk equations of motion remain second order. We also explore $f(R)$ gravity as a toy model where higher derivatives cannot be avoided. In this case, monoticity of the flow requires an additional condition related to the higher derivative nature of the theory. This is in contrast to the case of $f(R)$ black hole entropy, where the second law follows from application of the full Einstein equations and the null energy condition.

I. INTRODUCTION

Central charges in conformal field theories can often be thought of as a proxy for the number of degrees of freedom exhibited by the theory. In this context, the Zamolodchikov c-theorem [1] is a powerful result for two-dimensional conformal field theories. It states that there exists a c-function which is monotonically decreasing along flows from the UV to IR, and which is equal to the central charge at the fixed points of the flow. This is a direct indication that UV degrees of freedom of the CFT are removed as the theory flows to the IR.

While two dimensional conformal field theories are rather special, there have been numerous attempts to generalize the c-theorem to higher dimensions. However, one obstacle that needs to be surmounted in doing so is the realization that there may be multiple candidates for a satisfactory c-function. For example, in four dimensions, the Weyl anomaly has the well known form

$$\langle T^\mu_\mu \rangle = \frac{c}{16\pi^2} C^2_{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} E_4, \tag{1}$$

where $E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$ is the four-dimensional Euler density. In this context, Cardy demonstrated that, while the c anomaly coefficient may not be the proper object to investigate, the a coefficient appears to have the desired monotonicity property along flows [2]. This has subsequently been confirmed in various situations [3–6], and further investigated in the context of a-maximization [7–10].

While the above investigations have been carried out in a field theory context, AdS/CFT allows the possibility of a holographic version of the c-theorem. At large N, the a and c anomalies are equal, and can be computed holographically; for $\mathcal{N} = 4$ super-Yang-Mills, the result is simply $a = c = N^2/4$ [11]. This result may be extended to renormalization group flows, which correspond to radial flow in the bulk dual. A c-theorem can then be proven at leading order in large N by examining the flow equations for domain wall solutions interpolating between the UV and IR [12–15].

Recently, the leading order holographic c-theorem has been extended to the case where the bulk theory may contain higher order curvature terms, corresponding to moving away from the leading behavior in AdS/CFT [16–22]. In particular, the theories of interest include the addition of Gauss-Bonnet and 'quasi-topological' curvature-cubed terms to the bulk action. By generalizing the holographic a-anomaly, computed for a general bulk action in [23], Myers and Sinha constructed an appropriate a-function that is monotonic along radial flow [20–22]. A key element of this holographic c-theorem rests on the fact that the equations of motion contain no higher than second derivatives of the metric when expanded on the AdS background. Gauss-Bonnet gravity manifestly satisfies this requirement, as does the quasi-topological theory, which was explicitly constructed to have this property [17, 18].

Here we extend the investigation of Myers and Sinha in two directions. Firstly, we prove a holographic c-theorem for the case of Lovelock bulk theories. These theories generalize Gauss-Bonnet gravity in $d + 1$ dimensions by the addition of d' dimensional Euler densities where $d' < d$. While this result is mostly formal, in the sense that higher

Lovelock terms are only present in theories in dimensions too large for practical applications, it nevertheless suggests that the holographic c-theorem extends to arbitrary orders in the bulk curvature, so long as higher derivative terms in the equations of motion are able to be controlled.

Secondly, we address what happens when the bulk action is not restricted to second order equations of motion by examining $f(R)$ gravity as a toy model. In this case, we find that the constructed a-function may deviate from monotonicity by a term that is explicitly of higher derivative order. We suggest that this term, which may lead to a violation of the holographic c-theorem, is related to the additional ghost modes of the theory. From an AdS/CFT point of view, this would correspond to a breakdown of unitarity in the dual field theory.

While this work was being prepared, we became aware of [22], which overlaps with some of our results.

II.   A HOLOGRAPHIC c-THEOREM FOR LOVELOCK GRAVITY

While higher-curvature gravitational actions generically lead to higher derivative equations of motion and ensuing pathologies such as ghosts, a special class of higher-curvature actions may be constructed that nevertheless give rise to second order equations of motion for the metric. These are the Lovelock actions, which are constructed out of the $(d+1)$-dimensional continuation of lower dimensional Euler densities [24]

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} \sum_m \alpha_m L^{(m)} + S_{\text{matter}}. \tag{2}$$

Here the m-th Lovelock term is the Euler invariant in $2m$ dimensions

$$L^{(m)} = \frac{1}{2^m} \delta^{a_1 b_1 \cdots a_m b_m}_{c_1 d_1 \cdots c_m d_m} R_{a_1 b_1}{}^{c_1 d_1} \cdots R_{a_m b_m}{}^{c_m d_m}. \tag{3}$$

In particular, $L^{(0)} = 1$ is a cosmological constant, $L^{(1)} = R$ is the ordinary Einstein-Hilbert term and $L^{(2)} = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$ is the Gauss-Bonnet invariant. The equation of motion following from (2) is simply $G_{ab} = \kappa^2 T_{ab}$, where the generalized Einstein tensor is given by $G_{ab} = \sum_m \alpha_m G^{(m)}_{ab}$, with

$$G^{e\,(m)}_f = -\frac{1}{2^{m+1}} \delta^{e a_1 b_1 \cdots a_m b_m}_{f c_1 d_1 \cdots c_m d_m} R_{a_1 b_1}{}^{c_1 d_1} \cdots R_{a_m b_m}{}^{c_m d_m}. \tag{4}$$

With a suitable choice of the cosmological constant, we take the bulk Lovelock action (2) to be dual to a d-dimensional CFT. As demonstrated in [23], the d-dimensional type A trace anomaly (i.e. the term proportional to the Euler characteristic) is universal in holographic renormalization, and its coefficient may be expressed as

$$a_{\text{UV}} = -\frac{\pi^{d/2}}{2\kappa^2} \frac{\ell^{d+1}}{(d/2)!^2} f(\text{AdS}), \tag{5}$$

where

$$f(\text{AdS}) = \sum_m \alpha_m L^{(m)} \Big|_{\text{AdS}} \tag{6}$$

is the on-shell Lagrangian evaluated on the asymptotic AdS background with radius ℓ.

In order to construct a suitable a-function, we need to promote the a_{UV} central charge (5) at the UV fixed point into a function $a(r)$ of the radial flow. Here, for simplicity, we consider a radial slicing of the bulk space into flat slices of the form

$$ds^2 = e^{2A(r)}(-dt^2 + d\vec{x}^2_{d-1}) + dr^2, \tag{7}$$

with curvature components

$$R_{\mu\nu\rho\sigma} = -A'^2(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \qquad R_{\mu r \nu r} = -(A'' + A'^2)g_{\mu\nu}. \tag{8}$$

This allows us to evaluate the individual Lovelock terms

$$L^{(m)} = (-A'^2)^m \frac{(d+1)!}{(d+1-2m)!} - 2mA''(-A'^2)^{m-1} \frac{d!}{(d+1-2m)!}. \tag{9}$$

The second term is unimportant at AdS fixed points since we have

$$A \sim r/\ell, \qquad A' \sim 1/\ell, \qquad A'' \sim 0, \qquad \text{as} \qquad r \to \infty. \tag{10}$$

As a result, the on-shell value of $L^{(m)}$ takes the form

$$L^{(m)}\Big|_{\text{AdS}} = (-1)^m \frac{(d+1)!}{(d+1-2m)!\ell^{2m}}, \tag{11}$$

so that the a_{UV} central charge (5) may be expressed as

$$a_{\text{UV}} = -\frac{\pi^{d/2}}{2\kappa^2 (d/2)!^2} \sum_m \alpha_m (-1)^m \frac{(d+1)!\ell^{d+1-2m}}{(d+1-2m)!}. \tag{12}$$

The extension of $a_{\text{UV}} \to a(r)$ into the bulk is by no means unique. As a first attempt to do so, we make the substitution

$$\ell \to \ell_{\text{eff}}(r) \equiv \frac{1}{A'(r)}, \tag{13}$$

so that

$$a_0(r) = -\frac{\pi^{d/2}}{2\kappa^2 (d/2)!^2} \sum_m \alpha_m (-1)^m \frac{(d+1)!}{(d+1-2m)!(A')^{d+1-2m}}. \tag{14}$$

This satisfies the requirement that $a(r)$ reproduces the central charge at fixed points of the flow. In addition, it incorporates the metric function at the first derivative level, so that

$$a_0'(r) = -\frac{\pi^{d/2}}{2\kappa^2 (d/2)!^2} \frac{A''}{(A')^d} \sum_m \alpha_m (-1)^{m+1} \frac{(d+1)!(A')^{2m-2}}{(d-2m)!} \tag{15}$$

is linear in A''. Following [14, 20], we aim to demonstrate that $a_0'(r)$ is monotonic by appealing to the bulk equations of motion. To do so, we first compute the generalized Einstein tensor components (4) using the curvature components (8) for the bulk metric. The resulting two independent components are

$$
\begin{aligned}
G_t^{t\,(m)} &= -\frac{d!}{2(d-2m)!}(-A'^2)^m + \frac{m(d-1)!}{(d-2m)!}A''(-A'^2)^{m-1}, \\
G_r^{r\,(m)} &= -\frac{d!}{2(d-2m)!}(-A'^2)^m,
\end{aligned}
\tag{16}
$$

so that

$$G_t^t - G_r^r = A'' \sum_m m\alpha_m (-1)^{m+1} \frac{(d-1)!(A')^{2m-2}}{(d-2m)!}. \tag{17}$$

Comparison with (15) shows that, while the form of $a_0'(r)$ is suggestive, it nevertheless does not match with the difference $G_t^t - G_r^r$. However, as indicated above, $a_0(r)$ is not necessarily unique, and using the Einstein equation as a guide, we now construct a modified a-function which is monotonic.

To proceed, we first note that, at AdS fixed points where the bulk matter sector contributes vanishing vacuum energy, the background satisfies the vacuum Einstein equation

$$0 = G_r^r\Big|_{\text{AdS}} = \sum_m \alpha_m (-1)^{m+1} \frac{d!(A')^{2m}}{2(d-2m)!}\Big|_{\text{AdS}}. \tag{18}$$

(Recall that the cosmological constant term is included in the gravitational sector through α_0.) This allows us to add a vanishing on-shell contribution to (6), so that

$$f(\text{AdS}) = 2G_r^r + \sum_m \alpha_m L^{(m)}\Big|_{\text{AdS}} = \sum_m 2m\alpha_m (-1)^m \frac{d!}{(d+1-2m)!\ell^{2m}}. \tag{19}$$

In this case, we are led to the definition

$$a(r) = -\frac{\pi^{d/2}}{\kappa^2(d/2)!^2} \sum_m m\alpha_m(-1)^m \frac{d!}{(d+1-2m)!(A')^{d+1-2m}}. \tag{20}$$

Note that the shift removes the cosmological constant term α_0 from the definition of $a(r)$, and matches what is done in constructing a suitable a-function in the leading two-derivative gravity. This a-function can now be seen to satisfy

$$a'(r) = -\frac{d\pi^{d/2}}{\kappa^2(d/2)!^2}\frac{G_t^t - G_r^r}{(A')^d} = -\frac{d\pi^{d/2}}{(d/2)!^2}\frac{T_t^t - T_r^r}{(A')^d} \geq 0, \tag{21}$$

where the inequality corresponds to the null energy condition. Therefore we have found an appropriate extension of the a central charge which is indeed monotonic along flows from the UV to the IR. This result extends the observation of [20, 22] that a general holographic c-theorem may be obtained in the presence of higher order corrections, provided the (linearized) equations of motion remain second order.

III. $f(R)$ GRAVITY

In proving the holographic c-theorem for Lovelock gravity, we have constructed the $a(r)$ function (20) entirely out of the first derivative of the metric function, A'. This ensures that $a(r)$ reproduces the a central charge at fixed points of the flow where $A' \sim 1/\ell$. However, this construction also guarantees that $a'(r)$ is linear in the second derivative A'' so that it may be connected to the equations of motion. This connection suggests that having second order equations is an essential aspect of obtaining the c-theorem. On the other hand, bulk AdS duals are often considered to be effective theories where higher derivative corrections naturally arise (*e.g.* in the string α' expansion). Thus it is important to address whether any holographic c-theorem could hold in such higher derivative theories as well.

Here we take one step towards a fully general investigation by considering the case of $f(R)$ gravity. Such theories have been considered from a cosmological point of view, and are closely related to Brans-Dicke theory (see *e.g.* [25–27]). However, here we are mainly interested in $f(R)$ gravity as a toy model exhibiting higher derivative equations of motion. The action is given by

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} f(R) + S_{\text{matter}}, \tag{22}$$

where $f(R)$ is a fixed but arbitrary function of the scalar curvature R. The resulting equation of motion is

$$G_{ab} \equiv F R_{ab} - \tfrac{1}{2}f g_{ab} + (g_{ab}\Box - \nabla_a\nabla_b)F = \kappa^2 T_{ab}, \tag{23}$$

where

$$F(R) = \frac{df(R)}{dR}. \tag{24}$$

Since $F(R)$ is second order in derivatives, the equation of motion is in general fourth order.

In order to construct an appropriate a-function for $f(R)$ gravity, we follow [14, 20] and explore the difference in the Einstein equation components $G_t^t - G_r^r$. To proceed, we take the same metric (7) as used above, and compute the Ricci components

$$R_{\mu\nu} = -(A'' + dA'^2)g_{\mu\nu}, \qquad R_{rr} = -d(A'' + A'^2), \qquad R = -d(2A'' + (d+1)A'^2). \tag{25}$$

In this case, the Einstein equation (23) splits into

$$\begin{aligned} G_\nu^\mu &= -[F(A'' + dA'^2) + \tfrac{1}{2}f - (d-1)A'F' - F'']\delta_\nu^\mu &= \kappa^2 T_\nu^\mu, \\ G_r^r &= -F(dA'' + dA'^2) - \tfrac{1}{2}f + dA'F' &= \kappa^2 T_r^r. \end{aligned} \tag{26}$$

Taking the difference of G_t^t and G_r^r gives

$$G_t^t - G_r^r = (d-1)A''F - A'F' + F'' = \kappa^2(T_t^t - T_r^r). \tag{27}$$

Note that F is a function of R, which is in turn a function of A according to (25). Thus the higher derivatives of A are encoded in the $-A'F' + F''$ terms in this equation.

Our aim is to construct a suitable $a(r)$ which reproduces the a-anomaly at the UV boundary and which is subject to a flow governed by (27). We start with the anomaly coefficient itself, given by (5)

$$a_{\mathrm{UV}} = -\frac{\pi^{d/2}}{2\kappa^2}\frac{\ell^{d+1}}{(d/2)!^2}f(\mathrm{AdS}),$$
(28)

where this time $f(\mathrm{AdS})$ is simply the on-shell value of $f(R)$ at the asymptotic AdS fixed point

$$f(\mathrm{AdS}) = f(-d(d+1)\ell^{-2}).$$
(29)

As we saw above, the extension of a_{UV} to the interior is not unique. A straightforward choice would be to replace $f(\mathrm{AdS})$ by $f(R)$, so that

$$a_0(r) = -\frac{\pi^{d/2}}{2\kappa^2(d/2)!^2}\frac{f(R)}{(A')^{d+1}}.$$
(30)

This is similar to our choice of $a_0(r)$ in (14), although here we allow $f(R)$ to contain A'' through the dependence on the curvature scalar. However, differentiation of $a_0(r)$ with respect to r does not give any obvious correspondence with the difference (27). Thus we seek an improvement to $a_0(r)$, just as we did for the Lovelock case.

Since we do not want to destroy the matching of the a-function with the actual a-anomaly at AdS critical points, we can adjust $a_0(r)$ by at most functions which vanish at such points. A natural possibility for such a function is to take the equations of motion at a critical point. In this case, the stress tensor T_{ab} vanishes, and furthermore the functions f and F become constant. The rr equation of motion (26) then simplifies to

$$G^r_r\big|_{\mathrm{AdS}} = \left[-dA'^2F - \tfrac{1}{2}f\right]_{\mathrm{AdS}} = 0,$$
(31)

where $A' = 1/\ell$. This suggests that we shift $f(R)$ in (30) by $2G^r_r$, just as we did in the Lovelock case. The resulting a-function then takes the form

$$a(r) = \frac{d\pi^{d/2}}{\kappa^2(d/2)!^2}\frac{F(R)}{(A')^{d-1}},$$
(32)

where R is given in (25). Note that $F(R)$ in the numerator of this expression is essentially the derivative of the Lagrangian with respect to R (or equivalently with respect to $R^{tr}{}_{tr}$). This suggests a natural connection with an entropy function, as pointed out in [20, 22]. In fact, the appearance of $F(R)$ in the entropy function for $f(R)$ black holes was initially observed in [28].

With the definition (32), we now see that

$$\begin{aligned}
a'(r) &= \frac{d\pi^{d/2}}{\kappa^2(d/2)!^2}\frac{-(d-1)A''F + A'F'}{(A')^d}\\
&= \frac{d\pi^{d/2}}{(d/2)!^2}\frac{-(T^t_t - T^r_r) + F''/\kappa^2}{(A')^d},
\end{aligned}$$
(33)

where the second line follows from the equation of motion (27). If it were not for the F'' term, we would then use the null energy condition, $-(T^t_t - T^r_r) \geq 0$, to demonstrate that $a'(r) \geq 0$. This suggests that the higher derivative nature of $f(R)$ gravity directly impacts the fate of the holographic c-theorem. In particular, a non-trivial F'' contribution is a direct sign that the gravitational background incorporates up to four derivatives of A.

Taking a step back, it is perhaps not surprising that in this case monoticity of $a'(r)$ requires not just the weak energy condition on the matter sector, but also a further condition $F'' \geq 0$ on the gravity sector. While we do not see a direct connection with unitarity, it is certainly plausible that this $F'' \geq 0$ condition would be related to the absence of ghost modes in the background of the flow. One way to investigate this would be to map $f(R)$ gravity onto Brans-Dicke theory. In this case, F plays the role of the Brans-Dicke scalar. However, it is not clear to us how F'' may be related to any obvious pathologies of the theory.

In searching for a holographic c-theorem, we may also need to make a distinction between perturbative versus non-perturbative expansions in the higher derivative terms. For example, at the R^2 level, we may take $f = R + d(d-1)/\ell_0^2 + \alpha R^2$, so that $F = 2\alpha R$. The AdS vacuum with radius ℓ is given by the solution to (31), and in this case admits two branches

$$\left(\frac{\ell}{\ell_0}\right)^2 = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\alpha}{\ell_0^2}\frac{d(d+1)(d-3)}{d-1}}.$$
(34)

This may be expanded for small α

$$
\begin{aligned}
\ell_+ &= \ell_0 \left(1 - \frac{\alpha}{2\ell_0^2} \frac{d(d+1)(d-3)}{d-1} + \cdots \right), \\
\ell_- &= \frac{\alpha}{2\ell_0} \frac{d(d+1)(d-3)}{d-1} + \cdots .
\end{aligned}
\tag{35}
$$

We thus see that only the positive branch is smoothly connected to the finite radius background in the perturbative limit $\alpha \to 0$. In the context of AdS/CFT, it is natural to view α as an expansion parameter in an effective theory with higher curvature interactions. In this case, we ought to restrict to only the positive branch. Perhaps the sign of F'' can somehow be attributed to this choice. In particular, we have readily found numerical solutions for this example model coupled to a massless scalar where $a'(r)$ changes sign (from positive to negative) along the flow to the IR. However, all such resulting solutions involve a domain wall interpolating between the positive and negative branches of (34), and have the form

$$
A(r) \to \begin{cases} r/\ell_+, & r \to \infty \\ |r|/\ell_- & r \to -\infty. \end{cases}
\tag{36}
$$

This 'kink up' domain wall solution has the characteristic of a negative tension wall interpolating between two regions that both open up into AdS boundaries. Nevertheless, we have checked that the scalar matter is not responsible for this negative tension. Hence, it must have its origin in the higher derivative gravitational sector. This at least suggests that violation of the c-theorem is closely related to pathologies in the gravity sector of the theory that would not arise when treated in a proper perturbative expansion where the equations of motion can be perturbatively arranged to use no higher than second order derivatives in the expansion.

IV. DISCUSSION

In both the Lovelock and the $f(R)$ case, we have defined the a-function based on a shifted form of the Lagrangian

$$
a_{\mathrm{UV}} = -\frac{\pi^{d/2}}{2\kappa^2} \frac{\ell^{d+1}}{(d/2)!^2} f(\mathrm{AdS}) \qquad \Rightarrow \qquad a(r) = -\frac{\pi^{d/2}}{2\kappa^2 (d/2)!^2} \frac{f + 2G_r^r}{(A')^{d+1}},
\tag{37}
$$

where $f = e^{-1}\mathcal{L}$ is the Lagrangian density in the gravity sector (not including matter). Since we take the matter energy density to vanish at AdS fixed points, the addition of $2G_r^r$ does not affect the identification of $a(r)$ with the a anomaly coefficient. However, this improvement allows $a'(r)$ to be related to the difference of the Einstein equations $G_t^t - G_r^r$ along the flow.

As noted in [20], the a-function has a second interpretation in terms of entanglement entropy. This can be seen to arise in a natural manner, provided we perform the $2G_r^r$ shift. In particular, consider a general higher curvature action of the form

$$
S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} f(R^{ab}{}_{cd}) + S_{\mathrm{matter}}.
\tag{38}
$$

The corresponding Einstein equation may be written as

$$
G_{ab} \equiv F_{(a}{}^{cde} R_{b)cde} - \tfrac{1}{2} f g_{ab} + 2\nabla^c \nabla^d F_{acbd} = \kappa^2 T_{ab},
\tag{39}
$$

where

$$
F_{ab}{}^{cd} = \frac{\delta f(R^{ef}{}_{gh})}{\delta R^{ab}{}_{cd}}.
\tag{40}
$$

This generalizes the $f(R)$ equation of motion given in (23). The higher derivative terms are manifestly present in the Einstein tensor. However, they vanish on the AdS background where F_{abcd} is covariantly constant (as it is constructed out of the maximally symmetric Riemann tensor). Further taking T_{ab} to vanish in the asymptotic AdS region, we end up with

$$
G_r^r \big|_{\mathrm{AdS}} = \left[-2A'^2 F^{r\mu}{}_{r\mu} - \tfrac{1}{2} f \right]_{\mathrm{AdS}} = \left[-2dA'^2 F^{tr}{}_{tr} - \tfrac{1}{2} f \right]_{\mathrm{AdS}}.
\tag{41}
$$

The general a-function (37) then takes the form

$$a(r) = \frac{2d\pi^{d/2}}{\kappa^2(d/2)!^2}\frac{F^{tr}{}_{tr}}{(A')^{d-1}}. \tag{42}$$

If we were to consider black hole entropy in the presence of higher curvature corrections, it would be natural to use the Wald entropy formula [29–31]

$$S = -\frac{2\pi}{2\kappa^2}\int_\Sigma d^{d-1}x\sqrt{-g}\frac{\delta f}{\delta R_{abcd}}\epsilon_{ab}\epsilon_{cd} = \frac{4\pi}{\kappa^2}\int_\Sigma d^{d-1}x\sqrt{h}F^{tr}{}_{tr}, \tag{43}$$

where the integral is over the area of the horizon with unit binormal ϵ_{ab} along t and r. This reduces to the familiar one-quarter of the horizon area (in $G_N = \kappa^2/8\pi$ units) in the absence of higher curvatures, where $F^{ab}{}_{cd} = \frac{1}{2}\delta^{ab}_{cd}$. Although this expression is intended to be evaluated at the black hole horizon, it can nevertheless be generalized into an entropy function [32, 33]

$$\tilde{C}(r) = \frac{4\pi}{\kappa^2}F^{tr}{}_{tr}\sqrt{h} = \frac{4\pi}{\kappa^2}e^{(d-1)A}F^{tr}{}_{tr}, \tag{44}$$

where we have used the explicit form of the metric (7).

Flows of $\tilde{C}(r)$ have been investigated in the context of the second law of black hole thermodynamics in higher curvature gravity, including both Lovelock and $f(R)$ gravity [33–35]. In the case of $f(R)$ gravity, a c-theorem can be proven which generalizes the Hawking area theorem [36] by use of the Raychaudhuri equation [33, 34]. In particular, we consider an affinely parameterized null congruence given by the tangent vector $k^a\partial_a = d/d\lambda$ and define

$$\tilde{\theta} = \frac{d\log\tilde{C}}{d\lambda} = \theta + k^a\partial_a\log F, \tag{45}$$

where θ is the expansion of the null congruence. The Raychaudhuri equation then gives

$$\frac{d\tilde{\theta}}{d\lambda} = -\frac{1}{d-1}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - k^ak^bR_{ab} + k^ak^b\nabla_a\nabla_b\log F, \tag{46}$$

and further application of the Einstein equation (23) reduces this to

$$\frac{d\tilde{\theta}}{d\lambda} = -\frac{1}{d-1}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - \left(\frac{d\log F}{d\lambda}\right)^2 - \frac{\kappa^2k^ak^bT_{ab}}{F}. \tag{47}$$

Provided the congruence is twist-free, and assuming the null energy condition $k^ak^bT_{ab} \geq 0$ along with positivity of F, the terms on the right-hand-side are all non-positive, and as a result we may conclude that $d\tilde{\theta}/d\lambda \leq 0$, which is the statement of the second law in $f(R)$ gravity [33, 34].

For $f(R)$ gravity with the metric written in the explicit form (7), we define $k^a\partial_a = -e^{-2A}\partial_t + e^{-A}\partial_r$, in which case

$$\tilde{\theta} = e^{-A}\left[(d-1)A' + \frac{F'}{F}\right], \tag{48}$$

so that

$$\begin{aligned}\tilde{\theta}' &= e^{-A}\left[-(d-1)A'^2 - \left(\frac{F'}{F}\right)^2 + \frac{(d-1)A''F - A'F' + F''}{F}\right] \\ &= -e^{-A}\left[(d-1)A'^2 + \left(\frac{F'}{F}\right)^2 - \frac{\kappa^2(T^t_t - T^r_r)}{F}\right] \leq 0.\end{aligned} \tag{49}$$

We see here that, even with a higher order equation of motion, the terms involving higher derivatives arrange themselves in just the proper manner to match with the Einstein equation combination (27). This is in contrast with the holographic a-function, where $a'(r)$ given in (33) picks up an additional contribution proportional to F''. Nevertheless, it would be interesting to see if a connection can be made between the black hole entropy function (44) and the holographic a-function (42).

Ideally, it would be desirable to construct a general proof of a holographic c-theorem for the a-function of (42) using techniques such as the generalized Raychaudhuri equation (46). This would allow the c-theorem to be separated

from the particular bulk AdS flow parameterization of (7). However, unlike the Raychaudhuri equation itself, which is independent of dynamics, the incorporation of F^{tr}_{tr} into the generalized expansion $\tilde{\theta}$ necessarily brings higher derivative dynamics into the picture. Thus it appears unlikely that the proof of a holographic c-theorem for higher curvature gravity can be fully decoupled from the exact form of the higher order interactions. (In fact, we do not expect any c-theorem to hold unless additional unitarity constraints are imposed.) Nevertheless, we anticipate that it would be possible to make a general statement in the restricted case where the linearized equations of motion remain second order in the AdS background.

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