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Abstract

Any conformal field theory (CFT) on a sphere supports completely undamped collective oscillations. We discuss the implications of this fact for studies of thermalization using AdS/CFT. Analogous oscillations occur in Galilean CFT, and they could be observed in experiments on ultracold fermions.

I.   INTRODUCTION

Conformal field theories (CFTs) are interesting for many reasons: they arise in the study of critical phenomena, on the worldsheet of fundamental strings, and in the holographic dual of anti-de Sitter spacetimes [1–3]. It is plausible that all quantum field theories are relevant deformations of conformally invariant ultraviolet fixed points.

Here we describe an exotic property of any CFT in any number of dimensions. Any relativistic CFT whose spatial domain is a sphere contains a large class of non-stationary states whose time evolution is periodic, with frequencies that are integer multiples of the inverse radius of the sphere.

Nonrelativistic CFTs that realize the Schrödinger algebra also support undamped oscillations in the presence of a spherically symmetric harmonic potential. Cold fermionic atoms with tuned two-body interactions can provide an experimental realization of such a system, and the modes we discuss (which, in this context, have been discussed previously in [4]) could be (but have not yet been) observed.

The existence of these permanently oscillating many-body states is guaranteed by $sl(2, \mathbb{R})$ subalgebras of the conformal algebra, formed from the Hamiltonian and combinations of operators which act as ladder operators for energy eigenstates. It is striking that any CFT in any dimension, regardless of the strength or complication of its interactions, has states that undergo *undamped* oscillation.

We emphasize the distinction between these oscillating states and an energy eigenstate of an arbitrary Hamiltonian. Time evolution of an energy eigenstate is just multiplication of the wavefunction by a phase – nothing happens. Given knowledge of the exact energy eigenstates of any system, one may construct special operators whose correlations oscillate in time in certain states. In contrast, we show below that in the states described here, accessible physical quantities such as the energy density vary in time (at leading order in N^2 in holographic examples). Further, the oscillations arise and survive at late times starting from generic initial conditions[1].

The persistence of these oscillations conflicts with conventional expectations for thermalization of an interacting theory. The conventional wisdom is that an arbitrary initial state will settle down to an equilibrium stationary configuration characterized by its energy and angular momenta (for a recent discussion of this expectation, see [5]). This is not the case for CFT's due to the presence of these undamped oscillations.

The existence of these modes is due to the existence of extra conserved charges in CFT generated by conformal Killing vectors. Any conserved charge will partition the Hilbert space of a system into sectors which do not mix under time evolution. However, in the conventional situation there will be be a stationary state for each value of the charges which represents equilibrium in that sector. The states we describe do not approach a stationary state; rather, the *amplitude* of the oscillations is a conserved quantity, as we explain in Section III.

Any system with a finite number of degrees of freedom will exhibit *quasiperiodic* evolution: the time dependence of *e.g.* a correlation function in such a system is inevitably a sum of a finite number of Fourier modes. The class of oscillations we describe can be distinguished from such generic behavior in two respects: First, for each oscillating state the period is fixed to be an integer multiple of the circumference of the sphere. Second, and more importantly, the oscillations persist and remain undamped in a particular thermodynamic limit – namely, a large-N limit where the number of degrees of freedom per site diverges. Many such CFTs are described by classical gravity theories in asymptotically anti-de Sitter spacetime (AdS).

The latter fact points to a possible obstacle in studying thermalization of CFTs on a sphere using holography: when subjecting the CFT to a far-from-equilibrium process, if one excites such a mode of oscillation, this excitation will not go away, even at infinite N. (The simplest way to circumvent this obstacle is to study CFT on the plane. Then this issue does not arise, as the frequency of the modes in question goes to zero.)

In holographic theories, certain oscillating states with period equal to the circumference of the sphere have an especially simple bulk description. Begin with a Schwarzschild-AdS black hole in the bulk. This is dual to a CFT at finite temperature on the boundary. Now boost the black hole. This boost symmetry is an exact symmetry of AdS, and it creates a black hole that "sloshes" back and forth forever[2]. The dual CFT therefore has periodic correlators. The boosted black hole is related to the original black hole by a large diffeomorphism that acts nontrivially on the boundary. The boundary description of the boosting procedure is to act with a conformal Killing vector on the thermal state. The relevant CKV's are periodic in time, and they act nontrivially on the thermal state, so they produce oscillations.

This document is organized as follows: In section II we construct the oscillating states explicitly. In section III we describe conserved charges associated with conformal symmetry whose nonzero expectation value diagnoses oscillations

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[1] We explain the precise sense in which they are generic in III.

[2] Such oscillations are known to S. Shenker and L. Susskind as "sloshers". The action of conformal boosts on AdS black holes has been studied previously in [6].

in a given state. We also derive operator equations which demonstrate that certain $\ell = 1$ moments of the stress-energy tensor in CFT on a sphere behave like harmonic oscillators. In section IV we specialize to holographic CFTs and discuss the gravitational description of a specific class of oscillations around thermal equilibrium. In section V, we discuss the effect of the existence of oscillating states on correlation functions of local operators. We end with some explanation of our initial motivation for this work and some comments and open questions. Appendix A summarizes the conformal algebra. Appendix B gives some details on the normalization of the oscillating states. Appendix C extends Goldstone's theorem to explain the linearized oscillations of smallest frequency. Appendix D constructs one of the consequent linearized modes of the large AdS black hole, following a useful analogy with the translation mode of the Schwarzchild black hole in flat space. Appendix E constructs finite oscillations of an AdS black hole and calculates observables in the dual CFT. In the final Appendix F, we explain that an avatar of these modes has already been studied in experiments on cold atoms, and that it is in principle possible to demonstrate experimentally the precise analog of these oscillating states.

II. STABLE OSCILLATIONS IN RELATIVISTIC CFT ON A SPHERE

Consider a relativistic CFT on a d-sphere cross time. We set $c = 1$ and measure energies in units of the inverse radius of the sphere, R^{-1}. The conformal algebra acting on the Hilbert space of the CFT contains d (non-independent) copies of the $sl(2, \mathbb{R})$ algebra,

$$[H, L_+^i] = L_+^i, \quad [H, L_-^i] = -L_-^i, \quad [L_+^i, L_-^i] = 2H \tag{II.1}$$

(with no sum on i in the last equation) as reviewed in Appendix A. Here $i = 1...d$ is an index labeling directions in the $\mathbb{R}^d$ in which S^{d-1} is embedded as the unit sphere.

We construct the oscillating states in question using the $sl(2, \mathbb{R})$ algebras as follows. Let $|\varepsilon\rangle$ be an eigenstate of H, $H|\varepsilon\rangle = \varepsilon|\varepsilon\rangle$. Consider a state of the form

$$|\Psi(t = 0)\rangle = \mathcal{N}e^{\alpha L_+ + \beta L_-}|\varepsilon\rangle, \tag{II.2}$$

where $L_+ = L_+^i$ and $L_- = L_-^j$ for some $i, j \in \{1, ..., d\}$, $\mathcal{N}$ is a normalization constant given in Appendix B, and α and β are complex numbers. If α, β are chosen so that the operator in (II.2) is unitary, there is no constraint on α, β from normalizability; more generally, finite $\mathcal{N}$ constrains α, β as described in Appendix B. It evolves in time as[3]

$$|\Psi(t)\rangle = \mathcal{N}e^{-iHt}e^{\alpha L_+ + \beta L_-}|\varepsilon\rangle = \mathcal{N}e^{\alpha e^{-it}L_+ + \beta e^{it}L_-}e^{-iHt}|\varepsilon\rangle = \mathcal{N}e^{-i\varepsilon t}e^{\alpha(t)L_+ + \beta(t)L_-}|\varepsilon\rangle, \tag{II.3}$$

where $\alpha(t) = \alpha e^{-it}$ and $\beta(t) = \beta e^{it}$. $|\Psi\rangle$ has been constructed in analogy with coherent or squeezed states in a harmonic oscillator, where one also has ladder operators for the Hamiltonian. Restoring units, $|\Psi\rangle$ is seen to oscillate with frequency $1/R$.

More generally, the time evolution of a state

$$|\Psi(t = 0)\rangle = g(L_+^1, ..., L_+^d, L_-^1, ..., L_-^d)|\varepsilon\rangle \tag{II.4}$$

where g is any regular function of the $2d$ variables L_+^i, L_-^j, $1 \le i, j \le d$, is given by replacing $L_\pm^i$ with $L_\pm^i e^{\pm it} \, \forall \, i$. For example, an oscillating state with frequency $2/R$ is given by (II.4) with $g = \sum_{m=0}^{\infty} \frac{1}{(m!)^2}\left(\alpha L_+^2 + \beta L_-^2\right)^m$, where L_+ can be any of L_+^i, $1 \le i \le d$, and similarly for L_-.[4] Note, however, that not every such function g produces an oscillating state - a state in which physical quantities such as energy density display oscillation - as opposed to a state whose time evolution is given by a trivial but nonetheless periodic factor, as with an energy eigenstate. Take for example $g = L_+$, which merely produces another eigenstate. The criterion for oscillation which we establish in section III below ($\langle Q^{\pm,i} \rangle \ne 0$ for some i) can be applied to an initial state $|\Psi(t = 0)\rangle = g|\varepsilon\rangle$ to determine whether $|\Psi\rangle$ is indeed an oscillating state. In addition, normalizability of $|\Psi\rangle$ will constrain the function g in a manner which we have not determined.

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[3] This can be checked using the Baker-Campbell-Hausdorff formula $e^B A e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!}(ad_B)^n A$.

[4] The coefficient $(m!)^2$ is designed to give the state a finite norm. We note that a state of the form (II.4) with $g = e^{\alpha L_+^n + \beta L_-^n}$ is only normalizable for $n = 0, 1$; in particular, the direct analog of a squeezed state does not seem to exist.

To retain the simple time evolution in the above construction, the stationary state $|\varepsilon\rangle$ cannot be replaced by a nonstationary state $\sum_i c_i |\varepsilon_i\rangle$, with terms that evolve in time with distinct phases. However, it can be replaced by a stationary density matrix ρ, which evolves in time with a phase A (possibly zero),

$$[H, \rho] = A\rho. \tag{II.5}$$

Examples include an energy eigenstate $\rho = |\varepsilon\rangle\langle\varepsilon|$, and thermal density matrices $\rho = e^{-\beta H} Z_\beta^{-1}$. Given such a ρ, an initial ensemble

$$\tilde{\rho}(t=0) = \mathcal{N} e^{\alpha L_+ + \beta L_-} \rho e^{\beta^* L_+ + \alpha^* L_-} \tag{II.6}$$

evolves in time as

$$\tilde{\rho}(t) = \mathcal{N} e^{-iAt} e^{\alpha(t) L_+ + \beta(t) L_-} \rho e^{\beta(t)^* L_+ + \alpha(t)^* L_-}. \tag{II.7}$$

We refer to the collective oscillation $\tilde{\rho}$ as having been "built on" ρ, in the same way $|\Psi\rangle$ was built on $|\varepsilon\rangle$, by acting with a sufficiently constrained function of ladder operators g at $t = 0$. In section IV, we give a holographic description of the subset of oscillations built on a thermal density matrix for which $g = e^{\alpha L_+ + \beta L_-}$ and $\alpha L_+ + \beta L_-$ is anti-Hermitian.

III. DIAGNOSING THE AMPLITUDE OF OSCILLATIONS

It may be useful to be able to diagnose the presence of the above oscillations in a generic state of a CFT. Here we identify conserved charges associated with conformal Killing vectors, which correspond to the conserved amplitudes of possible oscillations.

Consider CFT on a spacetime M. Given a conformal Killing vector field (CKV) ξ^μ on M, there is an associated current and charge acting in the Hilbert space of the CFT

$$j_\xi^\mu = T^{\mu\nu} \xi_\nu, \qquad Q_\xi = \int_\Sigma d^{d-1}x \sqrt{g}\ j_\xi^\mu n_\mu, \tag{III.1}$$

where Σ is a spatial hypersurface and n^μ is a normal vector. $\nabla_\mu j_\xi^\mu$ is given by a state-independent but ξ-dependent constant

$$\nabla_\mu j_\xi^\mu = \nabla_\mu T^{\mu\nu} + T^{\mu\nu}(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) = \alpha_\xi T^\mu_\mu, \tag{III.2}$$

where $\alpha_\xi = \frac{2}{d} \nabla_\mu \xi^\mu$ is a c-number which vanishes when ξ^μ is an exact Killing vector field (KV)[5]. Then $\frac{d}{dt} Q_\xi = \int_\Sigma d^{d-1}x \sqrt{g}\ \alpha_\xi T^\mu_\mu\ n_t$, which vanishes for CFT on $\mathbb{R} \times S^{d-1}$ as follows. The trace anomaly has angular momentum $\ell = 0$. The proper CKVs have $\ell = 1$ and hence the associated α_ξ have $\ell = 1$. The integral over the sphere vanishes and Q_ξ is exactly conserved.

ξ^μ's can be obtained explicitly by projecting $J^{ab} = i(X^a \partial^b - X^b \partial^a)$, $a, b = -1, 0, 1, ..., d$ in $\mathbb{R}^{(2,d)}$ to the $r \to \infty$ boundary of AdS_{d+1}, the hypersurface $\sum_{i=1}^d \Omega^{i\,2} = 1$ in coordinates $(r, t, \Omega^1, ..., \Omega^d)$ with $r \geq 0$, $-\infty < t < \infty$, defined by

$$X_{-1} = R\sqrt{1 + r^2} \cos t, \tag{III.3}$$
$$X_0 = R\sqrt{1 + r^2} \sin t, \tag{III.4}$$
$$X_i = Rr\Omega_i. \tag{III.5}$$

In particular, CKVs which are not KVs, corresponding to boosts $J^{\pm,i} \equiv J^{-1,i} \pm iJ^{0,i}$ in $\mathbb{R}^{(2,d)}$, are

$$\xi^{\pm,i} = \mp ie^{\pm it}\, \Omega^i \partial_t - e^{\pm it}\left[(1 - \Omega^{i2})\partial_{\Omega^i} - \Omega^i \sum_{j \neq i} \Omega^j \partial_{\Omega^j}\right]\Bigg|_{\sum_k \Omega^{k\,2} = 1}. \tag{III.6}$$

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[5] Note that in (III.2) we have assumed $T^{\mu\nu}$ is both symmetric and traceless, up to a trace anomaly.

Letting $\left[(1 - \Omega^{i2})\partial_{\Omega^i} - \Omega^i \sum_{j \neq i} \Omega^j \partial_{\Omega^j}\right]\Big|_{\sum_k \Omega^{k\,2}=1} \equiv f^{i\alpha}(\theta)\partial_{\theta^\alpha}$ where θ^α, $\alpha = 1, ..., d-1$ are coordinates on S^{d-1}, the associated charges are

$$Q^{\pm,i} = \mp i e^{\pm it} \int_{S^{d-1}} d^{d-1}\theta \sqrt{g}\, T_{tt}\Omega^i - e^{\pm it}\int_{S^{d-1}} d^{d-1}\theta \sqrt{g}\, T_{t\alpha}f^{i\alpha}(\theta). \tag{III.7}$$

Now consider the quantities

$$\mathcal{X}^i \equiv \int_{S^{d-1}} d^{d-1}\theta \sqrt{g}\, T_{tt}\Omega^i, \qquad \mathcal{P}^i \equiv -\int_{S^{d-1}} d^{d-1}\theta \sqrt{g}\, T_{t\alpha}f^{i\alpha}(\Omega)\ . \tag{III.8}$$

These are the ith coordinate of the center of mass of the CFT state in the embedding space of S^{d-1}, and its momentum, respectively. From conservation of the charges $Q^{\pm,i}$, they satisfy the operator equations

$$\dot{\mathcal{X}}^i - \mathcal{P}^i = 0 \quad \dot{\mathcal{P}}^i + \mathcal{X}^i = 0 \tag{III.9}$$

– they undergo simple harmonic oscillation. From (III.7), the initial conditions for these oscillations are determined by the conformal charges.

It follows that given an arbitrary state, if any of the $2d$ expectation values $\langle Q^{\pm,i}\rangle$ are non-zero at $t = 0$, some of the non-complex, physical quantities $\langle \mathcal{X}^i \rangle$ and $\langle \mathcal{P}^i \rangle$ will oscillate with undying amplitude. The fact that $\langle Q^{\pm,i}\rangle \neq 0$ for some i is an open condition justifies our use of the word 'generic' in the Introduction. As a check that the condition is a good criterion for physical oscillation, note that if $|\Psi\rangle$ is an energy eigenstate, $\langle\Psi|Q^{\pm,i}|\Psi\rangle = 0$. This follows from $\langle\Psi|[H, Q^{\pm,i}]|\Psi\rangle = \langle\Psi|\pm Q^{\pm,i}|\Psi\rangle = 0$ [6] using $H|\Psi\rangle = E|\Psi\rangle$. On the other hand, $\langle\Psi|Q^{\pm,i}|\Psi\rangle = 0 \,\forall\, i$ does not imply that $|\Psi\rangle$ is an energy eigenstate, indicating that there are non-oscillating states which are not energy eigenstates. Take for example a superposition of energy eigenstates $|\Psi\rangle = |\Psi_1\rangle + |\Psi_2\rangle$, where $|\Psi_1\rangle$ and $|\Psi_2\rangle$ belong to different towers in the CFT spectrum, where each tower is built on ladder operators acting on a primary energy eigenstate. Then clearly $\langle\Psi|Q^{\pm,i}|\Psi\rangle = 0 \,\forall\, i$, although $|\Psi\rangle$ is not an eigenstate.

Finally, we clarify a potentially confusing point. By construction, all of the $\frac{(d+2)(d+1)}{2}$ charges Q^{AB} are time-independent. There is one associated with each generator of the conformal group in d dimensions; on general grounds of Noether's theorem, they satisfy the commutation relations of $so(2,d)$. Since the Hamiltonian for the CFT on the sphere H is one of these generators, and H is not central (*e.g.* $[H, L^i_\pm] = \pm L^i_\pm$), there may appear to be a tension between the two preceding sentences. Happily, there is no contradiction: the time evolution of the conformal charges $Q^{\pm,i}$ arising from their failure to commute with the time-evolution operator is precisely cancelled by the explicit time dependence of the CKVs $\xi^{\pm,i}$:

$$\frac{d}{dt}Q^{\pm,i} = \partial_t Q^{\pm,i} - i[H, Q^{\pm,i}] = 0. \tag{III.10}$$

Thus $Q^{\pm,i} = e^{\pm it}L^i_\pm$.

IV.   HOLOGRAPHIC REALIZATION OF OSCILLATIONS

The discussion in the previous sections applies to any relativistic CFT on the sphere. We now turn to the case of holographic CFTs and the dual gravitational description of a special class of collective oscillations built on thermal equilibrium, for which $g = e^{\alpha L_+ + \beta L_-}$ and is anti-Hermitian.

A.   Bouncing Black Hole

Consider a relativistic CFT_d with a gravity dual, on S^{d-1}. We focus on collective oscillations of the form in (II.7), built on the thermal density matrix $\rho = Z_\beta^{-1} \sum_\varepsilon e^{-\beta\epsilon}|\varepsilon\rangle\langle\varepsilon|$, with the parameter $\alpha L_+ + \beta L_-$ restricted to be anti-Hermitian. With this restriction, $e^{\alpha L_+ + \beta L_-}$ is a finite transformation in $SO(2,d)$.

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[6] $Q^{\pm,i} = e^{\pm it}L^i_\pm$, as explained in following paragraph.

The gravity dual of such a state can be constructed by a 'conformal boost' of a black hole, as follows. Begin with the static global AdS black hole dual to ρ [3, 7]. Now consider a non-normalizable bulk coordinate transformation, which reduces to the finite conformal transformation $e^{\alpha L_+ + \beta L_-}$ at the UV boundary of AdS. Such a transformation falls off too slowly to be gauge identification, but too fast to change the couplings of the dual CFT; it changes the state of the CFT.

For example, a coordinate transformation that corresponds to the boost J_{01} in the embedding coordinates $(X_{-1}, X_0, X_1, ..., X_d)$ of AdS_{d+1}, maps empty AdS_{d+1} to itself, but will produce a collective oscillation when acting on a global AdS black hole[7]. The CFT state at some fixed time is of the form

$$\mathcal{N} e^{\alpha(L^1_+ - L^1_-)} \left(\sum_\varepsilon e^{-\beta\varepsilon} |\varepsilon\rangle\langle\varepsilon| \right) e^{\alpha^*(L^1_- - L^1_+)} \tag{IV.1}$$

where α is the boost parameter and real, and evolves as in (II.7) with phase $A = 0$. In Appendix E, we discuss the bouncing black hole obtained from such a boost of the BTZ black hole. We exhibit oscillations of the stress-energy tensor expectation value and of entanglement entropy of a subregion in the corresponding mixed state.

B. Bulk Modes

In the CFT, excitations corresponding to oscillations of the form in (IV.1) are created by modes of the stress-energy tensor – linear combinations of L_+ and L_- – acting on the thermal ensemble. They can be interpreted as Goldstone excitations resulting from the breaking of conformal symmetry by the thermal state of the CFT. We elaborate on this point in Appendix C.

For simple large-N gauge theories, these $L_\pm$ are single-trace operators. The above excitations then translate holographically to single-particle modes in the form of solutions to the linearized equation of motion for the metric in the bulk. They have $\ell = 1$ on S^{d-1}, as all conformal generators including $L_\pm$ have $\ell = 1$. Their frequency is $\omega = \pm 1/R$ – oscillations of higher frequency are built with exponentials of higher powers of the stress-energy tensor, and correspond to multi-particle states.

In Appendix D, we construct one such $\ell = 1$ mode in the global AdS_4-Schwarzschild background, from which the others can be obtained by transformations in $SO(2) \times SO(3)$. In general d, the $d \cdot 2$ bulk modes in question will be related by symmetries in $SO(2) \times SO(d)$.[8]

Note that in empty global AdS, there are no analogous $\ell = 1$ modes, as all global conformal generators annihilate the vacuum in CFT. However, there are single-particle modes which are dual in the CFT to states created by single-trace primary operators acting on the vacuum, and descendant states obtained by acting with L^n_+ for some integer $n \geq 1$ on such primary states. The latter correspond to linearized oscillations built on energy eigenstates. Together these single-particle modes are a subset of normal modes found in AdS (see e.g. [13], [14]).

V. CONSEQUENCE FOR CORRELATION FUNCTIONS

What does the existence of these modes imply for thermalization as measured by correlation functions of local operators?

A generic correlation function $G(t)$ in a thermal state will decay exponentially in time to a mean value associated with Poincaré recurrences which is of order e^{-S}. In a large-N CFT, $S \sim T^{d-1} V N^2$. This Poincaré-recurrence behavior of G is therefore of order e^{-N^2}, and does not arise at any order in perturbation theory. We expect the contributions to generic correlators at any finite order in the $1/N$ expansion to decay exponentially in time like e^{-aTt} where a is some order-one numerical constant. From the point of view of a bulk holographic description, this is because waves propagating in a black hole background fall into the black hole; the amplitude for a particle *not* to have fallen into

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[7] The reader may be worried about ambiguities in the procedure of translating a KV on AdS_{d+1} into a vector field on the AdS black hole. The UV boundary condition and choice of gauge $g_{r\mu} = 0$ appears to make this procedure unique; this is demonstrated explicitly for the linearized modes in appendix D.

[8] These modes are not compatible with the Regge-Wheeler gauge for the metric [8] common in the literature on quasinormal modes. They are perturbations of the metric due to coordinate transformations that asymptote to conformal transformations on the boundary; these coordinate transformations do not preserve the Regge-Wheeler gauge. As such, they were not identified in the spectrum of quasinormal modes of the metric in AdS black holes studied in the Regge-Wheeler gauge [9–13].

the black hole after time t should decay exponentially, like e^{-aTt}. At leading order in N^2, *i.e.* in the classical limit in the bulk, $G(t)$ decays exponentially in time at a rate determined by the least-imaginary quasinormal mode of the associated field. A process whereby the final state at a late time $t \gg 1/T$ is correlated with the initial state is one by which the black hole retains information about its early-time state, and hence one which resolves the black hole information problem; this happens via contributions of order e^{-N^2} [15].

The preceding discussion of Poincaré recurrences is not special to CFT. In CFT, there are special correlators which are exceptions to these expectations. A strong precedent for this arises in hydrodynamics, where correlators of operators which excite hydrodynamic modes enjoy power-law tails $G_{special}(t) \sim \frac{1}{s} t^{-b}$, where s is the entropy density and b is another number. In deconfined phases of large-N theories, $s \propto N^2$ and the long-time tails arise as loop effects in the bulk [16]. For the high-temperature phase of of large-N CFTs with gravity duals, the necessary one-loop computation was performed by [17].

A similar situation obtains for large-N CFTs on the sphere. While correlation functions for generic operators have perturbative expansions in N which decay exponentially in time and (non-perturbatively) reach the Poincaré limit e^{-N^2} at late times,

$$G_{\text{generic}}(t) \sim \sum_{g=0}^{\infty} c_g e^{-a_g T t} N^{2-2g} + c_\infty e^{-N^2} \ , \tag{V.1}$$

special correlation functions are larger at late times.

One very special example is given by

$$G_{\text{very special}}(t) \equiv \langle L_-(t) L_+(0) \rangle_T.$$

Using $[H, L_-] = -\frac{1}{R} L_-$ we have

$$G_{\text{very special}}(t) = e^{-it/R} \langle L_-(0) L_+(0) \rangle_T$$

and hence

$$\left| \frac{G_{\text{very special}}(t)}{G_{\text{very special}}(0)} \right| = 1 \ ;$$

the amplitude of this correlation does not decay at all. This is analogous in hydrodynamics to correlations of the conserved quantities themselves, such as $P^i(k=0)$, which are time independent.

There are other operators, in particular certain modes of the stress-energy tensor, which can excite and destroy the oscillations we have described, and therefore have non-decaying contributions to their autocorrelation functions. These receive oscillating contributions at one loop, in close analogy with the calculation of [17], which finds

$$\langle T^{xy}(t, k=0) T^{xy}(0, k=0) \rangle_T = T^2 \int_k \left(G^{xtxt}(t,k) G^{ytyt}(t,k) + G^{xtyt}(t,k) G^{xtyt}(t,k) \right) \tag{V.2}$$

where $G^{abcd}(t,k) \equiv \langle T^{ab}(t,k) T^{cd}(0,-k) \rangle_T$. This is most easily understood (when a gravity dual is available) via the
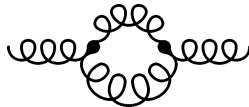
bulk one-loop Feynman diagram: The contribution from $k = 0$ in the momentum integral dominates at late times and produces the power-law tail in t.

The analog in CFT on S^{d-1} is given by the lowest angular-momentum mode of stress-energy tensor $\mathcal{O}_\pi = \int_{S^{d-1}} T_{ij} \pi^{ij}$ where π is a transverse traceless $J = 2$ ($\vec{J} = \vec{L} + \vec{S}$) tensor spherical harmonic. The leading contribution to $\langle \mathcal{O}_\pi(t) \mathcal{O}_\pi(0) \rangle_T$ at late times can again be described by the one-loop diagram above. The intermediate state now involves a sum over angular momenta rather than momenta; the term where both of the intermediate gravitons sit in the oscillating mode gives a contribution proportional to $e^{it/R}$ which does not decay; all other contributions decay exponentially in time.

Similarly, if we have additional conserved global charges in our CFT, we can construct non-decaying contributions where the graviton is in the oscillating mode and the bulk photon line carries the conserved charge, as follows:

This is analogous to the long-time tails in current-current correlators in hydrodynamics.

This paper [18] observes oscillations in real-time correlation functions in finite-volume CFT in 1+1 dimensions. Some of the explicit formulae are special to 1+1 dimensions. Also, [15] presents some such correlators.

VI. DISCUSSION

We provide some context for our thinking about these oscillating states, which could have been studied long ago[9], and in particular for thinking about collective oscillations dual to bouncing black holes.

Our initial motivation was to consider whether it is always the case that the entanglement entropy of subregions in QFT grows monotonically in time. If an entire system thermalizes, a subsystem should only thermalize faster, since the rest of the system can behave as a thermal bath.

However, it is well-known that a massive particle in global AdS oscillates about $\rho = 0$ in coordinates where

$$ds^2 = R^2(-\cosh \rho^2 dt^2 + d\rho^2 + \sinh \rho^2 d\Omega^2), \tag{VI.1}$$

with a period of oscillation $2\pi R$. This is because the geometry is a gravitational potential well. Massive geodesics of different amplitudes of oscillation about $\rho = 0$ are mapped to each other by isometries of AdS. Specifically, the static geodesic $\rho = 0$ for all t can be mapped to a geodesic oscillating about $\rho = 0$ by a special conformal transformation.

The effect of a massive object in the bulk on the entanglement entropy of a subregion in the dual CFT is proportional to its mass in Planck units (see section 6 of [22]). Only a very heavy object, whose mass is of order N^2, will affect the entanglement entropy at leading order in the $1/N^2$ expansion (which is the only bit of the entanglement entropy that we understand holographically so far). A localized object in AdS whose mass is of order N^2 is a large black hole. Therefore, by acting with an AdS isometry on a global AdS black hole, one can obtain a state in the CFT in which the entanglement entropy oscillates in time.

But this bouncing black hole is none other than the dual description of a collective oscillation built on thermal equilibrium, as introduced near (IV.1). (Recall that in systems with a classical gravity dual, the thermal ensemble at temperature of order N^0 is dominated by energy eigenstates with energy of order N^2.) The existence of this phenomenon is not a consequence of holography, but rather merely of conformal invariance.

An important general goal is to clarify in which ways holographic CFTs are weird because of holography and in which ways they are weird just because they are CFTs. Our analysis demonstrates that these oscillations are an example of the latter. The effect we have described arises because of the organization of the CFT spectrum into towers of equally-spaced states.

In holographic calculations, there is a strong temptation to study the global AdS extension because it is geodesically complete. This means compactifying the space on which the CFT lives by adding the 'point at infinity'. We have shown here that this seemingly-innocuous addition can make a big difference for the late-time behavior!

We close with some comments and open questions.

1. How are these oscillations deformed as we move away from the conformal fixed point? What does adding a relevant operator do to the oscillations? Such a relevant deformation should produce a finite damping rate for the mode. This damping rate provides a new scaling function – it has dimensions of energy and can depend only on R and the coupling of the relevant perturbation g, and must vanish as $g \to 0$. If the scaling dimension of g is ν, the damping rate is $\Gamma(g, R) = g^{1/\nu}\Phi(gR^\nu)$ where $\Phi(x)$ is finite as $x \to 0$. It may be possible to determine this function Φ holographically.

2. If we consider the special case of a CFT which is also a superconformal field theory, there are other stable oscillations that we can make by exponentiating the action of the fermionic symmetry generators $S_\pm^\alpha$. Since $[H, S_\pm^\alpha] = \pm S_\pm^\alpha$, these modes have frequency $\frac{1}{2R}$. Fermionic exponentials are simple, and these states take the form

$$|\mathfrak{a}(t)\rangle = e^{\mathfrak{a}_\alpha(t)S_+^\alpha}|\Delta\rangle = \left(1 + \mathfrak{a}_\alpha(t)S_+^\alpha\right)|\Delta\rangle \tag{VI.2}$$

with $\mathfrak{a}(t) = e^{it/2R}\mathfrak{a}(0)$.

---

[9] We are aware of the following related literature: Recently, very similar states were used as ground-like states for an implementation of an AdS_2/CFT_1 correspondence [19]. Coherent states for $SL(2, \mathbb{R})$ were constructed in [20]. The states we study are not eigenstates of the lowering operator L_-, but rather of linear combinations of powers of L_+ and L_-, and H. [19] builds pseudocoherent states which are annihilated by a linear combination of L_- and H. Other early work which emphasizes the role of $SL(2, \mathbb{R})$ representation theory in conformal quantum mechanics is [21].

3. We can define an analog for the bouncing black hole of the Aichelberg-Sexl shockwave [23] that results from a lightlike boost a Schwarzchild black hole in flat space. Here one takes the limit of lightlike boost $\beta \rightarrow \infty$, while simultaneously reducing the mass to keep the energy fixed. In this limit of the bouncing black hole, which merits further study, the profile of the *boundary* energy density is localized on the wavefront.

4. For which background geometries M can such states of CFT be constructed? A sufficient condition is the existence of a CKV ξ on M whose Lie bracket with the time-translation generator ∂_t is of the form $[\partial_t, \xi] = c\xi$ for some constant c. It would be interesting to decide whether there exist spacetimes M with such CKVs where the constant c remains finite as the volume of M is taken to infinity (unlike $S^{d-1} \times \mathbb{R}$ where $c = R^{-1} \rightarrow 0$). This would be interesting because in finite volume theories do not thermalize anyway (at finite N) because they are not in a thermodynamic limit.

5. It would be interesting to generalize these oscillations to CFTs on spacetimes with boundary, with conformal boundary conditions.

6. The recent paper [24] also identifies undamped oscillating states in holographic CFTs on S^2 using AdS gravity. These 'geons' are normalizible of classical AdS gravity with frequency of oscillation n/R. They clearly differ from the states constructed above in that they are excitations above the AdS vacuum, whereas our construction relies on broken conformal invariance. There has been also some interesting recent work on *damped* oscillations in the approach to equilibrium in scalar collapse in AdS [25].

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Appendix A: Conformal Algebra

Take the coordinates of the flat spacetime on which the global conformal group $SO(2,d)$ acts linearly to be $(X_{-1}, X_0, X_1, ..., X_d)$, with signature $(-, -, +, ..., +)$. This is the embedding space of AdS_{d+1} with boundary $\mathbb{R} \times S^{d-1}$. The compact subgroups $SO(2)$ and $SO(d)$ of $SO(2,d)$ correspond to time translation and spatial rotations in $\mathbb{R} \times S^{d-1}$, and are generated by rotations J_{-10} and $\{J_{ij}\}$, $i, j = 1, ..., d$. In order to identify ladder operators acting on eigenstates of the Hamiltonian on S^{d-1}, $H = -J_{-10}$, we will situate $\{J_{\mu\nu}\}$ in an $so(1, d+1)$ algebra [26] with generators adapted to the global conformal symmetry of $\mathbb{R}^d$.[10][11]

$$D' = -iJ_{-10} = iH, \quad M'_{ij} = J_{ij}, \quad P'_i = J_{i,-1} + iJ_{i0} \equiv L^i_+, \quad K'_i = J_{i,-1} - iJ_{i0} \equiv L^i_- \quad . \tag{A.1}$$

Note it was necessary to Euclideanize flat spacetime from $\mathbb{R}^{(1,d-1)}$ to $\mathbb{R}^d$ in order to identify the Hamiltonian in radially quantized $\mathbb{R} \times S^{d-1}$ with the dilation operator in flat spacetime. Now, from the familiar relations $[D', P'_i] = iP'_i$, $[D', K'_i] = -iK'_i$, and $[P'_i, K'_j] = -2i(g_{ij}D' - M'_{ij})$ in $\mathbb{R}^d$, one can easily identify d copies of the $SL(2, \mathbb{R})$ algebra with H as the central operator,

$$\begin{aligned} [H, L^i_+] &= L^i_+, \quad [H, L^i_-] = -L^i_-, \\ [L^i_+, L^j_-] &= 2H\delta^{ij} + 2iM'^{ij}, \quad [L^i_\pm, L^j_\pm] = 0. \end{aligned} \tag{A.2}$$

Note the raising (lowering) operators commute with raising (lowering) operators, but raising operators do not commute with lowering operators.

Appendix B: Norms of coherent states

Using formulas found in [27], we can determine the norm of a general coherent state of the form (II.2). For definiteness, will consider here coherent states built on energy eigenstates of the form $|m, \epsilon_0\rangle \propto L^m_+ |\epsilon_0\rangle$ with $|\epsilon_0\rangle$ a primary state. The norm-squared $|\mathcal{N}|^{-2}$ of the state $e^{\alpha L_+ + \beta L_-}|m, \epsilon_0\rangle$ is

$$|\mathcal{N}|^{-2} = \left(\left|\cosh\sqrt{\alpha\beta}\right|^2 + \left|\frac{\beta}{\alpha}\right| \left|\sinh\sqrt{\alpha\beta}\right|^2 \right)^{2(\epsilon+m)} \sum_{n=0}^{\infty} \frac{1}{n!^2} C^n \frac{(n+m)!}{m!} \frac{\Gamma(2\epsilon_0 + n + m)}{\Gamma(2\epsilon_0 + m)} \tag{B.1}$$

where

$$C = \cosh^2\left(\sqrt{\alpha\beta}\right) \left(\left|\frac{\beta\tanh^2\left(\sqrt{\alpha\beta}\right)}{\alpha}\right| + 1 \right)^2 \left(\frac{\sqrt{\frac{\beta}{\alpha}}\tanh\left(\sqrt{\alpha\beta}\right)}{\left|\frac{\beta\tanh^2\left(\sqrt{\alpha\beta}\right)}{\alpha}\right| + 1} + \frac{1}{2}\sqrt{\frac{\alpha}{\beta}}^* \sinh\left(2\sqrt{\alpha\beta}^*\right) \right)$$
$$\left(\frac{\sqrt{\frac{\beta}{\alpha}}^* \operatorname{sech}^2\left(\sqrt{\alpha\beta}\right)\tanh\left(\sqrt{\alpha\beta}^*\right)}{\left|\frac{\beta\tanh^2\left(\sqrt{\alpha\beta}\right)}{\alpha}\right| + 1} + \sqrt{\frac{\alpha}{\beta}}\tanh\left(\sqrt{\alpha\beta}\right) \right).$$

---

[10] Our conventions are $J_{ab} = i(X_a\partial_b - X_b\partial_a)$, $[J_{ab}, J_{cd}] = i(g_{ad}J_{bc} + g_{bc}J_{ad} - g_{ac}J_{bd} - g_{bd}J_{ac})$ for $a, b = -1, 0, 1, ..., d$. The $so(1, d+1)$ algebra is manifest in the basis $J'_{-10} = D'$, $J'_{ij} = J_{ij}$, $J'_{-1i} = (P'_i - K'_i)/2$, $J'_{0i} = (P'_i + K'_i)/2$ with signature $(-, +, +, ..., +)$.

[11] Generators of $so(1, d+1)$ adapted to the conformal symmetry of $\mathbb{R}^d$, which we denote using primed letters D', M'_{ij}, P'_i, K'_i, can be related to generators of $so(2, d)$ adapted to the conformal symmetry of $\mathbb{R}^{1,d-1}$, D, M_{ij}, P_i, K_i, by expressing the latter generators in terms of rotations J_{-10} and J_{ij} in AdS_{d+1}. In particular, in the isomorphism between $\{D, M_{ij}, P_i, K_i\}$ and $\{J_{-10}, J_{ij}\}$, $J_{0-1} = \frac{1}{2}(P_0 + K_0)$, from which we see $D' = iH = iJ_{0-1} = i\frac{1}{2}(P_0 + K_0)$.

Using the ratio test for convergence, $\sum_{n=0}^{\infty} a_n$ converges if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, we get the condition for convergence that $|C| < 1$. In the special case that $\beta = -\alpha^*$, i.e. $\alpha L_+ + \beta L_-$ is anti-Hermitian or $e^{\alpha L_+ + \beta L_-}$ is unitary, $C = 0$, so the norm squared is exactly 1 as it should be. When $\beta = \alpha^*$, i.e. $\alpha L_+ + \beta L_-$ is Hermitian, the condition becomes $\sinh^2(2|\alpha|) < 1$. When $\beta = 0$, we find $C = |\alpha|^2$, and the condition for convergence is $|\alpha|^2 < 1$.

Appendix C: Goldstone States

It is useful to interpret some of the states we have described by adapting Goldstone's theorem [28]. The remarks in the following two paragraphs are useful in developing intuition for this adaptation, but a reader impatient with discussion of holography can skip to the holography-independent argument which follows.

In the Schwarzchild black hole in flat space, there is a static $\ell = 1$ mode [8] which has a very simple interpretation. The black hole in flat space breaks translation invariance: the $\ell = 1$ mode is the Goldstone mode. It just shifts the center of mass of the BH.

There is a strong analogy between the breaking of translation invariance by the Schwarzchild black hole in flat space and the breaking of conformal invariance by the Schwarzchild black hole in AdS. But there is an important difference between momentum in flat space and the conformal charges in AdS: unlike $[\vec{p}, H_{\text{flat}}] = 0$, the conformal charges do not commute with the Hamiltonian. So there is a small modification of Goldstone's theorem which takes this into account and leads to definite time dependence $e^{-it/R}$, rather than no time dependence.

More generally, the oscillations we construct in (II.2) and (IV.1) can be viewed as Goldstone states arising from the breaking of conformal symmetry by the state on which the oscillation is built, the "base state". In fact there is such a mode for each of the $2d$ charges $Q^{\pm,i}$ in (III.10) which does not annihilate the base state.

The following algebraic argument shows that the state arising from spontaneous breaking of conformal symmetry associated with any of the charges $Q^{\pm,i}$ has frequency $\pm 1/R$, in agreement with the evolution found in (II.3) for $n = 1$.

To make the logic explicit, recall the usual Goldstone argument for a charge which commutes with H in a relativistic QFT. Proceed by noting that the broken current is an interpolating field for the Goldstone mode:

$$\langle \pi(k^\mu)|j^\mu(x)|\text{symmetry-broken groundstate}\rangle = i f_\pi k^\mu e^{-ikx} . \tag{C.1}$$

Then current conservation gives $0 = \partial_\mu j^\mu \propto k_\mu k^\mu$, and hence the long-wavelength $\vec{k} = 0$ Goldstone mode $|\pi\rangle$ has $\omega = 0$.

Here is the adaptation. Let Q be any of the charges $Q^{\pm,i}$. Significantly, $Q = Q(t)$ has explicit time-dependence. Let $|\Delta\rangle$ be a state of the CFT that breaks the conformal symmetry associated with $Q(t)$ but which is still stationary (the generalization to mixed states will be clear). Let $|\Delta, \pi(\omega)\rangle$ be the Goldstone state expected from spontaneous symmetry breaking. Then we can parametrize the matrix element

$$\langle \Delta, \pi(\omega)|Q(t)|\Delta\rangle = f_\pi e^{-i\omega t}, \tag{C.2}$$

where f_π is a constant that depends on the normalization of $Q(t)$, and ω the frequency of the Goldstone excitation, which is to be determined. Taking the partial derivative with respect to time on both sides,

$$\begin{aligned}
-i\omega f_\pi e^{-i\omega t} &= \langle \Delta, \pi(\omega)|\partial_t Q(t)|\Delta\rangle \\
&= \langle \Delta, \pi(\omega)|i[H, Q(t)]|\Delta\rangle \\
&= \pm \frac{i}{R}\langle \Delta, \pi(\omega)|Q(t)|\Delta\rangle = \pm \frac{i}{R} f_\pi e^{-i\omega t},
\end{aligned} \tag{C.3}$$

where in the second line we have used $\frac{d}{dt}Q = 0$. This shows $\omega = \pm 1/R$ for $Q^\pm$, as claimed.

Appendix D: $\ell = 1$ Mode in Global AdS_4-Schwarzchild Geometry

Here we construct a linearized gravity mode of frequency $\omega = 1/R$ in the AdS_4-Schwarzchild black hole, whose existence and frequency are guaranteed by conformal symmetry, and which corresponds to a particular collective oscillation in the dual CFT. We study AdS_4 for definiteness, but the generalization to other dimensions should be clear. We proceed by finding a vector field ξ in the spacetime which falls off too slowly at the AdS boundary to generate an equivalence of configurations, but falls off quickly enough to produce a normalizable metric perturbation

$$h_{ab} = \xi_{a;b} + \xi_{b;a} . \tag{D.1}$$

By the correspondence with flat-space Schwarzchild described in C, this mode is analogous to the Goldstone mode for broken translation invariance, *i.e.* the mode that translates the center-of-mass of the black hole.

We demand that near the AdS boundary (at $r = \infty$ in the coordinates we'll use here) ξ approaches a conformal isometry; in the empty global AdS_4 background

$$ds^2 = -f_0(r)dt^2 + \frac{1}{f_0(r)}dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2) , \quad f_0(r) = 1 + \frac{r^2}{R^2} , \tag{D.2}$$

one $\ell = 1, m_z = 0$ KV takes the form [29]

$$\xi_0 = e^{-it/R}\left(\frac{ir\cos(\theta)}{\sqrt{1 + \frac{r^2}{R^2}}}\frac{\partial}{\partial t} - R\cos\theta\sqrt{1 + \frac{r^2}{R^2}}\frac{\partial}{\partial r} + \frac{R}{r}\sin\theta\sqrt{1 + \frac{r^2}{R^2}}\frac{\partial}{\partial \theta} \right) .^{12} \tag{D.3}$$

The AdS_4-Schwarschild metric is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad f(r) = 1 + \frac{r^2}{R^2} - \frac{r_+}{r} . \tag{D.4}$$

We make the ansatz

$$\xi = e^{-i\omega t}\left(ig(r)\cos\theta\frac{\partial}{\partial t} + R\,h(r)\cos\theta\frac{\partial}{\partial r} + R\,j(r)\sin\theta\frac{\partial}{\partial \theta} \right) . \tag{D.5}$$

We demand that the resulting metric perturbation h_{ab}

- is in the 'gaussian normal' gauge $h_{ra} = 0$ customary for holography,
- is normalizable, corresponding to a state in the dual CFT,
- satisfies the b.c. $\xi^a \to \xi_0^a$ as $r \to \infty$.

Our boundary condition $\xi \overset{r \to \infty}{\to} \xi_0$ determines $\omega = 1$, but leaving ω arbitrary provides a check.

Imposing the gaussian normal gauge $h_{ra} = 0$, we find

$$h(r) = c_1\sqrt{f(r)} , \tag{D.6}$$
$$g(r) = c_1\int_r^\infty \frac{dr'}{f(r')^{3/2}} + c_2 ,$$
$$j(r) = -c_1\int_r^\infty \frac{dr'}{r'^2 f(r')^{1/2}} + c_3 .$$

Normalizability determines $\omega^2 = 1$. Demanding $\xi \to \xi_0$ as $r \to \infty$ gives

$$c_1 = -1 , \quad c_2 = R , \quad c_3 = 1/R , \tag{D.7}$$

after which one can check that the normalizability conditions h_{tt}, $h_{t\theta}$, $h_{\theta\theta}$, $h_{\phi\phi}$ $\overset{r \to \infty}{\sim} O(\frac{1}{r})$ are satisfied.

The resulting metric perturbation h_{ab} is

$$\begin{aligned} h_{ab}dx^a dx^b &= e^{-\frac{it}{R}}\left[-\cos\theta\left(2fg + h\left(2r + \frac{R^2 r_+}{r^2} \right) \right)dt^2 + i\sin\theta\left(fg - r^2 j \right)dtd\theta \right.\\ &\quad \left. + 2rR(h + rj)\cos\theta\left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right] . \end{aligned} \tag{D.8}$$

---

[12] The other $m_z = 0$ KV is obtained by complex conjugation, and $m_z = \pm 1$ KVs can be constructed using the Lie bracket with $so(3)$ raising and lowering operators

$$M^\pm = e^{\pm i\varphi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\varphi} \right) .$$

Appendix E: Oscillating Observables for a Bouncing Black Hole

For simplicity, we work with the non-rotating $2 + 1$-dimensional BTZ black hole, using coordinates in which its metric is

$$ds^2 = -(\frac{r^2}{R^2} - \frac{r_+^2}{R^2})dt^2 + \frac{dr^2}{\frac{r^2}{R^2} - \frac{r_+^2}{R^2}} + \frac{r^2}{R^2}dx^2. \tag{E.1}$$

Here R is the AdS radius, r_+ the radius of the black hole, and $\frac{x}{R} \sim \frac{x}{R} + 2\pi$.

The stress-energy tensor in the finite-temperature CFT corresponding to this large global AdS black hole can be obtained by varying the bulk Einstein action with respect to the boundary metric and using local counterterms [30]. With light-like coordinates $x_\pm = t \pm x$,

$$T_{++} = T_{--} = \frac{r_+^2}{32\pi R^3} \tag{E.2}$$

and $T_\mu^\mu \propto \mathcal{R} = 0$, where $\mathcal{R}$ is the scalar curvature of the boundary metric.

The stress-energy tensor for a BTZ black hole, after a coordinate transformation corresponding to a boost in $SO(2,2)$, is then obtained most easily by transforming T_{++} and T_{--} under the conformal transformation induced on the boundary by the $SO(2,2)$ boost. Note T_μ^μ remains zero. For the boost $e^{i\beta J^{01}}$, $v = \tanh\beta$, acting on embedding coordinates of AdS_3, the corresponding coordinate transformation on coordinates (t,x,r) in (E.1) is given by

$$\tan\frac{t'}{R} = \frac{\sqrt{1-v^2}\sqrt{1+(\frac{r}{R})^2}\sin\frac{t}{R}}{\sqrt{1+(\frac{r}{R})^2}\cos\frac{t}{R} + v\frac{r}{R}\cos\frac{x}{R}}, \tag{E.3}$$

$$\tan\frac{x'}{R} = \frac{\sqrt{1-v^2}\frac{r}{R}\sin\frac{x}{R}}{v\sqrt{1+(\frac{r}{R})^2}\cos\frac{t}{R} + \frac{r}{R}\cos\frac{x}{R}},$$

$$\frac{r'}{R} = \sqrt{\left(\frac{v}{\sqrt{1-v^2}}\sqrt{1+\left(\frac{r}{R}\right)^2}\cos\frac{t}{R} + \frac{1}{\sqrt{1-v^2}}\frac{r}{R}\cos\frac{x}{R}\right)^2 + \left(\frac{r}{R}\sin\frac{x}{R}\right)^2},$$
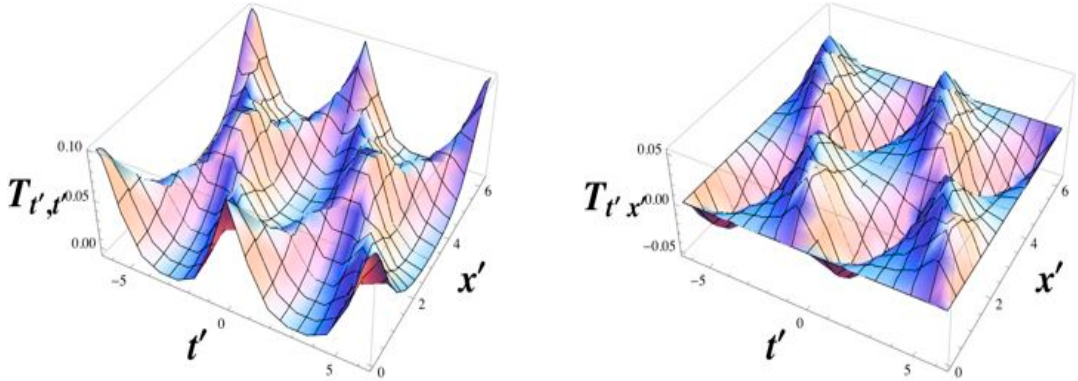

FIG. 1: Energy and momentum density as a function of position x' and time t' with $R = r_+ = G = 1$, $v = 0.5$.

from which follows

$$T_{t't'} = T_{x'x'} = \frac{\frac{(1-v^2)(R^2+r_+^2)}{(v\cos(\frac{t'-x'}{R})-1)^2} + \frac{(1-v^2)(R^2+r_+^2)}{(v\cos(\frac{t'+x'}{R})-1)^2} - 2R^2}{32\pi GR^3}, \tag{E.4}$$

$$T_{t'x'} = \frac{v(1-v^2)(R^2+r_+^2)\sin(\frac{t'}{R})\sin(\frac{x'}{R})(v\cos(\frac{t'}{R})\cos(\frac{x'}{R})-1)}{8\pi GR^3(v\cos(\frac{t'-x'}{R})-1)^2(v\cos(\frac{t'+x'}{R})-1)^2}. \tag{E.5}$$

Note $J^{01} = (L_+^1 - L_-^1)/(2i)$, so that the stress-energy tensor above is that of the collective oscillation in (IV.1) with $\alpha = \beta$. It indeed manifestly oscillates in time. Its two nonzero components are plotted in Fig. 1.
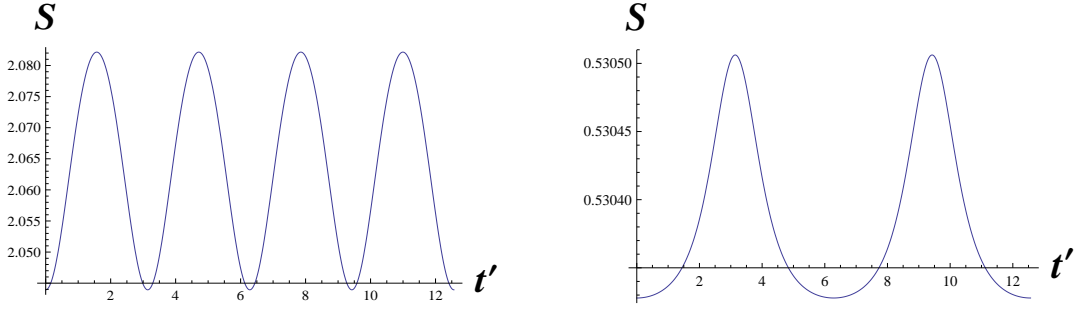
FIG. 2: Entanglement entropy of the interval $0 \leq x' \leq \pi$, left, and $0 \leq x' \leq \pi/64$ with $R = r_+ = G = c = 1$, $v = 0.5$, and cutoff $\epsilon = 0.01$.

We can also confirm that in the same state, the entanglement entropy calculated by the covariant holographic method in [22], of a spatial subregion with respect to the rest of S^1, oscillates.

Given the endpoints x'_1, x'_2 of such a spatial subregion, its entanglement entropy as a function of t' is given by

$$S(x'_1, x'_2, t') = \frac{c}{6} \log L(x'_1, x'_2, t'), \tag{E.6}$$

where $L(x'_1, x'_2, t')$ is the length of the space-like geodesic ending at points $p'_1 = (t', x'_1, r'_c), p'_2 = (t', x'_2, r'_c)$, with $r'_c = \frac{1}{\epsilon}$ an infrared cutoff in the $SO(2,2)$-boosted BTZ black hole.

The desired space-like geodesic may be obtained by first mapping p'_i to $p_i = (t_i, x_i, r_i)$, $i = 1, 2$, where (t, x, r) are non-boosted coordinates given by the inverse coordinates transformation of (E.3), and again mapping p_i to $q_i = (w_{+i}, w_{-i}, z_i)$, $i = 1, 2$, where (w_+, w_-, z) are coordinates in which the BTZ black hole has the manifestly AdS metric

$$ds^2 = R^2 \left(\frac{dw_+ dw_- + dz^2}{z^2} \right). \tag{E.7}$$

The coordinate transformation from (t, x, r) to (w_+, w_-, r) is given by

$$w_\pm \equiv X \pm T = \frac{\sqrt{r^2 - r_+^2}}{r} e^{\frac{(x \pm t) r_+}{R^2}}, z = \frac{r_+}{r} e^{\frac{x r_+}{R^2}}. \tag{E.8}$$

In the newest coordinates, the space-like geodesic with endpoints $q_i = (w_{+i}, w_{-i}, z_i)$, $i = 1, 2$ can be boosted by the mapping $w'_\pm = \gamma^{\pm 1} w_\pm$, $\gamma = \sqrt{\frac{1 - \beta}{1 + \beta}}$, with β the usual Lorentz boost parameter in coordinates (T, X), to lie on a constant T hypersurface. The resulting geodesic is a circular arc satisfying

$$(\frac{\gamma w_+ + \gamma^{-1} w_-}{2} - A)^2 + z^2 = B^2, \frac{\gamma w_+ - \gamma^{-1} w_-}{2} = C, \tag{E.9}$$

where constants γ, A, B, C can be determined by the two endpoints q_i, $i = 1, 2$, and which has length

$$L = R \log \frac{(w_{+2} - w_{+1})^2 (w_{-2} - w_{-1})^2 + 2(w_{+2} - w_{+1})(w_{-2} - w_{-1})(z_1^2 + z_2^2) + (z_2 - z_1)^2}{(w_{+2} - w_{+1})(w_{-2} - w_{-1}) z_1 z_2}. \tag{E.10}$$

Translating back to original coordinates (t', x', r'), $L = L(x'_1, x'_2, t')$, one has the holographic entanglement entropy (E.6) in a bouncing black hole geometry. The smoothly oscillating entanglement entropy is plotted for two different intervals in the spatial domain S^1 of the CFT in Fig. 2.

Appendix F: Oscillations in Galilean Conformal Field Theory

We point out that these oscillations can be observed in experiments on ultracold fermionic atoms at unitarity, and that related modes have already been studied in detail [31–34] (for reviews of the subject see [35–38]). The specific mode we discuss has been predicted previously by very different means in [4].

In the experiments, lithium atoms are cooled in an optical trap and their short-ranged two-body interactions are tuned to a Feshbach resonance via an external magnetic field. Above the superfluid transition temperature, this physical system is described by a Galilean-invariant CFT [39, 40]. The symmetries of such a system comprise a Schrödinger algebra, which importantly for our purposes contains a special conformal generator C. (This symmetry algebra has been realized holographically by isometries in [41, 42] (see also [43]) and more generally in [44].) The Hamiltonian for such a system in a spherically-symmetric harmonic trap H_{osc} is related to the free-space Hamiltonian H by [39, 40]

$$H_{\text{osc}} = H + \omega_0^2 C. \tag{F.1}$$

This H_{osc} is analogous to the Hamiltonian of relativistic CFT on the sphere in that its spectrum is determined by the spectrum of scaling dimensions of operators.

In the experiments of [31, 32], "breathing modes" of the fluid were excited by varying the frequency of the trap. One goal was to measure the shear viscosity of the strongly-coupled fluid (for a useful discussion, see §5.2 of [45]). Energy is dissipated via shear viscosity in these experiments because the trapping potential is not isotropic. The anisotropy of the trap breaks the special conformal generator. If the trap were spherical, our analysis would apply, and we predict that the mode with frequency $2\omega_0$ would not be damped, to the extent that our description is applicable (*e.g.* the trap is harmonic and spherical and the coupling to the environment can be ignored)[13].

This prediction is consistent with the linearized hydro analysis of [46, 47], and one can check that the sources of dissipation included in [34, 45] all vanish for the lowest spherical breathing mode. The mode we predict is adiabatically connected to the breathing mode studied in [31–34]. Note that an infinite-lifetime mode of frequency $2\omega_0$ is also a prediction for a free non-relativistic gas. Indeed, this is also a Galilean CFT, though a much more trivial one.

There is a large (theoretical and experimental) literature on the collective modes of trapped quantum gases, *e.g.* [46–49]. Much of the analysis of these collective modes in the literature relies on a hydrodynamic approximation. In this specific context of unitary fermions in 2+1 dimensions, *linearized* modes of this nature were described in a full quantum mechanical treatment at zero temperature by Pitaevskii and Rosch [50] and further studied in 3+1 dimensions by Werner and Castin [39]. Further, their existence was attributed to a hidden $so(2, 1)$ symmetry of the problem, which is the relevant part of the Schrödinger symmetry. The undamped nonlinear mode at $2\omega_0$ was described in [4]. Here we make several additional points:

- There are such stable modes at any even multiple of the frequency of the harmonic potential.

- The fully-nonlinear modes of finite amplitude can be explicitly constructed, and remain undamped.

- These modes are superuniversal – they can be generalized to oscillations in any conformal field theory.

In a Galilean CFT with a harmonic potential, the oscillations can be constructed as follows. Consider an eigenstate of $H_{osc} = H + \omega_0^2 C$ constructed from a primary operator $\mathcal{O}$ with dimension $\Delta_{\mathcal{O}}$ [40],

$$|\Delta_{\mathcal{O}}\rangle = e^{-H}\mathcal{O}^\dagger|0\rangle, \ H_{osc}|\Delta_{\mathcal{O}}\rangle = \Delta_{\mathcal{O}}|\Delta_{\mathcal{O}}\rangle. \tag{F.2}$$

Defining ladder operators $L_\pm \equiv H - \omega_0^2 C \pm i\omega_0 D$, the states

$$\exp\left(\alpha_0 L_+ + \beta_0 L_-\right)|\Delta_{\mathcal{O}}\rangle \tag{F.3}$$

with α_0, β_0 c-numbers, evolve under H_{osc} as

$$e^{-it\Delta_{\mathcal{O}}} \exp\left(e^{-2i\omega_0 t}\alpha_0 L_+ + e^{2i\omega_0 t}\beta_0 L_-\right)|\Delta_{\mathcal{O}}\rangle. \tag{F.4}$$

The algebraic manipulations which demonstrate this time evolution are identical to those for coherent states of a simple harmonic oscillator.

The Galilean generalization of (II.4) should be clear. The crucial property of the CFT spectrum is again the existence of equally-spaced levels connected to $|\Delta_{\mathcal{O}}\rangle$ by the ladder operators L_+, L_-.

Any real trap will be slightly anisotropic. Following [34, 45], estimates can be made using linearized hydrodynamics for the damping rate arising from the resulting shear of the fluid, in terms of the measured shear viscosity (see eqn. (159) of [45]). We are not prepared to estimate other sources of dissipation. It would be interesting to use softly-broken conformal invariance to predict the frequencies and damping rates of collective modes in slightly anisotropic traps in the nonlinear regime.

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[13] Note that this is not the lowest-frequency mode of the spherical trap; linearized hydrodynamic analysis predicts a linear mode with frequency $\sqrt{2}\omega_0$.

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