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Mechanics of universal horizons

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Modified gravity models such as Hořava-Lifshitz gravity or Einstein-æther theory violate local Lorentz invariance and therefore destroy the notion of a universal light cone. Despite this, in the infrared limit both models above possess static, spherically symmetric solutions with "universal horizons" - hypersurfaces that are causal boundaries between an interior region and asymptotic spatial infinity. In other words, there still exist black hole solutions. We construct a Smarr formula (the relationship between the total energy of the spacetime and the area of the horizon) for such a horizon in Einstein-æther theory. We further show that a slightly modified first law of black hole mechanics still holds with the relevant area now a cross-section of the universal horizon. We construct new analytic solutions for certain Einstein-æther Lagrangians and illustrate how our results work in these exact cases. Our results suggest that holography may be extended to these theories despite the very different causal structure as long as the universal horizon remains the unique causal boundary when matter fields are added.

I. INTRODUCTION

In the four decades since the seminal work of Bardeen, Carter, and Hawking [1] on the laws of black hole mechanics, a tremendous amount of effort has gone in to understanding horizon behavior. From the discovery of Hawking radiation [2], and the recognition that the four laws have a thermodynamic interpretation [3], to holography [4, 5] and its concrete realization through the AdS/CFT correspondence [6–8], the physics of horizons has provided useful information about quantum gravity. Using horizon thermodynamics and some mild assumptions about the behavior of matter, one can reverse the logic and reconstruct general relativity as the thermodynamic limit of a more fundamental theory of gravity [9–11]. Integral to horizon dynamics is the first law of black hole mechanics, which for the simplest Schwarzschild case, and the most similar to what we are interested in, is just

$$\delta M_{\rm ADM} = \frac{\kappa_{\rm KH}\,\delta A_{\rm KH}}{8\pi G_{\rm N}}\ . \tag{1}$$

Here $M_{\rm ADM}$ is the ADM mass of the spacetime and $\kappa_{\rm KH}$ and $A_{\rm KH}$ are the surface gravity and cross-sectional area evaluated on the Killing horizon, respectively. Identifying $(8\pi G_{\rm N})^{-1}\kappa_{\rm KH}$ as the temperature of the horizon and the area with the entropy allows one to make the analogy with the first law of thermodynamics, $\delta E = T\delta S$.

In order to have an appropriate formulation of thermodynamics for the black hole itself, as well as the combined system of the black hole and the exterior environment, it is necessary for the horizon to have an inherent entropy. If there was no entropy associated with the horizon, one could simply toss objects into the black hole and reduce the total entropy of the black hole and exterior system, thereby violating the second law of thermodynamics. If, however, the horizon has an associated entropy, then one can reformulate the usual second law into the generalized second law

$$\delta\left(S_{\rm outside} + S_{\rm horizon}\right) \geqq 0. \tag{2}$$

The generalized second law and the thought experiments behind it imply that *any* causal boundary in a gravitational theory should have an entropy associated with it. In general relativity, the entropy can be shown to be proportional to the area of a slice of the Killing horizon. However, if one includes higher curvature terms the entropy is still a function of the metric and matter fields evaluated on a slice of the Killing horizon, though no longer proportional to the surface area alone [12].

A much stronger departure from general relativity comes when one considers models that allow vacuum solutions with non-zero tensor fields besides the metric. In these models Lorentz symmetry and the equivalence principle are in general broken. A simple example of such a model is that of Einstein-æther theory [13], which introduces an æther vector field u^a and a dynamical constraint which forces u^a to be a timelike unit vector everywhere. The introduction of the æther vector preserves general covariance, but allows for novel effects such as matter fields travelling faster than the speed of light [14] and new gravitational wave polarizations that travel at different speeds [15]. Given certain choices of the action for the æther, the theory can be made phenomenologically viable [16, 17], have positive energy [18], and be ghost free [19]. In addition, the æther vector establishes a preferred frame and causality can be imposed in that frame [20] by requiring that all matter excitations propagate towards the future, even if the momentum vector of an excitation is spacelike. Since there is a preferred frame, Lorentz invariance does not hold, nor do the usual Lorentz invariance based arguments that a spacelike momentum vector in one frame immediately imply the existence of past directed momentum vectors. Thus, the propagation faster than the speed of light does *not* violate causality.

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Even though there is a notion of causality, it seems at first glance as if there would be no causal boundaries equivalent to an event horizon in the Einstein-æther theory – by coupling the æther vector u^a to matter kinetic terms, the matter Lagrangian can be chosen to make matter perturbations about flat space propagate arbitrarily fast. This is incorrect – causally separated regions of spacetime can exist even in this case. In the static, spherically symmetric and asymptotically flat solutions of Einstein-æther theory found by Eling and Jacobson [21] there exists such a region, and the boundary of this region has been dubbed the "universal horizon"[1]. Since the notion of a causal boundary and infinite speed modes is counterintuitive, we give a brief explanation of why they can occur here, postponing an in-depth discussion of universal horizons until later.

Consider a static, spherically symmetric spacetime, and cover it with Eddington-Finkelstein type coordinates such that the metric takes the form

$$\mathrm{d}s^2 = -e(r)\mathrm{d}v^2 + 2f(r)\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega_2^2 \ , \tag{3}$$

and the time translation Killing vector is $\chi^a = \{1,0,0,0\}$. Now let Σ_U denote a surface orthogonal to the æther vector u^a, so that U is the "æther time" generated by u^a that specifies each hypersurface in a foliation. At asymptotic spatial infinity χ^a and u^a coincide, but as one moves in towards $r = 0$ each Σ_U hypersurface bends down to the infinite past in v, eventually asymptoting to a 3 dimensional spacelike hypersurface on which $(u \cdot \chi) = 0$, which implies that the Killing vector χ^a becomes tangent to Σ_U. This hypersurface is the universal horizon. It is a causal boundary, as any signal must propagate to the future in U, which is necessarily towards decreasing r at the universal horizon. The surface is regular, and in fact Barausse, Jacobson and Sotiriou [22] have numerically continued the solution for metric and æther fields beyond the universal horizon.

Since such a causal boundary exists, it is natural to speculate that there must be an entropy associated with the universal horizon as well. In spherical symmetry, one does not need to worry about the zeroth law of black hole mechanics, as the symmetry enforces that all geometric quantities are constant over the universal horizon automatically. Hence one can immediately proceed to derive a Smarr formula and a corresponding first law. There are subtleties, however, as the boundary data of the theory at infinity naively contains two parameters. However, as noted in [21, 22], the boundary data can be reduced to one parameter, which is the total mass of the solution, by requiring that the solution is regular outside the universal horizon (see section IV for details). If one considers only regular solutions, we show that there is a Smarr relation between the total mass and geometric quantities evaluated on the universal horizon. In particular, the resulting Smarr relation contains a contribution from the extrinsic curvature of the Σ_U hypersurface in addition to the standard surface gravity term. We further show using a scaling argument that if one considers a transition between two regular solutions then there is also a first law that *may* admit a thermodynamic interpretation. With certain choices of the Lagrangian for Einstein-æther theory, we construct new, exact solutions and use those as examples to gain insight into the thermodynamics of the first law.

It is important to note that in previous works, black hole thermodynamics has failed when the limiting speed of matter fields is finite, but not necessarily the speed of light. In these cases one can construct perpetuum mobiles that violate the second law [23–25]. We will not consider any specific matter action in this paper, as our purpose is simply to determine whether a first law of mechanics holds for the universal horizon so that a thermodynamic interpretation *might* exist. However it is certainly possible that a true thermodynamic interpretation could only hold if the universal horizon is the only causal boundary for all fields. This could, for instance, be done if the Lagrangian for any matter fields contained higher derivatives that made the local speed of excitations infinite as the energy increased. From the point of view of effective field theory this is natural as one expects all terms consistent with the symmetries of the problem to appear in any operator expansion.

The paper is organized as follows. We first provide the background for Einstein-æther theory in Sec. II. In Sec. III we construct a Smarr formula and first law for spherically symmetric solutions. We then use two new exact, analytic solutions as examples in Sec. IV to verify the Smarr formula and the first law, as well as discuss the regularity of these solutions. We conclude with some more speculative comments on how to proceed to establish the thermodynamic connection and the obstacles that still remain.

II. THE EINSTEIN-ÆTHER THEORY

Einstein-æther theory was originally constructed [13] as a mechanism for breaking local Lorentz symmetry yet retaining as many of the other positive characteristics of general relativity as possible. In particular it is the most general action involving the metric and a unit timelike vector u^a that is two-derivative in fields and generally covariant. General covariance is maintained by enforcing the unit constraint on u^a via a Lagrange multiplier. Following the presentation in [18] the action of Einstein-æther theory is a sum of the usual Einstein-Hilbert action $\mathcal{S}_{\text{EH}}$ and the æther action $\mathcal{S}_{æ}$ [13]

$$\mathcal{S} = \mathcal{S}_{\text{EH}} + \mathcal{S}_{æ} = \frac{1}{16\pi G_{æ}} \int \mathrm{d}^4x \sqrt{-\mathtt{g}}\ (\mathcal{R} + \mathscr{L}_{æ}) \,. \tag{4}$$

[1] This point was first noted by Sergey Sibiryakov at Peyresq, 2010.

In terms of the tensor $Z^{ab}{}_{cd}$ defined as[2]

$$Z^{ab}{}_{cd} = c_1 \mathrm{g}^{ab}\mathrm{g}_{cd} + c_2\delta^a{}_c\delta^b{}_d + c_3\delta^a{}_d\delta^b{}_c - c_4 u^a u^b \mathrm{g}_{cd} \ , \quad (5)$$

where c_i, $i = 1, \ldots, 4$ are "coupling constants" (or couplings, for short) of the theory, the æther Lagrangian $\mathscr{L}_{æ}$ is given by

$$-\mathscr{L}_{æ} = Z^{ab}{}_{cd}(\nabla_a u^c)(\nabla_b u^d) - \lambda(u^2 + 1). \qquad (6)$$

The æther Lagrangian is therefore the sum of all possible terms for the æther field u^a up to mass dimension two, and a the constraint term $\lambda(u^2 + 1)$ with the Lagrange multiplier λ implementing the normalization condition[3]

$$u^2 = -1. \qquad (7)$$

An additional term, $\mathcal{R}_{ab}u^a u^b$ is a combination of the above terms when integrated by parts, and is not included here.

There exists a number of theoretical as well as observational bounds on the couplings c_i, $i = 1, \ldots, 4$ – see e.g. [16, 17] for a comprehensive review. In this work, we assume the following constraints to hold on these couplings

$$0 \leqq c_{14} < 2, \qquad 2 + c_{13} + 3c_2 > 0, \qquad c_{13} < 1 \ , \quad (8)$$

where we have defined $c_{13} = (c_1 + c_3)$ and $c_{14} = (c_1 + c_4)$. As we will see, these combinations of couplings, as well as $c_{123} = (c_1 + c_2 + c_3)$, play a more direct role in our analysis than the individual couplings c_i.

The constraints (8) come from the following conditions. If $c_{14} \geqq 2$ gravity becomes repulsive and one loses the proper Newtonian limit. Furthermore, in addition to the usual spin-2 gravitons, Einstein-æther theory also possesses two vector and one scalar modes (corresponding to the three degrees of freedom of u^a). If $c_{14} < 0$ or $2 + c_{13} + 3c_2 < 0$ then the scalar mode squared speed (see (50) below) about flat space becomes negative, signaling an instability of flat space to the production of scalar æther-metric excitations. Also, $2 + c_{13} + 3c_2$ cannot be strictly zero, as the G_{cosmo} appearing in the Friedmann equations derived from the Einstein-æther theory needs to be positive and finite [26]. Similarly, if $c_{13} \geqq 1$ then the squared speed of the usual spin-2 graviton in flat space becomes infinite or negative, which generates the same problem but with the usual spin-2 graviton modes. As we will also see, the Smarr formula and the first law of black hole mechanics that we derive below becomes unphysical if $c_{13} = 1$. There are other observational limits on the couplings, e.g., coming from the requirement that propagating high energy cosmic rays do not lose energy due to vacuum Čerenkov radiation of gravitons [27]. We will explicitly not impose this constraint here as we are interested in the behavior of the scalar mode, the interplay of any scalar mode horizon with the Killing and universal horizons, and the possible role of Čerenkov radiation from the universal horizon. Allowing the scalar mode to have any speed from almost zero to infinity is therefore theoretically useful.

The constant $G_{æ}$ in the action (4) is related to G_{N}, Newton's gravitational constant, via

$$G_{æ} = \left(1 - \frac{c_{14}}{2}\right) G_{\mathrm{N}} \ , \qquad (9)$$

as can be established using the weak field/slow-motion limit of the Einstein-æther theory [26].

The equations of motion, obtained by varying the action (4) with respect to the metric, u^a, and λ, are

$$\mathcal{G}_{ab} = \mathcal{T}^{æ}_{ab}, \qquad \text{Æ}_a = 0, \qquad u^2 = -1, \qquad (10)$$

respectively, where the æther stress tensor $\mathcal{T}^{æ}_{ab}$ is given by

$$\begin{aligned} \mathcal{T}^{æ}_{ab} =& \lambda u_a u_b + c_4 a_a a_b - \frac{1}{2}\mathrm{g}_{ab} Y^c{}_d \nabla_c u^d + \nabla_c X^c{}_{ab} \\ &+ c_1\left[(\nabla_a u_c)(\nabla_b u^c) - (\nabla^c u_a)(\nabla_c u_b)\right] \ , \end{aligned} \qquad (11)$$

and to make the notation more compact we have introduced the following

$$\begin{aligned} \text{Æ}_a &= \nabla_b Y^b{}_a + \lambda u_a + c_4(\nabla_a u^b)a_b \ , \\ Y^a{}_b &= Z^{ac}{}_{bd}\nabla_c u^d \ , \\ X^c{}_{ab} &= Y^c{}_{(a}\, u_{b)} - u_{(a}Y_{b)}{}^c + u^c Y_{(ab)} \ . \end{aligned} \qquad (12)$$

The acceleration vector a^a appearing in the expression for the æther stress tensor is defined as the parallel transport of the æther field along the æther field[4]

$$a^a = \nabla_u u^a \ . \qquad (13)$$

A. Static, spherically symmetric expansions

We now turn to the static, spherically symmetric case and provide some useful expansions which we will later use to analyze the equations of motion. Eling and Jacobson observed that for any spherically symmetric solution of Einstein-æther theory, u^a is hypersurface orthogonal[5] [21], which in turn implies that the twist of u^a vanishes, i.e.

$$u_{[a}\nabla_b u_{c]} = 0. \qquad (14)$$

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[2] Note the indicial symmetry $Z^{ba}{}_{dc} = Z^{ab}{}_{cd}$.

[3] We use a convention where the metric has signature $(-, +, +, +)$.

[4] We use the conventional notation ∇_X for the directional derivative $(X^a\nabla_a)$ along any vector field X^a. Once the normalization condition (7) is imposed, the acceleration is always orthogonal to the æther field (i.e. $u \cdot a = 0$), and therefore, is always spacelike.

[5] In fact they are also solutions of Hořava gravity [28, 29].

An immmediate consequence of the hypersurface orthogonality condition is that there exists a one-parameter redundancy among the couplings c_i. Using the unit norm constraint (7), the squared twist can be expressed as

$$\omega^2 = (\nabla_a u_b)(\nabla^a u^b) - (\nabla_a u_b)(\nabla^b u^a) + a^2 \ , \tag{15}$$

which also vanishes for hypersurface orthogonal solutions. We can, therefore, add any multiple of ω^2 to the action without affecting the solutions. In particular, by adding

$$\Delta\mathcal{S} = -\frac{c_0}{16\pi G_{æ}} \int \mathrm{d}^4x \sqrt{-\mathsf{g}}\ \omega^2 \tag{16}$$

to the action (4), where c_0 is an arbitrary real constant, the sole effect would be to obtain a new æther Lagrangian $\mathscr{L}'_{æ}$, otherwise identical to (6), except with a new set of coupling constants c'_i, $i = 1, ..., 4$, related to the unprimed c_i through

$$c'_1 = c_1 + c_0, \quad c'_2 = c_2, \quad c'_3 = c_3 - c_0, \quad c'_4 = c_4 - c_0. \tag{17}$$

Thus, by appropriately choosing c_0, one can set any one of the couplings c_1, c_3 and c_4, or any appropriate combinations of them, to any preassigned value. On the other hand, the coupling c_2, as well as combinations like c_{13}, c_{14} and c_{123} (8) stay invariant under the above redefinition of the couplings (17).

Our analysis will be further facilitated by defining a set of basis vectors at every point in spacetime so that we can project out various components of the equations of motion. Staticity and spherical symmetry implies the existence of a time translation Killing vector χ^a as well as three rotational Killing vectors $\zeta^a_{(i)}, i = 1, 2, 3$ (only two of the three are linearly independent). It is often convenient to choose χ^a as one of the basis vectors, but in this case it is actually more helpful to use a different basis. We first take u^a to be the (timelike) basis vector. We next pick any two spacelike unit vectors, call them m^a and n^a, both of which are normalized to unity, are mutually orthogonal and lie on the tangent plane of the two-spheres $\mathfrak{B}$ that foliate the hypersurface Σ_U. Finally, we use s^a, the spacelike unit vector orthogonal to u^a, m^a and n^a that points "outwards" along a Σ_U hypersurface. Note that the acceleration a^a only has a component along s^a by spherical symmetry, i.e.

$$a^a = (a \cdot s)s^a \ . \tag{18}$$

Thus, our tetrad consists of $\{u^a, s^a, m^a, n^a\}$. By spherical symmetry, any physical vector may have components along u^a and s^a, while any rank-two tensor may have components along the bi-vectors $u_a u_b$, $u_{(a}s_{b)}$, $u_{[a}s_{b]}$, $s_a s_b$ and $\hat{\mathsf{g}}_{ab}$, where $\hat{\mathsf{g}}_{ab}$ is the projection tensor onto the two-sphere $\mathfrak{B}$, bounding a section of a Σ_U hypersurface. For example, the basis-expansion of the extrinsic curvature of a Σ_U hypersurface is

$$K_{ab} = K_0 s_a s_b + \frac{\hat{K}}{2}\hat{\mathsf{g}}_{ab} \ , \tag{19}$$

where K_0 and $\hat{K}$[6] are scalar parameters related to each other through

$$K = K_0 + \hat{K} \ , \tag{20}$$

with K the trace of the extrinsic curvature of the Σ_U hypersurface. We ask the reader to refer to appendix A for further details on these points.

Finally, we work out some geometric quantities related to the Killing vector χ^a. We first use the spherical symmetry to write the timelike Killing vector χ^a as

$$\chi^a = -(u \cdot \chi)u^a + (s \cdot \chi)s^a \ . \tag{21}$$

The basis-expansion of $\nabla_a \chi_b$ takes the following form

$$\nabla_a \chi_b = -\kappa(u_a s_b - s_a u_b) \ . \tag{22}$$

Here κ is defined as the surface gravity on any two-sphere $\mathfrak{B}$ since, as we will now show, (at least outside the Killing horizon) κ is the acceleration of a static observer on $\mathfrak{B}$ as measured by an observer at asymptotic infinity. First consider the region outside the Killing horizon where there exists an outward pointing spacelike unit vector r^a, the radial unit vector, orthogonal to χ^a. The tangent vector $\hat{\chi}^a$ to the world line of a static observer on any two-sphere $\mathfrak{B}$ is simply the unit timelike vector along χ^a. In terms of the redshift factor ρ, we can then write $\chi^a = \rho\hat{\chi}^a$ (i.e. $\rho^2 = -\chi \cdot \chi$). Using (22), the directional derivative of $\hat{\chi}^a$ along itself is given by $\nabla_{\hat{\chi}}\hat{\chi}^a = a_\chi r^a$, where $a_\chi = (\kappa/\rho)$ is the local acceleration of the static observer. In other words, $\kappa = \rho a_\chi$ is the redshifted acceleration with respect to an observer at infinity, which prompts us to call κ the surface gravity on the two-sphere $\mathfrak{B}$. The mathematical analysis also follows through inside the Killing horizon[7] and the local acceleration is still given by (κ/ρ), but the interpretation of ρ as the redshift factor no longer holds. Nevertheless, we will continue to call κ the surface gravity. Using the results from appendix A, the surface gravity is explicitly given by[8]

$$\kappa = \sqrt{-\frac{1}{2}(\nabla_a \chi_b)(\nabla^a \chi^b)} = -(a \cdot s)(u \cdot \chi) + K_0 (s \cdot \chi) \ . \tag{23}$$

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[6] As a consequence of spherical symmetry and u^a being orthogonal on the hypersurface Σ_U, we have $s \wedge \mathrm{d}s = 0$ – compare with (14). Thus the vectors s^a are orthogonal to the hypersurfaces $\{\Sigma_s\}$, foliating the spacetime. It can then be shown that $\hat{K}$ is the trace of the extrinsic curvature of the two-spheres $\mathfrak{B}$ due to their embedding in Σ_s. See appendix A for further details.

[7] Note however that inside the Killing horizon $\hat{\chi}^a$ is spacelike and r^a is timelike. Associating a spacelike unit vector with an observer is allowed here since there is no local limiting speed.

[8] We thank Ted Jacobson for pointing out reference [30] in this context, where the surface gravity at the Killing horizon is generally shown to be proportional to the expansion of a congruence of timelike geodesics. However, we emphasize that in the present paper, equation (23) holds everywhere in spacetime rather than just at the Killing horizon. We also note that using $(u \cdot \chi)_{\text{KH}} = -(s \cdot \chi)_{\text{KH}}$ at the Killing horizon, we have $\kappa_{\text{KH}} = -(u \cdot \chi)_{\text{KH}}\{(a \cdot s) + K_0\}_{\text{KH}}$ from (23). However, this relation and the the central result of [30], although similar in appearance, actually differ since the æther does not define a geodesic flow.

As mentioned in the introduction, the universal horizon occurs when the Killing vector becomes tangent to a Σ_U hypersurface. Therefore as one travels inwards from spatial infinity along a Σ_U hypersurface, the universal horizon is actually reached only as a limit. Hence our quantities defined as "on the universal horizon" refer to this limit, rather than some actual intersection of Σ_U and the universal horizon which is a hypersurface of constant r (again in Eddington-Finkelstein coordinates). On the universal horizon, $(u\cdot\chi)_{\rm UH}=0$ and $(s\cdot\chi)_{\rm UH}=|\chi|_{\rm UH}$ where $|\chi|_{\rm UH}$ is the magnitude of the Killing vector χ^a on the universal horizon. Therefore the surface gravity on the universal horizon is

$$\kappa_{\rm UH}=K_{0,{\rm UH}}|\chi|_{\rm UH}\ . \tag{24}$$

By spherical symmetry, the surface gravity is constant over any two-sphere, and thus on the universal horizon as well.

B. Equations of motion for the static, spherically symmetric case

We next study Einstein's equations and the æther equations of motion (10) by explicitly using the time translational and spherical symmetries of the problem, in addition to hypersurface orthogonality. To set up the Einstein's equations, we need to know the basis-expansion of the æther stress tensor and the Ricci tensor. By spherical symmetry, they take the following form

$$\mathcal{T}^{æ}_{ab}=\mathcal{T}^{æ}_{uu}u_au_b-2\mathcal{T}^{æ}_{us}u_{(a}s_{b)}+\mathcal{T}^{æ}_{ss}s_as_b+\frac{\hat{\mathcal{T}}^{æ}}{2}\hat{\mathfrak{g}}_{ab}\ , \tag{25}$$

and

$$\mathcal{R}_{ab}=\mathcal{R}_{uu}u_au_b-2\mathcal{R}_{us}u_{(a}s_{b)}+\mathcal{R}_{ss}s_as_b+\frac{\hat{\mathcal{R}}}{2}\hat{\mathfrak{g}}_{ab}\ , \tag{26}$$

respectively. The coefficients of $\mathcal{T}^{æ}_{ab}$ in (25) are computed from the general expression (11) for the stress tensor, using the results in appendix A. The corresponding coefficients for $\mathcal{R}_{ab}$, on the other hand, are computed from the defining equation $\left[\nabla_a,\nabla_b\right]X^c=-\mathcal{R}_{abd}{}^cX^d$ by choosing $X^a=u^a$ or s^a, and then contracting the expression again with u^a and/or s^a appropriately[9]. For our present purpose, it is sufficient to show the components $\mathcal{T}^{æ}_{us}$ and $\mathcal{R}_{us}$, which are given as follows

$$\begin{aligned}\mathcal{T}^{æ}_{us}&=c_{14}(\hat{K}(a\cdot s)+\nabla_u(a\cdot s))\ ,\\ \mathcal{R}_{us}&=(K_0-\hat{K}/2)\hat{k}-\nabla_s\hat{K}\ ,\end{aligned} \tag{27}$$

where $\hat{k}$ is the extrinsic curvature of the two-spheres $\mathfrak{B}$ due to their embedding in Σ_U (see appendix A)[10]. Comparing (25) and (26), we see that there are altogether four[11] non-trivial components of the Einstein's equations.

The æther's equations of motion (10), on the other hand, reduce to the scalar equation[12]

$$\begin{aligned}0=(s\cdot Æ)=c_{13}\nabla_sK_0+c_{13}(K_0-\hat{K}/2)\hat{k}\\ +c_2\nabla_sK-\mathcal{T}^{æ}_{us}\ ,\end{aligned} \tag{28}$$

as a consequence of hypersurface orthogonality and spherical symmetry. Quite naturally, the coupling constants that appear in (28) above are precisely those which are invariant under (17).

A well-known fact in general relativity is that the Bianchi identities (a consequence of general covariance) can be used to show that a subset of the Einstein equations are actually constraint equations. In Einstein-æther theory, which is also generally covariant, there are generalized Bianchi identities, and projections of these give rise to constraint equations as well. As explained in [31] (see also [22]), the generalized Bianchi identities for Einstein-æther theory are

$$\nabla^a\left[\mathcal{G}_{ab}-\mathcal{T}^{æ}_{ab}+u_aÆ_b\right]+Æ_a\nabla_bu^a=0\ , \tag{29}$$

and the corresponding constraint equations for the Σ_U hypersurfaces are [22, 31]

$$\left(\mathcal{G}_{ab}-\mathcal{T}^{æ}_{ab}\right)u^a-Æ_b=0\ . \tag{30}$$

Projecting (30) along s^b and using (27) and (28) the explicit form of the constraint equation, adapted to the foliation Σ_U, is

$$\begin{aligned}0=c_{123}\nabla_sK_0-(1-c_{13})(K_0-\hat{K}/2)\hat{k}\\ +(1+c_2)\nabla_s\hat{K}\ .\end{aligned} \tag{31}$$

On the other hand, subtracting the equation $c_{13}(\mathcal{R}_{us}-\mathcal{T}^{æ}_{us})=0$[13] from the æther's equation of motion (28), we get

$$c_{123}\nabla_sK=(1-c_{13})\mathcal{T}^{æ}_{us}\ . \tag{32}$$

Equations (31) and (32) are two independent linear combinations of the æther's equation of motion (28) and the Einstein equation $\mathcal{R}_{us}=\mathcal{T}^{æ}_{us}$. Therefore the two sets of equations are equivalent. In section IV, we study these

[9] We note that the coefficient $\hat{\mathcal{R}}$ cannot be constructed in this method. This is not an obstruction for most part since we do not need the explicit expression for $\hat{\mathcal{R}}$. In section IV we use the explicit coordinates to construct $\hat{\mathcal{R}}$.

[10] Intimately related to this is the fact that $\hat{k}/2$ is the coefficient of $\hat{\mathfrak{g}}_{ab}$ in the basis-expansion of ∇_as_b.

[11] We have one extra Einstein's equation, because we do not assume staticity here. Thus these equations can also be used to study time dependent but spherically symmetric perturbations around static solutions.

[12] After solving for the Lagrange multiplier there is no non-trivial projection of $Æ^a$ (12) along u^a.

[13] This equation, obviously, follows from the Einstein equation $(\mathcal{R}_{us}-\mathcal{T}^{æ}_{us})=0$.

equations along with the uu, ss and the spherical components of the Einstein's equations.

Finally, considering the projection of the æther equations of motion along the Killing vector χ^a, we arrive at the following equation of central importance in this paper (as it will be used to derive a Smarr formula)

$$\nabla_b \mathcal{F}^{ab} = 0, \qquad \mathcal{F}_{ab} = q(u_a s_b - s_a u_b) \ , \tag{33}$$

where the quantity q is given by

$$q = -\left(1 - \frac{c_{14}}{2}\right)(a \cdot s)(u \cdot \chi) + (1 - c_{13}) K_0 (s \cdot \chi) + \frac{c_{123}}{2} K (s \cdot \chi) \ . \tag{34}$$

The derivation of (33) closely follows the manipulations leading to the Smarr formula in [1]. We also found the algebraic relations of appendix A, especially those discussed in the last part of the appendix, useful in arriving at (33).

The similarity between (33) and the source free Maxwell's equations allows us to solve (33) exactly once we adopt a particular coordinate system. A useful choice is Eddington-Finklestein like coordinates (see equation (44) in section IV), as these coordinates are good everywhere in the spacetime. Because of staticity and spherical symmetry, in this particular coordinate system $\mathcal{F}_{ab}$ has only one non-trivial component, namely $\mathcal{F}_{vr}$. Therefore, solving (33) amounts to solving for the electrostatic field of a point charge at $r = 0$ in this particular geometry. By Gauss's law, we conclude $q \sim r^{-2}$. Using the asymptotic series solutions (49) we then fix the constant of proportionality and obtain

$$q = \left(1 - \frac{c_{14}}{2}\right) \frac{r_0}{2r^2} \ , \tag{35}$$

where r_0 is *the* single free parameter which defines the regular Einstein-æther black hole solutions. We ask the reader to refer to section IV for a more detailed discussion of how the parameter counting works and related issues. The parameter r_0, as we show in the following section, defines the mass of the Einstein-æther black holes. With the aid of equations (33), (34) and (35) and the fact that the Einstein-æther black holes constitute a one-parameter family of solutions, one can provide very simple derivations of the Smarr relation and the first law for Einstein-æther black hole mechanics, which we now turn to.

III. THE SMARR FORMULA AND FIRST LAW

In this section we give very simple derivations of the Smarr formula and the first law of (universal) horizon mechanics for a general static and spherically symmetric Einstein-æther black hole.

To begin with, we need a suitable definition of the mass of the Einstein-æther black holes. In this case, the ADM mass of a solution is identical to its Komar mass. From the general definition of the Komar mass of a stationary solution

$$M_{\text{ADM}} = -\frac{1}{4\pi G_{æ}} \int_{\mathcal{B}_\infty} \mathrm{d}\Sigma_{ab} \nabla^a \chi^b \ , \tag{36}$$

where $\mathrm{d}\Sigma_{ab}$ is the integration measure on any two-sphere $\mathcal{B}$, explicitly given by

$$\mathrm{d}\Sigma_{ab} = -u_{[a} s_{b]} \mathrm{d}A \ ,$$

with $\mathrm{d}A$ the differential area element on the two-sphere $\mathcal{B}$, and $\mathcal{B}_\infty$ is the sphere at infinity. Note the apperance of $G_{æ}$ (as opposed to G_{N}) in (36) – this ensures the correct weak field/slow-motion limit of the Einstein-æther theory, as we will see below.

We can further express the right hand side of (36) in terms of the surface gravity on the sphere at infinity following (22). Using the asymptotic expressions of (49) – in which a particular choice of coordinates have been made – in (23), we find

$$\begin{aligned} M_{\text{ADM}} &= -\frac{1}{4\pi G_{æ}} \int_{\mathcal{B}_\infty} \mathrm{d}A \, (a \cdot s)(u \cdot \chi) \\ &= \frac{1}{4\pi G_{æ}} \int_{\mathcal{B}_\infty} \mathrm{d}A \, (a \cdot s) = \frac{r_0}{2G_{æ}} \ . \end{aligned} \tag{37}$$

Our starting point of the derivation of the Smarr relation is equation (33). As already noted, structurally (33) resembles the source free Maxwell's equations with $\mathcal{F}_{ab}$ akin to a purely electrostatic field. In particular, by Gauss' law the flux of $\mathcal{F}_{ab}$ through the sphere at asymptotic infinity equals the flux through sphere at the universal horizon[14]. Performing the flux integrals and using the expression for the ADM mass (37), we arrive at the promised Smarr relation

$$\left(1 - \frac{c_{14}}{2}\right) M_{\text{ADM}} = \frac{q_{\text{UH}} A_{\text{UH}}}{4\pi G_{æ}} \ , \tag{38}$$

where A_{UH} is the area of the universal horizon, q_{UH} is the value of q (34) on the universal horizon,

$$q_{\text{UH}} = (1 - c_{13}) \kappa_{\text{UH}} + \frac{c_{123}}{2} K_{\text{UH}} |\chi|_{\text{UH}} \ , \tag{39}$$

and we have used the expression (24) for the surface gravity at the universal horizon, κ_{UH}, to write q_{UH} as above. Following [32, 33] and [18] we can furthermore introduce $M_{æ}$, given by

$$M_{æ} = \left(1 - \frac{c_{14}}{2}\right) M_{\text{ADM}} \ , \tag{40}$$

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[14] According to (A10), the flux of $\mathcal{F}_{ab}$ through any two-sphere, $\mathcal{B}_r$, at radius r, is the surface integral of q over $\mathcal{B}_r$. By spherical symmetry, this flux is the value of q at r, i.e., $q(r)$, times the area of the sphere $\mathcal{B}_r$ itself.

which is the total mass of an asymptotically flat solution defined in the asymptotic æther rest frame. Using (37) and the relation (9) between $G_{\rm N}$ and $G_{æ}$, we then have [18, 32, 33]

$$M_{æ} = \frac{r_0}{2G_{\rm N}} \qquad \Leftrightarrow \qquad M_{æ} G_{\rm N} = M_{\rm ADM} G_{æ} \ . \tag{41}$$

The above relation between $M_{æ}$ and r_0 ensures that one gets the correct Newtonian limit of the Einstein-æther theory far away from the sources [26, 32, 33]. In terms of the total mass, one also obtains a more natural presentation of the Smarr formula (38), namely

$$M_{æ} = \frac{q_{\rm UH} A_{\rm UH}}{4\pi G_{æ}}. \tag{42}$$

To obtain the first law for the Einstein-æther black holes, we need to consider a variation which takes us from a given regular solution to a distinct nearby regular solution. The key observation leading to the first law is that the regular Einstein-æther black hole solutions depend on the single dimensionful parameter r_0 introduced in (35). As a result, the location of the universal horizon, $r_{\rm UH}$, is related to r_0 through a relation of the form $r_{\rm UH} = \mu r_0$, where μ is a dimensionless quantity which can depend only on the coefficients c_2, c_{13} and c_{14}. From the proportionality between $r_{\rm UH}$ and r_0, we now have $q_{\rm UH} \sim r_0^{-1}$ from (35), while $A_{\rm UH} = 4\pi r_{\rm UH}^2 \sim r_0^2$, and hence $\delta q_{\rm UH} A_{\rm UH} = -\frac{1}{2} q_{\rm UH} \delta A_{\rm UH}$. Considering then a variation of the Smarr relation (42), the first law for Einstein-æther black holes follows in a straightforward manner [34]

$$\delta M_{æ} = \frac{q_{\rm UH} \delta A_{\rm UH}}{8\pi G_{æ}} \ . \tag{43}$$

Note that our derivation of the Smarr formula and the first law makes it manifest that at least when spherical symmetry is present, we can always have a “first law” applied to any sphere[15], where a variation of the total mass is proportional to q evaluated on the sphere, times the variation of the area of the sphere. Furthermore, since on dimensional grounds $q \sim r_0^{-1}$ as well as $\kappa \sim r_0^{-1}$, where q and κ are the values of the respective quantities evaluated on the sphere in question, we can always write such a first law in terms of the surface gravity on the sphere. However, the importance of the universal horizon rests on the fact that it is *the* causal boundary in the spacetime and therefore, only (43) should possibly have a thermodynamic interpretation.

IV. THE SOLUTIONS

In this section, our main goal is to present two exact, asymptotically flat, static, spherically symmetric, single parameter families of æther black hole solutions. These solutions provide additional evidence for the general result [21, 22] that all asymptotically flat, static and spherically symmetric æther black holes depend on a single parameter after imposing a regularity condition (to be discussed below). This single-parameter dependence, as already emphasized earlier, is crucial in our derivation of the first law. With our exact solutions, we furthermore verify the Smarr formula (42) and the first law (43). As we will also see, the interesting piece in $q_{\rm UH}$ (39), that depends on the trace of the extrinsic curvature at the universal horizon, is absent (for separate reasons) for both these special solutions.

In the following, we first adopt a convenient coordinate system to express the equations of motion (see the paragraph following (32)). Next, we present an asymptotic series solution of these equations, valid for large r and for arbitrary nonzero values of the couplings c_2, c_{13}, c_{123} and c_{14}. The asymptotic solution has already been obtained in previus work [21, 22], and our purpose of presenting it here is three-fold: First of all, the asymptotic analysis determines the asymptotic (and sometimes the exact) nature of various relevant functions in the problem (e.g. the functional form q in (35)) and allows us to obtain the ADM mass (37). Secondly, the asymptotic analysis reveals that the general æther black hole solution can depend on at most two parameters, thereby providing a natural route to the topic of regularity of the solutions. Finally, the asymptotic analysis for two special choices of coupling constants lead to the exact solutions mentioned above. We discuss these special solutions in subsections IV A and IV B, respectively.

To perform an asymptotic analysis of the equations, we set up an Eddington-Finklestein-like coordinate system which naturally respects the symmetries of the problem. With this choice of coordinates, the metric takes the form

$$\mathrm{d}s^2 = -e(r)\mathrm{d}v^2 + 2f(r)\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega_2^2 \ , \tag{44}$$

and the timelike Killing vector is

$$\chi^a = \{1, 0, 0, 0\} \ . \tag{45}$$

The æther field can be parametrized as

$$u^a = \{\alpha(r), \beta(r), 0, 0\} \, , \quad \beta(r) = \frac{e(r)\alpha(r)^2 - 1}{2f(r)\alpha(r)} \ , \tag{46}$$

where the relation between $\alpha(r)$ and $\beta(r)$ takes care of the unit norm constraint (7). Therefore, to perform the asymptotic analysis, we only need the asymptotic behaviour of the three functions $e(r)$, $f(r)$ and $\alpha(r)$, which are given as follows

$$\begin{gathered} e(r) = 1 + \mathcal{O}(r^{-1}), \quad f(r) = 1 + \mathcal{O}(r^{-1}) \ , \\ \alpha(r) = 1 + \mathcal{O}(r^{-1}) \ . \end{gathered} \tag{47}$$

The boundary conditions on the metric coefficients are such that the solution is asymptotically flat, while those

[15] This point has also been stressed in [21], where a first law for the æther black holes applied to the *spin-0 horizon* was obtained. For earlier work on the first law for an æther black hole at the Killing horizon using the Noether approach [35], see [33].

for the æther components are such that (the boundary condition on $\alpha(r)$ implies $\beta(r) = \boldsymbol{\mathcal{O}}(r^{-1})$ asymptotically)

$$\lim_{r\to\infty} u^a = \{1,0,0,0\} \ . \tag{48}$$

It is worthwhile to make the following comments at this stage: we set up the asymptotic analysis for arbitrary non-zero values of the coefficients c_2, c_{13} and c_{14} such that $c_{123} \neq 0$. Although the results we present do not show this explicitly, if one takes the limit $c_{123} \to 0$ of this solution, most of the coefficients in the $1/r$ expansions of $e(r)$, $f(r)$ and $\alpha(r)$ diverge. Therefore, we need to set up a separate asymptotic analysis when $c_{123} = 0$ (but $c_{14} \neq 0$), and we then discover the exact solution presented in section IV B. On the other hand, if one takes the $c_{14} \to 0$ of the general asymptotic solution by keeping c_{123} non-zero, the exact solution presented in section IV A is obtained[16].

We can now solve the Einstein's equations and the æther's equation of motion order by order in $1/r$. We have solved the relevant equations to $\boldsymbol{\mathcal{O}}(r^{-10})$ which determines the functions $e(r)$, $f(r)$ and $\alpha(r)$ to $\boldsymbol{\mathcal{O}}(r^{-8})$[17]. Not surprisingly, the asymptotic forms of these functions, as well as those which depend on them are quite cumbersome and do not convey too much information beyond that they can be found. We thus quote the relevant results only up to $\boldsymbol{\mathcal{O}}(r^{-2})$. To begin with, the metric components are

$$\begin{aligned} e(r) &= 1 - \frac{r_0}{r} + \boldsymbol{\mathcal{O}}(r^{-3}) \ , \\ f(r) &= 1 + \frac{c_{14} r_0^2}{16 r^2} + \boldsymbol{\mathcal{O}}(r^{-3}) \ , \end{aligned} \tag{49a}$$

while the components of the æther are

$$\begin{aligned} \alpha(r) &= 1 + \frac{r_0}{2r} + \frac{3r_0^2 - 8r_{æ}^2}{8r^2} + \boldsymbol{\mathcal{O}}(r^{-3}) \ , \\ \beta(r) &= -\frac{r_{æ}^2}{r^2} + \boldsymbol{\mathcal{O}}(r^{-3}) \ . \end{aligned} \tag{49b}$$

Our results agree perfectly with those in [22] (see equations 24−26), under the identification $F_1 \leftrightarrow -r_0$ and $A_2 \leftrightarrow (\frac{3}{8} r_0^2 - r_{æ}^2)$.

Given the results in (49a) and (49b), we can compute the series expansion for everything else; for example the asymptotic forms of $(a\cdot s)$, $(u\cdot\chi)$ and $(s\cdot\chi)$ are

$$\begin{aligned} (a\cdot s) &= \frac{r_0}{2r^2} + \boldsymbol{\mathcal{O}}(r^{-3}) \ , \\ (u\cdot\chi) &= -1 + \frac{r_0}{2r} + \frac{r_0^2}{8r^2} + \boldsymbol{\mathcal{O}}(r^{-3}) \ , \\ (s\cdot\chi) &= \frac{r_{æ}^2}{r^2} + \boldsymbol{\mathcal{O}}(r^{-3}) \ , \end{aligned} \tag{49c}$$

respectively. The various components of the extrinsic curvature K_{ab}, as well as its trace, are likewise

$$\begin{aligned} K_0 &= \frac{2r_{æ}^2}{r^3} + \boldsymbol{\mathcal{O}}(r^{-5}) \ , \\ \hat{K} &= -\frac{2r_{æ}^2}{r^3} + \boldsymbol{\mathcal{O}}(r^{-5}), \\ K &= \boldsymbol{\mathcal{O}}(r^{-5}) \ . \end{aligned} \tag{49d}$$

These results are useful to compute the ADM mass (37).

The solution, at this stage, depends on two parameters namely r_0 and $r_{æ}$. Among these, the length scale r_0 is akin to the "Schwarzschild radius" and is related to the total mass of the black hole according to (41). The parameter $r_{æ}$ is essentially the $\boldsymbol{\mathcal{O}}(r^{-2})$ coefficient of $\alpha(r)$, and is defined in this way for convenience. From the asymptotic analysis, it appears that $r_{æ}$ is a second free parameter on which the solutions depend. This, as we now explain following [21, 22], is not the case after all; rather $r_{æ}$ is related to r_0 upon requiring that the solutions are regular everywhere outside the universal horizon.

We begin our discussion with the observation that the Einstein-æther theory admits a "ground state" solution where the spacetime is four dimensional Minkowski and the æther is $\{1,0,0,0\}$ with respect to an observer in the preferred frame (called the æther rest frame). In [15], the authors consider perturbations around this background and show, in particular, that there is a spin-0[18] mode which propogates with a speed s_0 given by

$$s_0^2 = \frac{c_{123}(2 - c_{14})}{c_{14}(1 - c_{13})(2 + c_{13} + 3c_2)} \ , \tag{50}$$

with respect to the æther rest frame. Because of general covariance of the Einstein-æther theory, perturbations around an æther black hole background will also give rise to a spin-0 mode with a local speed given by (50). The spin-0 horizon is a hypersurface beyond which any outward moving excitation travelling with s_0 (or less) gets trapped. More precisely, the spin-0 horizon is hypersurface where the timelike Killing vector becomes null with respect to the "effective spin-0 metric" $\mathsf{g}^{(0)}_{ab} = \mathsf{g}_{ab} - (s_0^2 - 1) u_a u_b$ [21, 22]. Equivalently, we can also define the spin-0 horizon as the hypersurface where $(s\cdot\chi)^2 = s_0^2 (u\cdot\chi)^2$.

For generic values of the couplings c_2, c_{13} and c_{14}, which respect (8), the spin-0 speed s_0 is a non-zero finite quantity. Consequently, the spin-0 horizon can be located

[16] The scalar equation of motion (32) becomes trivially satisfied if both c_{123} and c_{14} vanish simultaneously, and consequently the structure and solution space of the field equations changes significantly.

[17] All the equations are second order ODEs in r and hence at $\boldsymbol{\mathcal{O}}(r^{-(n+2)})$ the functions only up to $\boldsymbol{\mathcal{O}}(r^{-n})$ contribute.

[18] The results of [15] show that when the æther is not hypersurface orthogonal and if no symmetry is assumed, there are additional spin-1 and spin-2 modes. In our case, the condition of hypersurface orthogonality of the æther will prevent any spin-1 mode from propagating in the black hole backgrounds we consider. Likewise, any spin-2 mode will be excluded because of spherical symmetry.

anywhere outside the universal horizon. However, for finite non-zero s_0 one can always apply the field redefinitions introduced in [36] to set $s_0 = 1$, thereby making the spin-0 horizon coincide with the Killing horizon [21, 22]. This extra condition therefore reduces the number of independent couplings from three to two. However, we should emphasize that this does not mean we are exploring a restricted coupling space. Rather, we are using an extra freedom in the theory (the field redefinitions) to conveniently choose (by imposing $s_0 = 1$) typical sets of couplings $\{c_2, c_{13}, c_{14}\}$ which label larger equivalent classes. In [21, 22], the authors use the above logic to effectively scan a smaller coupling space in their numerical constructions of asymptotically flat, static and spherically symmetric æther black holes. Those studies clearly prove that a solution for generic values of r_0 and $r_æ$ is singular, precisely at the location of the spin-0 horizon. However, once the solution is required to be regular everywhere outside the universal horizon, the extra constraint automatically makes $r_æ$ dependent on r_0, i.e., the former cannot be an extra parameter. Since the general asymptotically flat solution can at most depend on two parameters, the regularity condition reduce the parameter space so that we have a one-parameter family of solutions. In this manner, [21, 22] obtain a unique asymptotically flat æther black hole solution for a given value of the parameter r_0 (and a given set of couplings) by making the solution regular at the corresponding spin-0 horizon.

The field redefinitions of [36] (see also [21]) mentioned above also scale c_{123} and $(1-c_{13})$ in the same way while keep c_{14} invariant. It is then clear that such a transformation does not exist when either of c_{123} or c_{14} vanish or when $c_{13} = 1$ (we can rule out this last possibility owing to (8)). At the same time, according to (50), the spin-0 speed diverges as $c_{14} \to 0$, while it vanishes as $c_{123} \to 0$[19]. In the context of a black hole solution, when $c_{14} = 0$, the spin-0 horizon coincides with the universal horizon, since as noted in the introduction, the latter is *the* causal boundary for arbitrarily fast excitations. On the other hand, when $c_{123} = 0$, the spin-0 horizon in a black hole solution is pushed all the way to spatial infinity and so overlaps with the asymptotic boundary. Owing to the absence of the field redefinitions for these cases however, the spin-0 horizons *cannot* be mapped on to the metric horizon. Remarkably, there exists exact solutions for these special cases, where the spin-0 regularity condition can be illustrated in an explicit manner. We present these solutions in the following two subsections.

A. Exact solution for $c_{14} = 0$

When the coupling c_{14} is set to zero, the system admits an exact solution given by

$$e(r) = 1 - \frac{r_0}{r} - \frac{c_{13} r_æ^4}{r^4}, \qquad f(r) = 1 \,, \tag{51a}$$

and

$$\begin{aligned} \alpha(r) &= \frac{1}{e(r)}\left(-\frac{r_æ^2}{r^2} + \sqrt{e(r) + \frac{r_æ^4}{r^4}}\right) \,, \\ \beta(r) &= -\frac{r_æ^2}{r^2} \,. \end{aligned} \tag{51b}$$

From the explicit solution, we further work out

$$\begin{aligned} (s\cdot\chi) &= -\beta(r) = \frac{r_æ^2}{r^2} \,, \\ (u\cdot\chi) &= -\sqrt{e(r)+\beta(r)^2} = -\sqrt{1 - \frac{r_0}{r} + \frac{(1-c_{13}) r_æ^4}{r^4}} \,. \end{aligned} \tag{51c}$$

As mentioned in the beginning (8), we always assume $c_{13} < 1$.

We now investigate the locations of the Killing and universal horizons in this solution. By definition, the Killing horizon is located where $(\chi\cdot\chi) = 0$, or equivalently, at the largest root of $e(r) = 0$, and the universal horizon is located at the largest root of $(u\cdot\chi) = 0$. From (51a) and (51c) this amounts to solving two quartic equations. Rather than doing this directly, we extract most of the important properties of these roots from simple arguments.

We begin by noting that $e'(r) > 0$ everywhere owing to $c_{13} > 0$. Therefore, $e(r)$ is a monotonically increasing function and we have a single real root[20] at $r = r_{\text{KH}}$, which is the location of the Killing horizon. Secondly, from (51c) $(u\cdot\chi)^2 = e(r) + (s\cdot\chi)^2$ – therefore from the monotonicity of $e(r)$, even the largest root of $(u\cdot\chi)^2$ is necessarily located at some $r = r_{\text{UH}} < r_{\text{KH}}$. We also conclude that $(u\cdot\chi)^2 = 0$ has at most two real roots by noting that the function has a single minimum at $r = \sqrt[3]{4(1-c_{13}) r_æ^4/r_0}$. Furthermore, the two roots are distinct when $r_æ < r_æ^*$, they coincide when $r_æ = r_æ^*$, and there are no real roots when $r_æ > r_æ^*$, where

$$r_æ^* = \frac{r_0}{4}\left[\frac{27}{1-c_{13}}\right]^{1/4} \,.$$

The situation should be contrasted with the existence of an event horizon for the usual charged (Reissner-Nordstrom) black hole. However, there is an important

[19] We note that there are other limits of (50) when s_0 can vanish or diverge: $c_{14} \to 2$ (s_0 vanishes), $c_{13} \to 1$ (s_0 diverges) and $(2 + c_{13} + 3c_2) \to 0$ (s_0 diverges). However, they all violate the constraints (8), and therefore are excluded on physical grounds.

[20] Of course, $e(r) = 0$ must have at least two real roots, but the second real root must be negative by the above argument, and hence unphysical.

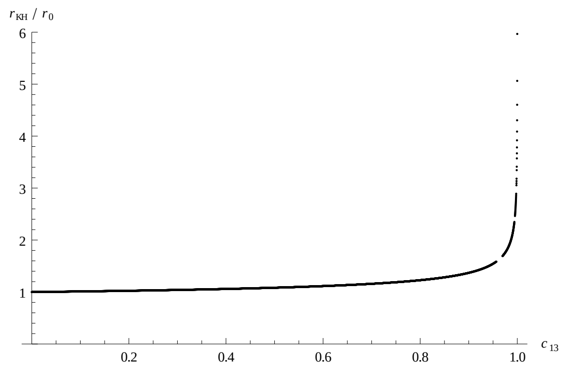

FIG. 1. r_{KH} in units of r_0 as a function of c_{13}.

difference between the present solution and the charged black hole solution in regards with the regularity of the solutions everywhere. In case of a charged black hole, the solutions are regular everywhere except at $r = 0$ and are physically allowed as long as the extremality condition is met. To examine the regularity of the present solution, we can see from the expressions of the curvature scalars for the present solution

$$\mathcal{R} = \frac{6c_{13}r_{\text{æ}}^4}{r^6}, \qquad \mathcal{R}_{ab}\mathcal{R}^{ab} = \frac{90c_{13}^2 r_{\text{æ}}^8}{r^{12}} ,$$

that the ambient spacetime is free of any curvature singularities except at $r = 0$. But, the æther field being a physical component of the theory, we also need to make sure that the solution for the æther is regular everywhere as well. A coordinate independent quantity associated with the æther, which can signal the existence of pathologies in the present solution is $(u\cdot\chi)$[21]. Indeed, when $r < r_{\text{UH}}$, $(u\cdot\chi)^2$ is negative based on our discussion above, and hence $(u\cdot\chi)$ is purely imaginary between $r = r_{\text{UH}}$ (the location of the universal horizon) and the smaller root of $(u\cdot\chi)^2 = 0$. In other words, for generic values of $r_{\text{æ}}$, the æther solution is irregular at the universal horizon. Naturally, this irregularity is prevented if $(u\cdot\chi)^2$ is never allowed to be negative. This regularity condition, along with the demand for the existence of at least one root[22] of $(u\cdot\chi)^2 = 0$, uniquely implies that the regular physical solution exists, iff

$$r_{\text{æ}} = r_{\text{æ}}^* = \frac{r_0}{4}\left[\frac{27}{1-c_{13}}\right]^{1/4} . \tag{52}$$

Thus, $r_{\text{æ}}$ is not an independent parameter. We have already argued that the spin-0 horizon for this solution overlaps with the universal horizon. Therefore, the regularity condition is indeed a regularity condition at the spin-0 horizon.

[21] $(s\cdot\chi)$ for this solution is another coordinate independent quantity. But it is nicely behaved everwhere except at $r = 0$, and is therefore incapable of signaling irregularities.

[22] As required by cosmic censorship – since we have superluminal propagation, the Killing horizon cannot save cosmic censorship.

Manifestly, the regular solution depends on a single parameter r_0. Here onwards, when we talk about this exact solution as well as about any quantity pertaining to it, the condition (52) will always be implied. The location of the universal horizon for this physical solution is very easy to find – it is a root of both $(u\cdot\chi)^2 = 0$ and $\mathrm{d}/\mathrm{d}r(u\cdot\chi)^2 = 0$, and is given by

$$r_{\text{UH}} = \frac{3r_0}{4} . \tag{53}$$

Quite interestingly, the result does not depend on the value of c_{13}. The location of the Killing horizon r_{KH} however does depend on c_{13}, but we did not attempt to find an analytical expression for it; instead, we solved for r_{KH} numerically and the result is presented in figure 1.

From its definition (27) $\mathcal{T}^{\text{æ}}_{us}$ vanishes when $c_{14} = 0$ and the æther stress tensor becomes diagonal, with the nontrivial components given by

$$\mathcal{T}^{\text{æ}}_{uu} = -\frac{3c_{13}r_{\text{æ}}^4}{r^6}, \quad \mathcal{T}^{\text{æ}}_{ss} = -\mathcal{T}^{\text{æ}}_{uu}, \quad \hat{\mathcal{T}}^{\text{æ}} = 4\mathcal{T}^{\text{æ}}_{uu} . \tag{54}$$

From the equation of motion (27) we then have $\nabla_s K = 0$, i.e., K is constant on a given hypersurface Σ_U. But K vanishes asymptotically for asymptotically flat spacetimes – this can be most readily seen from the fact that $u^a \sim \chi^a$ asymptotically, so that $K \sim \nabla\cdot\chi = 0$. Thus $K = 0$ on every hypersurface Σ_U, and therefore everywhere in spacetime. A related point is that, manifestly, the solution does not depend on the coupling c_2 in any way. According to [29], c_2 is the coupling of the K^2 term of the æther Lagrangian. Therefore we see that every reference to c_2 has been removed as K vanishes.

We are now in a position to derive a Smarr relation. Given the solution (51) we can compute the surface gravity on the universal horizon following (24) and this turns out to be

$$\kappa_{\text{UH}} = \frac{8}{9r_0(1-c_{13})} . \tag{55}$$

Using $A_{\text{UH}} = 4\pi r_{\text{UH}}^2 = 9\pi r_0^2/4$ as the area of the universal horizon, we can therefore derive a Smarr formula for the present solution[23]

$$M_{\text{æ}} = \frac{(1-c_{13})\kappa_{\text{UH}}A_{\text{UH}}}{4\pi G_{\text{æ}}} . \tag{56}$$

Varying the parameter r_0 we then also obtain a first law of black mechanics for the present solution explicitly

$$\delta M_{\text{æ}} = \frac{(1-c_{13})\kappa_{\text{UH}}\,\delta A_{\text{UH}}}{8\pi G_{\text{æ}}} . \tag{57}$$

Comparing (56) and (57) with the general Smarr relation (42) and the first law (43) respectively, we find perfect agreement. Naturally, the interesting piece proportional to $c_{123}K_{\text{UH}}$ is absent as $K = 0$ everywhere in this solution.

[23] Note: $M_{\text{æ}} = M_{\text{ADM}}$ and $G_{\text{æ}} = G_{\text{N}}$ for this solution since $c_{14} = 0$.

B. Exact solution for $c_{123} = 0$

When the coupling c_{123} is set to zero, we also have an exact solution given by

$$\begin{aligned} e(r) &= 1 - \frac{r_0}{r} - \frac{r_u(r_0 + r_u)}{r^2}, \qquad f(r) = 1\ , \\ \alpha(r) &= \left(1 + \frac{r_u}{r}\right)^{-1}, \qquad \beta(r) = -\frac{r_0 + 2r_u}{2r}\ , \end{aligned} \tag{58}$$

where r_u is a *positive* constant, given by

$$r_u = \left[\sqrt{\frac{2 - c_{14}}{2(1 - c_{13})}} - 1\right] \frac{r_0}{2}\ . \tag{59}$$

The requirement of positivity of r_u follows from demanding the function $\alpha(r)$ be regular everywhere. Consequently, this imposes the following bound on the couplings c_{13} and c_{14}

$$c_{14} \leqq 2c_{13}\ . \tag{60}$$

Therefore, for this special case we also need to ensure that c_{13} is non-negative in addition to $c_{13} < 1$ and $0 \leqq c_{14} < 2$ as we assume in general (8). The positivity of r_u criterion also rules out another possible solution for r_u

$$r_u = -\left[\sqrt{\frac{2 - c_{14}}{2(1 - c_{13})}} + 1\right] \frac{r_0}{2}\ ,$$

which solves all the equations of motion, but is manifestly negative for all values of c_{13} and c_{14}. The curvature invariants for this solution are

$$\mathcal{R} = 0, \qquad \mathcal{R}_{ab}\mathcal{R}^{ab} = \frac{4r_u^2(r_0 + r_u)^2}{r^8}\ ,$$

demonstrating that the geometry is free from any curvature singularity everywhere except at $r = 0$. Thus the solution is regular everywhere.

The reader might have spotted that the parameter $r_æ$ does not appear in this solution. Instead we have the parameter r_u, which is however not a free parameter. Rather, there are two possible choices for r_u as functions r_0 and this ambiguity is resolved (in the form of choosing r_u positive) by demanding that the solution be regular everwhere except at $r = 0$. One way to appreciate the difference between the present solution and all the solutions with $c_{123} \neq 0$, is to note that we need separate asymptotic analysis for the cases $c_{123} \neq 0$ and $c_{123} = 0$[24]. As it turns out, when $c_{123} \neq 0$, the equations of motion force the $\mathcal{O}(r^{-1})$ coefficient of $\alpha(r)$ to be $r_0/2$ (49b). When $c_{123} = 0$, this requirement no longer holds and we are left with a free parameter r_u. Furthermore, when $c_{123} \neq 0$, the parameter $r_æ$ appears as the $\mathcal{O}(r^{-2})$ coefficient of $\alpha(r)$ and stays as the second free parameter in the general asymptotic analysis. For the $c_{123} = 0$ case however, the $\mathcal{O}(r^{-2})$ coefficient is a function of r_u, and as the subsequent analysis reveals, the equations of motion eventually restrict r_u to take one of the two choices mentioned above. The requirement for regularity then removes one of the choices. We have already argued earlier that the spin-0 horizon for this solution "coincides" with the asymptotic boundary at infinity. We thus see that even with the correct boundary conditions imposed, there can be two different solutions corresponding to two different values of the paremeter r_u[25]. The actual regularity condition here comes in the form of choosing the correct value of r_u as given in (59). Manifestly, the regular solution depends on a single parameter r_0.

We next adress the issue of the locations of the Killing and universal horizons. From the explicit solution, we further work out

$$(s\cdot\chi) = -\beta(r) = \frac{r_0 + 2r_u}{2r}, \qquad (u\cdot\chi) = -1 + \frac{r_0}{2r}\ . \tag{61}$$

The roots of $e(r) = 0$ can be explicitly found in this case and they are $r = (r_0 + r_u)$ and $r = -r_u$ respectively. The second root is negative, i.e. unphysical, and the unique Killing horizon is at

$$r_{\text{KH}} = r_0 + r_u\ . \tag{62}$$

On the other hand, $(u\cdot\chi)$ has a unique root at $r = r_0/2$ which is therefore the location of the universal horizon in this solution

$$r_{\text{UH}} = \frac{r_0}{2}\ . \tag{63}$$

As in the case of the exact solution with $c_{14} = 0$, the location of the universal horizon does not depend on the coupling constants for the present case either, while the location of the Killing horizon does.

With $\mathcal{T}^{æ}_{us}$ vanishing according to (32) the stress tensor is also diagonal in this solution, with the non-trivial components given by

$$\begin{aligned} \mathcal{T}^{æ}_{uu} &= -\frac{2c_{13}(r_0 + 2r_u)^2 - c_{14}r_0^2}{8r^4}, \\ \mathcal{T}^{æ}_{ss} &= -\mathcal{T}^{æ}_{uu}, \qquad \hat{\mathcal{T}}^{æ} = 2\mathcal{T}^{æ}_{uu}\ . \end{aligned} \tag{64}$$

To derive a Smarr relation, we proceed as before and compute the surface gravity on the universal horizon from (24) using the present solution (58)

$$\kappa_{\text{UH}} = \frac{2 - c_{14}}{(1 - c_{13})r_0}\ . \tag{65}$$

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[24] There is no such need when the coupling in question is c_{14} as the general asymptotic solution (49) admits a smooth $c_{14} \to 0$ limit.

[25] The choice of the correct asymptotic boundary conditions with appropriate fall off conditions for the various fields should be treated as partly making the spin-0 horizon regular in this case. In particular, this seems to be the reason for no additional *continuous* parameter in the solution.

This time, using $A_{\rm UH} = 4\pi r_{\rm UH}^2 = \pi r_0^2$ as the area of the universal horizon, we arrive at the Smarr formula for the present solution

$$M_{æ} = \frac{(1-c_{13})\kappa_{\rm UH} A_{\rm UH}}{4\pi G_{æ}} \ . \tag{66}$$

Therefore, the first law of black mechanics for the present solution is

$$\delta M_{æ} = \frac{(1-c_{13})\kappa_{\rm UH}\, \delta A_{\rm UH}}{8\pi G_{æ}} \ . \tag{67}$$

We find perfect agreement, once again, upon comparing (66) and (67) with the general Smarr formula (42) and the first law (43) respectively. This time, the interesting piece proportional to $c_{123}K_{\rm UH}$ is absent for the obvious reason.

V. SUMMARY AND CONCLUSIONS

In this paper we have studied static, spherically symmetric, asymptotically flat black hole solutions of the Einstein-æther theory, a generally covariant modification of general relativity where a vector field, the æther, is forced to satisfy a unit normalization constraint. Since there is a preferred frame of reference defined by the æther, the solutions of the theory violate local Lorentz invariance and so matter fields do not necessarily have a finite local limiting speed. Even though at first sight the notion of a black hole seems impossible in such a situation, earlier work [21, 22] provided concrete evidence in support of the existence of a single parameter family of static, spherically symmetric, asymptotically flat black hole solutions. The causal boundary in question, called the universal horizon, is a hypersurface which traps excitations travelling even at infinite local speed.

In this work, we extend these earlier studies by demonstrating that a Smarr relation and a first law of black hole mechanics associated with the universal horizon can be found. We also provide analytical evidence for the existence of a universal horizon by constructing two exact solutions for special choices of the couplings of the Einstein-æther theory, and verify the Smarr formula and the first law with these special cases. Critical to our proof of the first law is the fact that the solutions depend on a single parameter, namely the mass of the solution [21, 22].

The Smarr formula (42) and the first law (43) suggest that in the present context, the quantity $q_{\rm UH}$ (39) plays the same role as surface gravity at the Killing horizon of a black hole in general relativity. Indeed, from (39) $q_{\rm UH}$ does include a contribution from the surface gravity at the universal horizon, but there is also an additional piece that depends on the extrinsic curvature of the preferred foliation as it asymptotes to the universal horizon. Based on the causal nature of the universal horizon, a thermodynamic interpretation of the first law (43) seems necessary, along the lines of [2].

Implementing such a thermodynamic interpretation in a concrete manner may, however, prove problematic. In deriving a Smarr formula and first law we have just dealt with classical, low energy physics. There is no need to worry about quantum effects in either the matter or gravitational sectors or the ultimately necessary ultraviolet completion of Einstein-æther theory. However, if one wants to provide an explicit and consistent thermodynamic interpretation of a horizon entropy one must introduce radiation from the causal horizon and now one runs into problematic issues. In the usual picture of Hawking radiation from a black hole, finite wavelength modes at infinity originates as infinitely short wavelengths near the horizon. If exact local Lorentz invariance holds then the ultraviolet near horizon modes are not an issue - the necessary microscopic physics is determined by the symmetry. This however, requires assumptions about Lorentz symmetry at untested energies and it is not conclusively proven that the microscopic structure of spacetime respects exact Lorentz invariance (although we have never seen a violation of Lorentz invariance [20, 37]). This is the so-called "transplanckian problem" of black hole physics [38]. Similarly, in the Einstein-æther case, modes that escape to infinity have infinitely high speeds/short wavelengths with respect to the æther frame near the universal horizon. Yet here we have neither local Lorentz symmetry nor a unique ultraviolet completion to the theory that would provide the necessary microscopic framework to unambiguously specify the near horizon mode behavior.

While there are only very few studies of the thermodynamics of universal horizons (c.f. the discussion on universal horizons in Einstein-æther theory in [39]), a number of authors have studied radiation from a Killing horizon when Lorentz symmetry is broken in the ultraviolet for the quantum field. The thermal nature of Hawking radiation from a Killing horizon has been shown to be fairly robust against ultraviolet modifications to Lorentz symmetry that yield subluminal propagation for quantum fields [40] but the behavior for superluminal fields is much less straightforward [41]. Hence the radiation spectrum from a universal horizon, which is not a Killing horizon and where modes are necessarily "superluminal", is completely unknown. Finally, if Einstein-æther is the low energy limit of a renormalizable theory such as Horava-Lifshitz gravity, then there are difficulties with assigning a holographic entropy to black holes, as this may interfere with the necessary ultraviolet behavior (c.f. the discussion in [42]). On the other hand, simple general thermodynamic arguments imply that there should be an entropy and our first law hints that the entropy is still a function of variables on a horizon area. These are puzzles that requires further investigation, which we leave for future work.

There is one more practical complication that arises when deriving thermodynamics as in Einstein-æther theory Čerenkov radiation generically occurs since æther-metric modes and matter fields all have different speeds.

In particular, a matter field mode propagating outwards from the universal horizon would emit spin-0 Čerenkov radiation, thereby modifying any thermal spectrum. Since no detailed examination of the radiation spectrum from a universal horizon has been made yet, it is unclear whether or not there is a Čerenkov-type component in addition to any presumed thermal flux. Note, however, that when $c_{14} \to 0$ the speed of the spin-0 mode goes to infinity, and so there is no possibility of emission of spin-0 Čerenkov radiation. On the other hand, when $c_{123} \to 0$, the speed of the spin-0 mode goes to zero, and so any spin-0 Čerenkov radiation would carry no energy away from the universal horizon. Thus, in both cases the reduction to a situation where the presumed temperature is proportional to the surface gravity is consistent with a limit where any Čerenkov radiation would naturally become unimportant.

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Appendix A: Some algebraic details

In this appendix, we provide some of the technical algebraic details behind our analysis. Of central importance to our analysis is the hypersurface orthogonality relation (14), as well as the assumptions of spherical symmetry and staticity. We already noted in section II (see the footnote around (20)) that as a consequence of the hypersurface orthogonality of u^a, spherical symmetry and staticity, the vector s^a is hypersurface orthogonal with respect to the foliations Σ_s of the spacetime, and satisfy $s \wedge \mathrm{d}s = 0$. Contracting these hypersurface orthogonality conditions with u^a and s^a respectively, we obtain

$$\nabla_{[a} u_{b]} = -(a \cdot s) u_{[a} s_{b]}, \qquad \nabla_{[a} s_{b]} = -K_0 u_{[a} s_{b]} \quad \text{(A1)}$$

Further contractions of these relations with any Killing vector η^a and with u^a and/or s^a lead to extremely useful scalar identities which play a major role in our analysis, and in particular, in the derivation of (33). One of our goals in this appendix is to briefly outline how these identities are obtained in a generalized and coherent fashion. To that end, we recall (as already noted in the paragagh following (18)) that by the spherical symmetry of the problem, any rank-two tensor can have components only along the bi-vectors $u_a u_b$, $u_{(a} s_{b)}$, $u_{[a} s_{b]}$, $s_a s_b$ and $\hat{\mathfrak{g}}_{ab}$. The most basic among these basis-expansions are those for $\nabla_a u_b$

$$\begin{aligned} \nabla_a u_b &= -(a \cdot s) u_a s_b + K_{ab}, \\ K_{ab} &= K_0 s_a s_b + \frac{\hat{K}}{2} \hat{\mathfrak{g}}_{ab} \end{aligned} \quad \text{(A2)}$$

and for $\nabla_a s_b$

$$\begin{aligned} \nabla_a s_b &= K_0 s_a u_b + K^{(s)}_{ab}, \\ K^{(s)}_{ab} &= -(a \cdot s) u_a u_b + \frac{\hat{k}}{2} \hat{\mathfrak{g}}_{ab} \end{aligned} \quad \text{(A3)}$$

where $K^{(s)}_{ab}$ is the extrinsic curvature of the hypersurfaces Σ_s due to their embedding in the spacetime, and $\hat{k}$ and $\hat{K}$ are the traces of the extrinsic curvatures of the two-spheres $\mathfrak{B}$ due to their embeddings in Σ_U and Σ_s respectively

$$\hat{k} = (1/2) \mathsf{g}^{ab} \pounds_s \hat{\mathfrak{g}}_{ab}, \qquad \hat{K} = (1/2) \mathsf{g}^{ab} \pounds_u \hat{\mathfrak{g}}_{ab} \quad \text{(A4)}$$

Our results in this paper make heavy uses of (A2) and (A3).

Now, consider an arbitrary vector of the following form

$$A_a = -f u_a + h s_a \quad \text{(A5)}$$

where f and h are arbitrary functions respecting the symmetries of the problem. A natural construct is the two-form

$$F_{ab} = \nabla_{[a} A_{b]} = Q\, u_{[a} s_{b]} \quad \text{(A6)}$$

where the second equality follows by the spherical symmetry of the problem, and the scalar Q is given by

$$Q = f(a \cdot s) + \nabla_s f - h K_0 - \nabla_u h \quad \text{(A7)}$$

Using the torsion-free condition of the covariant derivative, it can be shown that Q, like F_{ab}, is invariant under the "gauge transformations" $A_a \mapsto A'_a = A_a + \nabla_a \Lambda$.

Comparing (A1) with (A6), we see that the former relations are special cases of the latter. Contracting (A6) with a Killing vector η^a yields

$$\nabla_a (A \cdot \eta) = Q (s \cdot \eta) u_a - Q (u \cdot \eta) s_a \quad \text{(A8)}$$

Contracting (A8) further with u^a and s^a gives

$$\nabla_u (A \cdot \eta) = -Q (s \cdot \eta), \qquad \nabla_s (A \cdot \eta) = -Q (u \cdot \eta) \quad \text{(A9)}$$

We make ample use of the special cases of (A8) and (A9) for $A^a = u^a$ and $A^a = s^a$ throughout the analysis.

The flux of F_{ab} over any two-sphere $\mathfrak{B}$ is

$$\int_{\mathfrak{B}} \mathrm{d}\Sigma^{ab} F_{ab} = \int_{\mathfrak{B}} \mathrm{d}A\, Q \quad \text{(A10)}$$

Since F_{ab} is antisymmetric, we have a *kinematically* conserved current J^a through

$$\nabla_b F^{ab} = J^a \quad \text{(A11)}$$

We call J^a a kinematically conserved current since it satisfies the continuity equation $\nabla\cdot J = 0$ irrespective of any equations of motion.

By spherical symmetry, the current can only have components along u^a and s^a. These projections are given by

$$\begin{aligned}(u\cdot J) &= -(\hat{k}Q + \nabla_s Q) = -\vec{\nabla}\cdot[Q\vec{s}]\ ,\\ (s\cdot J) &= -(\hat{K}Q + \nabla_u Q) = -\underline{\nabla}\cdot[Q\underline{u}]\end{aligned} \tag{A12}$$

where $\vec{v}^a = \mathbb{P}^a{}_b v^b$ and $\underline{v}^a = \underline{\mathbb{P}}^a{}_b v^b$ are projections of any four vector v^a on the hypersurfaces Σ_U and Σ_s respectively, $\mathbb{P}^a{}_b = u^a u_b + \delta^a{}_b$ and $\underline{\mathbb{P}}^a{}_b = -s^a s_b + \delta^a{}_b$ are the corresponding projection tensors, and $\vec{\nabla}_a = \mathbb{P}_a{}^b\nabla_b$ and $\underline{\nabla}_a = \underline{\mathbb{P}}_a{}^b\nabla_b$ are the corresponding projections of the covariant derivatives.

The most important example of a kinematically conserved current, which is central in the derivation of (33), is that due to the timelike Killing vector χ^a. Choosing the vector $A_a = (1/2)\chi_a$ in (A5), so that from (A6) $F_{ab} = \nabla_a\chi_b$, we obtain $Q = -\kappa$. From the identity $\nabla_b\nabla^a\chi^b = \mathcal{R}^a{}_b\chi^b$, which forms the basis of the Smarr relation, we find that the kinematically conserved current is $J^a_\chi = \mathcal{R}^a{}_b\chi^b$ such that

$$(u\cdot J_\chi) = \vec{\nabla}\cdot[\kappa\vec{s}], \qquad (s\cdot J_\chi) = \underline{\nabla}\cdot[\kappa\underline{u}] \tag{A13}$$

We conclude this appendix by noting the following two identities

$$(u\cdot\eta)(\nabla\cdot v) = \vec{\nabla}\cdot\Big[\Big\{-(u\cdot v)(s\cdot\eta) + (s\cdot v)(u\cdot\eta)\Big\}\vec{s}\Big] \tag{A14}$$

and

$$(s\cdot\eta)(\nabla\cdot v) = \underline{\nabla}\cdot\Big[\Big\{-(u\cdot v)(s\cdot\eta) + (s\cdot v)(u\cdot\eta)\Big\}\underline{u}\Big] \tag{A15}$$

valid for any four-vector v^a. The identities (A14) and (A15) are indispensible for dealing with the contributions from the æther in the derivation of (33).

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