N=1 non-Abelian tensor multiplet in four dimensions

Hitoshi Nishino and Subhash Rajpoot

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N = 1 Non-Abelian Tensor Multiplet in Four Dimensions

Hitoshi Nishino[1] and Subhash Rajpoot[2]

Department of Physics & Astronomy
California State University
1250 Bellflower Boulevard
Long Beach, CA 90840

Abstract

We carry out the $N = 1$ supersymmetrization of a physical non-Abelian tensor with non-trivial consistent couplings in four dimensions. Our system has three multiplets: (i) The usual non-Abelian vector multiplet (VM) $(A_\mu{}^I, \lambda^I)$, (ii) A non-Abelian tensor multiplet (TM) $(B_{\mu\nu}{}^I, \chi^I, \varphi^I)$, and (iii) A compensator vector multiplet (CVM) $(C_\mu{}^I, \rho^I)$. All of these multiplets are in the adjoint representation of a non-Abelian group G. Unlike topological theory, all of our fields are propagating with kinetic terms. The $C_\mu{}^I$-field plays the role of a Stueckelberg compensator absorbed into the longitudinal component of $B_{\mu\nu}{}^I$. We give not only the component lagrangian, but also a corresponding superspace reformulation, reconfirming the total consistency of the system. The adjoint representation of the TM and CVM is further generalized to an arbitrary real representation of general $SO(N)$ gauge group. We also couple the globally $N = 1$ supersymmetric system to supergravity, as an additional non-trivial confirmation.

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Key Words: Non-Abelian Tensor, $N = 1$ Supersymmetry, Tensor Multiplet, Vector Field in Non-Trivial Representation, Consistency of Field Equations and Couplings.

[1] E-Mail: hnishino@csulb.edu
[2] E-Mail: rajpoot@csulb.edu

1. Introduction

Recently, the long-standing problem with non-Abelian tensors [1] has been solved by de Wit, Samtleben, and Nicolai [2][3]. The original motivation in [2] was to generalize the tensor and vector field interactions in manifestly $E_{6(+6)}$-covariant formulation of five-dimensional (5D) maximal supergravity by gauging non-Abelian sub-groups. In [3], this work was further related to M-theory [4] by confirming the representation assignments under the duality group of the gauge charges. The underlying hierarchies of these tensor and vector gauge fields are presented with the consistency of general gaugings.

The hierarchy in [2][3] has been further applied to the conformal supergravity in 6D [5]. In ref. [5], the 'minimal tensor hierarchy' as a special case of the more general hierarchy in [2][3] has been discussed. This hierarchy consists of $A_\mu{}^r$ and two-form gauge potentials $B_{\mu\nu}{}^I$, with two labels r and I. Also introduced is the 3-form gauge potentials $C_{\mu\nu\rho\, r}$ with the index $_r$ is dual to ^{r} of $A_\mu{}^r$. The field strengths of vector and two-form gauge potentials are defined by [5]

$$\mathcal{F}_{\mu\nu}{}^r \equiv 2\partial_{[\mu}A_{\nu]}{}^r + h_I{}^r B_{\mu\nu}{}^I \quad , \tag{1.1a}$$

$$\mathcal{H}_{\mu\nu\rho}{}^I \equiv 3D_{[\mu}B_{\nu\rho]}{}^I + 6d_{rs}{}^I A_{[\mu}{}^r \partial_\nu A_{\rho]}{}^s - 2f_{pq}{}^s d_{rs}{}^I A_{[\mu}{}^r A_\nu{}^p A_{\rho]}{}^q + g^{Ir} C_{\mu\nu\rho r} \quad . \tag{1.1b}$$

The prescription for tensor-vector system, which we will be based upon, is described with eq. (3.22) in [5]. To be more specific, we consider in the present paper the product of two identical gauge groups $G \times G$ [6], whose adjoint indices are respectively $r,\, s,\, \cdots$ and $r',\, s',\, \cdots$. Accordingly, we use the coefficients

$$f_{rs}{}^t = \mathrm{f}_{rs}{}^t \quad , \qquad f_{rs'}{}^{t'} = - f_{s'r}{}^{t'} = + \tfrac{1}{2} \mathrm{f}_{rs'}{}^{t'} \quad , \tag{1.2a}$$

$$d^t_{rs'} = d^t_{s'r} = - \tfrac{1}{2} \mathrm{f}_{rs'}{}^t \quad , \qquad h_s^{r'} = \delta_s^{r'} \quad , \tag{1.2b}$$

where $\mathrm{f}_{rs}{}^t$ is the structure constant of a non-Abelian gauge group. We use the same field content arising by this prescription.

Since the outstanding paper [5] gives the extensive details of how to get our system from [2][3][6], there is nothing new to explain, except for our notational preparation. In our notation, the field strengths of the B and C-fields are respectively G and H defined by

$$G_{\mu\nu\rho}{}^I \equiv +3D_{[\mu}B_{\nu\rho]}{}^I - 3f^{IJK} C_{[\mu}{}^J F_{\nu\rho]}{}^K \quad , \tag{1.3a}$$

$$H_{\mu\nu}{}^I \equiv +2D_{[\mu}C_{\nu]}{}^I + gB_{\mu\nu}{}^I \quad . \tag{1.3b}$$

The gauge transformations for B, C and A-fields are

$$\delta_\alpha (B_{\mu\nu}{}^I, C_\mu{}^I, A_\mu{}^I) \ = (-f^{IJK} \alpha^J B_{\mu\nu}{}^K, \ - f^{IJK} \alpha^J C_\mu{}^K, \ + D_\mu \alpha^I) \ , \tag{1.4a}$$

$$\delta_\beta (B_{\mu\nu}{}^I, C_\mu{}^I, A_\mu{}^I) \ = (+2 D_{[\mu} \beta_{\nu]}{}^I, \ - g \beta_\mu{}^I, \ 0) \ , \tag{1.4b}$$

$$\delta_\gamma (B_{\mu\nu}{}^I, C_\mu{}^I, A_\mu{}^I) \ = (-f^{IJK} F_{\mu\nu}{}^J \gamma^K, \ D_\mu \gamma^I, \ 0) \ . \tag{1.4c}$$

As (1.3b) or (1.4b) shows, $C_\mu{}^I$ is a vectorial Stueckelberg field, absorbed into the longitudinal component of $B_{\mu\nu}{}^I$. Due to the general hierarchy [2][3], all field strengths are invariant:

$$\delta_\alpha (G_{\mu\nu\rho}{}^I, \ H_{\mu\nu}{}^I, \ F_{\mu\nu}{}^I) = -f^{IJK} \alpha^J (G_{\mu\nu\rho}{}^K, \ H_{\mu\nu}{}^K, \ F_{\mu\nu}{}^K) \ , \tag{1.5a}$$

$$\delta_\beta (G_{\mu\nu\rho}{}^I, \ H_{\mu\nu}{}^I, \ F_{\mu\nu}{}^I) = 0 \ , \quad \delta_\gamma (G_{\mu\nu\rho}{}^I, \ H_{\mu\nu}{}^I, \ F_{\mu\nu}{}^I) = 0 \ . \tag{1.5b}$$

Since the hierarchy given in [2][3] guarantees the gauge invariance of all field strengths, the construction of purely bosonic lagrangian is straightforward. Consider the action $I_1 \equiv \int d^4 x \, g^2 \mathcal{L}_1$ [3] with

$$\mathcal{L}_1 \equiv - \tfrac{1}{12} (G_{\mu\nu\rho}{}^I)^2 - \tfrac{1}{4} (H_{\mu\nu}{}^I)^2 - \tfrac{1}{4} (F_{\mu\nu}{}^I)^2 \ . \tag{1.6}$$

The gauge invariances of all field strength also guarantee the consistency of the A, B and C-field equations, such as the divergence $D_\nu (\delta \mathcal{L}_1 / \delta B_{\mu\nu}{}^I) \doteq 0$.[4] Since we will do similar confirmation for supersymmetric system later, we skip the details for the purely bosonic system.

The purpose of our present paper is to supersymmetrize this system. The rest of our paper is organized as follows. In section 2, we give the component formulation of $N=1$ tensor multiplet (TM). In section 3, we give the superspace re-formulation of component result. In section 4, we give the generalization to non-adjoint representation of $G = SO(N)$ case. In section 5, we give the supergravity coupling to non-Abelian TM, as supporting evidence for the consistency of the global case. Section 6 is for concluding remarks. Appendix A is devoted to purely bosonic systems of non-Abelian tensors with much simpler structures than has been presented in arbitrary space-time dimensions with arbitrary signature. An example of tensor-vector duality $G = F^*$ in $D = 2+4$ dimensions, and its dimensional reduction (DR) into the self-dual YM $F = F^*$ in $D = 2+2$ is also presented.

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[3] The reason we need the factor g^2 in the action is due to the mass-dimension assignments of our fields.

[4] We use the symbol $\doteq$ for a field equation to be distinguished from an algebraic equation.

2. Component Formulation of N=1 TM

The supersymmetrization of the purely bosonic system (1.6) is rather straightforward, except for subtlety to be mentioned later. Our system has three multiplets: (i) A TM $(B_{\mu\nu}{}^I, \chi^I, \varphi^I)$, (ii) A compensating vector multiplet (CVM) $(C_\mu{}^I, \rho^I)$, and (iii) A Yang-Mills vector multiplet (YMVM) $(A_\mu{}^I, \lambda^I)$. Our total action $I \equiv \int d^4x\, \mathcal{L}$ has the lagrangian

$$\begin{aligned}\mathcal{L} = & -\tfrac{1}{12}(G_{\mu\nu\rho}{}^I)^2 + \tfrac{1}{2}(\overline{\chi}^I \not{\partial} \chi^I) - \tfrac{1}{2}(D_\mu \varphi^I)^2 - \tfrac{1}{2}g^2(\varphi^I)^2 - g(\overline{\chi}^I \rho^I) \\ & - \tfrac{1}{4}(H_{\mu\nu}{}^I)^2 + \tfrac{1}{2}(\overline{\rho}^I \not{\partial} \rho^I) - \tfrac{1}{4}(F_{\mu\nu}{}^I)^2 + \tfrac{1}{2}(\overline{\lambda}^I \not{\partial} \lambda^I) \\ & - \tfrac{1}{2}g f^{IJK}(\overline{\lambda}^I \chi^J)\varphi^K + \tfrac{1}{2}f^{IJK}(\overline{\lambda}^I \gamma^\mu \rho^J) D_\mu \varphi^K + \tfrac{1}{12}f^{IJK}(\overline{\lambda}^I \gamma^{\mu\nu\rho} \rho^J) G_{\mu\nu\rho}{}^K \\ & + \tfrac{1}{4}f^{IJK}(\overline{\rho}^I \gamma^{\mu\nu} \chi^J) F_{\mu\nu}{}^K - \tfrac{1}{4}f^{IJK}(\overline{\lambda}^I \gamma^{\mu\nu} \chi^J) H_{\mu\nu}{}^K - \tfrac{1}{2}f^{IJK} F_{\mu\nu}{}^I H^{\mu\nu\, J} \varphi^K \quad ,\end{aligned} \tag{2.1}$$

up to quartic-order terms $\mathcal{O}(\phi^4)$.

It is clear that the scalar φ^I has its mass g, while there is a mixture between χ^I and ρ^I, again with the asme mass g. As has been mentioned after (1.4), $C_\mu{}^I$ plays the role of Stueckelberg field [7], being absorbed into the longitudinal component of $B_{\mu\nu}{}^I$. Eventually, the kinetic term of the C-field becomes the mass term of $B_{\mu\nu}{}^I$. Accordingly, the degrees of freedom (DOF) for the massive TM fields are $B_{\mu\nu}{}^I$ (3), ρ^I (4) and $\varphi^I(1)$, up to the adjoint index I.

Our action I is invariant under global $N=1$ supersymmetry

$$\delta_Q B_{\mu\nu}{}^I = + (\overline{\epsilon}\gamma_{\mu\nu}\chi^I) - 2f^{IJK} C_{[\mu|}{}^J (\delta_Q A_{|\nu]}{}^K) \quad , \tag{2.2a}$$

$$\begin{aligned}\delta_Q \chi^I = & + \tfrac{1}{6}(\gamma^{\mu\nu\rho}\epsilon) G_{\mu\nu\rho}{}^I - (\gamma^\mu \epsilon) D_\mu \varphi^I \\ & + \tfrac{1}{2}f^{IJK}\Big[+ \epsilon(\overline{\lambda}^J \rho^K) - (\gamma_5\gamma^\mu\epsilon)(\overline{\lambda}^J \gamma_5 \gamma_\mu \rho^K) - (\gamma_5 \epsilon)(\overline{\lambda}^J \gamma_5 \rho^K)\Big] \quad ,\end{aligned} \tag{2.2b}$$

$$\delta_Q \varphi^I = + (\overline{\epsilon}\chi^I) \quad , \tag{2.2c}$$

$$\delta_Q C_\mu{}^I = + (\overline{\epsilon}\gamma_\mu \rho^I) + f^{IJK}(\overline{\epsilon}\gamma_\mu \lambda^J)\varphi^K \quad , \tag{2.2d}$$

$$\begin{aligned}\delta_Q \rho^I = & + \tfrac{1}{2}(\gamma^{\mu\nu}\epsilon) H_{\mu\nu}{}^I - g\epsilon\varphi^I - \tfrac{1}{2}f^{IJK}(\gamma^{\mu\nu}\epsilon) F_{\mu\nu}{}^J \varphi^K \\ & + \tfrac{1}{4}f^{IJK}\Big[+ \epsilon(\overline{\lambda}^J \chi^K) - (\gamma^\mu \epsilon)(\overline{\lambda}^J \gamma_\mu \chi^K) + \tfrac{1}{2}(\gamma^{\mu\nu}\epsilon)(\overline{\lambda}^J \gamma_{\mu\nu}\chi^K) \\ & \qquad - (\gamma_5 \gamma^\mu \epsilon)(\overline{\lambda}^J \gamma_5 \gamma_\mu \chi^K) - (\gamma_5 \epsilon)(\overline{\lambda}^J \gamma_5 \chi^K)\Big] \quad ,\end{aligned} \tag{2.2e}$$

$$\delta_Q A_\mu{}^I = + (\overline{\epsilon}\gamma_\mu \lambda^I) \quad , \tag{2.2f}$$

$$\delta_Q \lambda^I = + \tfrac{1}{2}(\gamma^{\mu\nu}\epsilon) F_{\mu\nu}{}^I + \tfrac{1}{2}f^{IJK}(\gamma_5 \epsilon)(\overline{\rho}^J \gamma_5 \chi^K) \quad , \tag{2.2g}$$

up to cubic terms $\mathcal{O}(\phi^3)$ in fields. The fermionic quadratic terms in (2.2b), (2.2e) and (2.2g) are fixed in superspace formulation, as will be explained later. In the *conventional* dimensions with all the bosonic (or fermionic) fields with 1 (or 3/2) mass dimensions,[5)] these terms lead to non-renormalizability. For example, the l.h.s. of (2.2b) has dimension 3/2, while its r.h.s. for the $\epsilon(\overline{\lambda}\gamma\rho)$ term has $(-1/2)+(3/2)+(3/2)=5/2$. In other words, there is an implicit coupling constant ℓ with the dimension of length in front of fermionic quadratic terms. This feature is also related to the existence of Pauli-terms which are non-renormalizable, already at a *globally* supersymmetric system. These features are similar to supergravity [8], even though our system so far has only *global* supersymmetry.

The usual non-Abelian gauge transformation δ_α and our tensorial gauge transformation δ_β, and δ_γ-transformation are exactly the same as (1.4), while all the fermionic fields are transforming only under δ_α, as the B and C-fields do, so that there arises no problem with the δ_β and δ_γ-invariances of the field strengths as in (1.5). These immediately lead to the invariances of our action $\delta_\alpha I=0$, $\delta_\beta I=0$ and $\delta_\gamma I=0$.

The Bianchi identities (BIds) for our field strengths G, H and F are:

$$D_{[\mu}G_{\nu\rho\sigma]}{}^I - \frac{3}{2} f^{IJK} F_{[\mu\nu}{}^J H_{\rho\sigma]}{}^K \equiv 0 \ , \tag{2.3a}$$

$$D_{[\mu}H_{\nu\rho]}{}^I - \frac{1}{3} g\, G_{\mu\nu\rho}{}^I \equiv 0 \ , \tag{2.3b}$$

$$D_{[\mu}F_{\nu\rho]}{}^I \equiv 0 \ . \tag{2.3c}$$

Relevantly, the non-trivial δ_Q-transformations of the field strengths are

$$\delta_Q G_{\mu\nu\rho}{}^I = +3(\overline{\epsilon}\gamma_{[\mu\nu}D_{\rho]}\chi^I) + 3f^{IJK}(\delta_Q A_{[\mu}{}^J)H_{\nu\rho]}{}^K - 3f^{IJK}(\delta_Q C_{[\mu}{}^J)F_{\nu\rho]}{}^K \ , \tag{2.4a}$$

$$\delta_Q H_{\mu\nu}{}^I = -2(\overline{\epsilon}\gamma_{[\mu}D_{\nu]}\rho^I) + g(\overline{\epsilon}\gamma_{\mu\nu}\chi^I) + 2f^{IJK}D_{[\mu|}\Big[(\delta_Q A_{|\nu]}{}^J)\varphi^K\Big] \ , \tag{2.4b}$$

$$\delta_Q F_{\mu\nu}{}^I = -2(\overline{\epsilon}\gamma_{[\mu}D_{\nu]}\lambda^I) \ , \tag{2.4c}$$

reflecting the presence of CS terms.

Note that our YMVM and CVM has *on-shell* DOF 2+2, while *off-shell* DOF 3+4, because we have *not* added the D-auxiliary field. On the other hand, our TM is in the *off-shell* formulation, because the total off-shell DOF is $4+4$, because the off-shell DOF of each field are $[(4-1)\cdot(4-2)]/2=3$ for $B_{\mu\nu}$, 4 for χ and 1 for φ.

5) Our bosonic (or fermionic) fields have dimensions 0 (or 1/2), in contrast to the conventional dimensions 1 (or 3/2).

The field equations for $\lambda^I,\ \chi^I,\ \rho^I,\ A_\mu{}^I,\ B_{\mu\nu}{}^I,\ \varphi^I$ and $C_\mu{}^I$ are respectively[6)]

$$+\not{D}\lambda^I - \tfrac{1}{2}gf^{IJK}\chi^J\varphi^K + \tfrac{1}{2}f^{IJK}(\gamma^\mu\rho^J)D_\mu\varphi^K - \tfrac{1}{4}f^{IJK}(\gamma^{\mu\nu}\chi^J)H_{\mu\nu}{}^K + \tfrac{1}{12}f^{IJK}(\gamma^{\mu\nu\rho}\rho^J)G_{\mu\nu\rho}{}^K \doteq 0 \ , \tag{2.5a}$$

$$+\not{D}\chi^I - g\rho^I + \tfrac{1}{2}gf^{IJK}\lambda^H\varphi^K - \tfrac{1}{4}f^{IJK}(\gamma^{\mu\nu}\lambda^J)H_{\mu\nu}{}^K + \tfrac{1}{4}f^{IJK}(\gamma^{\mu\nu}\rho^J)F_{\mu\nu}{}^K \doteq 0 \ , \tag{2.5b}$$

$$+\not{D}\rho^I - g\chi^I + \tfrac{1}{2}f^{IJK}(\gamma^\mu\lambda^J)D_\mu\varphi^K - \tfrac{1}{12}f^{IJK}(\gamma^{\mu\nu\rho}\lambda^J)G_{\mu\nu\rho}{}^K + \tfrac{1}{4}f^{IJK}(\gamma^{\mu\nu}\chi^J)F_{\mu\nu}{}^K \doteq 0 \ , \tag{2.5c}$$

$$+D_\nu F_\mu{}^{\nu\,I} + gf^{IJK}\varphi^J D_\mu\varphi^K + \tfrac{1}{2}gf^{IJK}(\overline{\lambda}^J\gamma_\mu\lambda^K) + f^{IJK}H_{\mu\nu}{}^J D^\nu\varphi^K - \tfrac{1}{2}f^{IJK}G_{\mu\rho\sigma}{}^J H^{\rho\sigma\,K} + \tfrac{1}{2}f^{IJK}(\overline{\chi}^J D_\mu\rho^K) + \tfrac{1}{2}f^{IJK}(\overline{\rho}^J D_\mu\chi^K) \doteq 0 \ , \tag{2.5d}$$

$$+D_\rho G^{\mu\nu\rho\,I} - gH^{\mu\nu\,I} - \tfrac{1}{2}f^{IJK}D_\rho(\overline{\lambda}^J\gamma^{\mu\nu\rho}\rho^K) + gf^{IJK}F^{\mu\nu\,J}\varphi^K - \tfrac{1}{2}gf^{IJK}(\overline{\lambda}^J\gamma^{\mu\nu}\chi^K) \doteq 0 \ , \tag{2.5e}$$

$$+D_\mu^2\varphi^I - gf^{IJK}(\overline{\lambda}^J\chi^K) - g^2\varphi^I - \tfrac{1}{2}f^{IJK}F_{\mu\nu}{}^J H^{\mu\nu\,K} \doteq 0 \ , \tag{2.5f}$$

$$+D_\nu H^{\mu\nu\,I} - \tfrac{1}{2}f^{IJK}F_{\rho\sigma}{}^J G^{\mu\rho\sigma\,K} - \tfrac{1}{2}f^{IJK}(\overline{\chi}^J D^\mu\lambda^K) - \tfrac{1}{2}f^{IJK}(\overline{\lambda}^J D^\mu\chi^K) + \tfrac{1}{2}gf^{IJK}(\overline{\lambda}^J\gamma^\mu\rho^K) - f^{IJK}F^{\mu\nu\,J}D_\nu\varphi^K \doteq 0 \ . \tag{2.5g}$$

In the derivation of these field equations, we have also used other field equations, in order to simply their final expressions, as a conventional prescription.

In the above computation, we do not attempt to fix the $\mathcal{O}(\phi^3)$-terms in field equations, or equivalently the fermionic $\mathcal{O}(\phi^4)$-terms in the lagrangian. There are several remarks about these terms. First, our system is non-renormalizable as supergravity theory [8], as has been mentioned after eq. (2.2). Accordingly, the (fermion)^2-terms in the fermionic transformations such as (2.2b), (2.2e) and (2.2g) are accompanied by the implicit constant ℓ carrying the dimension of (legnth). In supergravity theory [8], this is the gravitational coupling κ. In our lagrangian, all the quartic-fermion terms carry ℓ^2, so that the lagrangian has the mass dimension +4. Accordingly, a typical Noether-term has the structure $\ell\,\Psi^2\,\partial\,\Phi$, that

[6)] These equations are fixed up to $\mathcal{O}(\phi^3)$-terms, due to the quartic fermion terms in the lagrangian.

produces the terms of the form $\ell^2 \, \epsilon \, \Psi^3 \, \partial \, \Phi$ *via* $\delta_Q \, \Psi \approx \ell \, \epsilon \, \Psi^2$. Here Ψ (or Φ) is a general fermionic (or bosonic) fundamental field. These $\ell^2 \, \epsilon \, \Psi^3 \, \partial \, \Phi$-terms are cancelled by the variation of the fermionic quartic terms $\ell^2 \, \Psi^4$, *via* $\delta_Q \Psi \approx \epsilon \, \partial \Phi$. In other words, the structure of these cancellations associated with quartic-fermion terms is parallel to supergravity [8], since ℓ is analogous to κ.

However, in our peculiar system, this cancellation mechanism may be *not* simply parallel to conventional supergravity [8]. For example, there may be $\ell^2 \Psi^2 \Phi \partial \Psi$-type terms in the action, while $\ell^2 \epsilon \Psi^2 \Phi$-type terms in the transformation rules may exist, because both of them yield $\ell^2 \epsilon \Psi^3 \partial \Phi$-type terms, canceling each other in $\delta_Q I$. At the present time, we do not know, if such terms arise, because the $\ell^2 \epsilon \Psi^2 \Phi$-type terms in transformations are at $\mathcal{O}(\phi^3)$, while $\ell^2 \Psi^2 \Phi \partial \Psi$-type terms in the action are at $\mathcal{O}(\phi^4)$. In fact, even in the superspace re-confirmation in the next section, we have fixed only the $\mathcal{O}(\phi^1)$ and $\mathcal{O}(\phi^2)$-terms in the transformation rules for fermions, such as (3.2d), (3.2e) and (3.2f), but *not* cubic terms $\mathcal{O}(\phi^3)$. Our consistent principle in this paper is to fix only $\mathcal{O}(\phi^1)$, $\mathcal{O}(\phi^2)$ and $\mathcal{O}(\phi^3)$-terms in the lagrangian, $\mathcal{O}(\phi^1)$ and $\mathcal{O}(\phi^2)$-terms in all transformation rules, while $\mathcal{O}(\phi^1)$ and $\mathcal{O}(\phi^2)$-terms in all field equations. However, we try to fix *neither* $\mathcal{O}(\phi^4)$-terms in the lagrangian, *nor* $\mathcal{O}(\phi^3)$-term in all transformation rules, *nor* $\mathcal{O}(\phi^3)$-terms in all field equations. We do *not* specify each field meant by ϕ is fermionic or bosonic in this paper, either.

Second, as an additional difference from supergravity [8], the fermionic quartic terms do *not* contain any gravitino. This implies that we can not use the conventional technique of 'supercovariantizing' fermionic field equations. Due to this feature, as well as the above-mentioned possible non-purely-fermionnic $\ell^2 \Psi^2 \Phi \partial \Psi$-type terms, the quartic terms $\mathcal{O}(\phi^4)$ at $\mathcal{O}(\ell^2)$ will be more involved than conventional supergravity [8] which are tedious. For these reasons, we do not attempt to fix them in this paper.

Third, according to the past experience in supergravity theory [8], it is understood that the series in terms of κ in a lagrangian will stop at a finite order, such as the quartic-fermion terms at $\mathcal{O}(\kappa^2)$ [8]. However, at the present time, we do *not* know, whether this is also the case with our globally supersymmetric system. This is because of the above-mentioned differences of our system from supergravity [8], and therefore the analogy with supergravity might be not valid in our system. Fourth, since we have already fixed the cubic terms in the lagrangian, they seem sufficient for non-trivial and consistent couplings as a supersymmetric system.

3. Superspace Reformulation of N=1 TM

As a reconfirmation of the total consistency of our system, we re-formulate our theory in terms of superspace language. Our basic superspace BIds for the superfield strengths ${F_{AB}}^I$, ${G_{ABC}}^I$ and ${H_{AB}}^I$ are[7)]

$$+\frac{1}{6}\nabla_{[A}{G_{BCD)}}^I - \frac{1}{4}{T_{[AB|}}^E G_{E|CD)} - \frac{1}{4} f^{IJK} {F_{[AB}}^J {H_{CD)}}^K \equiv 0 \quad , \qquad (3.1a)$$

$$+\frac{1}{2}\nabla_{[A}{H_{BC)}}^I - \frac{1}{2}{T_{[AB|}}^D {H_{D|C)}}^I - g\,{G_{ABC}}^I \equiv 0 \quad , \qquad (3.1b)$$

$$+\frac{1}{2}\nabla_{[A}{F_{BC)}}^I - \frac{1}{2}{T_{[AB|}}^D {F_{D|C)}}^I \equiv 0 \quad . \qquad (3.1b)$$

These BIds are the superspace generalizations of the component BIds (2.3), with the super-torsion terms added for local Lorentz indices, as usual in superspace.

Our basic superspace constraints at mass dimensions $0 \le d \le 1$ are

$${T_{\alpha\beta}}^c = +2(\gamma^c)_{\alpha\beta} \quad , \qquad {G_{\alpha\beta c}}^I = +2(\gamma_c)_{\alpha\beta}\,\varphi^I \quad , \qquad (3.2a)$$

$${G_{\alpha bc}}^I = -(\gamma_{bc}\chi^I)_\alpha \quad , \qquad {H_{\alpha b}}^I = -(\gamma_b\rho^I)_\alpha - f^{IJK}(\gamma_b\lambda^J)_\alpha\,\varphi^K \quad , \qquad (3.2b)$$

$${F_{\alpha b}}^I = -(\gamma_b\lambda^I)_\alpha \quad , \qquad \nabla_\alpha\varphi^I = -{\chi_\alpha}^I \quad , \qquad (3.2c)$$

$$\begin{aligned} \nabla_\alpha{\chi_\beta}^I = & -\frac{1}{6}(\gamma^{cde})_{\alpha\beta}{G_{cde}}^I - (\gamma^c)_{\alpha\beta}\nabla_c\varphi^I \\ & -\tfrac{1}{2}f^{IJK}\Big[+C_{\alpha\beta}(\overline{\lambda}^J\rho^K) - (\gamma_5\gamma^c)_{\alpha\beta}(\overline{\lambda}^J\gamma_5\gamma_c\rho^K) - (\gamma_5)_{\alpha\beta}(\overline{\lambda}^J\gamma_5\rho^K)\Big] \quad , \end{aligned} \qquad (3.2d)$$

$$\begin{aligned} \nabla_\alpha{\rho_\beta}^I = & +\tfrac{1}{2}(\gamma^{cd})_{\alpha\beta}{H_{cd}}^I + g\,C_{\alpha\beta}\,\varphi^I - \tfrac{1}{2}f^{IJK}(\gamma^{cd})_{\alpha\beta}{F_{cd}}^J\varphi^K \\ & -\tfrac{1}{4}f^{IJK}\Big[+C_{\alpha\beta}(\overline{\lambda}^J\chi^K) + (\gamma^c)_{\alpha\beta}(\overline{\lambda}^J\gamma_c\chi^K) - \tfrac{1}{2}(\gamma^{cd})_{\alpha\beta}(\overline{\lambda}^J\gamma_{cd}\chi^K) \\ & \qquad - (\gamma_5\gamma^c)_{\alpha\beta}(\overline{\lambda}^J\gamma_5\gamma_c\chi^K) - (\gamma_5)_{\alpha\beta}(\overline{\lambda}^J\gamma_5\chi^K)\Big] \quad , \end{aligned} \qquad (3.2e)$$

$$\nabla_\alpha{\lambda_\beta}^I = +\tfrac{1}{2}(\gamma^{cd})_{\alpha\beta}{F_{cd}}^I - \tfrac{1}{2}(\gamma_5)_{\alpha\beta}\,f^{IJK}(\overline{\rho}^J\gamma_5\chi^K) \quad . \qquad (3.2f)$$

All other components, such as ${G_{\alpha\beta\gamma}}^I$, ${T_{\alpha\beta}}^\gamma$, ${T_{ab}}^c$, ${H_{\alpha\beta}}^I$ *etc.* at $d \le 1$ are zero. Note that (fermion)^2-terms in (3.2d) through (3.2f) have been determined in superspace by satisfying BIds at $d = 1$. Note that these results are valid up to $\mathcal{O}(\phi^3)$-terms, which we do not attempt to fix these terms in this paper. However, all the $\mathcal{O}(\phi^2)$-terms have been included, as has been also mentioned at the end of last section.

[7)] Only in this superspace section, we use the indices $A = (a,\alpha)$, $B = (b,\beta)$, $\cdots$ for superspace coordinates, where $a,\ b,\ \cdots = 0,1,2,3$ (or $\alpha,\ \beta,\ \cdots = 1,2,3,4$) are for bosonic (or fermionic) coordinates. In superspace, the (anti)symmetrization convention, *e.g.,* $X_{[AB)} \equiv X_{AB} - (-1)^{AB}X_{BA}$ is different from our component notation.

There are also useful relationships obtained from $d = +3/2$ BIds:

$$\nabla_\alpha G_{bcd} = -\tfrac{1}{2}(\gamma_{[bc}\nabla_{d]}\chi^I)_\alpha - \tfrac{1}{2}f^{IJK}(\gamma_{[b|}\lambda^J)_\alpha H_{|cd]}{}^K + \tfrac{1}{2}f^{IJK}(\gamma_{[b|}\rho^J)_\alpha F_{|cd]}{}^K \quad , \qquad (3.3a)$$

$$\nabla_\alpha H_{bc}{}^I = +(\gamma_{[b}\nabla_{c]}\rho^I)_\alpha - g(\gamma_{bc}\chi^I)_\alpha - f^{IJK}\nabla_{[b}\Big[(\gamma_{c]}\lambda^J)_\alpha\varphi^K\Big] \quad , \qquad (3.3b)$$

$$\nabla_\alpha F_{bc}{}^I = +(\gamma_{[b}\nabla_{c]}\lambda^I)_\alpha \quad , \qquad (3.3c)$$

up to $\mathcal{O}(\phi^3)$-terms. Note the existence of the $\mathcal{O}(\phi^2)$-terms in (3.3a) and (3.3b), reflecting the corresponding terms in the component results (2.4a) and (2.4b).

As usual, the satisfaction of all the BIds in superspace by the constraints (3.2) and (3.3) is straightforward to perform, from the dimension $d = 0$ to $d = 3/2$, as usual. In particular, the $(\text{Fermions})^2$-terms in (3.2d) through (3.2f) are the results of our superspace re-formulation.

The fermionic λ and ρ-field equations (2.5a) and (2.5c) are obtained as usual by computing $\{\nabla_\alpha, \nabla_\beta\}\,\lambda^{\beta I}$ and $\{\nabla_\alpha, \nabla_\beta\}\,\rho^{\beta I}$, while the χ-field equation is shown to be consistent with the component lagrangian. As has been mentioned, since the TM is *off-shell* multiplet, we can *not* get the χ-field equation (2.5b) in superspace directly, but we can show that (2.5b) is consistent in superspace. The bosonic field equations (2.5d) - (2.5g) are obtained by applying another fermionic derivative on the fermionic field equations (2.5a) - (2.5c).

4. Generalization to Non-Adjoint Representations of G = SO(N)

We have so far considered the case for the TM and CVM both carrying only the adjoint representation. We can generalize this result to other more general representations, such as an arbitrary real representation of a $SO(N)$-type gauge group.[8)]

To be more specific, we consider the TM $(B_{\mu\nu}{}^i, \chi^i, \varphi^i)$ and the CVM $(C_\mu{}^i, \rho^i)$, where the index i is for any real representation of a gauge group $G = SO(N)$. Let $(T^I)^{jk}$ be the generator of the group G. Then our action $I' \equiv \int d^4x\, \mathcal{L}'$ has the lagrangian[9)]

$$\begin{aligned}\mathcal{L}' = & -\tfrac{1}{12}(G_{\mu\nu\rho}{}^i)^2 + \tfrac{1}{2}(\overline{\chi}^i \not{D}\chi^i) - \tfrac{1}{2}(D_\mu\varphi^i)^2 - \tfrac{1}{2}g^2(\varphi^i)^2 - g(\overline{\rho}^i\chi^i) \\ & - \tfrac{1}{4}(H_{\mu\nu}{}^i)^2 + \tfrac{1}{2}(\overline{\rho}^i\not{D}\rho^i) - \tfrac{1}{4}(F_{\mu\nu}{}^I)^2 + \tfrac{1}{2}(\overline{\lambda}^I\not{D}\lambda^I) \\ & - \tfrac{1}{2}g(T^I)^{jk}(\overline{\lambda}^I\chi^j)\,\varphi^k + \tfrac{1}{2}(T^I)^{jk}(\overline{\lambda}^I\gamma^\mu\rho^j)D_\mu\varphi^k + \tfrac{1}{12}(T^I)^{jk}(\overline{\lambda}^I\gamma^{\mu\nu\rho}\rho^j)\,G_{\mu\nu\rho}{}^k \\ & + \tfrac{1}{4}(T^I)^{jk}(\overline{\rho}^j\gamma^{\mu\nu}\chi^k)F_{\mu\nu}{}^I - \tfrac{1}{4}(T^I)^{jk}(\overline{\lambda}^I\gamma^{\mu\nu}\chi^j)H_{\mu\nu}{}^k - \tfrac{1}{2}(T^I)^{jk}F_{\mu\nu}{}^I H^{\mu\nu\,j}\varphi^k \quad , \qquad (4.1)\end{aligned}$$

8) We can also consider the complex representation for $SU(N)$-type gauge groups.

9) Since the metric for the gauge group $G = SO(N)$ is positive definite, we do *not* distinguish the upper or lower indices for $i,\ j,\ \cdots = 1,\ 2,\ \cdots,\ \dim R$, where $\mathbf{R}$ is a real representation of G.

up to quartic terms $\mathcal{O}(\phi^4)$. Our action I' is invariant under global $N=1$ supersymmetry

$$\delta_Q B_{\mu\nu}{}^i = +(\overline{\epsilon}\gamma_{\mu\nu}\chi^i) - 2(T^J)^{ik} C_{[\mu|}{}^k (\delta_Q A_{|\nu]}{}^J) \ , \tag{4.2a}$$

$$\delta_Q \chi^i = +\tfrac{1}{6}(\gamma^{\mu\nu\rho}\epsilon) G_{\mu\nu\rho}{}^i - (\gamma^\mu\epsilon) D_\mu \varphi^i \\ - \tfrac{1}{2}(T^J)^{ik}\Big[+\epsilon(\overline{\lambda}^J\chi^k) - (\gamma_5\gamma^\mu\epsilon)(\overline{\lambda}^J\gamma_5\gamma_\mu\chi^k) - (\gamma_5\epsilon)(\overline{\lambda}^J\gamma_5\chi^k)\Big] \ , \tag{4.2b}$$

$$\delta_Q \varphi^i = +(\overline{\epsilon}\chi^i) \ , \tag{4.2c}$$

$$\delta_Q C_\mu{}^i = +(\overline{\epsilon}\gamma_\mu\rho^i) - (T^J)^{ik}(\overline{\epsilon}\gamma_\mu\lambda^J)\varphi^k \ , \tag{4.2d}$$

$$\delta_Q \rho^i = +\tfrac{1}{2}(\gamma^{\mu\nu}\epsilon) H_{\mu\nu}{}^i - g\epsilon\varphi^i + \tfrac{1}{2}(T^J)^{ik}(\gamma^{\mu\nu}\epsilon) F_{\mu\nu}{}^J\varphi^k \\ - \tfrac{1}{4}(T^J)^{ik}\Big[+\epsilon(\overline{\lambda}^J\chi^k) - (\gamma^\mu\epsilon)(\overline{\lambda}^J\gamma_\mu\chi^k) + \tfrac{1}{2}(\gamma^{\mu\nu}\epsilon)(\overline{\lambda}^J\gamma_{\mu\nu}\chi^k) \\ - (\gamma_5\gamma^\mu\epsilon)(\overline{\lambda}^J\gamma_5\gamma_\mu\chi^k) - (\gamma_5\epsilon)(\overline{\lambda}^J\gamma_5\chi^k)\Big] \ , \tag{4.2e}$$

$$\delta_Q A_\mu{}^I = +(\overline{\epsilon}\gamma_\mu\lambda^I) \ , \tag{4.2f}$$

$$\delta_Q \lambda^I = +\tfrac{1}{2}(\gamma^{\mu\nu}\epsilon) F_{\mu\nu}{}^I - \tfrac{1}{2}(T^I)^{jk}(\gamma_5\epsilon)(\overline{\rho}^j\gamma_5\chi^k) \ . \tag{4.2g}$$

The essential point is that all the cubic-order terms contain one component field $A_\mu{}^I$ or λ^I with the index I, and the remaining two component fields out of either TM or CVM carry the indices j and k. So the cancellation structure is parallel to the adjoint-representation case, *e.g.*, with the structure constant f^{IJK} replaced by the matrix $-(T^J)^{ik}$ in $D_\mu\chi^I = \partial_\mu\chi^I + gf^{IJK}A_\mu{}^J\chi^K \implies D_\mu\chi^i = \partial_\mu\chi^i - g(T^J)^{ik}A_\mu{}^J\chi^k$. Accordingly, the Stueckelberg mechanism [7] works in a parallel fashion, because $C_\mu{}^i$ is absorbed into the longitudinal component of $B_{\mu\nu}{}^i$, both in the same representation $\mathbf{R}$.

5. Coupling to $N=1$ Supergravity

Once we have established the $N=1$ global system of non-Abelian TM with non-trivial and consistent interactions, the next natural step is to make $N=1$ supersymmetry *local*, coupling to $N=1$ supergravity.

This coupling is rather straightforward, because most of the basic structure is parallel to the usual matter coupling to supergravity, except for certain couplings to be mentioned later. Our result for the lagrangian $\widetilde{\mathcal{L}}$ of our action is $\widetilde{I} \equiv \int d^4x\,\widetilde{\mathcal{L}}$:

$$e^{-1}\widetilde{\mathcal{L}} = -\tfrac{1}{4}R(\omega) - \Big[\overline{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu(\omega)\psi_\rho\Big] - \tfrac{1}{12}(G_{\mu\nu\rho}{}^I)^2 + \tfrac{1}{2}[\overline{\chi}^I \not{D}(\omega)\chi^I] - \tfrac{1}{2}(D_\mu\varphi^I)^2 \\ - \tfrac{1}{4}(F_{\mu\nu}{}^I)^2 + \tfrac{1}{2}[\overline{\lambda}^I\not{D}\lambda^I] - \tfrac{1}{4}(H_{\mu\nu}{}^I)^2 + \tfrac{1}{2}[\overline{\rho}^I\not{D}(\omega)\rho^I] - g(\overline{\chi}^I\rho^I) - \tfrac{1}{2}g^2(\varphi^I)^2$$

$$
\begin{aligned}
&- \tfrac{1}{2} g f^{IJK} (\overline{\lambda}^I \chi^J) \varphi^K - \tfrac{1}{4} f^{IJK} (\overline{\lambda}^I \gamma^{\mu\nu} \chi^J) H_{\mu\nu}{}^K \\
&+ \tfrac{1}{12} f^{IJK} (\overline{\lambda}^I \gamma^{\mu\nu\rho} \rho^J) G_{\mu\nu\rho}{}^K + \tfrac{1}{4} f^{IJK} (\overline{\rho}^I \gamma^\mu \chi^J) F_{\mu\nu}{}^K \\
&- \tfrac{1}{2} f^{IJK} F_{\mu\nu}{}^I H^{\mu\nu\, J} \varphi^K + \tfrac{1}{2} f^{IJK} (\overline{\lambda}^I \gamma^{\mu\nu} \rho^J) D_\mu \varphi^K \\
&+ (\overline{\psi}_\mu \gamma^\nu \gamma^\mu \chi^I) D_\nu \varphi^I + \tfrac{1}{6} (\overline{\psi}_\mu \gamma^{\rho\sigma\tau} \gamma^\mu \chi^I) G_{\rho\sigma\tau}{}^I \\
&- \tfrac{1}{2} (\overline{\psi}_\mu \gamma^{\rho\sigma} \gamma^\mu \lambda^I) F_{\rho\sigma}{}^I - \tfrac{1}{2} (\overline{\psi}_\mu \gamma^{\rho\sigma} \gamma^\mu \rho^I) H_{\rho\sigma}{}^I - g (\overline{\psi}_\mu \gamma^\mu \rho^I) \varphi^I \quad ,
\end{aligned} \tag{5.1}
$$

up to $\mathcal{O}(\phi^4)$ terms.

Our action $\widetilde{I}$ is now invariant under local $N = 1$ supersymmetry

$$\delta_Q e_\mu{}^m = -2(\overline{\epsilon} \gamma^m \psi_\mu) \quad , \tag{5.2a}$$

$$\delta_Q \psi_\mu = + D_\mu(\widehat{\omega}) \epsilon - \tfrac{1}{6} (\gamma_\mu{}^{\rho\sigma\tau} \epsilon) \widehat{G}_{\rho\sigma\tau}{}^I \varphi^I \quad , \tag{5.2b}$$

$$\delta_Q B_{\mu\nu}{}^I = + (\overline{\epsilon} \gamma_{\mu\nu} \chi^I) - 2 f^{IJK} C_{[\mu|}{}^J (\delta_Q A_{|\nu]}{}^K) - 4 (\overline{\epsilon} \gamma_{[\mu} \psi_{\nu]}) \varphi^I \quad , \tag{5.2c}$$

$$
\begin{aligned}
\delta_Q \chi^I = &+ \tfrac{1}{6} (\gamma^{\mu\nu\rho} \epsilon) \widehat{G}_{\mu\nu\rho}{}^I - (\gamma^\mu \epsilon) \widehat{D}_\mu \varphi^I \\
&+ \tfrac{1}{2} f^{IJK} \Big[+ \epsilon (\overline{\lambda}^J \rho^K) - (\gamma_5 \gamma^\mu \epsilon)(\overline{\lambda}^J \gamma_5 \gamma_\mu \rho^K) - (\gamma_5 \epsilon)(\overline{\lambda}^J \gamma_5 \rho^K) \Big] \quad ,
\end{aligned} \tag{5.2d}
$$

$$\delta_Q \varphi^I = + (\overline{\epsilon} \chi^I) \quad , \tag{5.2e}$$

$$\delta_Q C_\mu{}^I = + (\overline{\epsilon} \gamma_\mu \rho^I) + f^{IJK} (\overline{\epsilon} \gamma_\mu \lambda^J) \varphi^K \quad , \tag{5.2f}$$

$$
\begin{aligned}
\delta_Q \rho^I = &+ \tfrac{1}{2} (\gamma^{\mu\nu} \epsilon) \widehat{H}_{\mu\nu}{}^I - g\, \epsilon\, \varphi^I - \tfrac{1}{2} f^{IJK} (\gamma^{\mu\nu} \epsilon) \widehat{F}_{\mu\nu}{}^J \varphi^K \\
&+ \tfrac{1}{4} f^{IJK} \Big[+ \epsilon (\overline{\lambda}^J \chi^K) - (\gamma^\mu \epsilon)(\overline{\lambda}^J \gamma_\mu \chi^K) + \tfrac{1}{2} (\gamma^{\mu\nu} \epsilon)(\overline{\lambda}^J \gamma_{\mu\nu} \chi^K) \\
&\qquad - (\gamma_5 \gamma^\mu \epsilon)(\overline{\lambda}^J \gamma_5 \gamma_\mu \chi^K) - (\gamma_5 \epsilon)(\overline{\lambda}^J \gamma_5 \chi^K) \Big] \quad ,
\end{aligned} \tag{5.2g}
$$

$$\delta_Q A_\mu{}^I = + (\overline{\epsilon} \gamma_\mu \lambda^I) \quad , \tag{5.2h}$$

$$\delta_Q \lambda^I = + \tfrac{1}{2} (\gamma^{\mu\nu} \epsilon) \widehat{F}_{\mu\nu}{}^I + \tfrac{1}{2} f^{IJK} (\gamma_5 \epsilon)(\overline{\rho}^J \gamma_5 \chi^K) \quad , \tag{5.2i}$$

up to $\mathcal{O}(\phi^3)$ terms. The supercovariant field strengths are defined as usual in supergravity [8] by

$$\widehat{F}_{\mu\nu}{}^I \equiv + 2 \partial_{[\mu} A_{\nu]}{}^I + g f^{IJK} A_\mu{}^J A_\nu{}^K - 2 (\overline{\psi}_{[\mu} \gamma_{\nu]} \lambda^I) = F_{\mu\nu}{}^I - 2 (\overline{\psi}_{[\mu} \gamma_{\nu]} \lambda^I) \quad , \tag{5.3a}$$

$$
\begin{aligned}
\widehat{G}_{\mu\nu\rho}{}^I \equiv &+ 3 D_{[\mu} B_{\nu\rho]}{}^I - 3 f^{IJK} C_{[\mu}{}^J F_{\nu\rho]}{}^K - 3 (\overline{\psi}_{[\mu} \gamma_{\nu\rho]} \chi^I) + 6 (\overline{\psi}_{[\mu|} \gamma_{|\nu|} \psi_{|\rho]}) \varphi^I \\
= &+ G_{\mu\nu\rho}{}^I - 3 (\overline{\psi}_{[\mu} \gamma_{\nu\rho]} \chi^I) + 6 (\overline{\psi}_{[\mu|} \gamma_{|\nu|} \psi_{|\rho]}) \varphi^I \quad ,
\end{aligned} \tag{5.3b}
$$

$$\widehat{H}_{\mu\nu}{}^I \equiv + 2 D_{[\mu} C_{\nu]}{}^I + g B_{\mu\nu}{}^I - 2 (\overline{\psi}_{[\mu} \gamma_{\nu]} \rho^I) = H_{\mu\nu}{}^I - 2 (\overline{\psi}_{[\mu} \gamma_{\nu]} \rho^I) \quad , \tag{5.3c}$$

$$\widehat{D}_\mu \varphi^I \equiv + D_\mu \varphi^I - (\overline{\psi}_\mu \chi^I) \quad . \tag{5.3d}$$

Certain remarks are in order. First, the last term in (5.1) of the type $g(\overline{\psi}\gamma\rho)\varphi$ is related to the φ-linear term in $\delta_Q\rho$ in (5.2g). Second, the $\delta_Q B_{\mu\nu}$ contains the $(\overline{\epsilon}\gamma\psi)\varphi$-term. This is consistent with $G_{\alpha\beta c}{}^I = +2(\gamma_c)_{\alpha\beta}\,\varphi^I$ in (3.2a) in superspace. Third, for the $g\psi\rho\chi$-terms, we need non-trivial Fierz rearrangement. To be more specific, there are three contributions to this sector: (i) $g(\overline{\psi}\gamma\rho)\varphi$, (ii) $ge(\overline{\chi}\rho)$, and (iii) $(\overline{\psi}\gamma\gamma\rho)H$-terms. This rearrangement is highly non-trivial, showing the consistency of our total system.

As the couplings to supergravity in (5.1) show, our original *globally* supersymmetric system shares certain feature with supergravity, such as fermionic bilinear terms. Because such terms are common in supergravity [8], but *not* in conventional global supersymmetry. Our original *global* system already possessed the feature of *local* $N = 1$ supersymmetry. As has been mentioned after (2.2), the conventional dimensional analysis tells that such terms imply non-renormalizability. In other words, our *globally* supersymmetric system already had a hidden gravitational constant κ providing negative mass dimension. In a sense, this feature resembles σ-models with non-renormalizable couplings, sharing certain features with gravity interactions.

6. Concluding Remarks

In this paper, we have carried out the $N = 1$ supersymmetrization in 4D of a non-Abelian tensor with consistent couplings, as a special case [6] of the minimal tensor hierarchy discussed in [5], which is further a special case of more general hierarchy in [2][3]. We have given both the component and superspace formulations of our system, providing the non-trivial consistency of our system. Our CVM $(C_\mu{}^I, \rho^I)$ plays the role of a Stueckelberg [7] compensator multiplet, being absorbed into the TM $(B_{\mu\nu}{}^I, \chi^I, \varphi^I)$, making the latter massive.

We have also generalized the adjoint-representation case to the general real representation for $G = SO(N)$. The action invariance works in a fashion parallel to the former. We foresee no obstruction against generalizing these result further to the complex representation of, *e.g.,* $G = SU(N)$ group. Finally, we have also coupled the global $N = 1$ system to $N = 1$ supergravity up to quartic terms. This has provided a non-trivial confirmation for the total consistency of the non-Abelian TM.

It has been known that certain problem exists in the quantization of Stueckelberg model [7] for non-Abelian gauge groups [9]. The common problem is that the longitudinal components of the gauge field do not decouple from the physical Hilbert space, upsetting the renormalizability and unitarity of the system [9]. For this issue, we clarify our standpoints as follows: First of all, our theory is *not* renormalizable from the outset, due to Pauli couplings. Our theory makes stronger sense, when couplings to supergravity are also taken into account, as we have done in section 5. Moreover, there are certain theories in 4D, such as non-linear sigma models which are *not* renormalizable, but are *not* excluded from the outset. So we do *not* go into the renormalizability issue in this paper. Second, thanks to $N = 1$ supersymmetry, our system has good chance to have a better quantum behavior, compared with non-supersymmetric systems.

As will be shown in Appendix A, the purely bosonic part of our system can be generalized to arbitrary space-time dimensions with arbitrary signatures. The key ingredient is the tensor $B_{\mu_1\cdots\mu_{p+1}}{}^I$ and a Stueckelberg-type [7] compensator $C_{\mu_1\cdots\mu_p}{}^I$.

The potential importance of the result in this paper is $N = 1$ supersymmetry that has better quantum behavior compared with non-supersymmetric cases. We have presented a new *supersymmetric* physical system with Stueckelberg mechanism that solves both the problem with non-Abelian tensor, and the problem with extra vector fields in the non-singlet representation of a non-Abelian gauge group.

Appendix A: Higher-Dimensional Application of Purely Bosonic System

In this appendix, we generalize the purely bosonic part of our system in 4D into arbitrary space-time dimensions with arbitrary signatures. We also apply it to the case of tensor-vector duality in 6D, and perform a DR to 4D. Our field content is $(A_\mu{}^I, B_{[n-1]}{}^I, C_{[n-2]}{}^I)$.[10)]

We generalize the definitions of field strengths (2.1a) and (2.1b) to arbitrary space-time dimension D as

$$G_{\mu_1\cdots\mu_n}{}^I \equiv +nD_{[\mu_1}B_{\mu_2\cdots\mu_n]}{}^I - \frac{n(n-1)}{2} f^{IJK} C_{[\mu_1\cdots\mu_{n-2}}{}^J F_{\mu_{n-1}\mu_n]}{}^K \quad , \tag{A.1a}$$

$$H_{\mu_1\cdots\mu_{n-1}}{}^I \equiv +(n-1)D_{[\mu_1}C_{\mu_2\cdots\mu_{n-1}]}{}^I + gB_{\mu_1\cdots\mu_{n-1}}{}^I \quad . \tag{A.1b}$$

The YM field strength F is the same as in (1.2). The BIds for these field strengths are

$$D_{[\mu}F_{\nu\rho]}{}^I \equiv 0 \quad , \tag{A.2a}$$

$$D_{[\mu_1}G_{\mu_2\cdots\mu_{n+1}]}{}^I \equiv + \frac{n}{2} f^{IJK} F_{[\mu_1\mu_2|}{}^J H_{|\mu_3\cdots\mu_{n+1}]}{}^K \quad , \tag{A.2b}$$

$$D_{[\mu_1}H_{\mu_2\cdots\mu_n]}{}^I \equiv + \frac{1}{n}\, g\, G_{\mu_1\cdots\mu_n}{}^I \quad . \tag{A.2c}$$

10) We use the symbols like $[n]$ for totally antisymmetric indices $\mu_1\mu_2\cdots\mu_n$ in order to save space.

The α, β and γ-transformations for $A_\mu{}^I$, $B_{[n-1]}{}^I$ and $C_{[n-2]}{}^I$ are the generalizations of our 4D case:

$$\delta_\alpha(A_\mu{}^I,\ B_{[n-1]}{}^I,\ C_{[n-2]}{}^I) = (D_\mu\alpha^I,\ \ - gf^{IJK}\alpha^J B_{[n-1]}{}^K,\ \ - gf^{IJK}\alpha^J C_{[n-2]}{}^K) \ \ , \tag{A.3a}$$

$$\delta_\alpha(F_{\mu\nu}{}^I,\ G_{[n]}{}^I,\ H_{[n-1]}{}^I) = -gf^{IJK}\alpha^J(F_{\mu\nu}{}^K,\ G_{[n]}{}^K,\ H_{[n-1]}{}^K) \ \ , \tag{A.3b}$$

$$\delta_\beta B_{\mu_1\cdots\mu_{n-1}}{}^I = +(n-1)D_{[\mu_1}\beta_{\mu_2\cdots\mu_{n-1}]}{}^I \ \ , \qquad \delta_\beta A_\mu{}^I = 0 \ \ , \tag{A.3c}$$

$$\delta_\beta C_{\mu_1\cdots\mu_{n-2}}{}^I = -g\beta_{\mu_1\cdots\mu_{n-2}}{}^I \ \ , \tag{A.3d}$$

$$\delta_\beta(F_{\mu\nu}{}^I,\ G_{[n-1]}{}^I,\ H_{[n-2]}{}^I) = 0 \ \ , \tag{A.3e}$$

$$\delta_\gamma C_{\mu_1\cdots\mu_{n-2}}{}^I = +(n-2)D_{[\mu_1}\gamma_{\mu_2\cdots\mu_{n-2}]}{}^I \ \ , \qquad \delta_\gamma A_\mu{}^I = 0 \ \ , \tag{A.3f}$$

$$\delta_\gamma B_{\mu_1\cdots\mu_{n-1}}{}^I = + \frac{(n-1)(n-2)}{2}\, f^{IJK}\,\gamma_{[\mu_1\cdots\mu_{n-3}|}{}^J F_{|\mu_{n-2}\,\mu_{n-1}]}{}^K \ \ , \tag{A.3g}$$

$$\delta_\gamma(F_{\mu\nu}{}^I,\ G_{[n-1]}{}^I,\ H_{[n-2]}{}^I) = 0 \ \ . \tag{A.3h}$$

Eq. (A.3d) shows that the C-field is a Stueckelberg field absorbed into the longitudinal components of the B-field.

A typical action $I \equiv \int d^D x\, \mathcal{L}$ is given by the lagrangian

$$\mathcal{L} = \ -\ \frac{1}{2(n!)}\,(G_{[n]}{}^I)^2 - \frac{1}{2{\cdot}(n-1)!}\,(H_{[n-1]}{}^I)^2 - \frac{1}{4}\,(F_{\mu\nu}{}^I)^2 \ \ , \tag{A.4}$$

yielding the B and C-field equations

$$\frac{\delta\mathcal{L}}{\delta B_{[n-1]}{}^I} \ = \ \frac{1}{(n-1)!}\left(D_\mu G^{\mu[n-1]\,I} - gH^{[n-1]\,I}\right) \ \doteq \ 0 \ \ , \tag{A.5a}$$

$$\frac{\delta\mathcal{L}}{\delta C_{[n-2]}{}^I} \ = \ \frac{1}{(n-2)!}\left(D_\nu H^{\nu[n-2]\,I} + \frac{1}{2}\, f^{IJK} F_{\rho\sigma}{}^J G^{[n-2]\rho\sigma\,K}\right) \ \doteq \ 0 \ \ . \tag{A.5b}$$

As in the 4D case, it is straightforward to show the consistency

$$0 \overset{?}{\doteq} D_\mu\left(\frac{\delta\mathcal{L}}{\delta B_{\mu[n-2]}{}^I}\right) \ \equiv - \frac{1}{n-1}\, g\left(\frac{\delta\mathcal{L}}{\delta C_{[n-2]}{}^I}\right) \ \doteq \ 0 \ \ , \tag{A.6a}$$

$$0 \overset{?}{\doteq} D_\mu\left(\frac{\delta\mathcal{L}}{\delta C_{\mu[n-3]}{}^I}\right) \ \equiv + \frac{n-1}{2}\, f^{IJK} F_{\rho\sigma}{}^J\left(\frac{\delta\mathcal{L}}{\delta B_{[n-3]\rho\sigma}{}^K}\right) \ \doteq \ 0 \qquad (Q.E.D.) \tag{A.6b}$$

We next apply our result to $6D$ with the signature $(-,-,+,+,+,+)$, and consider the duality condition

$$F_{\mu\nu}{}^I \ \overset{*}{=} \ + \ \frac{1}{24}\, \epsilon_{\mu\nu}{}^{\rho\sigma\tau\lambda}\, G_{\rho\sigma\tau\lambda}{}^I \ \ , \qquad G_{\mu\nu\rho\sigma}{}^I \overset{*}{=} \ + \ \frac{1}{2}\, \epsilon_{\mu\nu\rho\sigma}{}^{\tau\lambda}\, F_{\tau\lambda}{}^I \ \ . \tag{A.7}$$

This duality looks similar to eq. (3.6) in [5], but the existence of the physical scalar field ϕ^I in the latter makes the fundamental difference.

We have to first confirm the consistency of (A.7) with the G and H-BIds. First, the rotation of the 2nd equation in (A.7) gives

$$\begin{aligned} 0 &\overset{?}{=} + \epsilon^{\mu\nu\rho\sigma\tau\lambda} D_\nu \left(G_{\rho\sigma\tau\lambda}{}^I - \frac{1}{2}\,\epsilon_{\rho\sigma\tau\lambda}{}^{\omega\psi} F_{\omega\psi}{}^I \right) \equiv + \epsilon^{\mu\nu\rho\sigma\tau\lambda} \left(2 f^{IJK} F_{\nu\rho}{}^J H_{\sigma\tau\lambda}{}^K \right) - 24 D_\nu F^{\mu\nu\, I} \\ &= -24 \left(D_\nu F^{\mu\nu\, I} - \frac{1}{12}\,\epsilon^{\mu\nu\rho\sigma\tau\lambda} f^{IJK} F_{\nu\rho}{}^J H_{\sigma\tau\lambda}{}^K \right) \quad . \end{aligned} \tag{A.8}$$

In the second identity in (A.8), we have used the G-BId (A.2b). The first term in the last line is the kinetic term of $A_\mu{}^I$, so that its last term is its source term. Second, in order to see if eq. (A.8) has consistent solutions, we can confirm the conservation of the source term, by applying D_μ on (A.8) based on H-BId (A.2c) and (A.7), but we skip the details here.

We next show that the usual self-duality relationship in $D = 2+2$

$$F_{\mu\nu}{}^I \overset{*}{=} + \frac{1}{2}\,\epsilon_{\mu\nu}{}^{\rho\sigma}\, F_{\rho\sigma}{}^I \tag{A.9}$$

is embedded into (A.7). To this end, we use *hat* symbols both on fields and indices in 6D, while *no hats* on 4D quantities from now on. We also use $\hat\mu,~\hat\nu,~\cdots = 1,~2,~3,~4,~5,~6$ and $\mu,~\nu,~\cdots = 1,~2,~3,~4$, while $\alpha,~\beta,~\cdots = 5,~6$. Our basic ansätze for the DR are

$$\widehat G_{\hat\mu\hat\nu\hat\rho\hat\sigma}{}^I \overset{*}{=} + \widehat F_{[\hat\mu\hat\nu}{}^I \widehat P_{\hat\rho\hat\sigma]} \quad , \qquad \widehat P_{\hat\mu\hat\nu} \equiv + \widehat\partial_{\hat\mu} \widehat X_{\hat\nu} - \widehat\partial_{\hat\nu} \widehat X_{\hat\mu} \quad , \qquad \widehat H_{\hat\mu\hat\nu\hat\rho}{}^I \overset{*}{=} + \frac{1}{2}\, g \widehat F_{[\hat\mu\hat\nu}{}^I \widehat X_{\hat\rho]} \quad , \tag{A.10a}$$

$$\widehat P_{\hat\mu\hat\nu} = \epsilon_{\alpha\beta} \quad (\text{for } \hat\mu = \alpha,~\hat\nu = \beta) \quad , \qquad \widehat F_{\hat\mu\hat\nu}{}^I = \widehat F_{\mu\nu}{}^I = F_{\mu\nu}{}^I \quad (\text{for } \hat\mu = \mu,~\hat\nu = \nu) \quad , \tag{A.10b}$$

$$\widehat\epsilon^{\,\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau\hat\lambda} = \widehat\epsilon^{\,\mu\nu\rho\sigma\alpha\beta} = \epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta} \quad (\text{for } [\hat\mu\hat\nu\hat\rho\hat\sigma\hat\tau\hat\lambda] = [\mu\nu\rho\sigma\alpha\beta]) \quad . \tag{A.10c}$$

Other components, such as $\widehat P_{\mu\beta}$ are all zero. We can confirm that (A.10) are consistent with the BIds (A.2b) and (A.2c). It is easy to show that the $[\alpha\beta]$ and $[\mu\alpha]$-components of the first equation in (A.7) are satisfied, while the $[\mu\nu]$-component gives directly the 4D self-duality (A.9). Thus the 4D self-duality $F \overset{*}{=} \widetilde F$ is indeed embedded in the 6D duality (A.7).

We next generalize the 6D result to the $D = 2m+2$ with the signature $(-,-,\overbrace{+,\cdots,+}^{2m})$. The duality condition (A.7) is generalized to

$$\widehat F_{\hat\mu\hat\nu}{}^I \overset{*}{=} + \frac{1}{(2m)!}\, \widehat\epsilon_{\hat\mu\hat\nu}{}^{\hat\rho_1\cdots\hat\rho_{2m}}\, \widehat G_{\hat\rho_1\cdots\hat\rho_{2m}}{}^I \quad , \qquad \widehat G_{\hat\rho_1\cdots\hat\rho_{2m}}{}^I \overset{*}{=} + \frac{1}{2}\, \widehat\epsilon_{\hat\rho_1\cdots\hat\rho_{2m}}{}^{\hat\mu\hat\nu}\, \widehat F_{\hat\mu\hat\nu}{}^I \quad . \tag{A.11}$$

As in the 6D case, we can first confirm the consistency with BIds. We can next confirm the current conservation, whose details are skipped here.

The previous ansätze for 6D case in (A.10) are generalized to

$$\widehat{G}_{\hat\mu_1\cdots\hat\mu_{2m}}{}^I \overset{*}{=} + c\widehat{F}_{[\hat\mu_1\hat\mu_2|}{}^I \widehat{P}^{(1)}_{|\hat\mu_3\hat\mu_4|}\cdots \widehat{P}^{(m-1)}_{|\hat\mu_{2m-1}\,\hat\mu_{2m}]} \quad , \qquad \widehat{P}^{(k)}_{\hat\mu\hat\nu} \equiv \widehat\partial_{\hat\mu}\widehat{X}^{(k)}_{\hat\nu} - \widehat\partial_{\hat\nu}\widehat{X}^{(k)}_{\hat\mu} \quad , \tag{A.12a}$$

$$\widehat{H}_{\hat\mu_1\cdots\hat\mu_{2m-1}}{}^I \overset{*}{=} + \frac{1}{m}\, cg\widehat{F}_{[\hat\mu_1\hat\mu_2|}{}^I \widehat{P}^{(1)}_{|\hat\mu_3\hat\mu_4|}\cdots \widehat{P}^{(m-2)}_{|\hat\mu_{2m-3}\,\hat\mu_{2m-2}|}\widehat{X}_{|\hat\mu_{2m-1}]} \quad , \tag{A.12b}$$

$$\widehat{P}^{(k)}_{\hat\mu\hat\nu} = \widehat{P}^{(k)}_{2k+3,\ 2k+4} = -\widehat{P}^{(k)}_{2k+4,\ 2k+3} = \epsilon^{(k)}_{2k+3,\ 2k+4} = -\epsilon^{(k)}_{2k+4,\ 2k+3} = +1$$

$$(\text{for } \ \hat\mu = 2k{+}3,\ \hat\nu = 2k{+}4;\ \ k = 1,\ \cdots,\ m{-}1) \ , \tag{A.12c}$$

$$\widehat{F}_{\hat\mu\hat\nu}{}^I = F_{\mu\nu}{}^I \qquad (\text{for } \ \hat\mu = \mu,\ \hat\nu = \nu) \quad , \tag{A.12d}$$

$$\widehat\epsilon^{\hat\mu_1\cdots\hat\mu_{2m+2}} = \epsilon^{\mu\nu\rho\sigma}\,\epsilon^{\alpha_1\cdots\alpha_{2m-2}} = \epsilon^{\mu\nu\rho\sigma}\,\epsilon^{[\alpha_1\alpha_2|}_{(1)}\cdots\epsilon^{|\alpha_{2m-3}\alpha_{2m-2}]}_{(m-1)}$$

$$(\text{for } \ [\hat\mu_1\cdots\hat\mu_{2m+2}] = [\mu\nu\rho\sigma\alpha_1\cdots\alpha_{2m-2}]) \ . \tag{A.12e}$$

where c is a constant to be fixed later.

As before, we can also confirm the G and H-BIds for (A.11). The constant c in (A.12a) is fixed by getting the 4D self-duality in the $[\mu\nu]$-component of the first equation in (A.11):

$$\begin{aligned} F_{\mu\nu}{}^I &\overset{*}{=} + \frac{1}{(2m)!}\,\widehat\epsilon_{\mu\nu}{}^{\hat\rho_1\cdots\hat\rho_{2m}}\,\widehat{G}_{\hat\rho_1\cdots\hat\rho_{2m}}{}^I = + \frac{\binom{2m}{2}}{(2m)!}\,\widehat\epsilon_{\mu\nu}{}^{\rho\sigma\alpha_1\cdots\alpha_{2m-2}}\,\widehat{G}_{\rho\sigma\alpha_1\cdots\alpha_{2m-2}}{}^I \\ &= +\frac{1}{2}\,c\left[\frac{1}{(m{-}1)!\cdot(2m{-}3)!!}\right]^2 \epsilon_{\mu\nu}{}^{\rho\sigma}\,F_{\rho\sigma}{}^I \quad . \end{aligned} \tag{A.13}$$

For this to agree with $F \overset{*}{=} \widetilde{F}$, we get $c = [\,(m-1)!\cdot(2m-3)!!\,]^2$. The remaining components $[\alpha\beta]$ and $[\mu\alpha]$ are trivially satisfied.

The above mechanism for $D = 2m+2$ is further generalized to $D = 2m+1$ with the signature $(-,-,\overbrace{+,+,\cdots,+}^{2m-1})$ with the duality condition

$$\widehat{F}_{\hat\mu\hat\nu}{}^I \overset{*}{=} + \frac{1}{(2m{-}1)!}\,\widehat\epsilon_{\hat\mu\hat\nu}{}^{\hat\rho_1\cdots\hat\rho_{2m-1}}\,\widehat{G}_{\hat\rho_1\cdots\hat\rho_{2m-1}}{}^I \quad , \qquad \widehat{G}_{\hat\rho_1\cdots\hat\rho_{2m-1}}{}^I \overset{*}{=} + \frac{1}{2}\,\widehat\epsilon_{\hat\rho_1\cdots\hat\rho_{2m-1}}{}^{\hat\mu\hat\nu}\,\widehat{F}_{\hat\mu\hat\nu}{}^I \quad . \tag{A.14}$$

The confirmation of G and H-BIds is just parallel to the $D = 2m+2$ case. The ansätze for DR is

$$\widehat{G}_{\hat\mu_1\cdots\hat\mu_{2m-1}}{}^I \overset{*}{=} + \frac{2c'}{3}\,\widehat{F}_{[\hat\mu_1\hat\mu_2|}{}^I \widehat{P}^{(1)}_{|\hat\mu_3\hat\mu_4|}\cdots \widehat{P}^{(m-3)}_{|\hat\mu_{2m-5}\,\hat\mu_{2m-4}|}\widehat{Q}_{|\hat\mu_{2m-3}\hat\mu_{2m-2}\hat\mu_{2m-1}]} \quad , \tag{A.15a}$$

$$\widehat{H}_{\hat\mu_1\cdots\hat\mu_{2m-2}}{}^I \overset{*}{=} + \frac{2c'g}{2m{-}1}\,\widehat{F}_{[\hat\mu_1\hat\mu_2|}{}^I \widehat{P}^{(1)}_{|\hat\mu_3\hat\mu_4|}\cdots \widehat{P}^{(m-3)}_{|\hat\mu_{2m-5}\,\hat\mu_{2m-4}|}\widehat{Y}_{|\hat\mu_{2m-3}\hat\mu_{2m-2}]} \quad , \tag{A.15b}$$

$$\widehat{P}^{(k)}_{\hat\mu\hat\nu} \equiv \widehat\partial_{\hat\mu}\widehat{X}^{(k)}_{\hat\nu} - \widehat\partial_{\hat\nu}\widehat{X}^{(k)}_{\hat\mu} \quad , \qquad \widehat{Q}_{\hat\mu\hat\nu\hat\rho} \equiv +\widehat\partial_{\hat\mu}\widehat{Y}_{\hat\nu\hat\rho} + \widehat\partial_{\hat\nu}\widehat{Y}_{\hat\rho\hat\mu} + \widehat\partial_{\hat\rho}\widehat{Y}_{\hat\mu\hat\nu} \quad , \tag{A.15c}$$

$$\widehat{P}^{(k)}_{\hat\mu\hat\nu} = \widehat{P}^{(k)}_{2k+3,\ 2k+4} = -\widehat{P}^{(k)}_{2k+4,\ 2k+3} = \epsilon^{(k)}_{2k+3,\ 2k+4} = -\epsilon^{(k)}_{2k+4,\ 2k+3} = +1 \quad , \tag{A.15d}$$

$$\widehat{Q}_{\hat{\mu}\hat{\nu}\hat{\rho}} = \widehat{Q}_{2m-3,2m-2,2m-1} = \epsilon_{2m-3,2m-2,2m-1} = +1 \quad \big(\text{for} \;\; {\scriptstyle [\hat{\mu}\hat{\nu}\hat{\rho}] \, = \, [2m-3, 2m-2, 2m-1]}\big) \quad , \tag{A.15e}$$

$$\widehat{F}_{\hat{\mu}\hat{\nu}}{}^I = F_{\mu\nu}{}^I \qquad \big(\text{for} \;\; {\scriptstyle \hat{\mu} \, = \, \mu, \; \hat{\nu} \, = \, \nu}\big) \quad , \tag{A.15f}$$

$$\widehat{\epsilon}^{\,\hat{\mu}_1 \cdots \hat{\mu}_{2m+1}} = \epsilon^{\mu\nu\rho\sigma} \, \epsilon^{\alpha_1 \cdots \alpha_{2m-3}} = \epsilon^{\mu\nu\rho\sigma} \, \epsilon_{(1)}^{[\alpha_1 \alpha_2|} \cdots \epsilon_{(m-3)}^{|\alpha_{2m-7}\alpha_{2m-6}|} \epsilon^{|\alpha_{2m-5}\alpha_{2m-4}\alpha_{2m-3}]} \quad . \tag{A.15g}$$

The totally antisymmetric constant tensor $\epsilon^{\alpha\beta\gamma}$ is for the last three coordinates in $D = 2m+1$. The satisfaction of the duality (A.14) fixes the constant $c' = [(m-3)! \cdot (2m-7)!!]^2$.

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