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Phys. Rev. D **85**, 125036 — Published 26 June 2012

DOI: 10.1103/PhysRevD.85.125036

SU-ITP-12/05
CERN-PH-TH/2012-006

$\mathcal{N} = 2$ Supersymmetry and $U(1)$-Duality

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Understanding the consequences of the $E_{7(7)}$ duality on the UV properties of $\mathcal{N} = 8$ supergravity requires unravelling when and how duality-covariant actions can be constructed so as to accommodate duality-invariant counter-terms. For non-supersymmetric abelian gauge theories exhibiting $U(1)$-duality, with and without derivative couplings, it was shown that such a covariant construction is always possible. In this paper we describe a similar procedure for the construction of covariant non-linear deformations of $U(1)$-duality invariant theories in the presence of rigid $\mathcal{N} = 2$ supersymmetry. This is a concrete step towards studying the interplay of duality and extended supersymmetry.

PACS numbers: 11.30.Pb, 04.65.+e, 11.15.Bt, 03.50.-z

I. INTRODUCTION

When the equations of motion of a classical field theory are said to respect a duality invariance, they are invariant under the rotation of an electric field into its magnetic field or of a field strength into its dual field strength. Among the theories exhibiting such properties are Maxwell's theory, whose duality group is $U(1)$, and extended $\mathcal{N} \geq 2$ supergravity theories, whose duality properties were first discussed in [1]. Of recent interest is the maximally supersymmetric supergravity theory [2], $\mathcal{N} = 8$ supergravity, whose duality group is $E_{7(7)}$. Such dualities could have rather nontrivial consequences. Indeed, it has been suggested that the UV-behavior of $\mathcal{N} = 8$ supergravity, known through four loops [3], could be explained by the absence of possible $E_{7(7)}$-invariant counterterms, as demonstrated through six-loops [4, 5].

The requirement that duality invariance be respected is very constraining, as it relates free and interaction terms. Perturbatively, the duality symmetry is expected to be continuous; as such, it has an associated conserved current, known as the Noether-Guillard-Zumino (NGZ) current [6]. The conservation of the NGZ current leads to non-trivial constraints on the possible deformations of the theories, as the addition of duality-invariant terms does not generically preserve the duality invariance of the equations of motion. An essential ingredient in these constraints is the fact that the dual field strengths are determined by the equations of motion which receive contributions from the deformation terms and thus modify the NGZ current. In $\mathcal{N} = 8$ supergravity, the relation between $E_{7(7)}$ invariants and the conservation of the duality current was raised in [7].

A very natural and interesting question is whether and in what sense classical duality symmetries are preserved at the quantum level. A direct answer to this question is not straightforward; for example, it is not clear how duality symmetries are generically visible in scattering amplitudes. While multi-soft scalar limits can probe [8] the coset structure of supergravity theories, it is not immediately clear how to similarly probe the transformation properties of vector fields. Several indirect approaches are possible. One could construct the (local part of the) effective action and see if the equations of motion derived from it obey the same duality symmetry as at the classical level. Another approach would be to construct rational functions of momenta obeying the properties of scattering amplitudes which also obey the multi-soft scalar limit constraints. Both approaches have been explored in [7] and [4, 5], respectively, for constraining and constructing possible counterterms of $\mathcal{N} = 8$ supergravity.

There exist examples of duality transformations which receive modifications at the quantum level. For two-dimensional sigma models, T-duality parallels electric/magnetic duality of four dimensional gauge gauge theories. In addition to replacing a field by its dual, invariance under duality transformation also requires changes of the parameters of the theory (*i.e.* of the target space supergravity fields). It was shown in [9] that the T-duality transformation rules [10] which guarantee the duality invariance of sigma models at one-loop level should be modified at higher loops. These higher-loop corrections may also be reinterpreted as higher-loop corrections to the relation between the sigma model fields and their duals. It is thus important to keep in mind the possibility of similar corrections in more general duality-satisfying interacting quantum field theories.

In order for a theory to preserve, at the quantum level, the duality of its classical equations of motion, while admitting a duality-invariant counterterm, it is necessary that it admits higher-order deformations that maintain the action's duality covariance. Recently, Bossard and Nicolai suggested [11] that there exist algorithms to perturbatively deform all duality-satisfying theories in a manner consistent with the classical duality transformations. If true, besides offering a possibility of constraining the finiteness of $\mathcal{N} = 8$ supergravity [7, 11, 12], it suggests the possibility of constructing non-trivial Born-Infeld-type supergravity theories the first of which would be $\mathcal{N} = 2$ supergravity, as proposed in [12].

The procedures outlined in [11] involved adding one nonlinear initial deformation source to the consistency relations imposed by the tree-level duality transformations and solving them – resulting in an infinite number of terms contributing to the effective action. Such consequences are in line with expectations based on soft scalar limits [5] in the case of $\mathcal{N} = 8$ supergravity. The unmodified covariant procedure of [11], however, does not reproduce known simpler duality-satisfying effective actions. In [12] three of the current authors explored the single-deformation-source approach and found that, in general, an infinite series of deformations of the consistency relations are required to reproduce known results. Indeed, obtaining even the venerable Born-Infeld (BI) model in the absence of supersymmetry requires an infinite sequence of deformation sources to the consistency relations. A generalized procedure was therefore proposed and analyzed for bosonic theories with no explicit derivatives. The algorithms developed in [12] were used in [13] to construct a class of self-dual models which, in addition to standard BI terms F^n, include higher derivatives terms $\partial^{4n} F^{2n+2}$.

A necessary step for the extension of this procedure to supergravity theories (and for showing that it is indeed possible to preserve the classical duality symmetry in the presence of quantum corrections) is the construction of supersymmetric theories of vector multiplets exhibiting duality symmetries. Such actions have been constructed previously through different methods: a manifestly supersymmetric Born-Infeld model was constructed in [14], non-linear superfield actions with spontaneously broken supersymmetry were studied in [15–17], and models with manifest supersymmetry and non-linear electromagnetic duality were developed in [18–21].

In this paper we describe the application of the constructions of [11, 12] to theories with rigid $\mathcal{N}=2$ supersymmetry when the duality is of $U(1)$-type. It is important to stress that this setup is different from the one of supergravity theories exhibiting duality symmetries. Indeed, here the starting action is free and the deformation may be tuned as desired; if the tree-level action is interacting, the deformation is generated by quantum corrections within this theory and cannot be freely adjusted. While the former setup is far less constraining than the latter, the construction of deformations of free actions has proved in the past not to be straightforward.

After setting up a convenient notation, in section II we briefly review the generalized procedure of [12], and the known duality-consistent models in the context of $\mathcal{N}=2$ supersymmetry. In section III we describe a new form of the action in terms of corrections generated by a deformation source. This allows us to write the action directly in terms of a recursively solved non-linear constraint. We provide examples of specific sources in section IV and discuss how they can be simply combined to generate a wide variety of actions, including the BI action found in the literature. We conclude and comment on the next steps required towards approaching the construction of a Born-Infeld-type $\mathcal{N}=2$ supergravity and potential ramifications for $\mathcal{N}=8$ supergravity in section V. In appendix A we provide, for completeness, information on $\mathcal{N}=0$ and $\mathcal{N}=1$ duality invariant models. In appendix B we argue that under $\mathcal{N}=2$ supersymmetry, in contrast with $\mathcal{N}=1$, the electromagnetic duality models require the presence of space-time derivatives acting on superfields. In appendix C we provide a summary of our $\mathcal{N}=2$ superspace conventions.

II. REVIEW

Let us begin by recalling the covariant construction [12] of duality-satisfying actions in terms of deformation sources. We will then proceed to summarize the known $\mathcal{N}=2$ supersymmetric theories whose equations of motion are duality-invariant.

A. Generalized duality covariant procedure

To efficiently formulate classical duality relations, and their corrections, it is useful to organize fields F and dual fields G such that the classical duality transformations act as simply as possible. For Maxwell's theory, as well as supersymmetric versions, this duality transformation can be expressed as a simple multiplication by a phase B and the relevant complex field basis is

$$T = F - \mathrm{i}\,G\,, \quad \overline{T} = F + \mathrm{i}\,G \tag{2.1}$$

or rather their self-dual and anti-self-dual components $T^{\pm} = \frac{1}{2}(T \pm \mathrm{i}\widetilde{T})$ and $\overline{T}^{\pm} = \frac{1}{2}(\overline{T} \pm \mathrm{i}\widetilde{\overline{T}})$ as follows

$$\delta T^{\pm} = \mathrm{i}\,B\,T^{\pm} \qquad \delta\overline{T}^{\pm} = -\mathrm{i}\,B\,\overline{T}^{\pm}\,. \tag{2.2}$$

Throughout this paper we suppress space-time indices whenever possible. However, introducing the tilde operation, as we do in defining $T^{\pm}$, involves normalized contraction with the Levi-Civita symbol,

$$\widetilde{A}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}A_{\rho\sigma}\,. \tag{2.3}$$

A similar organization of fields holds for supergravity theories; the main difference is the appearance of scalar field-dependent matrices in the analogs of the expressions above [6, 22]. In terms of these fields, the undeformed linear duality constraint on the equations of motion of field strengths T can be given quite simply by a "twisted self-duality" constraint:

$$T^{+} = \overline{T}^{-} = 0\,. \tag{2.4}$$

In these variables, the constraint [6] that the action be self-dual is

$$\overline{T}^+T^+ - \overline{T}^-T^- = 0 \;. \tag{2.5}$$

The twisted self-duality relation (2.4) may be interpreted as more fundamental than the action. Indeed, the action can be determined by its relation to the dual field strength. In the case of gauge theories this relation is just

$$G = 2\frac{\delta \mathcal{S}}{\delta F} \tag{2.6}$$

For supergravity theories the fields F and G (and thereby T) acquire further indices, specifying their transformation under the duality group, e.g. $T \mapsto T^{AB}$, *etc.* In the following we will not write such indices.

A covariant procedure proposed in [12], generalizing that of [11], parametrizes the possible deformations of an action in terms of a function $\mathcal{I}(T^-, \overline{T}^+, \lambda)$ where λ is a dimensionful coupling constant. We start with a duality conserving initial action $\mathcal{S}_{\text{initial}}$, and a duality-invariant counterterm (or deformation) $\Delta\mathcal{S}$, expressible as a function of the conjugate self-dual field-strength $\overline{T}^+$. We wish to arrive at an action $\mathcal{S}_{\text{final}}$ that incorporates the counterterm yet still conserves the duality current. We proceed as follows [12]:

1. Take the variation of the counterterm with respect to the field-strength, and express as a function of T^-, and $\overline{T}^+$,

$$\frac{\delta \Delta \mathcal{S}}{\delta \overline{T}^+} \to \frac{\delta \mathcal{I}(T^-, \overline{T}^+, \lambda)}{\delta \overline{T}^+} \,. \tag{2.7}$$

2. Introduce an ansatz for the deformation source $\mathcal{I}(T^-, \overline{T}^+, \lambda)$. In general, this may be taken to depend on all possible duality invariants.
3. Constrain the self-dual field strength to this variation:

$$T^+ = \frac{\delta \mathcal{I}(T^-, \overline{T}^+, \lambda)}{\delta \overline{T}^+} \,. \tag{2.8}$$

4. Translate eq. (2.8) to a differential constraint on $\mathcal{S}_{\text{final}}$.
5. Introduce an ansatz for $\mathcal{S}_{\text{final}}$, which is analytic around the origin, in terms of Lorentz invariants constructed from T^- and $\overline{T}^+$. For the case of $U(1)$, as we will see in section III, this will be straightforward for $\mathcal{N} = 2$ abelian gauge theories. In general this is unknown and can depend on other fields (e.g. scalars) in non-trivial ways.
6. Solve for both the $\mathcal{I}$ ansatz parameters, as well as the final action ansatz parameters, order by order in the coupling constant, enforcing the consistency of the relevant NGZ consistency equation and any additional desired symmetries of the target action, enlarging the ansatz if one runs into inconsistency.

The duality conservation relation (2.5) imposes constraints on the possible deformation sources. If the deformation source $\mathcal{I}(T, \overline{T}, \lambda)$ is hermitian this constraint is simply that $\mathcal{I}$ be invariant under the duality transformation, eq. (2.2),

$$\left(\overline{T}^+ \frac{\delta}{\delta \overline{T}^+} - T^- \frac{\delta}{\delta T^-}\right) \mathcal{I}(T^-, \overline{T}^+, \lambda) = 0 \;. \tag{2.9}$$

This differential operator "measures" the charge of $\mathcal{I}$ under the $U(1)$ duality transformation (2.2) and thus requires that $\mathcal{I}$ is invariant. As discussed in [12], an invariant deformation source does not imply that the deformation of the action is also invariant; rather, as discussed in [6], the complete deformed action should transform nontrivially under duality transformations.

In sections III and IV we will see that these steps have natural counterparts in models with $\mathcal{N} = 2$ rigid supersymmetry; they will allow us to easily superpose arbitrary $U(1)$-invariant deformation sources[1] and will lead to a straightforward generation of new classes of models – as well as a systematic reconstruction of known ones.

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[1] Here $U(1)$ invariance refers to invariance under duality transformations which, in terms of the field variables T and T^*, act as $U(1)$ transformations.

B. $\mathcal{N}=2$ Supersymmetry and Duality

The interplay between supersymmetry and duality invariance has been studied at length in the literature. A derivation and a review of the main results are given by Kuzenko and Theisen in [19]. While it is known how to promote essentially every duality-satisfying bosonic model to an $\mathcal{N}=1$ supersymmetric one, adding further supercharges proved to be relatively difficult. $\mathcal{N}=2$ supersymmetric extensions of the BI theory have been found [16, 17, 19–21]; one of their essential features is the presence of explicit spacetime superfield derivatives in the action. We will review here these actions; in appendix B we will argue that any such action necessarily contains spacetime derivatives, in addition to spinorial derivatives of the superfields.

In the absence of hypermultiplets, the standard $\mathcal{N}=2$ superspace provides an effective framework for organizing actions and their deformations. It is parametrized by four bosonic and eight fermionic coordinates $\mathcal{Z}^A=(x^a,\theta^\alpha_i,\bar\theta^i_{\dot\alpha})$, with a and α being a vector and Weyl spinor Lorentz indices and $i=1,2$ being the $SU(2)$ R-symmetry index. Actions describing the dynamics of $\mathcal{N}=2$ vector multiplets are written in terms of the (anti) chiral superfield strengths $\overline{\mathcal{W}}$ and $\mathcal{W}$ which satisfy the Bianchi identities [2]

$$\mathcal{D}^{ij}\,\mathcal{W}\;=\;\overline{\mathcal{D}}^{\,ij}\,\overline{\mathcal{W}}\,. \tag{2.10}$$

For an abelian gauge symmetry they can be solved by expressing the superfield strength in terms of an unconstrained prepotential V_{ij}:

$$\mathcal{W}=\overline{\mathcal{D}}^{\,4}\mathcal{D}^{ij}\,V_{ij}\,,\qquad\qquad \overline{\mathcal{W}}=\mathcal{D}^{\,4}\overline{\mathcal{D}}^{\,ij}\,V_{ij}\;. \tag{2.11}$$

The overall factors of $\overline{\mathcal{D}}^{\,4}$ and $\mathcal{D}^{\,4}$ guarantee that $\mathcal{W}$ and $\overline{\mathcal{W}}$ are chiral and anti-chiral, respectively, since $\overline{\mathcal{D}}{}^i_{\dot\alpha}\overline{\mathcal{D}}^{\,4}U=0$ for any superfield U.

Similarly to the $\mathcal{N}=0$ and $\mathcal{N}=1$ theories, duality transformations for $\mathcal{N}=2$ theories may be implemented [19] in the path integral as a Legendre transform. One starts with the action

$$\mathcal{S}_{\rm inv}=\mathcal{S}[\mathcal{W},\overline{\mathcal{W}}]-\frac{\rm i}{8}\int d^8\mathcal{Z}\,\mathcal{W}\mathcal{M}+\frac{\rm i}{8}\int d^8\,\overline{\mathcal{Z}}\,\overline{\mathcal{W}}\,\overline{\mathcal{M}}\,, \tag{2.12}$$

treating $\mathcal{W}$ and $\overline{\mathcal{W}}$ are unconstrained superfields (*i.e.* not obeying the Bianchi identity). $\mathcal{M}$ and its conjugate are determined by varying S with respect to $\mathcal{W}$ and $\overline{\mathcal{W}}$

$${\rm i}\,\mathcal{M}\equiv 4\,\frac{\delta}{\delta\mathcal{W}}\,\mathcal{S}[\mathcal{W},\overline{\mathcal{W}}]\,,\qquad\qquad -{\rm i}\,\overline{\mathcal{M}}\equiv 4\,\frac{\delta}{\delta\overline{\mathcal{W}}}\,\mathcal{S}[\mathcal{W},\overline{\mathcal{W}}]\,. \tag{2.13}$$

Their equations of motion

$$\mathcal{D}^{ij}\,\mathcal{M}\;=\;\overline{\mathcal{D}}^{\,ij}\,\overline{\mathcal{M}} \tag{2.14}$$

have the same functional form as the Bianchi identities (2.10).

The infinitesimal duality transformations are therefore very similar to the $\mathcal{N}=0$ and $\mathcal{N}=1$ ones:

$$\delta\mathcal{W}\;=\;B\,\mathcal{M}\,,\qquad\qquad \delta\mathcal{M}\;=\;-B\,\mathcal{W}\,. \tag{2.15}$$

The requirement that $S_{\rm inv}$ in eq. (2.12) is invariant under this transformation leads to the $\mathcal{N}=2$ analog of the duality conservation (NGZ) relation (2.5)

$$\int {\rm d}^8\mathcal{Z}\left(\mathcal{W}^2+\mathcal{M}^2\right)\;=\;\int {\rm d}^8\overline{\mathcal{Z}}\left(\overline{\mathcal{W}}^2+\overline{\mathcal{M}}^2\right)\,, \tag{2.16}$$

originally proven in [18], where it was called the "$\mathcal{N}=2$ self-duality equation". This is the direct analog of the $\mathcal{N}=1$ NGZ relation and reduces to it upon truncation of $\mathcal{N}=1$ chiral multiplet from the $\mathcal{N}=2$ vector multiplet and

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[2] The derivatives $\mathcal{D}^{ij}$ and $\overline{\mathcal{D}}^{\,ij}$ are defined as $\mathcal{D}^{ij}=\mathcal{D}^{i\alpha}\mathcal{D}^j_\alpha$ and $\overline{\mathcal{D}}^{\,ij}=\overline{\mathcal{D}}{}^i_{\dot\alpha}\overline{\mathcal{D}}^{j\dot\alpha}$. See also appendix C.

integration over two fermionic coordinates. One could in fact reconstruct the $\mathcal{N}=2$ constraint by starting from its $\mathcal{N}=1$ limit and requiring that it is manifestly supersymmetric and that it remains bilinear in superfields [12].

Solutions of this equation have proven fairly elusive. The free $\mathcal{N}=2$ supersymmetric Maxwell action

$$\mathcal{S}_{\text{free}} = \frac{1}{8}\int \mathrm{d}^8\mathcal{Z}\,\mathcal{W}^2 + \frac{1}{8}\int \mathrm{d}^8\overline{\mathcal{Z}}\,\overline{\mathcal{W}}^2 \tag{2.17}$$

satisfies this constraint. An interacting action

$$\mathcal{S} \;=\; \frac{1}{4}\int \mathrm{d}^8\mathcal{Z}\,\mathcal{X} + \frac{1}{4}\int \mathrm{d}^8\overline{\mathcal{Z}}\,\overline{\mathcal{X}}\,, \tag{2.18}$$

where the chiral superfield $\mathcal{X}$ is a functional of $\mathcal{W}$ and $\overline{\mathcal{W}}$ and is a solution of the constraint

$$\mathcal{X} \;=\; \mathcal{X}\,\overline{\mathcal{D}}^4\overline{\mathcal{X}} \;+\; \frac{1}{2}\mathcal{W}^2\,, \tag{2.19}$$

was proposed in [16, 17]; this action was proven in [18] to obey the $\mathcal{N}=2$ self-duality constraint (2.16). This system may be solved perturbatively in the number of fields [17, 19–21] and leads to an action of the form

$$\mathcal{S}_{\mathcal{N}=2} = S_{\text{free}} + \int \mathrm{d}^{12}\mathcal{Z}\,\mathcal{W}^2\,\overline{\mathcal{W}}^2\,\mathcal{Y}\,(\mathcal{D}^4\mathcal{W}^2,\;\overline{\mathcal{D}}^4\overline{\mathcal{W}}^2) + \mathcal{O}(\partial_\mu\mathcal{W}) \tag{2.20}$$

where $\mathcal{Y}$ is a Born-Infeld-type functional. The extra terms with space-time derivatives $\partial_\mu\mathcal{W}$ are required for $\mathcal{N}>1$ actions, see Appendix B. The system (2.18), (2.19) was introduced in [16, 17] as the $\mathcal{N}=2$ generalization of the Born-Infeld action.

An action exhibiting D3 brane type shift symmetry, exposing the spontaneous breaking of translational invariance in the directions transverse to the brane, was proposed in [19]. It simultaneously solves the $\mathcal{N}=2$ NGZ constraint (2.16).

$$\begin{aligned}
\mathcal{S}_{\text{BI}} \;&=\; \mathcal{S}_{\text{free}} \;+\; \mathcal{S}_{\text{int}} &(2.21)\\
\mathcal{S}_{\text{int}} \;&=\; \frac{1}{8}\int \mathrm{d}^{12}\mathcal{Z}\,\Bigg\{\mathcal{W}^2\,\overline{\mathcal{W}}^2\,\Bigg[\lambda + \frac{\lambda^2}{2}\left(\mathcal{D}^4\mathcal{W}^2 + \overline{\mathcal{D}}^4\overline{\mathcal{W}}^2\right)\\
&\quad + \frac{\lambda^3}{4}\left((\mathcal{D}^4\mathcal{W}^2)^2 + (\overline{\mathcal{D}}^4\overline{\mathcal{W}}^2)^2 + 3\,(\mathcal{D}^4\mathcal{W}^2)(\overline{\mathcal{D}}^4\overline{\mathcal{W}}^2)\right)\Bigg]\\
&\quad + \frac{1}{3}\left[\frac{\lambda^2}{3}\mathcal{W}^3\Box\overline{\mathcal{W}}^3 + \frac{\lambda^3}{2}\;\left((\mathcal{W}^3\Box\overline{\mathcal{W}}^3)\overline{\mathcal{D}}^4\overline{\mathcal{W}}^2 + (\overline{\mathcal{W}}^3\Box\mathcal{W}^3)\mathcal{D}^4\mathcal{W}^2 + \frac{1}{24}\mathcal{W}^4\Box^2\overline{\mathcal{W}}^4\right)\right]\\
&\quad + \mathcal{O}(\mathcal{W}^{10})\Bigg\}\,, &(2.22)
\end{aligned}$$

where we have introduced a dimensionful coupling λ, usually set to unity in the literature. The unique term with no fermionic or space-time derivatives, $\mathcal{W}^2\,\overline{\mathcal{W}}^2$, yields the known F^4 term of the Born-Infeld action, c.f. appendix A. The sixth-order terms, apart from the $\mathcal{W}^3\Box\overline{\mathcal{W}}^3$ terms with space-time derivatives, also correspond to a straightforward generalization of the bosonic BI model. This action was confirmed[3] in [20, 21]; moreover, it belongs to the class of actions constructed in [20] which also exhibit another nonlinearly realized $\mathcal{N}=2$ supersymmetry algebra.

The construction we will detail in later sections will allow us to recover the action (2.22) among infinitely many other actions. While all of these actions are expressible in the form of eq. (2.18), *i.e.* given by the sum of a chiral and anti-chiral actions, they will not generically satisfy eq. (2.19). Relaxing the requirement that the action has the form (2.19) leads, as we will see, to a variety of actions with different properties.

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[3] Ref. [20] assigns $\Box$ a factor of $-\frac{1}{2}$ relative to the convention of [19].

Combinations of superfields	Chirality	Charge
$T^+ = \mathcal{W} - \mathrm{i}\mathcal{M}$	$+$	$+$
$\overline{T}^+ = \mathcal{W} + \mathrm{i}\mathcal{M}$	$+$	$-$
$T^- = \overline{\mathcal{W}} - \mathrm{i}\overline{\mathcal{M}}$	$-$	$+$
$\overline{T}^- = \overline{\mathcal{W}} + \mathrm{i}\overline{\mathcal{M}}$	$-$	$-$

TABLE I: The four combinations of superfields have $\pm$ chirality and $\pm$ duality charge.

III. CONSTRUCTION OF DUALITY-SATISFYING ACTIONS

An important lesson from the construction of bosonic gauge theory duality-covariant actions was that the twisted self-duality constraint can be seen as more fundamental than the action. Indeed, the twisted self-duality constraint determines the action through the definition of the dual field G, see eq. (2.6). The supersymmetric generalization of this feature is that an $\mathcal{N}=2$ twisted self-duality constraint should determine the action through the definition (2.13) of the dual field $\mathcal{M}$. We will discuss the $\mathcal{N}=2$ case in the same language as the generalized procedure [12] and so begin by describing the suitable twisted self-duality constraint and its deformation sources. We will then proceed, for a generic deformation source $\mathcal{I}$, to construct an action with the desired properties.

A. Twisted self-duality.

As we saw in section II B, $\mathcal{W}$ and $\mathcal{M}$ are interchanged by infinitesimal duality transformations. Similarly to the $\mathcal{N}=0$ case, they may be combined into T-variables which are simply rescaled by such transformations: there are two chiral

$$T^+ = \mathcal{W} - \mathrm{i}\,\mathcal{M}\,, \qquad \overline{T}^+ = \mathcal{W} + \mathrm{i}\,\mathcal{M} \tag{3.1}$$

and two anti-chiral fields

$$T^- = \overline{\mathcal{W}} - \mathrm{i}\,\overline{\mathcal{M}}\,, \qquad \overline{T}^- = \overline{\mathcal{W}} + \mathrm{i}\,\overline{\mathcal{M}}\;, \tag{3.2}$$

each pair having one positive and one negative charge under the $U(1)$ duality transformation:

$$\begin{pmatrix} \delta T^+ \\ \delta\overline{T}^+ \end{pmatrix} = \begin{pmatrix} \mathrm{i}\,B & 0 \\ 0 & -\mathrm{i}\,B \end{pmatrix} \begin{pmatrix} T^+ \\ \overline{T}^+ \end{pmatrix}, \qquad \begin{pmatrix} \delta T^- \\ \delta\overline{T}^- \end{pmatrix} = \begin{pmatrix} \mathrm{i}\,B & 0 \\ 0 & -\mathrm{i}\,B \end{pmatrix} \begin{pmatrix} T^- \\ \overline{T}^- \end{pmatrix}. \tag{3.3}$$

The behavior of these fields are summarized in table I, which allows us to identify immediately the kind of superspace integral that is needed to turn some product of superfields into a supersymmetric action as well as identify its properties under duality transformations. The twisted self-duality constraint is the same as (2.4)

$$T^+ = \overline{T}^- = 0\,, \tag{3.4}$$

while the supersymmetric NGZ constraint (2.16) becomes

$$\int d^8\overline{\mathcal{Z}}\,\overline{T}^+ T^+ - \int d^8\mathcal{Z}\,\overline{T}^- T^- = 0\;, \tag{3.5}$$

whose solutions we would like to construct.

As in the bosonic case, we begin with an "initial source of deformation" $\mathcal{I}(T^-, \overline{T}^+)$ which is a function of the superfields not set to zero by (3.4). On dimensional grounds, any such $\mathcal{I}$ will depend on a dimensionful coupling λ. Moreover, since both chiral and anti-chiral superfields can appear as arguments, $\mathcal{I}$ is naturally a full superspace integral. We will further assume that $\mathcal{I}$ is hermitian, which will lead to a simple characterization of the solutions to (3.5). The deformation of the linear twisted self-duality constraint (3.4) will be given by

$$T^+ = \frac{\delta\mathcal{I}(T^-, \overline{T}^+)}{\delta\overline{T}^+}\,, \qquad (T^+)^* = \overline{T}^- = \frac{\delta\mathcal{I}(T^-, \overline{T}^+)}{\delta T^-}\,, \tag{3.6}$$

where, in the second equality, we used the assumption that $\overline{\mathcal{I}(T^-,\overline{T}^+)} = \mathcal{I}(T^-,\overline{T}^+)$. Any deformation source yields an action; the NGZ identity (3.5) identifies the deformation sources leading to actions with duality-invariant equations of motion. Indeed, by replacing (3.6) into (3.5) we find a differential equation for $\mathcal{I}$:

$$0 = \int d^8\overline{\mathcal{Z}}\,\overline{T}^+\frac{\delta}{\delta\overline{T}^+}\mathcal{I}(T^-,\overline{T}^+) - \int d^8\mathcal{Z}\,T^-\frac{\delta}{\delta T^-}\mathcal{I}(T^-,\overline{T}^+)\,. \tag{3.7}$$

It is worth noting that, since $\mathcal{I}$ is a full superspace integral, each of the two superficially chiral integrals above is, in fact, also an integral over the full superspace. A notable difference from the $\mathcal{N}=0$ case is that the supersymmetric NGZ constraint involves a space-time integral, which projects out possible total derivatives in its integrand. A solution to the equation (3.7) is that $\mathcal{I}$ is invariant under the $U(1)$ duality transformation (3.3). Indeed, as in the bosonic case, the operator

$$\left(\overline{T}^+\frac{\delta}{\delta\overline{T}^+} - T^-\frac{\delta}{\delta T^-}\right) \tag{3.8}$$

"measures" the charge under such transformations. A slightly more general solution to eq. (3.7) allows $\mathcal{I}$ to be invariant up to total derivatives.

The twisted self-duality equations (3.6) can be solved recursively and yield $\mathcal{M}$ as a series in λ with coefficients which are functions of $\mathcal{W}$ and $\overline{\mathcal{W}}$ of appropriate degrees of homogeneity:

$$\mathcal{M} = \mathcal{M}^{(0)} + \sum_{n\geq 1}\lambda^n\mathcal{M}^{(n)}(\mathcal{W},\overline{\mathcal{W}})\ . \tag{3.9}$$

Taking into account the fact that $\mathcal{I}$ is $\mathcal{O}(\lambda)$, the recursive solution for eqs. (3.6) is

$$\mathcal{M}^{(n)} \equiv \lambda^{-n}\left(\frac{\delta}{\delta\overline{T}^+}\mathcal{I}\left[T^-(\mathcal{W},\mathcal{M}^{(n-1)}),\overline{T}^+(\overline{\mathcal{W}},\overline{\mathcal{M}}^{(n-1)})\right] - \sum_{j=1}^{n-1}\lambda^j\mathcal{M}^{(j)}\right) \text{ with } \lambda^{m>n}\to 0\,, \tag{3.10}$$

with $\mathcal{M}^{(0)}$ being the solution to the linear twisted self-duality constraint, $\mathcal{M}^{(0)} = -i\mathcal{W}$.

With a solution in hand, the action may then be found by integrating the equations (2.13). This can be done directly on a case by case basis. We will show that it is in fact straightforward to carry out this integration for a general $\mathcal{I}$. The resulting action has a simple form as demonstrated by the examples detailed in the next section.

As reviewed in section II B, the NGZ consistency condition (2.16), (3.5) is simply the requirement that the right-hand side of eq. (2.12) – with $\mathcal{W}$, $\mathcal{M}$ and their conjugates treated as independent fields – remains invariant under duality transformations. While in general there exist many duality invariants, given an action $S[\mathcal{W},\overline{\mathcal{W}},\lambda]$, it is possible to construct a natural duality invariant expression[4]:

$$\mathcal{S}_{\rm inv} = -\lambda\frac{d}{d\lambda}\mathcal{S}[\mathcal{W},\overline{\mathcal{W}},\lambda]\ . \tag{3.11}$$

This construction is a particular example of the general fact (see e.g. [6, 19]) that the derivative of a duality-satisfying action with respect to a duality invariant parameter is duality-invariant. To see that this is indeed the case let us carry out an infinitesimal duality transformation of this relation:

$$\delta\mathcal{S}_{\rm inv} = -\lambda\frac{d}{d\lambda}\left(\int d^8\mathcal{Z}\,\delta\mathcal{W}\frac{\delta\mathcal{S}}{\delta\mathcal{W}} + \int d^8\overline{\mathcal{Z}}\,\delta\overline{\mathcal{W}}\frac{\delta\mathcal{S}}{\delta\overline{\mathcal{W}}}\right) \tag{3.12}$$

with $\delta\mathcal{W}$ and $\delta\overline{\mathcal{W}}$ given by (2.15). The variation of the action with respect to $\mathcal{W}$ and $\overline{\mathcal{W}}$ can then be expressed, using (2.13), in terms of $\mathcal{M}$ and its conjugate; thus, $\delta\mathcal{S}_{\rm inv}$ becomes

$$\delta\mathcal{S}_{\rm inv} = -\frac{\mathrm{i}}{4}B\,\lambda\frac{d}{d\lambda}\left(\int d^8\mathcal{Z}\,\mathcal{M}^2 - \int d^8\overline{\mathcal{Z}}\,\overline{\mathcal{M}}^2\right)\ . \tag{3.13}$$

[4] The overall sign is chosen for later convenience.

Since $\mathcal{W}$ and $\overline{\mathcal{W}}$ are independent of the coupling constant λ, we may freely add them to the parenthesis above:

$$\delta\mathcal{S}_{\rm inv} = -\frac{\rm i}{4}B\,\lambda\frac{d}{d\lambda}\left(\int d^8\mathcal{Z}\ \left(\mathcal{W}^2+\mathcal{M}^2\right) - \int d^8\overline{\mathcal{Z}}\ \left(\overline{\mathcal{W}}^2+\overline{\mathcal{M}}^2\right)\right)\ . \tag{3.14}$$

This expression vanishes identically for $\mathcal{M}$ and $\overline{\mathcal{M}}$ satisfying the NGZ duality constraint (2.16), implying that (3.11) indeed represents a valid choice of $\mathcal{S}_{\rm inv}$, though of course not the only possible one[5].

Clearly this choice of $S_{\rm inv}$ allows us, through eq. (2.12), to reconstruct the action in terms of $\mathcal{W}$, $\mathcal{M}$ and their conjugates. It is indeed easy to see that, upon use of (3.11), eq. (2.12) becomes a first-order differential equation for $\mathcal{S}[\mathcal{W},\overline{\mathcal{W}},\lambda]$

$$-\lambda\frac{d}{d\lambda}\mathcal{S}[\mathcal{W},\overline{\mathcal{W}},\lambda] = \mathcal{S}[\mathcal{W},\overline{\mathcal{W}},\lambda] - \frac{\rm i}{8}\int d^8\mathcal{Z}\ \mathcal{W}\mathcal{M}[\mathcal{W},\overline{\mathcal{W}},\lambda] + \frac{\rm i}{8}\int d^8\overline{\mathcal{Z}}\ \overline{\mathcal{W}}\overline{\mathcal{M}}[\mathcal{W},\overline{\mathcal{W}},\lambda] \tag{3.15}$$

with the condition that at $\lambda = 0$ its solution should be the free action. Its solution provides a form of the reconstructive identity for the action

$$\mathcal{S} = \frac{\rm i}{8\lambda}\int d\lambda\Big[\int {\rm d}^8\mathcal{Z}\,\mathcal{W}\mathcal{M}[\mathcal{W},\overline{\mathcal{W}},\lambda] - \int {\rm d}^8\overline{\mathcal{Z}}\,\overline{\mathcal{W}}\,\overline{\mathcal{M}}[\mathcal{W},\overline{\mathcal{W}},\lambda]\Big]\,, \tag{3.16}$$

where $\mathcal{M}[\mathcal{W},\overline{\mathcal{W}},\lambda]$ and $\overline{\mathcal{M}}[\mathcal{W},\overline{\mathcal{W}},\lambda]$ are simultaneous solutions of the NGZ duality constraint (2.16) and of the deformed twisted self-duality equation. The latter introduces the dependence on the dimensionful coupling λ through the initial deformation source. Given such a solution (3.9), (3.10), the action we are looking for is:

$$\mathcal{S} = {\rm i}\int {\rm d}^8\mathcal{Z}\,\mathcal{W}\sum_{n=0}\frac{\lambda^n}{8(n+1)}\mathcal{M}^{(n)}[\mathcal{W},\overline{\mathcal{W}}] + {\rm h.c.}\,. \tag{3.17}$$

One may then check, on a case by case basis, that both the dual field $\mathcal{M}$ and its conjugate (also constructed from this action as in eq. (2.13)) reproduce the dual field that was used to construct the action through eq. (3.16).

This class of actions encompasses both the free action (obtained for $\mathcal{I} = 0$) as well as the interacting actions reviewed in the previous section. The relevant $\mathcal{X}$ function is just

$$\mathcal{X} \equiv \frac{\rm i}{2\lambda}\int {\rm d}\lambda\,\mathcal{W}\,\mathcal{M}\ . \tag{3.18}$$

This requires a very specific $\mathcal{I}_{BI}$ – one that involves a superposition of an infinite number of initial source terms, much like the $\mathcal{N} = 0$ Born-Infeld models discussed in [12, 13]. We will present this aggregate deformation source through order λ^3 in section IV. It is not difficult to construct higher-order terms which reproduce the action (2.18)-(2.19).

B. Covariant construction of $\mathcal{N} = 2$-SUSY duality-satisfying actions

Let us summarize here the $\mathcal{N} = 2$ generalization of the bosonic covariant construction [12, 13] reviewed in section II A beginning with some initial deformation or counterterm $\delta\mathcal{S}$. While some of the steps are very similar, others depart from the bosonic ones due mainly to the compact action introduced in eq. (3.16).

1. Take the variation of the counterterm with respect to the field-strength superfield, and express as a function of T^- and $\overline{T}^+$,

$$\frac{\delta\Delta\mathcal{S}}{\delta\overline{T}^+} \to \frac{\delta\mathcal{I}(T^-,\overline{T}^+,\lambda)}{\delta\overline{T}^+}\,. \tag{3.19}$$

[5] For example, any function of λ and $dS/d\lambda$ will lead to a possible candidate for the action. It is possible that all such choices are in fact equivalent through a change of initial deformation source, perhaps through a field redefinition.

2. Introduce an ansatz for the deformation source $\mathcal{I}(T^-, \overline{T}^+, \lambda)$. In general, this may be taken to depend on all possible duality invariants.

3. Constrain the dual field strength to this variation:

$$T^+ = \frac{\delta \mathcal{I}(T^-, \overline{T}^+, \lambda)}{\delta \overline{T}^+} \tag{3.20}$$

4. Solve eq. (3.20) iteratively for the dual field $\mathcal{M} = \mathcal{M}[\mathcal{W}, \overline{\mathcal{W}}, \lambda]$ and its conjugate while checking that the NGZ duality constraint is satisfied. Any $U(1)$-invariant hermitian deformation source will automatically lead to solutions which pass this test.

5. Use $\mathcal{M}$ and its conjugate found at step 4 to construct the action

$$\mathcal{S} = \frac{\mathrm{i}}{8\lambda} \int d\lambda \Big[\int \mathrm{d}^8 \mathcal{Z}\, \mathcal{W}\, \mathcal{M}[\mathcal{W}, \overline{\mathcal{W}}, \lambda] - \int \mathrm{d}^8 \overline{\mathcal{Z}}\, \overline{\mathcal{W}}\overline{\mathcal{M}}[\mathcal{W}, \overline{\mathcal{W}}, \lambda] \Big] \tag{3.21}$$

while checking for additional desired properties and enlarging the ansatz for $\mathcal{I}$ if necessary.

6. Verify that $\mathcal{M}$ and its conjugate used at step 5 are reproduced as

$$\mathrm{i}\, \mathcal{M}[\mathcal{W}, \overline{\mathcal{W}}, \lambda] \equiv 4\, \frac{\delta}{\delta \mathcal{W}}\, \mathcal{S}[\mathcal{W}, \overline{\mathcal{W}}]\,, \qquad -\mathrm{i}\, \overline{\mathcal{M}}[\mathcal{W}, \overline{\mathcal{W}}, \lambda] \equiv 4\, \frac{\delta}{\delta \overline{\mathcal{W}}}\, \mathcal{S}[\mathcal{W}, \overline{\mathcal{W}}] \tag{3.22}$$

It is important to point out that the last step above is not a substitute for any of the earlier steps. For example, it is possible to construct deformation sources $\mathcal{I}$ which, while not solving the NGZ constraint lead nevertheless through eq. (3.20) to dual fields $\mathcal{M}$ and an action which reproduces them.

We will proceed in the next section to apply this construction to recover the actions reviewed in section II B as well as duality covariant actions with manifest $\mathcal{N} = 2$ supersymmetry and quite novel structure.

IV. DUALITY EXAMPLES

In this section we will follow the steps outlined above and discuss four examples of deformation sources and their corresponding duality-covariant actions. We present each source with an arbitrary scalar pre-factor (a, b, c, d) . In each subsection we will mention the relevant value necessary to match terms present in the BI action given in eq. (2.22). At higher orders in the number of fields, the nested super derivatives can become quite lengthy. To shorten the expressions we introduce the notation

$$\mathcal{H}^{(n)} \; = \; \mathcal{D}^4(\mathcal{W}^2 \overline{\mathcal{H}}^{(n-1)}) \tag{4.1}$$

$$\overline{\mathcal{H}}^{(n)} \; = \; \overline{\mathcal{D}}^4(\overline{\mathcal{W}}^2 \mathcal{H}^{(n-1)})\,, \tag{4.2}$$

with $\mathcal{H}^{(0)} \equiv \mathcal{D}^4(\mathcal{W}^2)$ and $\overline{\mathcal{H}}^{(0)} \equiv \overline{\mathcal{D}}^4(\overline{\mathcal{W}}^2)$. This notation will also eliminate the explicit space-time derivatives, unless they appear already in the deformation source.

A. $(T^-)^2(\overline{T}^+)^2$

The lowest dimension "initial source of deformation" which is manifestly invariant under duality transformations is

$$\mathcal{I}_1 = a\, \lambda \int \mathrm{d}^{12} \mathcal{Z}\; (T^-)^2 (\overline{T}^+)^2 \;; \tag{4.3}$$

it also has a direct counterpart in the bosonic theory. Upon use of the definition of $T^{\pm}$ and its conjugate in terms of $\mathcal{W}$ and $\mathcal{M}$, eq. (3.6) can be written as:

$$\mathcal{M} = -\mathrm{i}\, \mathcal{W} + 2a\lambda\, \mathrm{i} \left(\overline{\mathcal{D}}^4 (\overline{\mathcal{W}} - \mathrm{i}\overline{\mathcal{M}})^2 \right) (\mathcal{W} + \mathrm{i}\, \mathcal{M})\,. \tag{4.4}$$

This equation, and its conjugate involving $\overline{\mathcal{M}}$, can be solved recursively order by order in λ. The solution is relatively compact[6] with the coefficients of $\mathcal{M} = \sum_n \lambda^n \mathcal{M}^{(n)}$ being given by

$$\mathcal{M}^{(0)} = -\mathrm{i}\,\mathcal{W}\,, \tag{4.5}$$

$$\mathcal{M}^{(n)}|_{n>0} = (-2)^{5-n} a \sum_{l=0}^{n-1} \sum_{q=0}^{n-(1+l)} \alpha(l,q;n) \overline{\mathcal{D}}^4 [\, \overline{\mathcal{M}}^{(n-(1+q+l))} \overline{\mathcal{M}}^{(q)} \mathcal{M}^{(l)} \,] \tag{4.6}$$

with

$$\alpha(q,l;n) \equiv \xi_2(q)\xi_2(l)\xi_2(n-l-q-1)\,. \tag{4.7}$$

$$\xi_2(x)|_{x>0} \equiv (-2)^x/2\,, \tag{4.8}$$

$$\xi_2(x)|_{x=0} \equiv 1\,. \tag{4.9}$$

With the notation introduced in eq. (4.2), the first few terms in the expansion of $\mathcal{M}$ are

$$\mathcal{M} = -\mathrm{i}\,\mathcal{W} + 16a\,\mathrm{i}\,\lambda \mathcal{W}\overline{\mathcal{H}}^{(0)} - \frac{\mathrm{i}}{2}(16a)^2\lambda^2\,\mathcal{W}\,\left((\overline{\mathcal{H}}^{(0)})^2 + 2\overline{\mathcal{H}}^{(1)} \right) + \cdots\,. \tag{4.10}$$

The action is then given directly by eq. (3.17); since it is linear in $\mathcal{M}$, that a recursive solution for $\mathcal{M}$ automatically translates into a recursive expression for the action. Here we choose to solve the recursion and express it through λ^4 in the form of a Hermitian action:

$$\begin{aligned}
\mathcal{S}_1^{\mathrm{int}} = \int \mathrm{d}^{12}\mathcal{Z}\mathcal{W}^2\overline{\mathcal{W}}^2 \Bigg\{ & -2a\lambda + 16a^2\lambda^2(\mathcal{H}^{(0)} + \overline{\mathcal{H}}^{(0)}) \\
& - 128a^3\lambda^3 \left({\mathcal{H}^{(0)}}^2 + 2\mathcal{H}^{(1)} + {\overline{\mathcal{H}}^{(0)}}^2 + 2\overline{\mathcal{H}}^{(1)} \right) \\
& + 1024a^4\lambda^4 \left(6(\mathcal{H}^{(0)}\mathcal{H}^{(1)} + \overline{\mathcal{H}}^{(0)}\overline{\mathcal{H}}^{(1)}) + 4(\mathcal{H}^{(2)} + \overline{\mathcal{H}}^{(2)}) + {\mathcal{H}^{(0)}}^3 + {\overline{\mathcal{H}}^{(0)}}^3 \right) \\
& - 8192a^5\lambda^5 \Bigg(6\left({\mathcal{H}^{(0)}}^2{\overline{\mathcal{H}}^{(0)}}^2 + 2(\mathcal{H}^{(0)}\mathcal{H}^{(2)} + \overline{\mathcal{H}}^{(0)}\overline{\mathcal{H}}^{(2)}) + 2{\mathcal{H}^{(1)}}^2 + 2{\overline{\mathcal{H}}^{(1)}}^2 \right) \\
& + 8\left(\mathcal{H}^{(1)}{\mathcal{H}^{(0)}}^2 + \mathcal{H}^{(3)} + {\overline{\mathcal{H}}^{(0)}}^2\overline{\mathcal{H}}^{(1)} + \overline{\mathcal{H}}^{(3)} \right) + {\mathcal{H}^{(0)}}^4 + {\overline{\mathcal{H}}^{(0)}}^4 \Bigg) + \mathcal{O}(\lambda^6) \Bigg\}\,.
\end{aligned} \tag{4.11}$$

Note that setting $a = -2^{-4}$ recovers the terms in the BI action, eq. (2.22) through λ^2 which do not contain the space-time Laplacian, as well as relevant contributions at higher orders. While some sequences of terms – such as those depending only on powers of $\mathcal{H}^{(0)}$ – can be resummed, it does not appear that this action has a closed-form expression.

B. $(T^-)^3\Box(\overline{T}^+)^3$

As reviewed in section II B, the term $(\mathcal{W}^3\Box\overline{\mathcal{W}}^3)$ in the BI action (2.22) is required if that action is to be interpreted as the supersymmetric D3-brane action. Such a term will not appear in an action of the type (3.17) unless we add a term like $(T^-)^3\Box(\overline{T}^+)^3$ to the initial deformation source. Such terms may be obtained from those discussed in [12] by dressing them with space-time derivatives. Let us therefore consider the duality-invariant

$$\mathcal{I}_2 = b\lambda^2 \int \mathrm{d}^{12}z\;(T^-)^3\Box(\overline{T}^+)^3\,. \tag{4.12}$$

[6] It reflects the fact that when the source of deformation is a single quartic term, we have a cubic deformation of the linear constraint, as was also noticed in the "Model A" explored in [13].

The twisted self-duality equation eq. (3.6) can be written as

$$\mathcal{M} = -\mathrm{i}\,\mathcal{W} + 3\,\mathrm{i}\,b\,\lambda^2\,(\mathcal{W} + \mathrm{i}\,\mathcal{M})^2\,\left(\overline{\mathcal{D}}^4\,\Box(\overline{\mathcal{W}} - \mathrm{i}\overline{\mathcal{M}})^3\right) \tag{4.13}$$

and, together with its conjugate, can be solved for $\mathcal{M}$ and $\overline{\mathcal{M}}$. The first few terms in their solution are

$$\begin{aligned}
\mathcal{M} = -\mathrm{i}\,\mathcal{W} &+ 2^5 \times 3\,\mathrm{i}\,b\,\lambda^2\,\mathcal{W}^2\,(\overline{\mathcal{D}}^4\,\Box\,\overline{\mathcal{W}}^3\Big) \\
&- 2^9 \times 3^2\,\mathrm{i}\,b^2\,\lambda^4\,\mathcal{W}^2\Big[2\mathcal{W}\,\left(\overline{\mathcal{D}}^4\,\Box\,\overline{\mathcal{W}}^3\right)^2 + 3\,\overline{\mathcal{D}}^4(\Box\,(\overline{\mathcal{W}}^4\mathcal{D}^4\bar{\mathcal{W}}^3)\Big] \\
&+ 2^{13} \times 3^3\,\mathrm{i}\,b^3\,\lambda^6\mathcal{W}^2\Big[12\mathcal{W}\overline{\mathcal{D}}^4\left(\Box\left(\overline{\mathcal{W}}^4\mathcal{D}^4\left(\Box\left(\mathcal{W}^3\right)\right)\right)\right)\overline{\mathcal{D}}^4\left(\Box\left(\overline{\mathcal{W}}^3\right)\right) \\
&+ 9\left(\overline{\mathcal{D}}^4\left(\Box\left(\overline{\mathcal{W}}^5\mathcal{D}^4\left(\Box\left(\mathcal{W}^3\right)\right)^2\right)\right) + \overline{\mathcal{D}}^4\left(\Box\left(\overline{\mathcal{W}}^4\mathcal{D}^4\left(\Box\left(\mathcal{W}^4\overline{\mathcal{D}}^4\left(\Box\left(\overline{\mathcal{W}}^3\right)\right)\right)\right)\right)\right)\right) \\
&+ 5\mathcal{W}^2\overline{\mathcal{D}}^4\left(\Box\left(\overline{\mathcal{W}}^3\right)\right)^3\Big] + \cdots\,.
\end{aligned} \tag{4.14}$$

Due to the presence of the Laplacian, it is inconvenient to use the notation (4.2) for this deformation; we will instead express the action in terms of $\mathcal{W}$ and its conjugate. Through order λ^4 it is

$$\begin{aligned}
\mathcal{S}_2^{\mathrm{int}} = \int \mathrm{d}^{12}\mathcal{Z}\Big\{ &- 4b\lambda^2\left(\mathcal{W}^3\Box\left(\overline{\mathcal{W}}^3\right) + \overline{\mathcal{W}}^3\Box\left(\mathcal{W}^3\right)\right) \\
&+ 2^6\frac{9}{5}b^2\lambda^4\Big(3\mathcal{W}^3\Box\left(\overline{\mathcal{W}}^4\mathcal{D}^4\left(\Box\left(\mathcal{W}^3\right)\right)\right) + 2\overline{\mathcal{W}}^4\Box\left(\mathcal{W}^3\right)\mathcal{D}^4\left(\Box\left(\mathcal{W}^3\right)\right) \\
&+ 2\mathcal{W}^4\Box\left(\overline{\mathcal{W}}^3\right)\overline{\mathcal{D}}^4\left(\Box\left(\overline{\mathcal{W}}^3\right)\right) + 3\overline{\mathcal{W}}^3\Box\left(\mathcal{W}^4\overline{\mathcal{D}}^4\left(\Box\left(\overline{\mathcal{W}}^3\right)\right)\right)\Big) + \cdots\Big\}\,.
\end{aligned} \tag{4.15}$$

Here we chose the interaction terms in the action as a full superspace integral, making use of the fact that the nonlinear terms in $\mathcal{M}$ contain the appropriate chiral projector. It is not difficult to see that, by choosing $b = -\frac{1}{576} = -2^{-6}\,3^{-2}$ we recover the $\Box$-dependent term in the λ^2 contribution to the BI action eq. (2.22) as well as relevant contributions at higher orders.

C. $(T^-)^4\Box^2(\overline{T}^+)^4$

Invariants of type eq. (4.3) naturally generalize to higher orders – for example

$$\mathcal{I}_3 = c\lambda^3\int \mathrm{d}^{12}z\;(T^-)^4\Box^2(\overline{T}^+)^4\,. \tag{4.16}$$

With this deformation source, the twisted self-duality equation eq. (3.6) is

$$\mathcal{M} = -\mathrm{i}\,\mathcal{W} + 4\,\mathrm{i}\,c\,\lambda^3\;\;(\mathcal{W} + \mathrm{i}\,\mathcal{M})^3\left(\overline{\mathcal{D}}^4\,\Box^2(\overline{\mathcal{W}} - \mathrm{i}\overline{\mathcal{M}})^4\right)\,. \tag{4.17}$$

As in the previous two cases, this equation and its conjugate can be solved recursively for $\mathcal{M}$ and $\overline{\mathcal{M}}$, leading to

$$\begin{aligned}
\mathcal{M} = -\mathrm{i}\,\mathcal{W} &+ 2^9\,\mathrm{i}\,c\lambda^3\mathcal{W}^3\overline{\mathcal{D}}^4(\Box^2(\overline{\mathcal{W}}^4)) \\
&- 2^{17}\,\mathrm{i}\,c^2\lambda^6\mathcal{W}^3(4\overline{\mathcal{D}}^4(\Box^2(\overline{\mathcal{W}}^6\mathcal{D}^4(\Box^2(\mathcal{W}^4)))) + 3\mathcal{W}^2\overline{\mathcal{D}}^4(\Box^2(\overline{\mathcal{W}}^4))^2) \\
&+ 2^{26}\,\mathrm{i}\,c^3\lambda^9\mathcal{W}^3\Big(12\mathcal{W}^2\overline{\mathcal{D}}^4(\Box^2(\overline{\mathcal{W}}^4)))\overline{\mathcal{D}}^4(\Box^2(\overline{\mathcal{W}}^6\mathcal{D}^4(\Box^2(\mathcal{W}^4)))) \\
&+ 9\overline{\mathcal{D}}^4(\Box^2(\overline{\mathcal{W}}^8\mathcal{D}^4(\Box^2(\mathcal{W}^4))^2)) \\
&+ 8\overline{\mathcal{D}}^4(\Box^2(\overline{\mathcal{W}}^6\mathcal{D}^4(\Box^2(\mathcal{W}^6\overline{\mathcal{D}}^4(\Box^2(\overline{\mathcal{W}}^4))) + 6\mathcal{W}^4\overline{\mathcal{D}}^4(\Box^2(\overline{\mathcal{W}}^4))^3\Big)\cdots\,.
\end{aligned} \tag{4.18}$$

Absorbing the overall chiral projector in the nonlinear terms into the integration measure and thus expressing the action as a hermitian full superspace integral we find, through $\mathcal{O}(\lambda^6)$ that

$$\mathcal{S}_3^{\rm int} = \int \mathrm{d}^{12}\mathcal{Z}\Big\{ -16c\lambda^3\left(\mathcal{W}^4\Box\left(\Box\left(\overline{\mathcal{W}}^4\right)\right)+\overline{\mathcal{W}}^4\Box\left(\Box\left(\mathcal{W}^4\right)\right)\right)$$
$$+\,2^{14}\tfrac{1}{7}c^2\lambda^6\Big(4\mathcal{W}^4\Box\left(\Box\left(\overline{\mathcal{W}}^6\mathcal{D}^4\left(\Box\left(\Box\left(\mathcal{W}^4\right)\right)\right)\right)\right)+3\overline{\mathcal{W}}^6\Box\left(\Box\left(\mathcal{W}^4\right)\right)\mathcal{D}^4\left(\Box\left(\Box\left(\mathcal{W}^4\right)\right)\right)$$
$$+\,3\mathcal{W}^6\Box\left(\Box\left(\overline{\mathcal{W}}^4\right)\right)\overline{\mathcal{D}}^4\left(\Box\left(\Box\left(\overline{\mathcal{W}}^4\right)\right)\right)$$
$$+\,4\overline{\mathcal{W}}^4\Box\left(\Box\left(\mathcal{W}^6\overline{\mathcal{D}}^4\left(\Box\left(\Box\left(\overline{\mathcal{W}}^4\right)\right)\right)\right)\right)\Big)+\cdots\Big\}\,. \tag{4.19}$$

To recover the $\Box^2$ term in the λ^3 contribution to the BI action eq. (2.22), as well as relevant terms at higher orders, we can set $c = -\,2^{-12}\,3^{-2}$.

D. $(T^-)^2(\overline{T}^+)^2\overline{\mathcal{D}}^4((T^-)^2)\mathcal{D}^4((\overline{T}^+)^2)$

Another interesting invariant constructed out of four T^- and four $\overline{T}^+$ factors and a number of super-derivatives is:

$$\mathcal{I}_4 = \lambda^3 d\int \mathrm{d}^{12}\mathcal{Z}(T^-)^2(\overline{T}^+)^2\overline{\mathcal{D}}^4((T^-)^2)\mathcal{D}^4((\overline{T}^+)^2)\,. \tag{4.20}$$

The resulting constraint equation is:

$$\mathcal{M} = -\mathrm{i}\,\mathcal{W}+2\,\mathrm{i}\,d\,\lambda^3(\mathcal{W}+\mathrm{i}\,\mathcal{M})\Big\{\overline{\mathcal{D}}^4\left(\left(\overline{\mathcal{W}}-\mathrm{i}\,\overline{\mathcal{M}}\right)^2\right)\overline{\mathcal{D}}^4\left(\left(\overline{\mathcal{W}}-\mathrm{i}\,\overline{\mathcal{M}}\right)^2\mathcal{D}^4\left((\mathcal{W}+\mathrm{i}\,\mathcal{M})^2\right)\right)$$
$$+\,\overline{\mathcal{D}}^4\left(\left(\overline{\mathcal{W}}-\mathrm{i}\,\overline{\mathcal{M}}\right)^2\mathcal{D}^4\left((\mathcal{W}+\mathrm{i}\,\mathcal{M})^2\overline{\mathcal{D}}^4\left(\left(\overline{\mathcal{W}}-\mathrm{i}\,\overline{\mathcal{M}}\right)^2\right)\right)\right)\Big\}\,. \tag{4.21}$$

Solving it recursively leads, order by order in λ, to the following interaction terms in the action:

$$\mathcal{S}_4^{\rm int} = \int \mathrm{d}^{12}\mathcal{Z}\,\mathcal{W}^2\overline{\mathcal{W}}^2\Big\{-16d\lambda^3(\mathcal{H}^{(1)}+\overline{\mathcal{H}}^{(1)})+4096\,d^2\,\lambda^6\Big(2(\mathcal{H}^{(1)}\mathcal{H}^{(2)}+\overline{\mathcal{H}}^{(1)}\overline{\mathcal{H}}^{(2)})+$$
$$\mathcal{H}^{(0)}{\mathcal{H}^{(1)}}^2+\mathcal{H}^{(4)}+\overline{\mathcal{H}}^{(0)}{\overline{\mathcal{H}}^{(1)}}^2+\overline{\mathcal{H}}^{(4)}\Big)+\mathcal{O}(\lambda^9)\Big\}\,. \tag{4.22}$$

The leading λ^3 term recovers the final λ^3 terms in the BI action (2.22) with d chosen to be $d = 2^{-10}$.

E. BI action through λ^3

By suitably combining the deformation sources discussed above with the mentioned relative coefficients

$$\mathcal{I}_{BI} = -\int \mathrm{d}^{12}\mathcal{Z}\Big(\lambda\,2^{-4}\,(T^-)^2(\overline{T}^+)^2+\lambda^2\,2^{-6}\,3^{-2}\,(T^-)^3\Box(\overline{T}^+)^3$$
$$+\,\lambda^3\,2^{-12}3^{-2}(T^-)^4\Box^2(\overline{T}^+)^4-\lambda^3\,2^{-10}\,(T^-)^2(\overline{T}^+)^2\overline{\mathcal{D}}^4((T^-)^2)\mathcal{D}^4((\overline{T}^+)^2)$$
$$+\,\mathcal{O}(\lambda^4)\Big) \tag{4.23}$$

and carrying out the procedure of section III B we recover the Born-Infeld action from the literature given above in eq. (2.22). While the solution to the twisted self-duality constraint is inherently nonlinear in the initial deformation source, to this order only a small number of cross terms are actually relevant. At higher orders in λ more such terms will become important together with the appearance of new invariants that may be added to the initial deformation source. However, not all cross terms can be modified by adding higher order invariants. For example, at λ^4 the only new source should likely be $(T^-)^5\Box^3(\overline{T}^+)^5$ which clearly cannot modify the majority of $\mathcal{O}(\lambda^4)$ terms in the action.

F. More general models

Any $U(1)$ duality invariant source can be used in this procedure, and the new form of the action makes it rather trivial to combine various sources to tailor craft duality-consistent $\mathcal{N}=2$ actions incorporating such sources – such as recovering eq. (2.22). Consider for example:

$$\begin{aligned}
\mathcal{I}^{(n,m)}_{\text{gen A}} &= \lambda_{(n,m)}(\Box^n\overline{T}^{(+)2})\Box^m(\Box^n T^{(-)2})\,, &(4.24)\\
\mathcal{I}^{(n,m)}_{\text{gen A}'} &= \lambda_{(n,m)}(\Box^n\mathcal{D}^4\overline{T}^{(+)2})\Box^m(\Box^n\overline{\mathcal{D}}^4 T^{(-)2})\,, &(4.25)\\
\mathcal{I}^{(n,m)}_{\text{gen A}''} &= \lambda_{(n,m)}(\Box^n\partial^\mu\overline{T}^{(+)}\partial^\nu\overline{T}^{(+)})\Box^m(\Box^n\partial_\mu T^{(-)}\partial_\nu T^{(-)})\,, &(4.26)\\
\mathcal{I}^{(n,m)}_{\text{gen B}} &= \lambda^2_{(n,m)}(\Box^n\overline{T}^{(+)3})\Box^m(\Box^n T^{(-)3})\,, &(4.27)\\
\mathcal{I}^{(n,m)}_{\text{gen B}'} &= \lambda^2_{(n,m)}(\Box^n\mathcal{D}^4\overline{T}^{(+)3})\Box^m(\Box^n\overline{\mathcal{D}}^4 T^{(-)3})\,, &(4.28)\\
\mathcal{I}^{(n_1,n_2,m)}_{\text{gen C}} &= \lambda^3_{(n_1,n_2,m)}(\Box^{n_1}\overline{T}^{(+)2})(\Box^{n_2}\overline{T}^{(+)2})\Box^m(\Box^{n_1}T^{(-)2})(\Box^{n_2}T^{(-)2})\,, &(4.29)
\end{aligned}$$

for $n_i, m \geq 0$. It is important to note that the $\lambda_{n_i,m}$ above will have different dimensions based upon the value of n_i, m. Each of these is a duality invariant initial source, and generates novel duality-covariant actions through the procedure specified above. It is not difficult to see how these patterns generalize to an infinite set of other initial sources.

The BI action is non-renormalizable by power counting. As a test of the preservation of duality symmetries at the quantum level one may construct counterterms in the BI model and check whether they preserve the classical $U(1)$ duality symmetry as well as whether the higher-order terms generated by our procedure reproduce those obtained by direct calculation. Compared to supergravity theories, the simplicity of the BI model provides a clear advantage as a testing ground for such questions. One-loop calculations in the $\mathcal{N}=2$ BI theory have been already carried out in [23, 24] where it was found that the relevant momentum space on-shell counterterm is

$$\Gamma^{\text{div}}_1 = c(\epsilon)\int dp_1dp_2dp_3dp_4\delta^4\left(\sum p_i\right)\int d^8\theta\left(s^2+\frac{4}{3}t^2\right)\mathcal{W}(p_1)\mathcal{W}(p_2)\overline{\mathcal{W}}(p_3)\overline{\mathcal{W}}(p_4)\,. \tag{4.30}$$

Here $c(\epsilon)$ is a divergent coefficient. This counterterm may be written in position space in several ways, related by use of on-shell conditions $p_i^2=0$. One way, chosen in [24] places one derivative on each superfield factor:

$$\Gamma^{\text{div}}_1 = c(\epsilon)\int d^4xd^8\theta\left(\partial^\mu\mathcal{W}\partial_\mu\mathcal{W}\partial^\nu\overline{\mathcal{W}}\partial_\nu\overline{\mathcal{W}}+\frac{4}{3}\partial^\mu\mathcal{W}\partial^\nu\mathcal{W}\partial_\mu\overline{\mathcal{W}}\partial_\nu\overline{\mathcal{W}}\right)\,. \tag{4.31}$$

Using the on shell conditions, $p_i^2=0$, the first term may also be written as

$$\int d^4xd^8\theta\,\Box(\mathcal{W}^2)\Box(\overline{\mathcal{W}}^2)\,, \tag{4.32}$$

which identifies it as as arising from $\mathcal{I}^{(1,0)}_{\text{gen }A}$. The second term, which is similar to bosonic terms considered in [13], arises from $\mathcal{I}^{(0,0)}_{\text{gen }A''}$. Through our procedure, these deformation sources make definite predictions about some of the higher order terms that should appear in perturbative higher loop calculations. In particular, these terms will necessarily be accompanied by higher powers of the coefficient $c(\epsilon)$ in eq. (4.30) thus implying, apart from the field dependence, also a definite strength of the corresponding divergence. It should be interesting to check explicitly whether these predicted higher order terms correspond to the results of multi-loop and multi-leg perturbative calculations and, if they do not, whether the deformation source may be suitably modified to accommodate the difference. New deformation sources will, however, always be necessary at each loop order. Indeed, the nonlinearity of eq. (3.20) implies that all higher-order terms which are generated will contain more fields than the deformation source. Consequently, new deformation sources will be required to all orders in perturbation theory at least for the four-superfield counterterm.

As we explained above, as long as the action depends only on a chiral and anti-chiral superfields $\mathcal{W}$, $\overline{\mathcal{W}}$ and their spinorial and space-time derivatives, one is free to make any choice of $\mathcal{I}$. When one such choice is made eq. (3.6) supplies a recursive procedure for obtaining $\mathcal{M}$. While it may not always be as simple as the quartic deformation given in eq. (4.3) the method described above, generalized from [12], and as demonstrated also in [13], allows one to recursively produce the action to any desired level. At each order the number of invariants that can be used is limited, given a fixed engineering dimension of the coupling constant. We see, therefore, choices made at lower orders typically have definite consequences at higher orders in the dimensional coupling.

It is in principle possible to impose additional requirements of our construction, in particular the existence of additional symmetries beyond duality. In general however, it is not immediately clear how to encode such requirements in the choice of initial deformation source. It may be possible to first identify the properties of $\mathcal{M}[\mathcal{W}, \overline{\mathcal{W}}, \lambda]$ from the action (3.16) and then require that the twisted self-duality equation is also invariant under the same transformations. This could prove too strong a requirement, as many algebraic equations can have the same solution. Alternatively, one may start with the most general deformation source with arbitrary coefficients and determine them by requiring that the resulting action is invariant under the desired symmetries. An important example in this direction is the construction of the action in [19] – which we have reproduced with our method – where in addition to duality symmetry the action has to satisfy a certain constant shift symmetry[7], associated with the D3 brane action:

$$\delta\mathcal{W} = \sigma + \mathcal{O}(\mathcal{W}, \overline{\mathcal{W}}) \ . \tag{4.33}$$

Similarly, in [20] the $\mathcal{N} = 2$ supersymmetric Born-Infeld action was required, apart from duality invariance, to also have a partially broken $\mathcal{N} = 4$ supersymmetry. It is, however, not obvious that the construction of the action is algorithmic and whether actions of a different structure may be obtained by relaxing any one of these properties. Given any self-dual hermitian function $\mathcal{I}$ of $\mathcal{W}$, $\overline{\mathcal{W}}$ and their derivatives, our construction directly constructs, order by order, an action which is self-dual and covariant, thus showing that there exist infinitely many solutions to the self-duality constraints. Further symmetry requirements may also be imposed by a suitable choice of deformation source $\mathcal{I}$.

V. DISCUSSION

In this article we explore the space of $U(1)$-duality invariant actions with rigid $\mathcal{N} = 2$ supersymmetry employing and extending the methods developed and explored by three of the current authors in the previous publication [12]. For these models we identify a useful presentation of the action and the $U(1)$ duality constraint such that—when the latter is solved perturbatively—the construction of the former follows immediately. Namely, when a certain choice of the manifestly duality invariant source of the deformation of the linear twisted self-duality constraint is made, $\mathcal{I}(T, \overline{T}, \lambda)$, its derivative provides a dual superfield $\mathcal{M} = \mathcal{M}(\mathcal{W}, \overline{\mathcal{W}}, \lambda)$ as the function of the original superfields $\mathcal{W}, \overline{\mathcal{W}}$ and as a power series in the coupling λ, as one can see from eqs. (3.6) and (3.9). The chiral part of the action is reconstructed by the integration over λ of the product $\mathcal{W}\mathcal{M}(\mathcal{W}, \overline{\mathcal{W}}, \lambda)$, and the conjugate to it provides the anti-chiral part of the action, as shown in eq. (3.16).

Employing this approach, we have identified several initial deformations, which, after applying the recursive method, collectively reproduce the actions found in [16–19]. Beyond those, there are further classes of deformations, which lead to a rich variety of duality-invariant models with $\mathcal{N} = 2$ supersymmetry. One important observation has been described in [12] for the non-supersymmetric and $\mathcal{N} = 1$ supersymmetric theories: when the initial source of deformation is quartic in F, the deformation of the linear twisted self-duality condition leads to a *Born-Infeld type* action, that is an action containing all powers of F up to infinity. The higher order terms are necessary to maintain the duality invariance of the equations of motion order by order. Using the corresponding construction for $\mathcal{N} = 2$ supersymmetric models detailed in this paper leads to the same conclusion.

The next natural step towards understanding the implications of $E_{7(7)}$ duality for the UV behavior of $\mathcal{N} = 8$ supergravity is the construction of simpler examples, such as U(1) duality-consistent deformed $\mathcal{N} = 2$ supergravity theories. While the jump from rigid-supersymmetry to supergravity is non-trivial, we expect $\mathcal{N} = 2$ supergravity to be an excellent proving ground. It is possible that a suitable covariantization of the twisted self-duality constraint

[7] The shift symmetry of this type is reminiscent of the shift symmetry part of the $E_{7(7)}$ in $\mathcal{N} = 8$ supergravity, acting on scalar fields.

(3.20) coupled with an appropriate choice of deformation source will allow us to construct the analog of the actions discussed in this paper in the presence of local $\mathcal{N}=2$ supersymmetry.

We expect [25] that a suitable covariantization of the twisted self-duality constraint (3.20) coupled with an appropriate choice of deformation source will allow us to construct the analog of the actions discussed in this paper in the presence of local $\mathcal{N}=2$ supersymmetry.

Let us briefly comment on the possible consequences of such covariant constructions for UV divergences in $\mathcal{N}=8$ supergravity. If existent, the first UV divergence for a four-point amplitude (which is sufficient to consider) will be the $\mathcal{N}=8$ supersymmetric completion of a term of the form $f(s,t,u)R^4$, which necessarily contains a local quartic term of the form $f(s,t,u)(dF)^4$ in momentum space. Here f is a polynomial of the usual Mandelstam variables whose degree depends on the loop order at which the divergence arises.

Assuming $E_{7(7)}$-symmetry to persist unmodified at the quantum level, there exist $E_{7(7)}$-invariant counterterms at sufficiently high loop order, as originally shown in [26] and more recently reinforced in [5]. The sufficiency of this reasoning was questioned in [7]: as $E_{7(7)}$ is a continuous global symmetry, it requires the conservation of the corresponding NGZ current[6] in addition to the $E_{7(7)}$-invariance of the counterterm candidates. It was, however, subsequently suggested in [11] that there exists a procedure of perturbative deformation of the linear self-duality constraint which always allows the addition to the action of the candidate counterterm along with all other higher terms required for the conservation of the NGZ current.

The covariant procedure introduced in [11] required generalization to recover the venerable bosonic Born-Infeld action [12]. This class of generalizations has been applied to consider higher derivative terms [13] and in this paper we have presented an application to $\mathcal{N}=2$ global supersymmetry. Discovering the way to meaningfully apply such generalizations to $\mathcal{N}=8$ supergravity could shed light on the UV-finiteness question. In principle, one would start from the classical $\mathcal{N}=8$ theory, in which the supersymmetrization of the Ricci scalar contains terms quadratic in vector fields:

$$\mathcal{S}_{\mathcal{N}=8}(g_{\mu\nu}, F_{\mu\nu}, ...; \kappa^2) = \int \frac{1}{2\kappa^2}(R - F\mathcal{N}(\phi)F + ...)\ . \tag{5.1}$$

One could imagine that attempting to construct a Born-Infeld type $\mathcal{N}=8$ supergravity, by adding suitable deformation sources and applying the covariant duality construction, results in two possible scenarios:

- construction of an $\mathcal{N}=8$ Born-Infeld type supergravity is possible, either for a general value g of the four-vector terms

$$\mathcal{S}^{BI}_{\mathcal{N}=8}(g_{\mu\nu}, F_{\mu\nu}, ...; \kappa^2, g^2) = \int \frac{1}{2\kappa^2}(R - F\mathcal{N}(\phi)F + \cdots) + g^2F^4f_4(s,t,u) + \cdots + g^{2m}F^nf_n(s,t,u,...) + \cdots \tag{5.2}$$

  or only for specific coefficients $g_n(\kappa)$ of the n–vector terms

$$\begin{aligned}\mathcal{S}^{BI'}_{\mathcal{N}=8}(g_{\mu\nu}, F_{\mu\nu}, ...; \kappa^2, g^2) = \int \frac{1}{2\kappa^2}(R - F\mathcal{N}(\phi)F + \cdots) + g_4(\kappa^2)F^4f_4(s,t,u) + \cdots \\ + g_m(\kappa^2)F^nf_n(s,t,u,...) + \cdots\end{aligned} \tag{5.3}$$

  It is however only this second option which has the possibility of being consistent with perturbation theory of the undeformed $\mathcal{N}=8$ supergravity[8]. Superficially, the existence of such an action may suggest that the $E_{7(7)}$ duality symmetry is consistent with the existence of UV divergences in this theory (5.1). One must however make sure that the higher-order terms predicted by the covariant duality construction are the same as those obtained from direct calculations. Not being able to render them consistent (*e.g.* by adjusting the deformation source) would imply that some of the assumptions of the construction (*e.g.* that the tree-level duality transformations are unmodified) may need to be relaxed.

- construction of a Born-Infeld type $\mathcal{N}=8$ supergravity is not possible. If it is possible to simultaneously prove that the classical $E_{7(7)}$ transformations do not receive modifications at the quantum level, then $E_{7(7)}$ would predict UV finiteness of $\mathcal{N}=8$ supergravity in four dimensions.

[8] While the κ dependence of n-vector couplings is fixed by dimensional analysis, their precise numerical coefficients may only be fixed by direct calculations.

Either outcome would expose more of the quantum properties of $\mathcal{N}=8$ supergravity and the consequence classical duality symmetries have on them. Similar scenarios exist in all theories exhibiting duality-invariant equations of motion. Along the way to $\mathcal{N}=8$ supergravity, consideration of such symmetries may lead to the identification of other supergravity theories with unexpectedly good ultraviolet properties or may point to a mechanism that makes duality symmetries consistent with the existence of counterterms/UV divergences.

Acknowledgments

We are grateful to S. Ketov and S. Kuzenko and S. Theisen for their help in understanding the issues of $\mathcal{N}=2$ supersymmetry and self-duality and for providing us with important details of the work [18]. We are grateful to P. Aschieri, W. Chemissany, T. Ortin and A. Tseytlin for the stimulating discussions. We are grateful to A. Linde for raising the issue of the predictive power of the two-coupling model (5.2) with regard to the one coupling model in (5.1). J.B. acknowledges support from the Alexander-von- Humboldt foundation within the Feodor-Lynen program. J.J.M.C. and R.K. gratefully acknowledge the Stanford Institute for Theoretical Physics and the NSF Grant No. 0756174 for their support. The work of S.F. has been supported by the ERC Advanced Grant no. 226455, Supersymmetry, Quantum Gravity and Gauge Fields (SUPERFIELDS), and in part by DOE Grant DE-FG03-91ER40662. R.R. also acknowledges the support of NSF under Grant No. 0855356.

Appendix A: Review of $\mathcal{N}=0$ and $\mathcal{N}=1$ duality

A manifestly $\mathcal{N}=1$ supersymmetric NGZ-type identity derived by Kuzenko and Theisen in [18], where it was called "$\mathcal{N}=1$ self-duality equation," is

$$\int \mathrm{d}^6 z \left(W^\alpha W_\alpha + M^\alpha M_\alpha\right) \;=\; \int \mathrm{d}^6 \bar{z} \left(\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha} + \overline{M}_{\dot\alpha}\overline{M}^{\dot\alpha}\right) . \tag{A1}$$

where the chiral and antichiral $\mathcal{N}=1$ superfield strengths are defined as

$$W_\alpha = -\frac{1}{4}\overline{D}^2 D_\alpha V \quad \text{and} \quad \overline{W}_{\dot\alpha} = \frac{1}{4}D^2\overline{D}_{\dot\alpha}V \tag{A2}$$

in terms of a real unconstrained prepotential V. In analogy to eq. (2.13) one defines

$$\mathrm{i}M_\alpha \equiv 2\,\frac{\delta}{\delta W^\alpha}\,S[W,\overline{W}]\,, \qquad\qquad -\mathrm{i}\overline{M}^{\dot\alpha} \equiv 2\,\frac{\delta}{\delta \overline{W}_{\dot\alpha}}\,S[W,\overline{W}]\ . \tag{A3}$$

$\mathcal{N}=1$ duality invariant models can be obtained by considering a general action of the form

$$S \;=\; \frac{1}{4}\int \mathrm{d}^6 z\, W^2 + \frac{1}{4}\int \mathrm{d}^6 \bar{z}\,\overline{W}^2 + \frac{1}{4}\int \mathrm{d}^8 z\, W^2\,\overline{W}^2\,\mathscr{L}\Big(\frac{1}{8}D^2\,W^2, \frac{1}{8}\overline{D}^2\,\overline{W}^2\Big)\ , \tag{A4}$$

where $\mathscr{L}(u,\bar{u})$ is a real analytic function of the complex variable $u \equiv \frac{1}{8}D^2\,W^2$ and its conjugate. With [9]

$$\frac{\delta S}{\delta W^\alpha} = \frac{1}{2}W_\alpha\Big(1 - \frac{1}{4}\overline{D}^2\Big[\overline{W}^2\Gamma\Big]\Big) \quad \text{where} \quad \Gamma = \mathscr{L} + \frac{1}{8}D^2\Big[W^2\frac{\delta\mathscr{L}}{\delta u}\Big] \tag{A5}$$

one finds with eq. (A3)

$$\mathrm{i}\,M_\alpha \;=\; W_\alpha\Big(1 - \frac{1}{4}\overline{D}^2\Big[\overline{W}^2\Gamma\Big]\Big) \tag{A6}$$

[9] Here, as well as in [19], the $\mathcal{N}=1$ functional superderivative is defined as $\frac{\delta W^\beta(z')}{\delta W^\alpha(z)} = -\frac{1}{4}\delta_\alpha{}^\beta\overline{D}^2\,\delta^8(z-z')$.

or, equivalently,

$$\mathrm{i}\, M_\alpha \;=\; W_\alpha \left\{ 1 - \frac{1}{4}\,\overline{D}^2 \Big[\overline{W}^2 \Big(\mathscr{L} + \frac{1}{8} D^2 \Big(W^2 \frac{\partial \mathscr{L}(u,\bar{u})}{\partial u} \Big)\Big) \Big] \right\} . \tag{A7}$$

Plugging eq. (A7) into the NGZ constraint eq. (A1) leads to a functional equation for Γ

$$\int \mathrm{d}^8 z W^2 \overline{W}^2 \mathrm{Im}[\Gamma - \overline{u}\Gamma^2] = 0 \;, \tag{A8}$$

where we have used that for any $\mathcal{N}=1$ superfield Y

$$W^2 \overline{W}^2 \overline{D}^2 [\overline{W}^2\, Y] = W^2 \overline{W}^2 \overline{D}^2 [\overline{W}^2]\, Y \tag{A9}$$

because $W^3 = W_\alpha W_\beta W_\gamma = 0$ for the two-component spinor W_α. One can rewrite the above constraint in terms of $\mathscr{L}$ as

$$\int \mathrm{d}^8 z W^2 \overline{W}^2 \mathrm{Im}\Big[\partial_u (u \mathscr{L}) - \overline{u} (\partial_u (u \mathscr{L}))^2 \Big] = 0 \;. \tag{A10}$$

This partial differential equation has infinitely many solutions, parametrized e.g. by the coefficients of the terms $(u\overline{u})^n$ with $n > 2$ in the expansion around $u = 0$ (as well as the coefficient of $u\overline{u}^2$), as was shown in [27] in the non-supersymmetric case.

The relation to the non-supersymmetric case discussed in [12] is straightforward: taking the integral over the fermionic superspace coordinates and setting the gauginos and auxiliary fields to zero, one finds

$$L = -\frac{1}{2}(\mathbf{u} + \overline{\mathbf{u}}) + \mathbf{u}\overline{\mathbf{u}}\, \mathscr{L}(\mathbf{u}\overline{\mathbf{u}}), \qquad \mathbf{u} \,\equiv\, \frac{1}{8} D^2 W^2 \big|_{\theta=0, D=0, \psi=0} \Rightarrow \frac{1}{4} F^2 + \frac{i}{4} F\tilde{F} \equiv \omega \;. \tag{A11}$$

In the non-supersymmetric cases the Born-Infeld (BI) and Bossard-Nicolai (BN) examples are reproduced by functions [12]:

$$\begin{aligned}
\mathscr{L}_{\mathrm{BI}} \;&=\; \frac{g^2}{1 + \frac{1}{2} g^2 (\omega + \bar{\omega}) + \sqrt{1 + g^2(\omega + \bar{\omega}) + \frac{1}{4} g^4 (\omega - \bar{\omega})^2}} && \text{(A12)} \\
&=\; \frac{g^2}{2} - \frac{g^4}{4}(\omega + \bar{\omega}) + \frac{g^6}{8}((\omega + \bar{\omega})^2 + \omega\bar{\omega}) - \frac{g^8}{16}((\omega + \bar{\omega})^3 + 3(\omega^2 \bar{\omega} + \omega \bar{\omega}^2)) + \ldots && \text{(A13)}
\end{aligned}$$

and

$$\mathscr{L}_{\mathrm{BN}} \;=\; \frac{g^2}{2} - \frac{g^4}{4}(\omega + \bar{\omega}) + \frac{g^6}{8}((\omega + \bar{\omega})^2 + 2\omega\bar{\omega}) - \frac{g^8}{16}\big((\omega + \bar{\omega})^3 + 7(\omega^2 \bar{\omega} + \omega \bar{\omega}^2)\big) + \ldots$$

where the first deviation occurs at $\mathcal{O}(g^6)$.

With the above identifications, the same is true for the model with $\mathcal{N}=1$ supersymmetry. At $\mathcal{O}(g^6)$ a deviation between the Born-Infeld-type $\mathcal{N}=1$ model and the $\mathcal{N}=1$ supersymmetrization of the Bossard-Nicolai model will occur:

$$\mathscr{L}_{\mathrm{BN}}|_{\mathcal{O}(g^6)} - \mathscr{L}_{\mathrm{BI}}|_{\mathcal{O}(g^6)} = \frac{g^6}{8} \omega\bar{\omega} \equiv \frac{g^6}{8} D^2 W^2 \overline{D}^2 \overline{W}^2 \,. \tag{A14}$$

Comparing now the $\mathcal{O}(\lambda^3)$ terms in eq. (4.11), which corresponds to the $\mathcal{N}=2$ supersymmetrization of the BN-initial deformation, with the corresponding terms in the Born-Infeld action eq. (2.22), one finds again the difference

$$\frac{\lambda^3}{32} \mathcal{H}^{(0)} \overline{\mathcal{H}}^{(0)} = \frac{\lambda^3}{32} \mathcal{D}^4 \mathcal{W}^2 \overline{\mathcal{D}}^4 \overline{\mathcal{W}}^2 \,, \tag{A15}$$

where we have set $a = -\frac{1}{16}$ in eq. (4.11) in order to connect to the BI result as discussed in subsection IV A.

Appendix B: $\mathcal{N}=2$ SUSY, $U(1)$ duality, and derivative requirements

Here we consider a generalization of the $\mathcal{N}=0$ and $\mathcal{N}=1$ actions assuming—as in [18]—dependence on the $\mathcal{N}=2$ superfields $\mathcal{W}^2$ and $\overline{\mathcal{W}}^2$. While the free action reads

$$\mathcal{S}_{\text{free}} = \frac{1}{8}\int \mathrm{d}^8\mathcal{Z}\,\mathcal{W}^2 + \frac{1}{8}\int \mathrm{d}^8\overline{\mathcal{Z}}\,\overline{\mathcal{W}}^2\,, \tag{B1}$$

it is obvious to try, whether an ansatz similar[10] to the interaction part of eq. (A4),

$$\mathcal{S}_{\text{int}} = -\frac{1}{4}\int \mathrm{d}^{12}\mathcal{Z}\,\mathcal{W}^2\,\overline{\mathcal{W}}^2\,\mathscr{L}(\mathcal{D}^4\mathcal{W}^2, \overline{\mathcal{D}}^4\overline{\mathcal{W}}^2) \tag{B2}$$

suffices to produce a duality-invariant action. We will show in the following, that this is not the case: one has to include new terms depending on spacetime derivatives in order to satisfy conservation of the NGZ current eq. (2.16).

Using eq. (2.13) in the above equation yields

$$\frac{\delta\mathcal{S}_{\text{int}}}{\delta\mathcal{W}} = -\frac{1}{2}\mathcal{W}\overline{\mathcal{D}}^4(\overline{\mathcal{W}}^2\Gamma) \quad \text{where} \quad \Gamma = \mathscr{L} + \mathcal{D}^4\Big[\mathcal{W}^2\frac{\delta\mathscr{L}}{\delta u}\Big]\,. \tag{B3}$$

where here $u=\mathcal{D}^4\mathcal{W}^2$. In terms of Γ one finds

$$i\,\mathcal{M} = \mathcal{W}\Big(1 - 2\mathcal{W}\overline{\mathcal{D}}^4[\overline{\mathcal{W}}^2\Gamma]\Big)\,. \tag{B4}$$

Plugging eq. (B4) $\mathcal{M}$ into the self-duality equation (2.16) yields

$$\int \mathrm{d}^{12}\mathcal{Z}\,\mathcal{W}^2\,\overline{\mathcal{W}}^2\,\mathrm{Im}[\Gamma - \Gamma\overline{\mathcal{D}}^4(\overline{\mathcal{W}}^2\Gamma)] = 0\,. \tag{B5}$$

Rewriting the above equation in terms of $\mathscr{L}$ does not give the same beautiful result as in the $\mathcal{N}=1$ situation: there is no $\mathcal{N}=2$-analogue to eq. (A9). As the $\mathcal{N}=2$ superfield $\mathcal{W}$ is a scalar, terms containing $\mathcal{W}^3$ will not vanish. Thus the constraint analogue to eq. (A10) reads

$$\int \mathrm{d}^{12}\mathcal{Z}\,\mathcal{W}^2\,\overline{\mathcal{W}}^2\,\mathrm{Im}\Big[\partial_{\mathrm{u}}(\mathrm{u}\mathscr{L}) - \bar{\mathrm{u}}(\partial_{\mathrm{u}}(\mathrm{u}\mathscr{L}))^2 + \Delta\mathscr{L}\Big] = 0\,. \tag{B6}$$

The correctional term $\Delta\mathscr{L}$ contains terms like $\mathcal{D}^4[\overline{\mathcal{D}}^4\overline{\mathcal{W}}]$ which, after carrying out the derivatives, yields terms proportional to $\partial_\mu\overline{\mathcal{W}}$ which do not cancel. Thus, the ansatz eq. (B2) is not sufficient, instead one needs

$$\mathcal{S}_{\text{int}} = \frac{1}{2}\int \mathrm{d}^{12}\mathcal{Z}\,\mathcal{W}^2\,\overline{\mathcal{W}}^2\,\mathscr{L}(\mathcal{D}^4\mathcal{W}^2, \overline{\mathcal{D}}^4\overline{\mathcal{W}}^2) + \mathcal{O}(\partial\mathcal{W}, \partial\overline{\mathcal{W}})\,. \tag{B7}$$

Comparing with eq. (2.20), one finds that those terms do indeed appear. Already at $\mathcal{O}(\lambda^2)$ there is a term $\frac{\lambda^2}{9}\mathcal{W}^3\Box\overline{\mathcal{W}}^3$, which would have vanished in a $\mathcal{N}=1$ supersymmetric theory. At the next order, $\mathcal{O}(\lambda^3)$, one finds terms of the form $\mathcal{W}^2\overline{\mathcal{W}}^2\mathcal{D}^4[\mathcal{W}^2\overline{\mathcal{D}}^4[\overline{\mathcal{W}}^2]]$, which are not covered by an ansatz of the form $\mathcal{W}^2\overline{\mathcal{W}}^2\mathscr{L}(\mathcal{D}^4\mathcal{W}^2, \overline{\mathcal{D}}^4\overline{\mathcal{W}}^2)$.

Appendix C: Conventions of $\mathcal{N}=2$ superspace

Superderivatives are defined as

$$\mathcal{D}^i_\alpha = \partial^i_\alpha + \mathrm{i}\bar\theta^{\dot\alpha i}\partial_{\alpha\dot\alpha} \qquad \text{and} \qquad \overline{\mathcal{D}}_{\dot\alpha i} = -\partial_{\dot\alpha i} - \mathrm{i}\theta^\alpha_i\partial_{\alpha\dot\alpha} \tag{C1}$$

[10] The prefactor is chosen to allow for straightforward comparison of the resulting differential equation with the $\mathcal{N}=1$ model in the last section. Of course, any prefactor can be absorbed in the definition of $\mathscr{L}$.

where $\alpha, \dot\alpha$ are usual $SU(2)$ spinor indices and latin indices are super-indices in the range from $\{1,\ldots 4\}$. Anticommutation relations read

$$\{\mathcal{D}^i_\alpha, \overline{\mathcal{D}}_{\dot\alpha i}\} = -2\,\mathrm{i}\,\delta^i_j\,\partial_{\alpha\dot\alpha}\,. \tag{C2}$$

Derivatives can be combined into

$$\mathcal{D}^{ij} = \mathcal{D}^{\alpha i}\mathcal{D}^j_\alpha \qquad \text{and} \qquad \overline{\mathcal{D}}^{ij} = \overline{\mathcal{D}}^i_{\dot\alpha}\overline{\mathcal{D}}^{\dot\alpha j} \tag{C3}$$

and finally

$$\mathcal{D}^4 = \frac{1}{48}\mathcal{D}^{ij}\mathcal{D}_{ij} \qquad \text{and} \qquad \overline{\mathcal{D}}^4 = \frac{1}{48}\overline{\mathcal{D}}^{ij}\overline{\mathcal{D}}_{ij}\ . \tag{C4}$$

Chiral and antichiral superfields $\mathcal{W}(x,\theta)$ and $\overline{\mathcal{W}}(x,\bar\theta)$ are defined as

$$\overline{\mathcal{D}}_{\dot\alpha i}\mathcal{W} = 0 \qquad \text{and} \qquad \mathcal{D}^i_\alpha\overline{\mathcal{W}} = 0\,. \tag{C5}$$

For a full superfield $\mathcal{V}(x,\theta,\bar\theta)$,

$$\int \mathrm{d}^8\mathcal{Z}\,\overline{\mathcal{D}}^4\mathcal{V} = \int \mathrm{d}^8\mathcal{Z}\,\mathcal{W} = \int \mathrm{d}^{12}\mathcal{Z}\,\mathcal{V} \tag{C6}$$

Correspondingly, the functional derivative for chiral and antichiral superfields are defined via

$$\frac{\delta\mathcal{W}(\mathcal{Z})}{\delta\mathcal{W}(\mathcal{Z}')} = \overline{\mathcal{D}}^4\delta^{12}(\mathcal{Z}-\mathcal{Z}') \qquad \text{and} \qquad \frac{\delta\overline{\mathcal{W}}(\mathcal{Z})}{\delta\overline{\mathcal{W}}(\mathcal{Z}')} = \mathcal{D}^4\delta^{12}(\mathcal{Z}-\mathcal{Z}'). \tag{C7}$$

Because of anticommutativity, powers higher than four in the superderivatives vanish (here we write the chiral part only, the antichiral is completely equivalent),

$$\mathcal{D}^n = 0 \quad \forall n > 4\,, \tag{C8}$$

which leads to

$$\mathcal{D}^4(\mathcal{D}^4(x)) = 0 \qquad \text{and thus} \qquad \mathcal{D}^4(X\mathcal{D}^4(Y)) = \mathcal{D}^4(X)\mathcal{D}^4(Y) \tag{C9}$$

for arbitrary X and Y. Besides linearity and scaling

$$\mathcal{D}^4(X+Y) = \mathcal{D}^4(X) + \mathcal{D}^4(Y) \quad \text{and} \quad \mathcal{D}^4(cX) = c\mathcal{D}^4(X) \quad \forall\,\text{scalar}\,c \tag{C10}$$

the chain rule does not apply trivially due to the product structure of $\mathcal{D}^4$.

For the space-time d'Alambert operator appearing in the examples in subsections 4.3 and 4.16, the following commutation relation holds:

$$\Box\mathcal{D}^4(X) = \mathcal{D}^4\Box(X)\,. \tag{C11}$$

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