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Non-radial oscillations of anisotropic neutron stars in the Cowling approximation

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Abstract

One of the most common assumptions in the studies of neutron star models and their oscillations is that the pressure is isotopic but there are arguments that this may not be correct. Thus in the present paper we make a first step towards studying the non-radial oscillations of neutron stars with an anisotropic pressure. We adopt the so-called Cowling approximation where the spacetime metric is kept fixed and the oscillation spectrum for the first few fluid modes is obtained. The effect of the anisotropy on the frequencies is apparent, although with the present results it might be hard to distinguish it from the changes in the frequencies caused by different equations of state.

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1 Introduction

The discovery of gravitational waves is one of the most important goals of the astrophysics nowadays. A lot of efforts are devoted to this problem worldwide. Ground based experiments [1]– [4] as well as space missions [5] are planned and some of them are expected to be able to give results in the near future. Parallel to these efforts the first steps towards a third generation gravitational wave telescope (the so called Einstein telescope) which is supposed to have much higher sensitivity are undertaken [6]. The reason why the gravitational waves are so difficult to be detected is that they are extremely weak which requires detectors with very high sensitivity and also accurate waveforms of the signal emitted from the astrophysical objects.

One of the promising sources of gravitational waves is the oscillations of neutron stars [7]. A lot of efforts were spent in studying the gravitational wave emission of these objects but still there are a lot of important open questions. The gravitational wave emission by neutron stars is ultimately connected to their interior structure. In order to predict accurately enough the characteristics of the gravitation waves we need adequate relativistic models of the neutron star interior. However, at present little is known about the properties and the behaviour of matter at very high densities and pressures. So, in modeling the neutron star interior we are forced to make certain assumptions about the properties of the neutron star matter. Some of these assumptions seem to be natural from a physical point of view, however, there are always uncertainties and suspicions that the assumptions may be not fully correct. As the science history shows there are surprises sometimes – Nature does not always share our notions for "natural". That is why the alternatives should also be investigated.

One of the widely accepted assumptions in studying the equilibrium configurations of neutron stars and their oscillations is that the pressure of the neutron star matter is isotropic. There are however arguments that the pressure could be anisotropic[†] [8]. Some theoretical investigations [9,10] show that the nuclear matter may be anisotropic at very high densities where the nuclear interactions must be treated relativistically. Anisotropy in the fluid pressure can be caused by many other factors. Anisotropy can be yielded by the existence of a solid core or by the presence of superfluid [11]– [13], by pion condensation [14], by different kind of phase transitions [16], by the presence of strong magnetic field [17] or by other factors [8]. From a formal point of view the mixture of two fluids is mathematically equivalent to an anisotropic fluid [8], [18].

During the last decades, starting with the pioneering work [19] there have been many papers studying anisotropic spherically symmetric static configurations within general relativity [20]– [43]. These studies show that the anisotropy may have non-negligible effects on the neutron star structure and properties. For example the anisotropy may influence notably the maximal equilibrium mass, maximum redshift and maximum compactness of the stars [20], [35]. It is worth noting also that even for stable configurations the anisotropy can support outwardly increasing energy density in the star core [42].

[†]Generally speaking the anisotropic fluid has pressures which can differ among the spacial directions.

The fact that the anisotropy can affect seriously the interior structure and the properties of the stellar configurations makes us think that the anisotropy may also have serious influence on the gravitational wave emission and more precisely on the gravitational wave spectrum of the stellar configurations. Therefore, in the context of the current efforts to detect the gravitational waves, it is important to study the gravitational wave spectrum of the anisotropic neutron stars. Such study is twofold. On the one hand it can reveal the basic characteristics of the gravitational wave spectrum of the anisotropic stars and the differences with the spectrum of the isotropic stars. On the other hand, such a study provides us with a tool to study the reverse problem – to put constraints on the amount of neutron star anisotropy using the observed gravitational wave spectrum in the future.

In the present paper we undertake the first step towards the study of the gravitational wave spectrum of anisotropic neutron stars. More precisely we investigate the spectrum of the non-radial oscillations of anisotropic neutron stars in Cowling approximation. The paper is organized as follows. In section 2 we numerically construct equilibrium configurations describing anisotropic stars. Section 3 is devoted to the derivation of the perturbation equations of the anisotropic neutron stars in Cowling approximation and the formulation of the boundary value problem for the oscillation spectrum. In section 4 we present the numerical results for the oscillation frequencies. The paper ends with conclusions.

2 Equilibrium anisotropic configurations of neutron stars

In the spherically symmetric case[‡] which we will consider in the present paper, the fluid anisotropy means that the radial pressure p differs from the transverse pressure q. The mathematical description of an anisotropic fluid in spherical symmetry is given by the following energy-momentum tensor

$$T_{\mu\nu} = \rho u_\mu u_\nu + p k_\mu k_\nu + q \left(g_{\mu\nu} + u_\mu u_\nu - k_\mu k_\nu \right), \tag{1}$$

where $g_{\mu\nu}$ is the spacetime metric, u^μ is the fluid 4-velocity, ρ is the fluid energy density and k^μ is the unit radial vector ($k_\mu k^\mu = 1$) with $u^\mu k_\mu = 0$. Note that $g_{\mu\nu} + u_\mu u_\nu - k_\mu k_\nu$ is the projection tensor onto the 2-surfaces orthogonal to both u^μ and k^μ. At the center of symmetry the anisotropic pressure must vanish since k^μ is not defined there.

For spherically symmetric spacetimes the metric can be written in the well-known form

$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{2}$$

[‡]The assumption of spherical symmetry is applicable to the static case of matter sources with an energy momentum tensor satisfying $|T^\theta_\theta - T^\phi_\phi| << T^\theta_\theta$. So the assumption of spherical symmetry is applicable for example to the cases when the anisotropy is yielded by the existence of a solid core, by the presence of superfluid or by the presence of pion condensation. Also the spherical symmetry can be used when the anisotropy is yielded by weak enough magnetic field. However if the magnetic field is very strong, as in the case of magnetars, the spherical symmetry assumption may not be a good approximation [17].

The Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \tag{3}$$

then reduce to

$$\frac{2\Lambda'}{r} e^{-2\Lambda} + \frac{1}{r^2}\left(1 - e^{-2\Lambda}\right) = 8\pi\rho, \tag{4}$$
$$\frac{2\Phi'}{r} e^{-2\Lambda} - \frac{1}{r^2}\left(1 - e^{-2\Lambda}\right) = 8\pi p, \tag{5}$$

while the contracted Bianchi identity

$$\nabla^{\mu} T_{\mu\nu} = 0 \tag{6}$$

gives

$$p' = -(\rho + p)\Phi' - \frac{2\sigma}{r}, \tag{7}$$

where $\sigma = p - q$. Introducing the local mass $m(r) = \frac{r}{2}\left(1 - e^{-2\Lambda}\right)$ and expressing Φ' from (5) we can write the dimensionally reduced equations in the Tolman-Oppenheimer-Volkoff form

$$m' = 4\pi\rho r^2, \tag{8}$$
$$p' = -(\rho + p)\,\frac{4\pi p r^3 + m}{r\,(r - 2m)} - \frac{2\sigma}{r}. \tag{9}$$

In order to close our system we should specify the equations of state for p and for σ. For the radial pressure we will consider a barotropic equation of state and more precisely

$$\rho = p_0 \left(\frac{p}{K p_0}\right)^{1/\Gamma} + \frac{p}{\Gamma - 1}, \tag{10}$$

where $p_0 = 1.67 \times 10^{14}$ g/cm^3 in units where $c = 1$ and we have chosen $\Gamma = 2.34$ and $K = 0.0195$ obtained when fitting the tabulated data for EOS II [44]. The results are qualitatively the same for other values of K and Γ.

As explained in [42] we can not just take $\sigma = \sigma(\rho)$ because this equation of state is too restrictive. Instead we should consider quasi-local equation of state $\sigma = \sigma(p, \mu)$ where μ denotes a quasi-local variable. In principle the equation of state $\sigma = \sigma(\rho, \mu)$ should be determined by the microscopic theory. Unfortunately, at present we do not have a good enough microscopic theory to allow us to find the explicit form of the dependence

$\sigma = \sigma(\rho, \mu)$. A drawback of the available microscopic models is the fact that the models are developed in flat spacetime and then the results are transferred to curved spacetime which is not completely satisfactory [12]. Within the framework of this type of microscopic models it is impossible to find the influence of the curved geometry on the equation of state. That is why our approach in the present paper is phenomenological. Following [42] for the quasi-local variable we take the local compactness $\mu = \frac{2m(r)}{r} = 1 - e^{-2\Lambda}$ and we consider the following equation of state

$$\sigma = \lambda p \mu, \tag{11}$$

where λ is a parameter. Since the local compactness is zero at the center, this guarantees that $\sigma(0) = 0$. In order to roughly estimate the range of the parameter λ we use the results of [15] where the anisotropy is caused by a pion condensation. In [15] it is found that $0 \leq \sigma/p \leq 1$ and therefore we could expect that the maximum value of λ is of order of 1. In the present paper we adopt the range $\lambda \in [-2, 2]$.

In order to obtain the background solutions which will be perturbed we solve the reduced field equations (4),(5) and (7) with the appropriate boundary conditions

$$\Lambda(0) = 0, \quad \rho(0) = \rho_0, \quad \Phi(\infty) = 0. \tag{12}$$

The normalized density ρ, the radial p and the anisotropic σ pressure as functions of the radial coordinate r are shown in Figs. 1 and 2 for several neutron star solutions. The central energy density ρ_0 is the same for all of the solutions presented in the figures and the results for several values of the parameter λ, which controls the anisotropic pressure, are shown. It is interesting to note that for large values of λ and for large masses, neutron star solutions exist for which the density ρ is not a monotonic function of the radial coordinate but has a maximum and these solutions are dynamically stable [42].

In Fig. 3 the mass M of the anisotropic neutron stars is shown as a function of the central density ρ_0 and of the radius R for several values of the parameter λ. As we can see the properties of the star vary significantly when the anisotropic pressure is varied, i.e. when we vary the parameter λ. The dynamical stability analysis shows that the solutions are stable up to the maximum mass of the sequences [42].

3 Perturbation equations in Cowling approximation

In this section we derive the equations describing the non-radial perturbations of the anisotropic stars in the so-called Cowling approximation [45], [46]. In Cowling approximation the spacetime metric is kept fixed. Despite of this simplification the Cowling formalism turns out to be accurate enough and reproduces the oscillation spectrum with good accuracy. In fact the comparison of the oscillation frequencies obtained by a fully general relativistic numerical approach and by the Cowling approximation shows that the discrepancy is less than 20% for the typical stellar models [47].

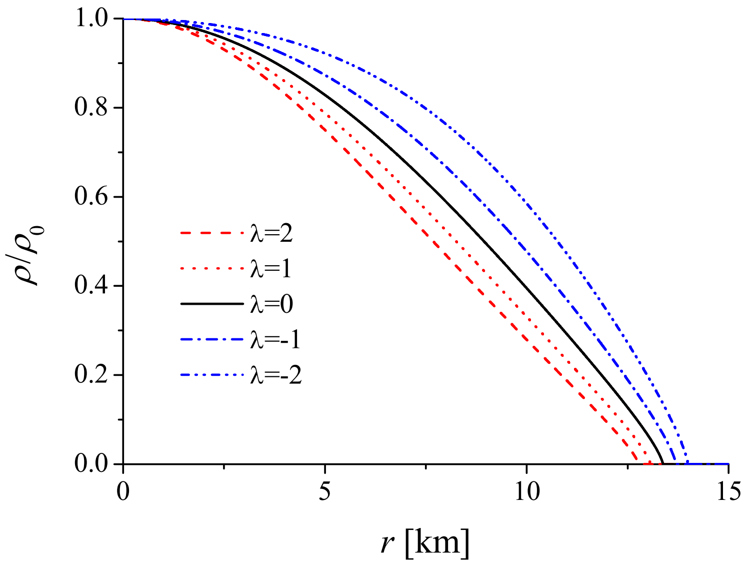

Figure 1: The normalized energy density ρ as a function of the radial coordinate r. The results for several neutron-star solutions with the same central energy density $\rho_0 = 7.455 \times 10^{14}$ g/cm^3 and different values of the parameter λ are shown (this central energy density gives neutron star with mass $M = 1.4M_\odot$ in the case with zero anisotropic pressure, i.e. when $\lambda = 0$).

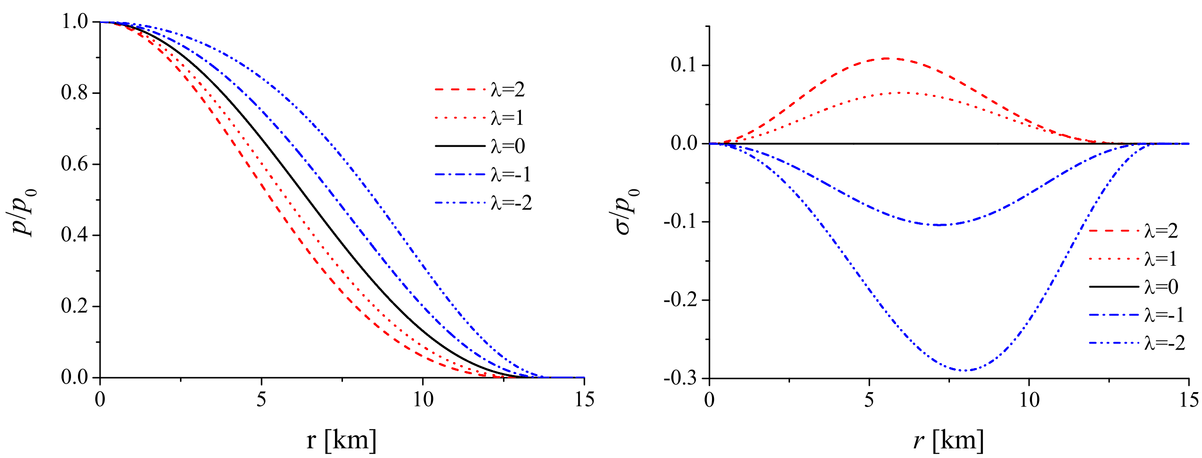

Figure 2: The radial p and the anisotropic pressure σ as functions of the radial coordinate r, normalized to the value of the radial pressure at the center of the star p_0. The results are for the same solutions as in Fig. 1.

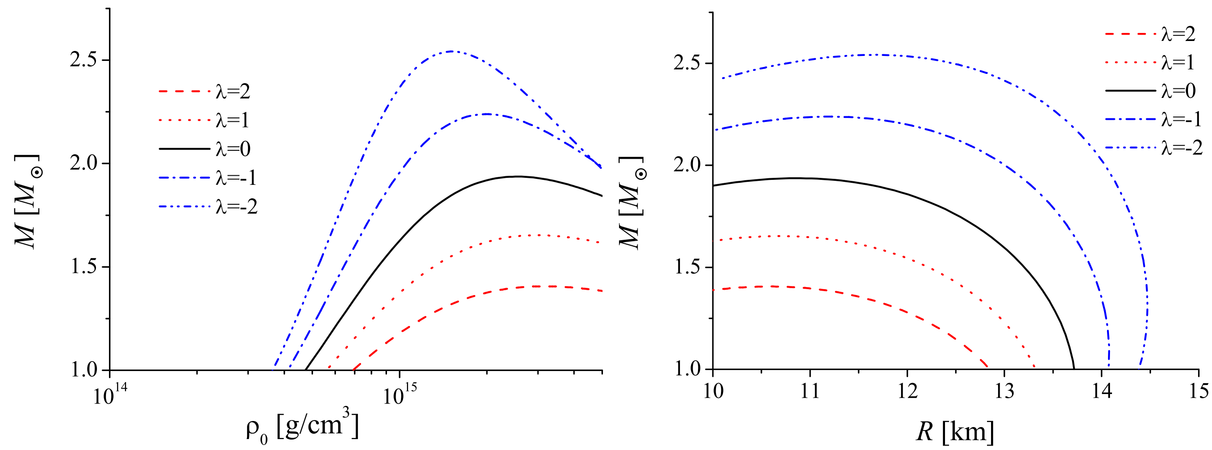

Figure 3: The mass of the neutron stars as a function of the central density (left panel) and of the radius (right panel) for several values the parameter λ.

The equations describing the perturbations in Cowling formalism are obtained by varying the equations for the conservation of the energy-momentum tensor (6). Taking into account that the metric is kept fixed, we find $\nabla_\nu \delta T^\nu_\mu = 0$ where

$$\delta T^\nu_\mu = (\delta\rho + \delta q)\, u_\mu u^\nu + (\rho + q)\,(u_\mu \delta u^\nu + \delta u_\mu u^\nu) + \delta q \delta^\nu_\mu + \delta\sigma k^\nu k_\mu + \sigma \delta k^\nu k_\mu + \sigma k^\nu \delta k_\mu. \quad (13)$$

Projecting equation $\nabla_\nu \delta T^\nu_\mu = 0$ along the background 4-velocity u^μ we have

$$u^\nu \nabla_\nu \delta\rho + \nabla_\nu \left\{ \left[(\rho + q)\delta^\nu_\mu + \sigma k^\nu k_\mu \right] \delta u^\mu \right\} + (\rho + q)\, a_\nu \delta u^\nu + \nabla_\nu u_\mu \delta\left(\sigma k^\nu k^\mu\right) = 0. \quad (14)$$

Projecting orthogonally to the background 4-velocity by using the operator $\mathcal{P}^\nu_\mu = \delta^\nu_\mu + u^\nu u_\mu$, we obtain

$$(\delta\rho + \delta q)\, a_\mu + (\rho + q)\, u^\nu \left(\nabla_\nu \delta u_\mu - \nabla_\mu \delta u_\nu\right) + \nabla_\mu \delta q + u_\mu u^\nu \nabla_\nu \delta q + \mathcal{P}^\nu_\mu \nabla_\alpha \delta\left(\sigma k^\alpha k_\nu\right) = 0, \quad (15)$$

where $a_\mu = u^\nu \nabla_\nu u_\mu$ is the background 4-acceleration.

At this stage we can express the perturbations of the 4-velocity via the Lagrangian displacement vector ξ^i, namely

$$\frac{\partial \xi^i}{\partial t} = \frac{\delta u^i}{u^t}, \quad (16)$$

where $i = 1, 2, 3 = r, \theta, \phi$.

Now let us consider eq.(15) for $\mu = \theta$ and $\mu = \phi$. Since $a_\theta = a_\phi = 0$ and $u^\mu = (u^t, 0, 0, 0)$ we find

$$(\rho + q)(u^t)^2 \partial_t^2 \xi_\theta + \partial_\theta \delta q = 0, \tag{17}$$

$$(\rho + q)(u^t)^2 \partial_t^2 \xi_\phi + \partial_\phi \delta q = 0. \tag{18}$$

Taking into account that ρ, q and u^t depend on r only, the integrability condition for the above equations gives

$$\partial_\theta \xi_\phi = \partial_\phi \xi_\theta. \tag{19}$$

From this condition and the fact that the background is spherically symmetric we find that ξ_θ and ξ_ϕ are of the form

$$\xi_\theta = -\sum_{lm} V_{lm}(r,t)\partial_\theta Y_{lm}(\theta,\phi), \tag{20}$$

$$\xi_\phi = -\sum_{lm} V_{lm}(r,t)\partial_\phi Y_{lm}(\theta,\phi), \tag{21}$$

where $Y_{lm}(\theta,\phi)$ are the spherical harmonics. From now on, in order to simplify the notations, we will just write $\xi = -VY_{lm}$ when we have expansion in spherical harmonics.

We proceed further with finding the expressions for the density and pressure perturbations. From eq.(14) after some algebra we find

$$\begin{aligned}\delta\rho = &-\frac{1}{\sqrt{-g}}\partial_i \left\{\sqrt{-g}\left[(\rho+q)\xi^i + \sigma(k_j\xi^j)k^i\right]\right\} - \\ &\left[(\rho+q)\xi^i + \sigma(k_j\xi^j)k^i\right]\partial_i \ln(u^t) - (\rho+p)a_i\xi^i.\end{aligned} \tag{22}$$

It is convenient to express ξ^r in the form

$$\xi^r = e^{-\Lambda}\frac{W}{r^2}Y_{lm} \tag{23}$$

and substituting in the above equations, after some algebra we find[§]

$$\delta\rho = -(\rho+p)\left[e^{-\Lambda}\frac{W'}{r^2} + \frac{l(l+1)}{r^2}V\right]Y_{lm} - \frac{d\rho}{dr}e^{-\Lambda}\frac{W}{r^2}Y_{lm} + \frac{2\sigma}{r^3}e^{-\Lambda}WY_{lm} + \sigma\frac{l(l+1)}{r^2}VY_{lm}, \tag{24}$$

where in the last step we have taken into account that $a_r = \Phi'$ and eq. (7).

[§]The derivative with respect to the radial coordinate r will be denoted by prime or by the standard symbol interchangeably.

In order to find the perturbation of the radial pressure we first use the relation between the Eulerian and Lagrangian variations, namely

$$\delta p = \Delta p - \xi^r \partial_r p \tag{25}$$

with Δp being the Lagrangian variation. From the equation of state we have

$$\Delta p = \frac{dp}{d\rho}\Delta\rho = \frac{dp}{d\rho}\left(\delta\rho + \xi^r \partial_r \rho\right). \tag{26}$$

In this way we obtain the following formula for the perturbation of the radial pressure

$$\delta p = -\frac{dp}{d\rho}\left\{(\rho+p)\left[e^{-\Lambda}\frac{W'}{r^2} + \frac{l(l+1)}{r^2}V\right] - 2\frac{\sigma}{r^3}e^{-\Lambda}W - \sigma\frac{l(l+1)}{r^2}V\right\}Y_{lm} - \frac{dp}{dr}e^{-\Lambda}\frac{W}{r^2}Y_{lm}. \tag{27}$$

For the perturbation of the anisotropic pressure $\sigma = \sigma(p,\mu)$ we have

$$\delta\sigma = \frac{\partial\sigma}{\partial p}\delta p, \tag{28}$$

where we have taken into account that $\delta\mu = 0$.

The dynamical equations for W and V follow from eq. (15), namely

$$(\rho+p)\frac{e^{\Lambda-2\Phi}}{r^2}\partial_t^2 W + \partial_r\hat{\delta p} + (\hat{\delta\rho} + \hat{\delta p})a_r + \frac{2}{r}\hat{\delta\sigma} = 0, \tag{29}$$

$$(\rho+p-\sigma)e^{-2\Phi}\partial_t^2 V - \hat{\delta p} + \hat{\delta\sigma} = 0, \tag{30}$$

where $\hat{\delta p}$ are the coefficients in the expansion in the spherical harmonics Y_{lm}, i.e $\delta p = \hat{\delta p}Y_{lm}$.

From now on we will assume for the perturbation functions a harmonic dependence on time, i.e. $W(r,t) = W(r)e^{i\omega t}$ and $V(r,t) = V(r)e^{i\omega t}$. Then the above equations become

$$-(\rho+p)\frac{e^{\Lambda-2\Phi}}{r^2}\omega^2 W + \partial_r\hat{\delta p} + (\hat{\delta\rho} + \hat{\delta p})a_r + \frac{2}{r}\hat{\delta\sigma} = 0, \tag{31}$$

$$-(\rho+p-\sigma)e^{-2\Phi}\omega^2 V - \hat{\delta p} + \hat{\delta}\sigma = 0. \tag{32}$$

The system (31)–(32) can be considerably simplified by combining the equations in an appropriate manner. Differentiating equation (32) and adding it to equation (31), and also using eq.(7), we find

$$\begin{aligned} V' &= 2V\Phi' - \left(1 - \frac{\partial\sigma}{\partial p}\right)\frac{\rho+p}{\rho+p-\sigma}\frac{e^{\Lambda}}{r^2}W \qquad (33)\\ &+ \left[\frac{\sigma'}{\rho+p-\sigma} + \frac{\frac{d\rho}{dp}+1}{\rho+p-\sigma}\sigma\left(\Phi' + \frac{2}{r}\right) - \frac{2}{r}\frac{\partial\sigma}{\partial p} - \left(1-\frac{\partial\sigma}{\partial p}\right)^{-1}\left(\frac{\partial^2\sigma}{\partial p^2}p' + \frac{\partial^2\sigma}{\partial p\partial\mu}\mu'\right)\right]V. \end{aligned}$$

This equation together with equation (31) solved for W', form a system which is equivalent to (31)–(32) but much more tractable:

$$\begin{aligned}
W' &= \frac{d\rho}{dp}\left[\omega^2 \frac{\rho+p-\sigma}{\rho+p}\left(1-\frac{\partial\sigma}{\partial p}\right)^{-1} e^{\Lambda-2\Phi} r^2 V + \Phi' W\right] - l(l+1)e^{\Lambda}V \qquad (34)\\
&+ \frac{\sigma}{\rho+p}\left[\frac{2}{r}\left(1+\frac{d\rho}{dp}\right)W + l(l+1)e^{\Lambda}V\right],\\
V' &= 2V\Phi' - \left(1-\frac{\partial\sigma}{\partial p}\right)\frac{\rho+p}{\rho+p-\sigma}\frac{e^{\Lambda}}{r^2}W \qquad (35)\\
&+ \left[\frac{\sigma'}{\rho+p-\sigma} + \left(\frac{d\rho}{dp}+1\right)\frac{\sigma}{\rho+p-\sigma}\left(\Phi'+\frac{2}{r}\right)\right.\\
&- \left.\frac{2}{r}\frac{\partial\sigma}{\partial p} - \left(1-\frac{\partial\sigma}{\partial p}\right)^{-1}\left(\frac{\partial^2\sigma}{\partial p^2}p' + \frac{\partial^2\sigma}{\partial p\partial\mu}\mu'\right)\right]V.
\end{aligned}$$

The boundary condition at the star surface is that the Lagrangian perturbation of the radial pressure vanishes

$$\omega^2\frac{\rho+p-\sigma}{\rho+p}\left(1-\frac{\partial\sigma}{\partial p}\right)^{-1} e^{-2\Phi}V + \left(\Phi'+\frac{2}{r}\frac{\sigma}{\rho+p}\right)e^{-\Lambda}\frac{W}{r^2} = 0. \qquad (36)$$

The boundary conditions at the star center can be obtained by examining the behaviour in the vicinity of $r=0$. For this purpose it is convenient to introduce the new functions $\tilde{W}$ and $\tilde{V}$ defined by

$$W = \tilde{W} r^{l+1}, \quad V = \tilde{V} r^{l}. \qquad (37)$$

Then one can show that at $r=0$ the following boundary condition is satisfied

$$\tilde{W} = -l\tilde{V}. \qquad (38)$$

4 Oscillation spectrum of the anisotropic neutron stars

The oscillation spectrum of the anisotropic neutron stars in Cowling approximation can be obtained by solving the differential equations (34)–(35) together with the boundary conditions (36) and (38). We have calculated the frequencies of the f-modes and the higher fluid modes p_1 and p_2. All of the presented dependences are shown up to the maximum mass for the corresponding parameters where the solutions become unstable [42].

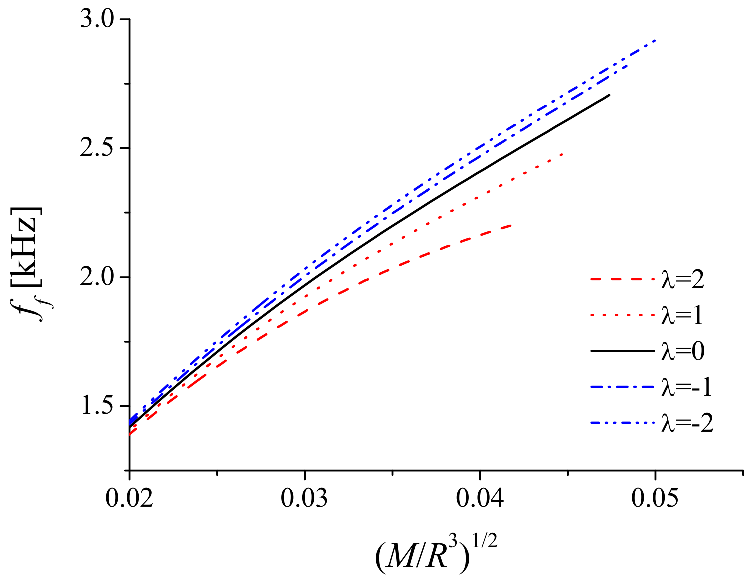

Figure 4: The f-mode frequency as a function of the average density of the neutron star $\sqrt{M/R^3}$. The results for several values of the parameter λ are shown.

An empirical dependence between the f-mode frequencies and the average density was found in [48, 49] for the case of isotropic neutron stars and it is interesting to see if it changes in our case. The f-mode oscillation frequencies as a function of the average density are presented in Fig. 4 for several values of the parameter λ. The graph shows that the dependence does not change significantly for small values of the average density. Only for large values of the average density and for large absolute values of λ, the deviation from the isotropic neutron stars (i.e. when $\lambda = 0$) is more significant. But still the uncertainties in obtaining the coefficients in the empirical dependence in [48], which come from varying the equation of state, are comparable with the deviation due to the anisotropic pressure.

The frequency f and the normalized frequency $\omega\sqrt{R^3/M}$ as a function of the mass are shown in Figs. 5 and 6 for the f and the p_1 modes. Depending on the sign of λ the frequencies can be larger or smaller than in the case of isotropic neutron stars. As we can see the frequencies can change considerably when we increase the absolute value of λ. Also for fixed value of λ the differences with the isotropic neutron stars are bigger for larger masses because in this case the compactness μ, which enters the EOS for the anisotropic pressure (11), is larger. Therefore for large absolute values of λ and for large masses the oscillation frequencies can differ significantly from the isotropic neutron star case.

The normalized frequencies of the f, p_1 and p_2 modes as a function of λ are shown in Fig. 7 where the mass $M = 1.4M_\odot$ is the same for all the solutions. As we can see the changes in the frequencies as we vary λ, are similar for all the modes. This behavior is qualitatively different from some of the alternative models of neutron stars where the oscillations frequencies vary more significantly as a function of the corresponding parameter,

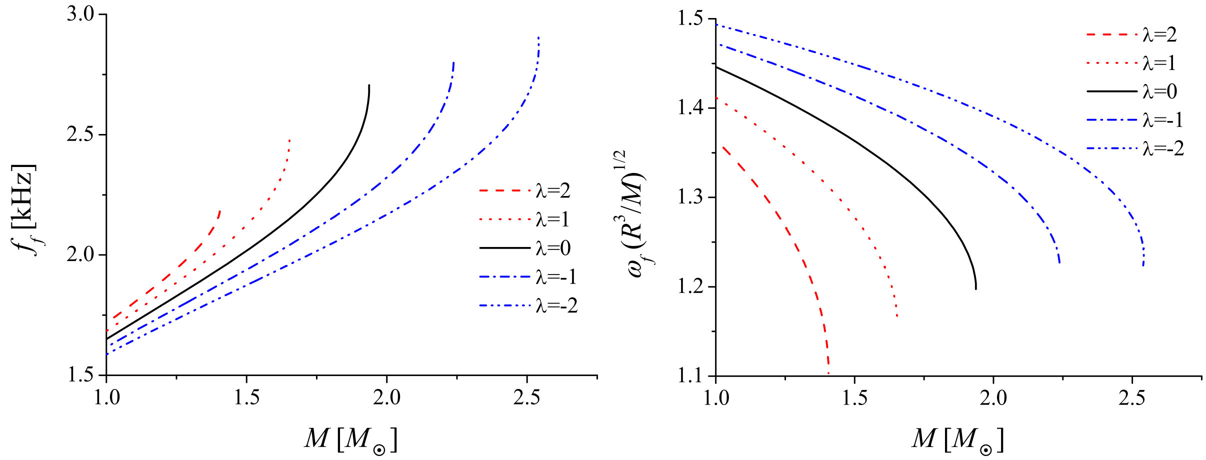

Figure 5: The frequency f as a function of the mass M (left panel) and the normalized frequency ω as a function of M (right panel) for the f mode. The results for several values of the parameter λ are shown.

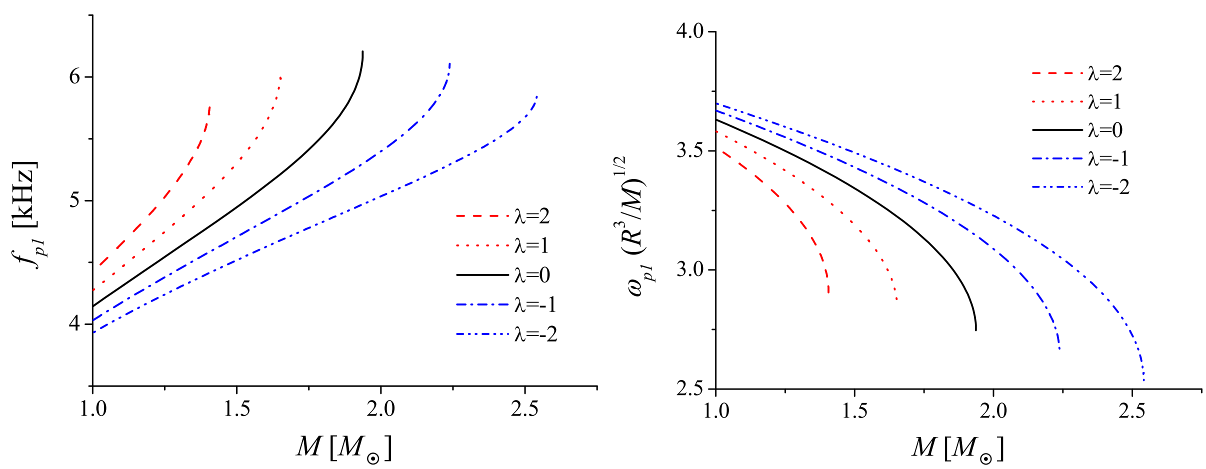

Figure 6: The results for the p_1-mode of the same solutions as shown on figure 5.

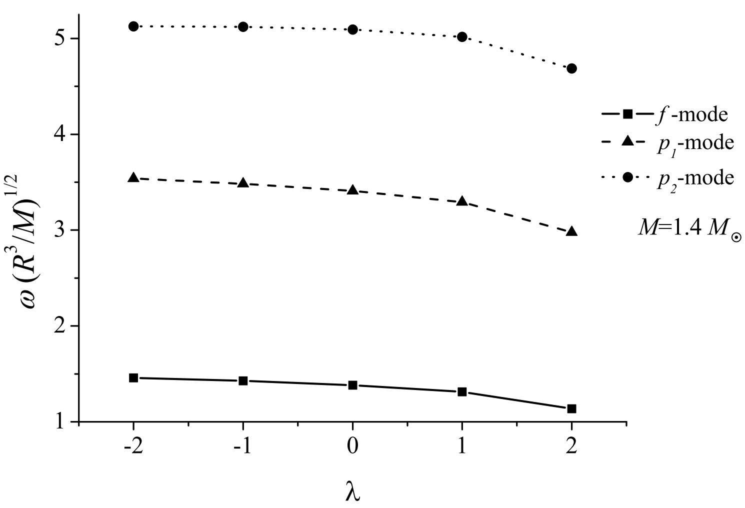

Figure 7: The normalized frequency ω as a function of λ for fixed value of the mass $M = 1.4M_{\odot}$. The results for the f, p_1 and p_2 modes are shown.

for the higher fluid modes [50]– [52].

It is interesting to compare the effects on the oscillation spectrum caused by varying the anisotropic pressure and by changing the equation of state of the radial pressure. As we said before the dependence between the f-mode frequencies and the average density does not change much when we vary the equation of state. The same is true also when we vary the anisotropic pressure. But the normalized frequency ω as a function of the mass changes significantly when we vary λ and the equations of state. This can be seen on Fig. 8 where the results for the f-mode oscillation frequencies are presented for two equations of state of the radial pressure and for several values of λ. The EOS II is the standard polytropic equation of state which we used up to now with $\Gamma = 2.34$ and $K = 0.0195$. The EOS A is again a polytropic equation of state where the coefficients $\Gamma = 2.46$ and $K = 0.00936$ are obtained when fitting the tabulated data for EOS A [53] and EOS II is stiffer than EOS A. As we can see the presence of an anisotropic pressure changes the frequencies in a similar way as changing the EOS, and more precisely positive values of λ lead to frequencies similar to a softer EOS, and negative values of λ lead to frequencies similar to a stiffer EOS. Thus the oscillation spectrum of neutron stars with anisotropic pressure can mimic to a certain extent the oscillation spectrum of neutron stars with softer/stiffer equation of state. But still if we consider strong anisotropic pressure the frequencies can change a lot which is hard to be achieved by the standard nuclear equations of state. Thus observing more than one fluid mode of a neutron star can help us to prove or at least set limits on the possible existence of anisotropic pressure in the neutron stars.

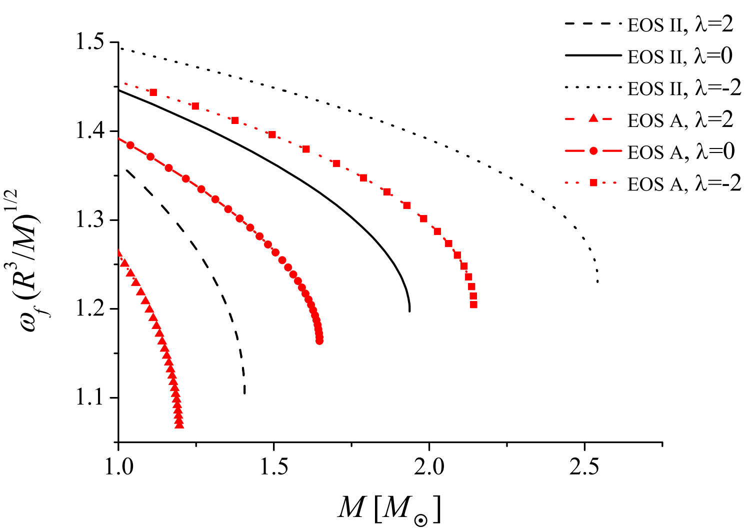

Figure 8: The normalized frequency ω as a function of M for the f mode. The results for different values of λ and for two polytropic equations of state of the radial pressure are shown – a soft equation of state EOS A and a stiff equation of state EOS II.

5 Conclusions

In the present paper we study how the possible existence of an anisotropic pressure inside a neutron star can change the oscillation spectrum. As a first step we examine the oscillations in Cowling approximation where the metric is kept fixed and within this approximation the perturbation equations for the anisotropic neutron stars are derived.

The background solution are obtained numerically by solving the reduced field equations where the equation of state for the radial pressure is polytropic and we use a quasi-local equation of state for the anisotropic pressure [42]. The properties of the obtained solutions can differ significantly from the isotropic neutron stars.

The oscillation spectrum of the anisotropic neutron stars is obtained when solving the perturbation equations with the appropriate boundary condition and the results for the f-mode and the higher fluid modes are obtained. It turns out that the dependence between the f-mode frequencies and the average density which was obtained in [48] does not change much in the presence of an anisotropic pressure. The effect of the anisotropy is more evident on other dependences, for example the normalized frequency as a function of the mass changes considerably for anisotropic stars. Thus the observation of more than one fluid mode can serve as a test for the existence of an anisotropic pressure in the neutron stars.

We have also compared the effect on the oscillation spectrum caused by the anisotropic pressure and by changing the equation of state of the radial pressure. It turns out that for negative values of the anisotropic pressure, i.e. for $\lambda < 0$, the changes in the frequencies are similar to what we will obtain in the case of isotopic neutron star with a stiffer equation of state, and for $\lambda > 0$ the results are similar to the case of an isotopic neutron star with a softer equation of state. A more detailed analysis, for example if we drop the Cowling approximation or if we consider rotating solutions, may show more differences between the oscillation spectrum of isotropic and anisotropic neutron stars and we plan to make such a study in the future.

It would be also interesting to check how a change in the equation of state of the anisotropic pressure influences the results because the choice of the quasi-local EOS (11) is by no means the only possible one. Up to now, however, little is know about the EOS of the anisotropic pressure and that is why further studies on the possible quasi-local equations of state and their effect on the stellar structure and oscillations are needed.

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