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In this paper we consider the relation between the super-renormalizable theories of quantum gravity (SRQG) studied in [1, 2] and an underlying non-commutativity of spacetime. For one particular super-renormalizable theory we show that at linear level (quadratic in the Lagrangian) the propagator of the theory is the same we obtain starting from a theory of gravity endowed with θ-Poincaré quantum groups of symmetry. Such a theory is over the so called θ-Minkowski non-commuative spacetime. We shed new light on this link and show that among the theories considered in [1, 2], there exist only one non-local and Lorentz invariant super-renormalizable theory of quantum gravity that can be described in terms of a quantum group symmetry structure. We also emphasize contact with pre-existent works in the literature and discuss preservation of the equivalence principle in our framework.

I. INTRODUCTION

In the recent papers [1, 2] it has been introduced a modified theory of gravity assuming a synthesis of minimal requirements: (i) regularity of classical solutions; (ii) Einstein-Hilbert action should be the correct low energy limit; (iii) the spacetime dimension has to decrease with the energy; (iv) the theory has to be perturbatively renormalizable at quantum level; (v) the theory has to be unitary, with no other pole beyond the graviton in the propagator.

The theory we are going to summarize in the next section is power counting super-renormalizable at the quantum level at least perturbatively and at classical level the gravitational potential [1], black hole solutions [2, 3, 6–28] and the cosmological solutions are singularity free [1, 4, 5]. The Lagrangian is a "nonlocal" extension of the renormalizable quadratic Stelle theory [29] but the non locality only involve positive powers of the D'Alembertian covariant operator. In other words there are not operators like $1/\Box^p$ $(p > 0)$. The theory is not unique (we thus refer to super-renormalizable "theories"), but all the freedom present in the action can be read in an "entire function" of the D' Alembertian operator, $H(-\Box/\Lambda^2)$ [30] (Λ is a physical mass-invariant scale

introduced in the classical action).

The reason of this paper is not only to find an elegant reason for the nonlocal nature of the action, but it is to find a way to fix uniquely the entire function which is mentioned above. In this paper we show that the propagator of the theory, for a particular choice of the entire function $H(-\Box/\Lambda^2)$, has exactly the same form of the propagator we obtain starting from a theory of gravity endowed with θ-Poincaré quantum groups of symmetry. The right choice is much easier we could think, i.e. $H(-\Box/\Lambda^2) = -\Box/\Lambda^2$. Any other entire function gives of course a well defined super-renormalizable theory of gravity (consistently with some particular properties [1, 2]) but is not compatible with the requirement of having a non-trivial Hopf-algebra-like symmetry regulating the super-renormalizability of the theory. In particular, the Hopf-algebra underlying the super-renormalizable model we discuss below is a quantum-group associated to an associative non-commutative space-time. In particular, this is the only quantum group of (space-time) symmetry that can be accounted within the model presented in [2], if we do not relax the associativity of the space-time points' coordinates. What emerges is therefore a new symmetric structure underlying the theory.

II. THE THEORY

A simplified version of the theory is a nonlocal generalization of the Stelle quadratic action for gravity [29] and

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can be written in the following compact form,

$$\mathcal{L}_g = -\sqrt{-g}\left[\frac{\beta}{\kappa^2}R + R_{\mu\nu}\,F(-\Box_\Lambda)^{\mu\nu\rho\sigma}R_{\rho\sigma}\right], \quad (1)$$

where the tensor $F(-\Box_\Lambda)$ is a function of the covariant D'Alembertian operator $-\Box_\Lambda := -\Box/\Lambda^2$, Λ is a physical mass scale and $\kappa^2 = 32\pi G$. To fix the notation we can write more explicitly the tensor $F(-\Box_\Lambda)$ in terms of two entire functions h_2 and h_0 that we are going to define in this same section,

$$F(-\Box_\Lambda)^{\mu\nu\rho\sigma} := -\left(\beta_2 - h_2(-\Box_\Lambda)\right)g^{\mu\rho}g^{\nu\sigma}$$

$$+\left(\frac{\beta_2}{3} + \beta_0 - h_0(-\Box_\Lambda) - \frac{h_2(-\Box_\Lambda)}{3}\right)g^{\mu\nu}g^{\rho\sigma}. \quad (2)$$

The complete Lagrangian including also the gauge fixing and ghost terms is

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{\mathrm{GF,GH}}, \quad (3)$$

where the gauge fixing and ghost Lagrangian terms are

$$\mathcal{L}_{\mathrm{GF,GH}} = -\frac{1}{2\xi}F^\mu\omega(-\Box_\Lambda^\eta)F_\mu + \bar{C}^\mu M_{\mu\nu}C^\nu. \quad (4)$$

The operator $\Box_\Lambda^\eta$ encapsulates the D'Alembertian of the flat fixed background, whereas F_μ is the gauge fixing function with the weight functional ω. The two functions h_2 and h_0 have not to be polynomial but "entire functions without poles or essential singularities" to avoid ghosts (states with negative norm) in the spectrum ω. $\bar{C}^\mu, C^\mu$ are the ghosts fields and $M_\alpha^\tau = F^\tau_{\mu\nu}D_\alpha^{\mu\nu}$. The gauge fixing function $F^\tau_{\mu\nu}$ and the operator $D_\alpha^{\mu\nu}$ will be defined shortly, in (6).

We calculate now the graviton propagator. For this purpose we start by considering the quadratic expansion of the Lagrangian (1) in the graviton field fluctuation without specifying the explicit form of the functionals h_2 and h_0 (if not necessary). Following the Stelle paper [29], we expand around the Minkowski background $\eta^{\mu\nu}$ in power of the graviton field $h^{\mu\nu}$ defined in the following way

$$\sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}. \quad (5)$$

The form of the propagator depends not only on the gauge choice but also on the definition of the gravitational fluctuation [32]. The gauge choice is the familiar "harmonic gauge" $\partial_\nu h^{\mu\nu} = 0$ and in (4), $F^\tau = F^\tau_{\mu\nu}h^{\mu\nu}$ with $F^\tau_{\mu\nu} = \delta^\tau_\mu\partial_\nu$. $D_\alpha^{\mu\nu}$ is the operator which generates the gauge transformations in the graviton fluctuation $h^{\mu\nu}$. Given the infinitesimal coordinates transformation $x^{\mu'} = x^\mu + \kappa\xi^\mu$, the graviton field transforms as follows

$$\delta h^{\mu\nu} = D_\alpha^{\mu\nu}\xi^\alpha = \partial^\mu\xi^\nu + \partial^\nu\xi^\mu - \eta^{\mu\nu}\partial_\alpha\xi^\alpha$$

$$+\kappa(\partial_\alpha\xi^\mu h^{\alpha\nu} + \partial_\alpha\xi^\nu h^{\alpha\mu} - \xi^\alpha\partial_\alpha h^{\mu\nu} - \partial_\alpha\xi^\alpha h^{\mu\nu}). \quad (6)$$

We Taylor-expand now the gravitational part of the action (1) to the second order in the gravitational perturbation $h^{\mu\nu}(x)$ to obtain the graviton propagator. In the momentum space, the action which is purely quadratic in the gravitational field, reads

$$\mathcal{L}^{(2)} = \frac{1}{4}h^{\mu\nu}(-k)K^{\mu\nu\rho\sigma}h^{\rho\sigma}(k) + \mathcal{L}_{\mathrm{GF}}, \quad (7)$$

where $\mathcal{L}_{\mathrm{GF}}$ is the gauge fixing Lagrangian at the second order in the graviton field

$$\mathcal{L}_{\mathrm{GF}} = \frac{1}{4\xi}h^{\mu\nu}(-k)\Big(\omega(k^2/\Lambda^2)k^2 P^{(1)}_{\mu\nu\rho\sigma}(k)$$

$$+2\omega(k^2/\Lambda^2)k^2\}P^{(0-\omega)}_{\mu\nu\rho\sigma}(k)\Big)h^{\rho\sigma}(k). \quad (8)$$

The kinetic operator $K_{\mu\nu\rho\sigma}$ is defined by

$$K_{\mu\nu\rho\sigma} := -\bar{h}_2(z)\,k^2\,P^{(2)}_{\mu\nu\rho\sigma}(k) + \frac{3}{2}\,k^2\,\bar{h}_0(z)\,P^{(0-\omega)}_{\mu\nu\rho\sigma}(k)$$

$$+\frac{k^2}{2}\,\bar{h}_0(z)\,\{P^{(0-s)}_{\mu\nu\rho\sigma}(k) + \sqrt{3}[P^{(0-\omega s)}_{\mu\nu\rho\sigma}(k) + P^{(0-s\omega)}_{\mu\nu\rho\sigma}(k)]\},$$

and we have introduced the following notation

$$\bar{h}_2(z) := \beta - \beta_2\kappa^2\Lambda^2 z + \kappa^2\Lambda^2 z\,h_2(z),$$
$$\bar{h}_0(z) := \beta - 6\beta_0\kappa^2\Lambda^2 z + 6\kappa^2\Lambda^2 z\,h_0(z), \quad (9)$$

where $z := -\Box_\Lambda$. Notice that in (7) $\Box_\Lambda$ has to be identified with the D'Alembertian operator in flat spacetime $-\Box_\Lambda^\eta$. We have used the gauge $F^\tau = \partial_\mu h^{\mu\tau}$ and introduced the projectors $P^{(2)}, P^{(1)}, P^{(0-s)}, P^{(0-s\omega)}, P^{(0-\omega s)}$ [45] (see also appendix B). Using the orthogonality properties of the projectors we can now invert the kinetic matrix in (7) and obtain the graviton propagator. In the following expression the graviton propagator is expressed in the momentum space according to the quadratic Lagrangian (7),

$$D_{\mu\nu\rho\sigma}(k) = D^{\xi=0}_{\mu\nu\rho\sigma}(k) + D^\xi_{\mu\nu\rho\sigma}(k), \quad (10)$$

where the propagator in the gauge $\xi = 0$ is

$$D^{\xi=0}_{\mu\nu\rho\sigma}(k) = \frac{-i}{(2\pi)^4}\frac{2}{k^2 + i\epsilon}\left(\frac{P^{(2)}_{\mu\nu\rho\sigma}(k)}{\bar{h}_2(k^2/\Lambda^2)} - \frac{2P^{(0-s)}_{\mu\nu\rho\sigma}(k)}{\bar{h}_0(k^2/\Lambda^2)}\right)$$

and $D^\xi_{\mu\nu\rho\sigma}(k)$ is the gauge dependent part of the propagator.

We are now in the position to find an upper bound to the divergences in quantum gravity. We consider a particular theory in which the two general entire functions $h_i(z)$ introduced in the action have the following asymptotic exponential behavior,

$$h_2(z) = \frac{\alpha(e^z - 1) + \alpha_2 z}{\kappa^2\Lambda^2 z},$$

$$h_0(z) = \frac{\alpha(e^z - 1) + \alpha_0 z}{6\kappa^2\Lambda^2 z}, \quad (11)$$

for three general parameters α, α_2 and α_0.

Given the ultraviolet exponential behavior of the two functions $h_i(z)$, let us study the high energy behavior of the quantum theory. The ultraviolet behavior of the propagator in momentum space (actually we will see that this is the correct scaling of the propagator at any energy scale), omitting the tensorial structure, reads

$$D(k) \sim \frac{e^{-k^2/\Lambda^2}}{k^2}\,. \tag{12}$$

But also the n-graviton interaction has the same scaling in the momentum space, since it can be written in the following schematic way

$$\mathcal{L}^{(n)} \sim h^n \, \Box_\eta h \; h_i(-\Box_\Lambda) \, \Box_\eta h$$
$$\rightarrow h^n \, \Box_\eta h \, \frac{e^{-\Box_\eta}}{\Box_\eta} \, \Box_\eta h + \dots \,, \tag{13}$$

in which "..." indicates other interaction terms coming from the covariant D'Alembertian and $\Box_\eta = \eta^{\mu\nu}\partial_\mu\partial_\nu$.

Placing an upper bound to the amplitude with L-loops, we find

$$A(L) \leqslant \int (d^4 p)^L \left(\frac{e^{-p^2/\Lambda^2}}{p^2}\right)^I \left(e^{p^2/\Lambda^2} p^2\right)^V$$
$$= \int (dp)^{4L} \left(\frac{e^{-p^2/\Lambda^2}}{p^2}\right)^{I-V}$$
$$= \int (dp)^{4L} \left(\frac{e^{-p^2/\Lambda^2}}{p^2}\right)^{L-1}\,. \tag{14}$$

In the last step we used again the topological identity $I = V + L - 1$. The L-loops amplitude is UV finite for $L > 1$ and it diverges as "p^4" for $L = 1$.

Thus only 1-loop divergences exist and the theory is super-renormalizable[1]. In these SRQG theories the quantities β, β_2, β_0 and eventually the cosmological constant are renormalized, namely

$$\mathcal{L}_{\text{Ren}} = \mathcal{L} - \sqrt{-g}\Big\{\frac{\beta(Z-1)}{\kappa^2}R + \lambda(Z_\lambda - 1)$$
$$-\beta_2(Z_2 - 1)(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2) + \beta_0(Z_0 - 1)R^2\Big\}\,, \tag{15}$$

in which all the coupling must be understood as renormalized at an energy scale μ. On the other hand, the functions h_i are not renormalized because the upper limit $A(L) \leqslant 4$ in (14).

We assume that the theory is renormalized at an energy scale scale μ_0. If we want the bare propagator to possess no other gauge-invariant pole than the transverse physical graviton pole, we have to set

$$\alpha = \beta(\mu_0)\,, \quad \frac{\alpha_2}{\kappa^2\Lambda^2} = \beta_2(\mu_0)\,, \quad \frac{\alpha_0}{6\kappa^2\Lambda^2} = \beta_0(\mu_0)\,. \tag{16}$$

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[1] A *local* super-renormalizable quantum gravity with a large number of metric derivatives was for the first time introduced in [46].

If the energy scale μ_0 is taken as the renormalization point, then $\bar{h}_2 = \bar{h}_0 = \beta(\mu_0)\exp(z)$, and only the physical massless spin-2 graviton pole occurs in the bare propagator. In the gauge $\xi = 0$ the propagator in (10) reads

$$D_{\mu\nu\rho\sigma}(k) = \frac{-i}{(2\pi)^4} \frac{e^{-k^2/\Lambda^2}}{\alpha\,(k^2 + i\epsilon)}\Big(2P^{(2)}_{\mu\nu\rho\sigma}(k) - 4P^{(0-s)}_{\mu\nu\rho\sigma}(k)\Big).$$

If we choose another renormalization scale μ, then the bare propagator acquires poles; however, these poles cancel in the dressed physical propagator because the renormalization group invariance preserves unitarity in the dressed physical propagator at any energy scale and no other physical pole emerges at any other scale.

III. NONCOMMUTATIVE SPACETIME & QUANTUM GROUPS

We unveil in this section the link between one of the SRQG theories analyzed above and the quantum-group structure of spacetime-symmetries proper to noncommutative space-times. The key point is that the two-point function of the super-renormalizable theory can be re-expressed in such a way to exhibit the hidden quantum-group-like structure in the momentum space through the Fourier transform of $\bar{h}_i(-\Box_\Lambda)$ $(i = 0, 2)$. We present in particular two procedures accounting for this result and leading to a particularly simple example of non-commuativity that is well known and has been studied mathematically in depth, namely the θ-Minkowski space-time with its associated θ-Poincaré Hopf algebra of symmetries. We then move to scrutinize possible generalizations within the framework of spacetimes with noncommutativity of the type

$$[\hat{X}^\mu, \hat{X}^\nu] = i\theta^{\mu\nu}(\hat{X}^\alpha)$$

and conclude with the theorem that, focusing on associative space-time algebras, there is only one possible choice of $\bar{h}_n(\Box_\Lambda)$ compatible with a non-trivial Hopf-algebra structure of space-time symmetries.

A. Emergence of the quantum θ-structure

Starting from the expression of the two-point function (10), we easily obtain, within an appropriate choice of the gauge, the scalar structure for the graviton propagator to be

$$D^{\xi=0}_{\mu\nu\rho\sigma}(k) \sim \frac{-i}{(2\pi)^4} \frac{e^{-H(k^2/\Lambda^2)}}{k^2 + i\epsilon} \times \text{TS}\,, \tag{17}$$

where we start considering $\bar{h}_2(z) = \bar{h}_0(z) := \exp H(z)$ as to be as general as possible and where $H(k^2/\Lambda^2)$ is an entire function of the argument. TS means "tensorial structure". In order to make explicit the mechanism underlying the result we are going to show, we focus in this

first part of the section on a Euclidean *2D* space-time and then consider a phase-space non-commutativity involving $\hat{X}_i$ space-coordinates operators and $\hat{P}_j$ momentum operators characterized by the following Lie-brackets

$$[\hat{X}^i, \hat{X}^j] = i\theta^{ij}\,, \quad [\hat{X}^i, \hat{P}_j] = i\delta^i_j\,, \quad [\hat{P}_i, \hat{P}_j] = 0\,, \quad (18)$$

namely the Heisemberg non-commutativity between conjugated variables and the Moyal-plane noncommutativity between space-coordinates. It has been shown in Refs. [31, 33] that for a particular choice of θ^{ij} involving noncommutativity in two of the space-coordinates (*e.g.* say $\theta^{3i} = -\theta^{i3} = 0$ and $\theta^{ab} = \theta\,\epsilon^{ab}$ with $a,b = 1,2$) it is possible to cast the space-noncommutativity on the *2D* plane in terms of Ladder operators and coherent states digonalizing these latter. For instance, assuming $\theta^{ab} = \theta\,\epsilon^{ab}$ one can define $\sqrt{2}\hat{Z} = \hat{X}^1 + i\hat{X}^2$ and $\sqrt{2}\hat{Z}^\dagger = \hat{X}^1 - i\hat{X}^2$. These new operators fulfill the algebra $[\hat{Z}, \hat{Z}^\dagger] = \theta$, and their eigenstates are labelled as $|z\rangle$ and are such that $\hat{Z}|z\rangle = z|z\rangle$ and $\langle z|\hat{Z}^\dagger = \langle z|\bar{z}$, namely

$$|z\rangle = \exp(-\frac{z\bar{z}}{\theta})\,\exp(-\frac{z}{\theta}\hat{Z}^\dagger)|0\rangle\,. \quad (19)$$

These coherent states of the non-commutative plane satisfy the completeness relation $\int dz d\bar{z}|z\rangle\langle z| = \pi\theta$. In quantum field theory the basic non-commutative variables are fields and their conjugated momenta. Coordinates are represented as labels and are commutative. Differently, for a quantum filed theory grounded on (18) we must consider the expectation value of fields over coherent states (19) encoding space non-commutativity, in order to relate quantization results to standard commutative quantum field theory. Indeed, within the context of (18), non-commutativity emerges already at the level of classical fields not subjected to the canonical quantization of the symplectic phase-space. In order to make contact with classical fields, we must therefore recur to the procedure of extracting the expectation value over semi-classical coherent states, which we called here $|z\rangle$, following the lines of Refs. [31] and [33]. Non-commuting coordinates, which may be treated as operators within the scheme of (18), are evaluated on the coherent states (19), as we show explicitly in the forthcoming relation, equation (20). We should now consider that non-commutative functions can be Fourier expanded using complex exponential bases, such as

$$\exp(i \sum_j p_j \hat{X}^j) \quad \text{or} \quad \prod_j \exp(ip_j \hat{X}^j).$$

The two bases that we are mentioning among many others are equal on the standard commutative *2D* space, but differ among each other within the case specified by (18), because of the underlying non-commutativity of $\hat{X}^j$ coordinates. This basic fact allows us to expand the expectation value over coherent states of non-commutative fields on the expectation value $\langle z|\exp(ip_j\hat{X}^j)|z\rangle$ of the Fourier basis elements $\exp(i \sum_j p_j \hat{X}^j)$, yielding the cru-

cial result [31] [33],

$$\langle z|\exp(ip_1\hat{X}^1 + ip_2\hat{X}^2)|z\rangle = \quad (20)$$
$$= \langle z|\exp(ip_+\hat{Z}^\dagger)\,\exp(ip_-\hat{Z})\,\exp\left(\frac{p_-p_+}{2}[\hat{Z}^\dagger, \hat{Z}]\right)|z\rangle\,,$$

in which the Baker-Campbell-Hausdorff formula has been used (see Appendix A) and the quantities $\sqrt{2}p_\pm = (p_1 \pm ip_2)$ have been defined. Notice also that shrinking to zero the deformation parameter θ accounts for considering the "classical limit" toward standard-commutative quantum filed theory.

We can now generalize this procedure to a noncommutative *4D* space-time and find an energy-momentum exponential-dumping behavior as in (17), but only if $H(k^2/\Lambda^2) \sim k^2$. We start considering a phase-space involving spacetime coordinates and conjugated momenta of the type

$$[\hat{X}^\mu, \hat{X}^\nu] = i\theta^{\mu\nu}\,, \quad [\hat{X}^\mu, \hat{P}_\nu] = i\delta^\mu_\nu\,, \quad [\hat{P}_\mu, \hat{P}_\nu] = 0\,. \quad (21)$$

We recall that for $\theta_{\mu 0} \neq 0$, any Lorentzian theory constructed on (21) is non-unitary [34]. For the moment we disregard this problem, perform a Wick rotation to the Euclidean spacetime, and show that assuming the only non-zero components $\theta^{03} = -\theta^{30} \equiv \xi \neq 0$ and $\theta^{12} = -\theta^{21} \equiv \theta \neq 0$ it is possible to give sense to a graviton propagator whose scalar structure is expressed by (17). Let us see here below how it is possible to achieve this result. Together with the Ladder operators $\hat{Z}$ and $\hat{Z}^\dagger$, we consider the choice of $\theta^{\mu\nu}$ specified above and of another class of Ladder operators involving $\hat{X}_1$ and $\hat{X}_3$ coordinates, namely $\sqrt{2}\hat{T} = \hat{X}^0 + iX^3$ and $\sqrt{2}\hat{T}^\dagger = \hat{X}^0 - iX^3$. It follows that $[\hat{T}, \hat{T}^\dagger] = \xi$, and from the type of space-time non-commutativity we assumed above that $[\hat{T}, \hat{Z}] = [\hat{T}, \hat{Z}^\dagger] = 0$, *i.e.* the two sectors of Ladder operators can be simultaneously diagonalized. The coherent states for the $\hat{T}$-sector can be constructed in the same way as for the $\hat{Z}$-sector, yielding eigenstates $|t\rangle$ such that $\hat{T}|t\rangle = t|t\rangle$ and $\langle t|\hat{T}^\dagger = \langle t|\bar{t}$, namely

$$|t\rangle = \exp(-\frac{t\bar{t}}{\theta})\,\exp(-\frac{t}{\theta}\hat{T}^\dagger)|0\rangle\,, \quad (22)$$

which is provided with the completeness relation $\int dt d\bar{t}|t\rangle\langle t| = \pi\theta$. We can therefore consider the coherent states $|z, t\rangle = |z\rangle\,|t\rangle$. The relevant formula for expanding quantum fields on a Fourier basis is given by the manipulation of $\langle z, t|\exp(ip_\mu\hat{X}^\mu)|z, t\rangle$. This is the expectation value over the coherent state $|z, t\rangle$ of wave-exponentials entering the Fourier-modes expansion of quantum fields on non-commutative space-time, and yields the crucial result

$$\langle z, t|\exp(ip_\mu\hat{X}^\mu)|z, t\rangle = \quad (23)$$
$$= \langle z|\exp(ip_+\hat{Z}^\dagger)\,\exp(ip_-\hat{Z})\,\exp\left(\frac{p_-p_+}{2}[\hat{Z}^\dagger, \hat{Z}]\right)|z\rangle \times$$
$$\times \langle t|\exp(i\tilde{p}_-\hat{T}^\dagger)\,\exp(i\tilde{p}_+\hat{T})\,\exp\left(\frac{\tilde{p}_-\tilde{p}_+}{2}[\hat{T}^\dagger, \hat{T}]\right)|t\rangle\,.$$

In (23) we have introduced the quantities $\sqrt{2}\tilde{p}_\pm = (p_0 \pm ip_3)$ and used the Baker-Campbell-Hausdorff formula. Once again we emphasize that, having performed a Wick rotation, our analysis focuses on the Euclidean signature $\text{sign}(\eta_{\mu\nu}) = (+, +, +, +)$, for which problems of unitarity do not appear. Indeed, thanks to the analysis that has been carefully developed in [34, 35] about unitarity being preserved in the Lorentzian case once $\theta^{0i} = 0$, based on the symmetry arguments we can convince ourselves that the Euclidean theory will not suffer for the lack of unitarity[2]. Among the super-renormalizable theories previously considered there exist one that can be recast when $\xi = \theta$ as a theory over a non-commutative space-time, with non-commutativity of the type

$$[\hat{X}^\mu, \hat{X}^\nu] = i\theta^{\mu\nu}. \tag{24}$$

Indeed, when $\xi = \theta = 1/\Lambda$, the damping factor in (17), specialized to the case $H(z) = z$, is automatically recreated from the kinematical manipulations reviewed in (23). Moreover, the choice of $\xi = \theta$ preserves Lorentz covariance when we Wick rotate back to the non-commutative space-time with Lorentzian signature. It is well known in literature that the algebra of symmetries for the non-commutative space-time in (24) is a twisted Hopf algebra $\mathcal{P}_\theta$ called θ-Poincaré Hopf algebra, and that a theory with a covariant action gives rise in this framework to conserved Noether charges (see *e.g.* [37], which extends the work done in [38] for the case of the κ-Poincaré Hopf algebra). Therefore θ-Poincaré truly represents an external symmetry (Hopf) algebra, and the same holds for its Lorentz sub-algebra. Moreover, we can construct a covariant theory under the action of the generators of $\mathcal{P}_\theta$ that has a new scale invariant, *i.e.* θ. We will discuss later these implications. For the moment, we want just to make clear the point that the θ-Poincaré symmetry does not imply a breakdown or a modification of the Poincaré symmetries in the algebraic sector, or better at the level of the Hilbert space of the theory, but rather an enlargement of the Lie-algebra-type Poincaré symmetry, which can be recast as a trivial Hopf-algebra, to a non-trivial Hopf-algebra, precisely a θ-Poincaré Hopf-algebra in our case. This latter reflects[3] in the deformation of the trivial co-algebraic structure of the Poincaré algebra (Leibnitz rule) to a twisted co-product structure (non-cocommutativity or θ-deformed

Leibnitz rule), leaving unchanged the algebraic structure of the Hopf-algebra[4]. In other words, the enlargement of the symmetry structure we are dealing with, will be affect a modification of the Fock space of the theory (see for instance the last reference of [38]), without chancing the Hilbert space of the theory. This is also reflected in the fact that the quantum groups structure emerges at tree-level, though the evaluation of the graviton propagator. We will come back later, in Sec. III D, to the issue of the relation between the θ-Poincaré and the Poincaré Hopf algebras.

We complete the discussion on the emergence of the θ-Poincaré symmetry noticing that the tensorial structure in (10) does not affect the result of our analysis, as indeed this can be made fully consistent with the θ-Poincaré symmetry of this theory [39]. We emphasize that this property relies on a remarkable feature of the θ-Poincaré quantum groups, namely that the Lorentz subalgebra and the whole Poincaré sector are unmodified with respect to the θ-Poincaré Hopf-algebra, as stated here above and as will be clarified in Sec. III D. This property allows to define linear Lorentz transformation and conservation laws following the standard recipe. From now on, we will mention the Lorentz sector, without specifying that it belongs to the θ-Poincaré or Poincaré algebra Hopf algebra.

B. A different philosophy to unveil $\mathcal{P}_\theta$

The emergence of the θ-Poincaré symmetry-structure does not rely on the particular procedure we adopted in the preceding subsection. For instance, we could have chosen to adopt the "Weyl system" procedure [40], as it has been done in [37] at the purpose of analyzing the symmetry-structure of a scalar field theory on θ-Minkowski spacetime. The "Weyl map" Ω associates to any function $f(\hat{x})$ of θ-Minkowski an auxiliary commutative function $f^{(c)}(x)$. The easiest way of implementing this map is to consider the Fourier transform $\tilde{f}(p)$ of $f(\hat{x})$ and then apply on the Fourier modes the Weyl map, namely

$$f(\hat{x}) = \Omega\left(f^{(c)}(x)\right) \equiv \Omega\left(\int d^4p\, \tilde{f}(p)e^{ipx}\right)$$
$$= \int d^4p\, \tilde{f}(p) : e^{ip\hat{x}} :, \tag{25}$$

in which ": $\cdot$:" denotes an ordering of the non-commutative coordinates associated to Ω. The inverse of the Weyl map, namely Ω^{-1}, is also well defined and is called the Wigner map. Our Wigner map is expressed

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by the semi-classical limit of the measurement-procedure involving coherent states $|z, t\rangle$, namely

$$\Omega^{-1}(\cdots) = \langle z, t| \cdots |z, t\rangle. \qquad (26)$$

This procedure also provides a physical picture in our context of the Wigner and Weyl maps. As reminded above, because of the peculiar features of spacetime non-commutativity, a Weyl map selects a particular normal ordering for the spacetime coordinates. Suppose to choose the "semiclassical-state" ordering, which is defined in the $x_1 - x_2$ plane by the following action of the Weyl map on the Fourier-modes basis elements

$$\Omega|_{x-y}(e^{ip_1 x_1 + ip_2 x_2}) = \Omega|_{x-y}(e^{ip_+ \bar{z} + ip_- z})$$
$$= e^{ip_+ \hat{Z}^\dagger + ip_- \hat{Z}} = e^{ip_+ \hat{Z}^\dagger} e^{ip_- \hat{Z}} e^{\frac{p_- p_+}{2} [\hat{Z}^\dagger, \hat{Z}]}. \quad (27)$$

On the whole θ-Minkowski spacetime (24), in which $\theta_{03} = \theta_{12} = \theta = -\theta_{30} = -\theta_{21}$ and $\theta_{13} = \theta_{23} = \theta_{01} = \theta_{02} = 0$, we might consider the definition of "semiclassical-state" ordering and the related Weyl map by adopting the coordinates $\hat{T}$ and $\hat{T}^\dagger$,

$$\Omega \left(e^{ip_\mu x^\mu} \right) = e^{ip_+ \hat{Z}^\dagger} e^{ip_- \hat{Z}} e^{\frac{p_- p_+}{2} [\hat{Z}^\dagger, \hat{Z}]} \times$$
$$\times e^{i\tilde{p}_- \hat{T}^\dagger} e^{i\tilde{p}_+ \hat{T}} e^{\frac{\tilde{p}_- \tilde{p}_+}{2} [\hat{T}^\dagger, \hat{T}]}. \qquad (28)$$

Now define the integration map $\int$ on the non-commutative θ-Minkowski spacetime as the map such that

$$\int \Omega(f(x))\Omega(g(x)) = \int d^4 p \tilde{f}(p)\tilde{g}(-p) e^{-\theta(p_\mu p^\mu)^2}. \quad (29)$$

A quantum theory of non commutative fields can now be constructed following the same steps as in [41], namely considering an expansion of the quantum field, fulfilling a Lorentz covariant equation of motion, on a non-commutative Fourier basis

$$\Psi_r(\hat{X}) = \int d\mu_p \left[a_p \, \Omega\left(e^{-ip_\mu x^\mu} \right) + a_p^\dagger \, \Omega\left(e^{ip_\mu x^\mu} \right) \right]$$
$$= \int \frac{d^3 \vec{p}}{2p_0} \left[a_{\vec{p}} \, \Omega\left(e^{-ip_\mu x^\mu} \right) + a_{\vec{p}}^\dagger \, \Omega\left(e^{ip_\mu x^\mu} \right) \right], \qquad (30)$$

and then imposing braiding relations on the ladder operators by means of the bi-algebra twisting element

$$\mathcal{F}_\theta = \exp(\frac{1}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu).$$

Notice that in (30) we have introduced the notation for the Lorentz invariant measure $d\mu_p = d^4 p \, \delta(p^2) = d^3 \vec{p}/2p_0$. The requirement of compatibility of the covariant action of symmetries on tensor product of states with the tensor product of state on which symmetries have already acted, yields indeed the braiding in the multi-particle states. This peculiar feature of non-commutative quantum field theory enjoying θ-Poincaré symmetries are originated by the action of the twisting element, which we

define here by means of $\mathcal{F}_\theta \triangleright (|p\rangle \otimes |q\rangle) = \mathcal{F}_\theta(p, q) |p\rangle \otimes |q\rangle$, *i.e.* through

$$a_p a_q = \mathcal{F}_\theta^{-2}(q, p) a_q a_p,$$
$$a_p a_q^\dagger = \mathcal{F}_\theta^{-2}(-q, p) a_q^\dagger a_p + 2p_0 \delta^4(p - q),$$
$$a_p^\dagger a_q^\dagger = \mathcal{F}_\theta^{-2} a_q^\dagger a_p^\dagger. \qquad (31)$$

The vacuum state of the Fock space is defined by $a_p|0\rangle = 0$, and states of the Hilbert by $|p\rangle = a_p^\dagger|0\rangle$. The second quantization procedure hence defined can be applied to the geometric two-tensor field, as defined in [39] and perturbative-expanded as in [2]. The second relation in (31) is what we need in order to compute the graviton propagator in the non-commutative theory, namely

$$D_{\mu\nu\rho\sigma}(\hat{X}^\alpha - \hat{Y}^\alpha) = \int d\mu_p d\mu_k$$
$$\left\{ \left[\langle 0| \left(a_p a_k^\dagger \, \Omega(e^{-ip_\mu x^\mu})\Omega(e^{ik_\mu y^\mu}) \right. \right. \right.$$
$$\left. \left. + a_p^\dagger a_k \Omega(e^{ip_\mu x^\mu})\Omega(e^{-ik_\mu y^\mu}) \right) |0\rangle \, \theta(\hat{X}_0 - \hat{Y}_0) \right] +$$
$$\left[\hat{X}_0 \leftrightarrow \hat{Y}_0 \text{ and } x \leftrightarrow y \right] \right\} \times \text{TS}$$
$$= \int d\mu_p \times \text{TS} \times \Omega \left(e^{ip_\mu(x^\mu - y^\mu)} \right) \times \text{pole structure}.$$

We emphasize that $D_{\mu\nu\rho\sigma}(\hat{X}^\alpha - \hat{Y}^\alpha)$ is differs from the expectation value (on the coherent states $|z, t\rangle$) of the propagator of the quantum theory, namely $D_{\mu\nu\rho\sigma}(x^\alpha - y^\alpha)$. We would have obtained $D_{\mu\nu\rho\sigma}(x^\alpha - y^\alpha)$ if we had followed the same strategy as in Ref. [31, 33]. In our notation, in terms of the Wigner map, this accounts for

$$\Omega^{-1}\left(D_{\mu\nu\rho\sigma}(\hat{X}^\alpha - \hat{Y}^\alpha) \right) = D_{\mu\nu\rho\sigma}(x^\alpha - y^\alpha)$$

that leads to the graviton propagator

$$D_{\mu\nu\rho\sigma}(x^\alpha - y^\alpha) = \int d\mu_p d\mu_k \qquad (32)$$
$$\times \left\{ \Omega^{-1} \left[\langle 0| \left(a_p a_k^\dagger \Omega(e^{-ip_\mu x^\mu}) \times \Omega(e^{ik_\mu y^\mu}) \right. \right. \right.$$
$$\left. \left. + a_p^\dagger a_k \Omega(e^{ip_\mu x^\mu})\Omega(e^{-ik_\mu y^\mu}) \right) |0\rangle \, \theta(\hat{X}_0 - \hat{Y}_0) \right] +$$
$$+ \Omega^{-1} \left[x \leftrightarrow y \right] \right\} \times \text{TS} = \int d^4 p \frac{e^{-p^2/\Lambda^2}}{p^2} \times \text{TS}.$$

The Fourier transform of the graviton propagator sketched in (32) by using the ordering introduced in (28) turns out to give the same value determined henceforth at the beginning of this section, in (17).

The procedure incorporated in this second section is more general than that one based on the expectation value on coherent states and must be in general considered as distinct. Nevertheless, this reduces to the one exposed in the preceding section whenever we consider (26) as a concrete definition for the Wigner map.

C. Uniqueness of the link between quantum groups and SRQG and falsifiability of the theory

In this section we prove a simple theorem stating the uniqueness of the link between quantum groups and SRQG. Specifically, we prove that the only non trivial Hopf algebra connected to SRQG is the θ-Poincaré Hopf algebra, and that this latter selects only one among the many possible theories (namely, the theory defined by the choice $\bar{h}_2(z) = \bar{h}_0(z) = \exp z$, with $z = -\Box/\Lambda^2$) described in [2]. Thus, in what follows we scrutinize the possibility of generalizing results previously exposed to a wider class of non-commutative spacetimes and prove the impossibility of achieving this goal if we decide not to relax the requirement of associativity for the non-commutative space-time algebra.

The natural place in order to seek for the generalization of previous results is represented by (20) and the implementation within it of the Baker-Campbell-Hausdorff formula and of its inverse formula, the Zassenhaus formula. Suppose indeed to consider in (17) the integer function to be $H(k^2/\Lambda^2) = c_1 k^2/\Lambda^2 + c_2(k^2/\Lambda^2)^2$. We address the search for a suitable Lie algebra reproducing this structure for $H(-\Box/\Lambda^2)$ in terms of generic functions θ_{34}, depending[5] on $\hat{Z}$ and $\hat{Z}^\dagger$, and θ_{12} depending on $\hat{T}$ and $\hat{T}^\dagger$. Commutation relations for the Ladder operators now read

$$
\begin{aligned}
[\hat{Z}, \hat{Z}^\dagger] &= \theta_{34}(\hat{Z}, \hat{Z}^\dagger)\,, \\
[\hat{T}, \hat{T}^\dagger] &= \theta_{12}(\hat{T}, \hat{T}^\dagger)\,.
\end{aligned}
\tag{33}
$$

Maintaining unchanged the definition of $\hat{Z}$ and $\hat{T}$, formulas (33) yields a space-time non-commutativity of the form

$$
\begin{aligned}
[\hat{X}^1, \hat{X}^2] &= i\tilde{\theta}_{12}(\hat{X}_1, ...\hat{X}_4)\,, \\
[\hat{X}^3, \hat{X}^4] &= i\tilde{\theta}_{34}(\hat{X}_1, ...\hat{X}_4)\,.
\end{aligned}
\tag{34}
$$

Notice that in general both the θ-Minkowski type and κ-Minkowski [43] type of non-commutativity are present in the expansion of the functions $\tilde{\theta}_{12}$ and $\tilde{\theta}_{34}$. Such a co-presence of space-time non-commutativities has been considered in literature [44] in light of its relation with string-theory scenarios[6]. But (34) is not sufficient in or-

der to ensure the desired behavior for the Fourier transform of the integer function $H(k^2/\Lambda^2)$ appearing in the graviton propagator calculation. In other words, we can not achieve the Fourer transform

$$
H(k^2/\Lambda^2) = c_1 k^2/\Lambda^2 + c_2(k^2/\Lambda^2)^2
\tag{35}
$$

on a perturbed background of the form (5) if we still require the non-commutative algebra to be associative. We can prove this theorem considering that two requirements should be fulfilled as necessary conditions in order to add a term like $c_2(k^2/\Lambda^2)^2$ in (17). The first one reads

$$
\frac{\partial \tilde{\theta}_{12}}{\partial \hat{Z}} = \frac{\partial \tilde{\theta}_{12}}{\partial \hat{Z}^\dagger} = \frac{\partial \tilde{\theta}_{34}}{\partial \hat{T}} = \frac{\partial \tilde{\theta}_{34}}{\partial \hat{T}^\dagger} = 0
\tag{36}
$$

and ensures that momenta are not redefined at linear order in $\sqrt{\theta}$, i.e. for linear Planck mass corrections. The second condition is

$$
\frac{\partial^2 \tilde{\theta}_{12}}{\partial \hat{Z} \partial \hat{Z}^\dagger} = \frac{\partial \tilde{\theta}_{12}}{\partial \hat{Z}^\dagger \partial \hat{Z}} = \frac{\partial^2 \tilde{\theta}_{34}}{\partial \hat{T}^\dagger \partial \hat{T}} = \frac{\partial \tilde{\theta}_{34}}{\partial \hat{T} \partial \hat{T}^\dagger} = \theta
\tag{37}
$$

and ensures the existence of two terms summing in $H(p^2/\Lambda^2)$ within (17) that are $\theta(p_1^2 + p_2^2)^2$ from the $\hat{T} - \hat{T}^\dagger$ sector and $\theta(p_3^2 + p_4^2)^2$ from the $\hat{Z} - \hat{Z}^\dagger$ sector. But once summed, these contributions are not sufficient in recreating a covariant $(p^2)^2$ term, which in stead would come from the Fourier transform of $\Box^2$ as it appears in the second term of (35). Therefore we should consider now interaction between the two sectors, $\hat{T} - \hat{T}^\dagger$ and $\hat{Z} - \hat{Z}^\dagger$, which would now determine the appearance of mixed terms in p_2^2 and p_3^2, from one side, and p_4^2 and p_1^2 from the other side. In the euclidean spacetime, we now label operators within the $\hat{Z} - \hat{Z}^\dagger$ sector as

$$
\hat{Z} = \hat{Z}_{34}\,, \qquad \hat{Z}^\dagger = \hat{Z}_{34}^\dagger\,,
\tag{38}
$$

and momenta as

$$
p_+ = p_{34}\,, \qquad p_- = p_{34}^*\,.
\tag{39}
$$

In the $\hat{T} - \hat{T}^\dagger$ sector, operators are now labeled as

$$
\hat{T} = \hat{Z}_{12}\,, \qquad \hat{T}^\dagger = \hat{Z}_{12}^\dagger\,,
\tag{40}
$$

while momenta are labeled as follows

$$
\tilde{p}_+ = p_{12}\,, \qquad \tilde{p}_- = p_{12}^*\,.
\tag{41}
$$

We emphasize that, in order to obtain a dumping exponential phase-term $\exp\left[-2\theta^2(p_1^2 + p_2^2)(p_3^2 + p_4^2)\right]$ multiplying the other dumping phase-term $\exp\{-\theta^2\left[(p_1^2 + p_2^2)^2 + \right.$

---

[5] We recall that the BCH formula and its inverse have been developed considering only Lie-algebra cases. Thus θ_{34} and θ_{12} could be rigorously expanded only up to linear order in the generators of the algebra.

[6] Expansion of formulae (34) make sense up to second order in a scale κ having dimension of inverse energy. Following dimensional arguments, for a spacetime non commutativity of the type

$$
[\hat{X}^\mu, \hat{X}^\nu] = i\theta^{\mu\nu}(\hat{X}^\alpha)
$$

the only class of deformations of spacetime having classical limit for $\kappa \to 0$ are of the type

$$
[\hat{X}^\mu, \hat{X}^\nu] = i\kappa^2\,\theta_0^{\mu\nu} + i\kappa\,\theta_{(1)}^{\mu\nu}{}_\rho \hat{X}^\rho + \theta_{(1)}^{\mu\nu}{}_{\rho\sigma} \hat{X}^\rho \hat{X}^\sigma\,,
$$

with $\theta_{(0)}^{\mu\nu}$ and $\theta_{(1)}^{\mu\nu}{}_\rho$ and $\theta_{(1)}^{\mu\nu}{}_{\rho\sigma}$ dimensionless quantities. Within this class of deformations of spacetime, previous expansion describes (for open string first-quantized in $D=10$) noncommutative coordinates on D-branes providing the localizations of the ends of the strings

$(p_3^2 + p_4^2)^2]\}$, and hence recreating a covariant exponential dumping phase-factor $\exp -\theta^2 (p^2)^2$, new conditions must be fulfilled about the non-commutativity in the $\hat{X}_1 - \hat{X}_3$ plane and in the $\hat{X}_1 - \hat{X}_4$ plane, as well as in the $\hat{X}_2 - \hat{X}_3$ and $\hat{X}_2 - \hat{X}_4$ planes. These can be derived looking at the exponential

$$\exp i(\hat{X}_1 p_1 + \hat{X}_2 p_2 + \hat{X}_3 p_3 + \hat{X}_4 p_4)$$
$$= \exp i(\hat{Z}_{34}^\dagger p_{34}^* + \hat{Z}_{34} p_{34} + \hat{Z}_{12}^\dagger p_{12}^* + \hat{Z}_{12} p_{12}), \quad (42)$$

and at its decomposition by means of the Zassenhaus formula, and then imposing that

$$e^{i(p_1 \hat{X}_1 + p_2 \hat{X}_2 + p_3 \hat{X}_3 + p_4 \hat{X}_4)} \qquad (43)$$
$$= e^{i(p_1 \hat{X}_1 + p_2 \hat{X}_2)} e^{i(p_3 \hat{X}_3 + p_4 \hat{X}_4)} \times$$
$$\times e^{\frac{1}{2}\Gamma[p_1, p_2, p_3, p_4]} e^{-\frac{1}{6}\{2\Phi[p_1, p_2, p_3, p_4] + \Sigma[p_1, p_2, p_3, p_4]\}} \times$$
$$\times e^{-\frac{1}{24}\{\Xi[p_1, p_2, p_3, p_4] + 3\Psi[p_1, p_2, p_3, p_4] + 3\Theta[p_1, p_2, p_3, p_4]\}},$$

with

$$\Gamma[p_1, p_2, p_3, p_4] = [(p_1 \hat{X}_1 + p_2 \hat{X}_2), (p_3 \hat{X}_3 + p_4 \hat{X}_4)],$$
$$\Phi[p_1, p_2, p_3, p_4] = [(p_3 \hat{X}_3 + p_4 \hat{X}_4), [(p_1 \hat{X}_1 + p_2 \hat{X}_2),$$
$$(p_3 \hat{X}_3 + p_4 \hat{X}_4)]],$$
$$\Sigma[p_1, p_2, p_3, p_4] = [(p_1 \hat{X}_1 + p_2 \hat{X}_2), [(p_1 \hat{X}_1 + p_2 \hat{X}_2),$$
$$(p_3 \hat{X}_3 + p_4 \hat{X}_4)]],$$
$$\Xi[p_1, p_2, p_3, p_4] = [[[(p_1 \hat{X}_1 + p_2 \hat{X}_2), (p_3 \hat{X}_3 + p_4 \hat{X}_4)],$$
$$(p_1 \hat{X}_1 + p_2 \hat{X}_2)], (p_1 \hat{X}_1 + p_2 \hat{X}_2)],$$
$$\Psi[p_1, p_2, p_3, p_4] = [[[(p_1 \hat{X}_1 + p_2 \hat{X}_2), (p_3 \hat{X}_3 + p_4 \hat{X}_4)],$$
$$(p_1 \hat{X}_1 + p_2 \hat{X}_2)], (p_3 \hat{X}_3 + p_4 \hat{X}_4)],$$
$$\Theta[p_1, p_2, p_3, p_4] = [[[(p_1 \hat{X}_1 + p_2 \hat{X}_2), (p_3 \hat{X}_3 + p_4 \hat{X}_4)],$$
$$(p_3 \hat{X}_3 + p_4 \hat{X}_4)], (p_3 \hat{X}_3 + p_4 \hat{X}_4)].$$

For arbitrary values of p_μ, the requirement on the Lorentz-invariance of the algebraic sector (and thus on the Lorentz-invariance of the Fourier space) implies from enforcing $\Gamma[p_1, p_2, p_3, p_4] = 0$ that

$$[\hat{X}_1, \hat{X}_3] = [\hat{X}_1, \hat{X}_4] = [\hat{X}_2, \hat{X}_3] = [\hat{X}_2, \hat{X}_4] = 0. \quad (44)$$

Namely, Lorentz invariance requires that the only type of affordable non-commutativity is the one we considered above on the $\hat{X}_1 - \hat{X}_2$ plane and on the $\hat{X}_3 - \hat{X}_4$ plane. We would have reached the same conclusion from (44) by imposing Lorentz-invariance on the Fourier space and hence the simultaneous vanishing of $\Phi[p_1, p_2, p_3, p_4]$ and $\Sigma[p_1, p_2, p_3, p_4]$. Notice that also (36) and (37), once expressed in terms of $\hat{X}^\mu$ operator, would lead to the same type of inconsistencies. Finally, requirements in (44) impose the vanishing of both $\Xi[p_1, p_2, p_3, p_4]$, $\Psi[p_1, p_2, p_3, p_4]$ and $\Theta[p_1, p_2, p_3, p_4]$. As a consequence, it is manifest the impossibility of recovering a term which goes like $(k^2/\Lambda)^2$ for $H(k^2/\Lambda^2)$ in (17), if we start from a Lie algebra type of non-commutativity. This result could be in part anticipated. We know indeed that only the twisted θ-Poincaré Hopf algebra at the same time preserves Lorentz symmetry, at least in the algebraic sector

and in the Fourier space, and consistently realizes the associativity in the module algebra (of space-time coordinates functions).

Finally, the argument developed here above and based on the choice of the particular SRQG theory defined by (35), can be repeated for any entire function $H(z)$, thus for any generic SRQG theory. This ends our proof about the uniqueness of the link between SRQG and non-trivial Hopf algebra, specifically the θ-Poincaré quantum group. It is not overwhelming to emphasize that the uniqueness, *i.e.* having fixed $H(z)$ to a unique function, traces back to the definition of a unique theory of SRQG among the many allowed in the framework of [2]. We also emphasize that such a result relies on the construction of the phase-space with Heisenberg-type of non-commutativity between spacetime and momenta, thus it is consistent with the associativity of the space-time coordinates considered in previous sections[7]. Extending the analysis to deformed phase-space, and hence to a non-associative spacetimes, would not have allowed us to conclude with the same statement.

D. Poincaré and θ-Poincaré symmetries

In this section we want to recall some basic facts about the Poincaré and the θ-Poincaré Hopf-algebras and show explicitly their relation. We start by reviewing how the Poincaré Lie-algebra can be recast as a Hopf-algebra. Before doing that, we remind the Lie-multiplication rules for the Poincaré algebra $\mathcal{P}$, whose elements are denoted as P_μ (generators of translations) and $M_{\mu\nu}$ (generators of Lorentz transformations),

$$[P_\mu, P_\nu] = 0,$$
$$[M_{\mu\nu}, M_{\alpha\beta}] = -i(\eta_{\mu\alpha} M_{\nu\beta} - \eta_{\mu\beta} M_{\nu\alpha} +$$
$$- \eta_{\nu\alpha} M_{\mu\beta} + \eta_{\nu\beta} M_{\mu\alpha}),$$
$$[M_{\mu\nu}, P_\alpha] = -i(\eta_{\mu\alpha} P_\nu - \eta_{\nu\alpha} P_\mu). \quad (45)$$

The Poincaré Lie algebra $\mathcal{P}$ can be easily promoted to a Hopf algebra if we introduce same basic definitions. In particular, in order to deal with deformation of the Leibnitz rule, it is convenient from an abstract algebraic point of view to introduce the "coproduct" map Δ as the application from elements of the algebra $\mathcal{P}$ to elements of the tensor product $\mathcal{P} \otimes \mathcal{P}$, namely $\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$. If the module space, *i.e.* the space on which generators of the algebra act, is thought to be the space of functions on commutative space, it is of immediate evidence that $\Delta(1) = 1 \otimes 1$, and $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$ for each generator $Y = \{M_{\mu\nu}, P_\beta\}$ of the Poincaré Lie algebra. This last property, which plays a key role in allowing a description

---

[7] A notable example of nonassociative spacetime is the Snyder noncommutative space-time (see *e.g.* Ref. [51] and references therein).

of the symmetries at the simple Lie-algebraic level — without any true need to resort to a full Hopf-algebra description — is actually connected with the commutativity of functions in Minkowski space-time $\mathcal{M}$, and thus fully reflects the Leibnitz rule. Consider in fact two generic functions f and g. From $f \cdot g = g \cdot f$, one easily finds for each $U \in \mathcal{U}(\mathcal{P})$ that[8] Δ is symmetric, *i.e.* $U_{(1)} \otimes U_{(2)} = U_{(2)} \otimes U_{(1)}$ for all U. Adopting math jargon, Δ is "cocommutative". We say that a cocommutative coproduct is, in some sense, "trivial" in order to emphasize its simple structure with respect to the coproduct of the generators of the symmetry-transformations in a non-commutative space-time. These latter coproducts may be in stead "non-cocommutative".

The axiomatic definition of a Hopf-algebra requires also the definition of a "co-unit" map $\epsilon : \mathcal{P} \to \mathbb{C}$, such that for any function $f(x)$ it results that the action $\int d^4 x \, U \, f(x) = \epsilon(U) \int d^4 x f(x)$ can be defined. It is straightforward to verify that $\epsilon(1) = 1$ and $\epsilon(Y) = 0$. We may also define the "unit map" $\eta : \mathbb{C} \to \mathcal{P}$ and the "multiplication map" $m : \mathcal{P} \otimes \mathcal{P} \to \mathcal{P}$. Another ingredient we might add to this construction concerns how to construct generators of inverse transformations. At this purpose, we can define $S(1) = 1$, $S(Y) = -Y$ for each generator $Y \in \mathcal{P}$ and $S(UU') = S(U')S(U)$ for each element of the corresponding enveloping algebra. We then obtain a map satisfying $U_{(1)} S(U_{(2)}) = S(U_{(1)}) U_{(2)} = \epsilon(U)$. The map S so far introduced is called "antipode". This makes $\mathcal{P}$ a Hopf algebra provided that some axioms are satisfied[9].

The universal enveloping algebra of a Lie algebra is then equivalent to a Hopf algebra generated by the primitive elements Y:

$$\Delta(Y) = Y \otimes 1 + 1 \otimes Y, \quad \epsilon(Y) = 0, \quad S(Y) = -Y .$$

This algebraic structure is usually referred to as a "trivial Hopf algebra".

---

[8] $\mathcal{U}(\mathcal{P})$ represents the universal enveloping algebra of the Poincaré Lie-algebra $\mathcal{P}$. The universal enveloping algebra $\mathcal{U}(\mathcal{A})$ of an algebra $\mathcal{A}$ is the non-commutative algebra generated by 1 and elements of $\mathcal{A}$, modulo the relations that specify the Lie-multiplication rules for the elements of $\mathcal{A}$.

[9] We refer to [42] for a detailed review of Hopf-algebras and of the Hopf-algebra axioms that must be fulfilled. Here, we briefly mention that for an algebra $\mathcal{A}$, the axioms to be fulfilled are the *algebra axioms*

$$\begin{aligned} m(m \otimes 1) &= m(1 \otimes m) \quad \text{(associativity)}, \\ m(1 \otimes \eta) &= m(\eta \otimes 1) = 1 \quad \text{(unit)}, \end{aligned}$$

and the *co-algebra axioms*

$$\begin{aligned} (\Delta \otimes 1)\Delta &= (1 \otimes \Delta)\Delta \quad \text{(co} - \text{associativity)}, \\ (1 \otimes \varepsilon)\Delta &= (\varepsilon \otimes 1)\Delta = 1 \quad \text{(co} - \text{unit)}, \end{aligned}$$

which together specify a *bialgebra* $(A, m, \eta, \Delta, \varepsilon)$; and finally the *antipode axioms*, which specify a *Hopf-algebra*, namely

$$m(S \otimes 1)\Delta = m(1 \otimes S)\Delta = \eta \circ \varepsilon .$$

The new symmetry structure, whose emergence in the framework of SRQG theories we are claiming, by looking at the tree-level for the graviton propagator, is in stead the θ-Poinacré algebra. This is a "non-triavial" Hopf-algebra, in that the coproducts do not reflect anymore the Leibnitz rule, *i.e.* the module space of the algebra is that one of non-commutative functions on non-commutative θ-Minkowski space-time. The algebra is still defined by (45), but the co-algebra is now deformed in the tensor θ. Far from deriving the deformation of the coproducts in (46) and discussing the whole θ-Poncaré algebra, we just mention few basic properties of $\mathcal{P}_\theta$. Coprodcuts Δ_θ are obtained by an element of the bialgebra $\mathcal{P} \otimes \mathcal{P}$ called twist element,

$$\mathcal{F}_\theta = e^{\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu} , \tag{46}$$

which satisfies

$$\mathcal{F}_\theta (\Delta_0 \otimes 1)\mathcal{F}_\theta = \mathcal{F}_\theta (1 \otimes \Delta_0)\mathcal{F}_\theta . \tag{47}$$

Given the generators $Y \in \mathcal{P}$, the twist element $\mathcal{F}$ "modifies" the coproduct of $U(\mathcal{P})$ in the following way [36]

$$\Delta_0(Y) \mapsto \Delta_\theta(Y) = \mathcal{F}\Delta_0(Y)\mathcal{F}^{-1} ,$$

$\Delta_0(Y)$ denoting the "trivial" coproduct in (46). Moreover, in the limit $\theta \to 0$ the co-algebraic structure of $\mathcal{P}$ is recovered from that one of $\mathcal{P}_\theta$, *i.e.* $\Delta_\theta(Y) \to \Delta_0(Y)$.

Finally, being the generators of translations P_α commutative among each others, it can be easily recovered that their coproduct is not deformed (in the math jargon, $\Delta_\theta = \Delta_0$ is "primitive" or also the subalgebra of translation is "co-commutative"):

$$\Delta_\theta(P_\alpha) = \Delta_0(P_\alpha) = P_\alpha \otimes 1 + 1 \otimes P_\alpha . \tag{48}$$

Whilst it is more laborious but nevertheless easy to check that

$$\begin{aligned} \Delta_\theta(M_{\mu\nu}) &= Ad \; e^{\frac{i}{2}\theta^{\alpha\beta} P_\alpha \otimes P_\beta}\Delta_0(M_{\mu\nu}) = \\ &= M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}[(\eta_{\alpha\mu}P_\nu - \eta_{\alpha\nu}P_\mu) \otimes P_\beta \\ &+ P_\alpha \otimes (\eta_{\beta\mu}P_\nu - \eta_{\beta\nu}P_\mu)], \end{aligned} \tag{49}$$

which shows the modification in $\theta^{\mu\nu}$ of the co-algebraic structure of the θ-Poincaré Hopf-algebra, reflecting the modification in θ of the Leibnitz rule.

E. SRQG and Non-commutative Gravity

In preparation for the conclusions, we want to address in this section a brief comparison of the model above with the theory of non-commutative geometry and gravity developed in [39], and with seminal works on the relation between differential calculi over a given noncommutative associative algebra and space-time metrics addressed in [47, 48]. We first emphasize the differences between the model presented in this letter and the works in [47–49]

and [39], and then conclude with a list of points to be investigated in forthcoming works in order to gain a clearer physical picture.

The most striking point we are confronted with is the in-equivalence of our model with the ones addressed in [47, 48] and [39]. This feature indeed is already evident at the level of the linearized equation of motion for the SRQG theories described in [1, 2]. A first heuristic analysis based on the work reported in references [52] reveals indeed that linearized equation for the model here treated would involve a non-local operator $H_{\mu\nu}{}^{\rho\sigma}(\nabla_\alpha)$ acting on the Ricci scalar R and the Ricci tensor $R_{\mu\nu}$ in the form

$$G_{\mu\nu} + \kappa^2 H_{\mu\nu}{}^{\rho\sigma}(\nabla_\alpha)R_{\rho\sigma} = 0\,, \qquad (50)$$

in which $G_{\mu\nu}$ denotes the Einstein tensor and $\kappa^2 = 8\pi G$ in natural units. In (50) $H_{\mu\nu}{}^{\rho\sigma}(\nabla_\alpha)$ acts as a total derivative only on the Ricci tensor and therefore the associativity condition of an eventual star-produc would not be satisfied. Thus it would be completely meaningless even try to make sense of $H_{\mu\nu}{}^{\rho\sigma}(\nabla_\alpha)$ in terms of a star-product and of an underlying non-commutative of space-time, even in situation in which the background has been fixed and gravity has been linearized at the first order, as for instance when considering $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. Moreover is matter of fact that equation (50) differs from the equation of motion derived in the model studied in [39], where the non-commutative Einstein equations read

$$\mathbf{Ric} - \frac{1}{2}\mathbf{g} \star \mathfrak{R} = 0\,. \qquad (51)$$

In (51) the $\star$ product is defined in order to be consistent with the twist element that modifies the diffeomorphism algebra. The non-commutative metric tensor reads locally as $\mathbf{g} = \theta^j \otimes_\star \theta^i \star g_{ij}$, with θ^i basis one-forms and $\otimes_\star$ the associative $\star$-tensor product associated to the deformed algebra of non-commutatve tensor fields. The Ricci tensor $\mathbf{Ric}$ and the Ricci scalar $\mathfrak{R}$ are derived as the contraction of the Curvature tensor, that is defined in terms of the $\star$-covariant derivative $\nabla_u^\star$ (along any vector field u of the module algebra). In brief, the equation (51) would read as a modification of the only Einstein tensor, whilst the equation of motion of the SRQG model we have studied involve higher derivatives applied to the square of the curvature tensor and their contractions. However, as this heuristic argument does not provide a solid proof, in order to be confident about the in-equivalence of the two models it would be appropriate to analyze some particular symmetry-reduced solutions, which we will do in the future providing accurate equation of motions in the general curved case.

Another point of difference we should single out is the absence in our framework of a consistent interpretation of the non-locality within the action (1) in terms of a twisted star-product. Following for instance a common procedure (see *e.g.* Ref. [50] and references therein), we can express any field theory on non-commutative space-time as a non-local field theory on a commutative space-time, provided that non-locality is described in terms of a star-product. Thus in principle we can ask whether it is possible to do the converse in our framework, recovering a star-product. But we should also consider a twist-element which leave undeformed the Lorentz sector of the Poincaré algebra, because of the particular dependence on the D'Alambertian covariant operator $\Box$ in the non-local function F. This feature represents a strong constraint for the theories studied in [2]. A twist element would naturally achieve this goal, but it is quite easy to see that from the particular form of F we would not be able to derive the associativity of the star-product, neither the normalization condition for it (see *e.g.* section II of Ref. [39]), both of them necessary requirements to recover a twist-element. Therefore we would be naturally lead to search for a generalization of our framework, and more in general of the theories presented in [2], in order to account for a consistent twist-element. We emphasize that in this latter theoretical framework we would be able to address interesting conceptual questions. Indeed, although in the seminal works in [47, 48] cases in which non-commutativity singled out a preferred metric were considered, in [39] any moving frame has been treated on equal footing and it has been shown that there are infinitely many metrics compatible with a given non-commutative differential geometry. Moreover, as a consequence of the bicovariant differential calculus and of the framework single out in [39], torsion appears also in the vacuum. This scheme hence implies a deformation of the geodesic motion, and consequences for the equivalence principle should be also investigated in detail. Conversely, the compatibility of the metric and the validity of the equivalence principle are imposed from the beginning in [1, 2] and not quested.

IV. CONCLUSIONS

Moving from the work in [2] defining a class of SRQG theories, we have shown in this paper that is possible to define a unique SRQG theory provided with a nontrivial Hopf algebra of space-time symmetries. The associated phase space is Heisenberg type, and associativity must be preserved in the non-commutative theory. Specifically, the non-trivial Hopf algebra connected to SRQG is the twisted algebra of θ-Poincaré, which shows as a remarkable feature that one of having a Poincaré algebra, and thus a Lorentz subalgebra, that are unmodified in the dimension-full parameter θ. For θ-Poincaré, deformation emerges in the co-algebra structure and in the other mathematical structures defining the concept of Hopf-algebra, which is a bi-algebra fulfilling certain consistency relations [42]. Therefore, Lorentz transformation and Lorentz covariance is defined in the standard way in this quantum-group symmetric SRQG theory, and locally it makes still sense to say that the Lorentzian theory is invariant under action of the generators of $\mathfrak{so}(3,1)$. And we emphasize that the analysis we have developed at tree-level for the graviton propa-

gator must be developed at high order in the perturbation theory and generalized to n-point functions in the quantum theory, in order to show a fully correspondence between the non-commutative field theory and the model of SRQG here investigated. Furthermore, the analysis of the non-commutative quantum theory provided with interactions, which can be achieved by following the lines of [31, 53–55], and the development of possible intriguing features of the multi-particle states in the Fock-space of the quantum theory are points strictly intertwined that may have a relation to the model we have focused on. Indeed, one might argue that the $\theta \sim 1/\Lambda^2$ modifications to the co-algebraic sector induces a θ modification to the many-particle states at the quantum level, and establishing firmly this point would open the path to the study of entanglement effects, as studied in the last one of Refs. [38] for the case of the κ-Poincaré algebra. Finally, a clear understanding of the non-commutative space-time reformulation of the models studied in may be derived through a comparison of the interacting terms contained in the SRQG theory at the perturbative level with an appropriate expansion of a selected non-commutative theory of gravity, which may be described by [39] or some other model. Nevertheless, at the present stage of development of the theory, the only knowledge of the two point function is sufficient to derive an appropriate physical description at least of the gravitational potential.

The one listed here above are all suggestive questions, which can be addressed only at the level of a full quantized non-commutative theory; we leave therefore them for future developments.

Appendix A: BCH formulae

We summarize in this appendix some useful formulae that we have used in the above sections. We first consider a linear operator A, which is defined by means of

$$\exp A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k. \tag{A1}$$

As a consequence, $\partial_\tau e^{\tau A} = A e^{\tau A} = e^{\tau A} A$. Let us consider another linear operator B, and let be $B(\tau) = e^{\tau A} B e^{-\tau A}$. The *Sophus-Lie* formula then provide us with the following series representation for $B(\tau)$:

$$B(\tau) = \sum_{m=0}^{\infty} \frac{\tau^m}{m!} B_m, \tag{A2}$$

in which $B_m = [A, B]_m := [A, [A, B]_{m-1}]$ and $B_0 := B$. The *Baker-Campbell-Hausdorff* (BCH) formula is a particular case of the *Sophus-Lie* formula. Setting $\tau = 1$, one obtains indeed

$$e^A B e^{-A} = \sum_{m=0}^{\infty} \frac{1}{m!} B_m. \tag{A3}$$

This latter expression can be re-manipulated in the form

$$[B, e^{-A}] = e^{-A} \Big([A, B] + [A, [A, B]]/2 + \cdots \Big), \tag{A4}$$

or

$$[e^A, B] = \Big([A, B] + [A, [A, B]]/2 + \cdots \Big) e^A. \tag{A5}$$

Furthermore, in addition to the BCH formula, there is another expression which is also referred to as BCH formula, but which is due to Eugene Dynkin. This latter expression provides us with the multiplication law for two exponentials of linear operators within the assumptions $[A, [A, B]] = [B, [B, A]] = 0$, corresponding to a central algebra in our θ-Minkowski case. It follows that

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}, \tag{A6}$$

and reshuffling this latter expression, one obtain the Zassenhaus formula at the second order

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}. \tag{A7}$$

As for practical reasons we were mostly interested to the Zassenhaus formula up to the fourth order, here below we furnish it for completeness (see *e.g.* [56] and references therein)

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} e^{\frac{1}{3!}\{2[B,[A,B]]+[A,[A,B]]\}} \times$$
$$\times e^{-\frac{1}{4!}\{[[[A,B],A],A]+3[[[A,B],A],Y]+3[[[A,B],B],B]\}}. \tag{A8}$$

We emphasize that the general BCH expansion is not a special case of the Hadamard lemma. Indeed the Dynkin formula is exact, but not closed, and in general there is no explicit closed form for the BCH expansion, except in the "degenerate cases". Nevertheless, those latter cases are the most interesting for the application to physics [57].

Appendix B: $P^{(2)}, P^{(1)}, P^{(0-s)}, P^{(0-s\omega)}$-Tensors

We furnish here below the expression for some quantities introduced in Section II, namely

$$P^{(2)}_{\mu\nu\rho\sigma}(k) = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma},$$

$$P^{(1)}_{\mu\nu\rho\sigma}(k) = \frac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}),$$

$$P^{(0-s)}_{\mu\nu\rho\sigma}(k) = \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \ , \ P^{(0-\omega)}_{\mu\nu\rho\sigma}(k) = \omega_{\mu\nu}\omega_{\rho\sigma},$$

$$P^{(0-s\omega)}_{\mu\nu\rho\sigma} = \frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\rho\sigma}, \ P^{(0-\omega s)}_{\mu\nu\rho\sigma} = \frac{1}{\sqrt{3}}\omega_{\mu\nu}\theta_{\rho\sigma},$$

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \ , \ \omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}. \tag{B1}$$

[1] T. Biswas, E. Gerwick, T. Koivisto, A. Mazumdar, accepted in Phys. Rev. Letters [arXiv:1110.5249v2].

[2] L. Modesto, arXiv:1107.2403 [hep-th]; L. Modesto [arXiv:1202.0008 [hep-th]].

[3] A. S. Koshelev [arXiv:1112.6410 [hep-th]].

[4] Alexey S. Koshelev, Sergey Yu. Vernov [arXiv:1202.1289 [hep-th]].

[5] A. S. Koshelev, AIP Conf. Proc. 1241 (2010) 630-638 [arXiv:0912.5457 [hep-th]].

[6] P. Nicolini, A. Smailagic and E. Spallucci, [arXiv:hep-th/0507226].

[7] P. Nicolini, J. Phys. A 38, L631 (2005) [arXiv:hep-th/0507266].

[8] P. Nicolini, A. Smailagic and E. Spallucci, Phys. Lett. B 632, 547 (2006) [arXiv:gr-qc/0510112].

[9] T. G. Rizzo, JHEP 0609, 021 (2006) [arXiv:hep-ph/0606051].

[10] E. Spallucci, A. Smailagic and P. Nicolini, Phys. Rev. D 73, 084004 (2006) [arXiv:hep-th/0604094].

[11] S. Ansoldi, P. Nicolini, A. Smailagic and E. Spallucci, Phys. Lett. B 645, 261 (2007) [arXiv:gr-qc/0612035].

[12] E. Spallucci, A. Smailagic and P. Nicolini, Phys. Lett. B 670, 449 (2009) [arXiv:0801.3519 [hep-th]].

[13] P. Nicolini, Int. J. Mod. Phys. A 24, 1229 (2009) [arXiv:0807.1939 [hep-th]].

[14] R. Casadio and P. Nicolini, JHEP 0811, 072 (2008) [arXiv:0809.2471 [hep-th]].

[15] I. Arraut, D. Batic and M. Nowakowski, Class. Quant. Grav. 26, 245006 (2009) [arXiv:0902.3481 [gr-qc]].

[16] P. Nicolini and E. Spallucci, Class. Quant. Grav. 27, 015010 (2010) [arXiv:0902.4654 [gr-qc]].

[17] A. Smailagic and E. Spallucci, Phys. Lett. B 688, 82 (2010) [arXiv:1003.3918 [hep-th]].

[18] D. M. Gingrich, JHEP 1005, 022 (2010) [arXiv:1003.1798 [hep-ph]].

[19] I. Arraut, D. Batic and M. Nowakowski, J. Math. Phys. 51, 022503 (2010) [arXiv:1001.2226 [gr-qc]].

[20] R. Banerjee, S. Gangopadhyay and S. K. Modak, Phys. Lett. B 686, 181 (2010) [arXiv:0911.2123 [hep-th]].

[21] P. Nicolini, Phys. Rev. D 82, 044030 (2010) [arXiv:1005.2996 [gr-qc]].

[22] L. Modesto, A. Randono, [arXiv:1003.1998 [hep-th]].

[23] F. Caravelli, L. Modesto, Phys. Lett. B 702 (2011) 307-311 [arXiv:1001.4364 [gr-qc]].

[24] L. Modesto and P. Nicolini, Phys. Rev. D 82, 104035 (2010) [arXiv:1005.5605 [gr-qc]].

[25] R. B. Mann and P. Nicolini, Phys. Rev. D 84 (2011) 064014 arXiv:1102.5096 [gr-qc].

[26] J. R. Mureika and P. Nicolini, Phys. Rev. D 84 (2011) 044020 [arXiv:1104.4120 [gr-qc]].

[27] L. Modesto, J. W. Moffat, P. Nicolini, Phys. Lett. B 695, 397-400 (2011) [arXiv:1010.0680 [gr-qc]].

[28] P. Gaete, J. A. Helayel-Neto, E. Spallucci, Phys. Lett. B 693, 155-158 (2010) [arXiv:1005.0234 [hep-ph]].

[29] K.S. Stelle, Phys. Rev. D 16, 953-969 (1977).

[30] E. T. Tomboulis, [hep-th/9702146v1].

[31] A. Smailagic and E. Spallucci, J.Phys. A36 (2003) L517-L521, [hep-th/0308193v2].

[32] I. L. Buchbinder, Sergei D. Odintsov, I. L. Shapiro, "Effective action in quantum gravity", IOP Publishing Ltd 1992.

[33] E. Spallucci, A. Smailagic and P. Nicolini, Phys. Rev. D 73, 084004 (2006) [hep-th/0604094v1].

[34] M. Chaichian, A. Demichev, P. Presnajder and A. Tureanu, Eur. Phys. J. C20 (2001) 767 [hep-th/0007156].

[35] J. Gomis and T. Mehen, Nucl. Phys. B **591**, 265 (2000) [hep-th/0005129].

[36] V. G. Drinfeld, Leningrad Math. J. 1, 321 (1990) .

[37] G. Amelino-Camelia, F. Briscese, G. Gubitosi, A. Marciano, P. Martinetti and F. Mercati, Phys.Rev. D78 (2008) 025005 [arXiv:0709.4600[hep-th]]; G. Amelino-Camelia, G. Gubitosi, A. Marciano, P. Martinetti, F. Mercati, D. Pranzetti and R. A. Tacchi, Prog. Theor. Phys. Suppl. 171 (2007) 65-78 [arXiv:0710.1219[gr-qc]]; A. Marciano, Proceedings of the XII Marcel Grossmann Meeting (Paris 2009) [arXiv:1003.0395 [hep-th]].

[38] A. Agostini, G. Amelino-Camelia, M. Arzano, A. Marciano and R. A. Tacchi, Mod. Phys. Lett. A **22**, 1779 (2007) [arXiv:hep-th/0607221]; M. Arzano and A. Marciano, Phys. Rev. D **75**, 081701 (2007) [arXiv:hep-th/0701268]; M. Arzano and A. Marciano, Phys. Rev. D **76**, 125005 (2007) [arXiv:0707.1329 [hep-th]].

[39] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp and J. Wess, Class. Quant. Grav. 22 (2005) 3511-3532 [hep-th/0504183]; P. Aschieri, M. Dimitrijevic, F. Meyer and J. Wess, Class. Quant. Grav. 23 (2006) 1883-1912 [hep-th/0510059].

[40] J. Baez, I.E.Segal and Z. Zhou, Introduction to Algebraic and constructive Quantum Field Theory, Princeton University Press (1992).

[41] A. P. Balachandran, A. Pinzul and B. Qureshi, arXiv:hep-th/0508151; A. P. Balachandran, T. R. Govindarajan, G. Mangano, A. Pinzul, B. A. Qureshi and S. Vaidya, Phys. Rev. D75, 045009 (2007) [hep-th/0608179].

[42] V. Chari and A. Pressley, A guide to quantum groups, (Cambridge University Press, Cambridge UK, 1994); S. Majid, Foundations of Quantum Group Theory, (Cambridge University Press, Cambridge UK, 2000).

[43] S. Majid and H. Ruegg, Phys. Lett. B 334 (1994) 348 [hep-th/9405107].

[44] N. Seiberg and E. Witten, JHEP 09 (1999) 032, hep-th/9908142; Chong-Sun Chu, [hep-th/0502167].

[45] P. Van Nieuwenhuizen, Nuclear Physics B 60 478-492 (1973).

[46] M. Asorey, J.L. Lopez, I.L. Shapiro, Intern. Journal of Mod. Phys. A12, 5711-5734 (1997) [hep-th/9610006].

[47] J. Madore and J. Mourad, J. Math. Phys. **39**, 423 (1998) [gr-qc/9607060].

[48] B. L. Cerchiai, G. Fiore and J. Madore, Commun. Math. Phys. 217, 521 (2001) [math/0002007 [math-qa]].

[49] P. Schupp and S. Solodukhin, [arXiv:0906.2724 [hep-th]]; T. Ohl and A. Schenkel, JHEP **0910**, 052 (2009) [arXiv:0906.2730 [hep-th]]; P. Aschieri and L. Castellani, J. Geom. Phys. **60**, 375 (2010) [arXiv:0906.2774 [hep-th]]; A. Schenkel, Ph.D Thesis, http://opus.bibliothek.uni-wuerzburg.de/volltexte/2011/6582/ .

[50] L. Freidel, J. Kowalski-Glikman and S. Nowak, Phys. Lett. B 648, 70 (2007) [hep-th/0612170].

[51] M. V. Battisti and S. Meljanac, Phys. Rev. D 82, 024028 (2010) [arXiv:1003.2108 [hep-th]].

[52] A. O. Barvinsky [arXiv:1107.1463 [hep-th]; A. O. Barvinsky, Phys. Rev. D 71 (2005) 084007 [hep-th/0501093v2]; A. O. Barvinsky, Phys. Lett. B 572 (2003) 109-116 [hep-th/0304229v3]; H. W. Hamber and R. M. Williams, Phys. Rev. D72, 044026 (2005) [hep-th/0507017]; N. Arkani-Hamed, S. Dimopoulos, G. Dvali and G. Gabadadze, [hep-th/0209227v1]; G. Calcagni and G. Nardelli, JHEP 1002, 093 (2010) [arXiv:0910.2160 [hep-th]].

[53] X. Calmet, B. Jurco, P. Schupp, J. Wess and M. Wohlgenannt, Eur. Phys. J. C **23**, 363 (2002) [hep-ph/0111115].

[54] A. Micu and M. M. Sheikh Jabbari, JHEP **0101**, 025 (2001) [hep-th/0008057].

[55] A. Smailagic and E. Spallucci, J. Phys. A A **37**, 1 (2004) [Erratum-ibid. A **37**, 7169 (2004)] [hep-th/0406174].

[56] C. K. Zachos, *Crib Notes on Campbell-Baker-Hausdorff expansions*, http://www.hep.anl.gov/czachos/CBH.pdf .

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