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Phys. Rev. D **85**, 086006 — Published 23 April 2012

DOI: 10.1103/PhysRevD.85.086006

FTPI-MINN-12/01, UMN-TH-3027/12
ITEP-TH- 41/11

More on the Tensor Response of the QCD Vacuum to an External Magnetic Field

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Abstract

In this Letter we discuss a few issues concerning the magnetic susceptibility of the quark condensate and the Son-Yamamoto (SY) anomaly matching equation. It is shown that the SY relation in the IR implies a nontrivial interplay between the kinetic and WZW terms in the chiral Lagrangian. It is also demonstrated that in a holographic framework an external magnetic field triggers mixing between scalar and tensor fields. Accounting for this, one may calculate the magnetic susceptibility of the quark condensate to all orders in the magnetic field.

PACS numbers: 11.25.Tq, 11.40.Ha, 12.38.Aw, 12.39.Fe

I. INTRODUCTION

The magnetic susceptibility χ, introduced in [1] in the sum rules framework, is an interesting characteristic of the vacuum response to an external magnetic field in the confinement phase. It measures the induced tensor current in the QCD vacuum. The expression

$$\chi_v = -\frac{N_c}{4\pi^2 f_\pi^2} \tag{1}$$

has been obtained analytically in [2] using the OPE and pion dominance for the $\langle VVA \rangle$ correlator of two vector currents and one axial current in the kinematics where two virtualities of the external legs are large while one vector current represents the constant external field strength. Surprisingly it differs from the sum rule fit by a factor of 3 which implies that some qualitative essential effect responsible for the disagreement has been overlooked yet. The several phenomenological estimates yield the low value of the susceptibility while the large N_c consideration [3] fits the Vainshtein relation (1).

Hence it is natural to look for alternative derivations of χ to identify the missed ingredients. The problem was discussed in the holographic hard wall model involving the 5D Yang-Mills and Chern-Simons (CS) terms. The Lagrangian corresponds to the gauge theory on the flavor branes extended along the radial coordinate in the AdS space. It turns out that in this model the Vainshtein relation (1) is not exact, however it is fulfilled with good accuracy [4]. Moreover it was shown in [4] that the whole answer follows from the CS term which implies that we are dealing with a sort of "anomalous" phenomena.

Other more refined holographic models have been considered by Son and Yamamoto (SY) in [5]. They derived a new relation between two-point and three-point correlators which yields nontrivial matching conditions for the low-energy QCD parameters of the mesons. The SY relation is assumed to be valid at any momentum transfer, for instance, Vainshtein expression (1) follows from the SY relation at large virtualities if one assumes that operator product expansion of QCD is applicable. This 'if' is important because the SY model does not support OPE *per se*: dependence on momentum transfer is exponential and does not contain power terms required by OPE. Different aspects of the SY relation were discussed in [6–8].

The situation looks a little bit puzzling since there is no field theory derivation of the SY relation yet. The expression for χ in terms of pion decay coupling suggests that it can be obtained purely in terms of the chiral Lagrangian together with the SY relation. With the holographic experience it could be expected that the Wess-Zumino-Witten (WZW) term in the chiral Lagrangian related to the CS term in 5D should be responsible for the nontrivial answer.

In this note motivated by the comments above we consider the additional arguments concerning the derivation of χ. Since the holographic model of QCD is nothing but the extended chiral Lagrangian it is natural to look more carefully at the place of the SY relation in the ChPT per se. The small Q^2 region is the most comfortable to be analyzed in ChPT hence we shall look at the first terms in small Q^2 expansion. The relation between the ChPT parameters at the tree level has been discussed in [6] and we extend it to the one loop level focusing at the chiral logs. It will be shown that the SY relation holds true for the log terms.

A simple argument involving the calculation of the quark determinant in the tensor source background implies that the nonvanishing magnetic susceptibility corresponds to the peculiar additional mixed term in the Lagrangian. In the improved holographic model for QCD [9–11] the tensor source in $D = 4$ theory is promoted into the tensor field in $D = 5$. We shall analyze the improved model in a magnetic field focusing at the scalar-tensor mixing. It turned out that in the improved model the magnetic susceptibility can be obtained to all orders in the magnetic field. In a small field the magnetization grows linearly with the field, in accordance with its generally established properties, while in a large field it does not depend on the magnetic field.

The paper is organized as follows. In Section 2 we consider the SY relation in ChPT and show the matching of the chiral logs in this relation. Some general comments concerning the SY relation are also presented. In Section 3 we consider the improved holographic model for QCD with the tensor field and demonstrate how the scalar-tensor mixing in the magnetic field yields the magnetic susceptibility. Some discussion concerning the proper degrees of freedom can be found in the last Section.

II. SY RELATION WITHIN THE CHIRAL LAGRANGIAN

A. SY relation and chiral logs

Let us analyze the SY relations in the framework of ChPT. The SY relation has been obtained holographically and is based on the simple $D = 5$ action on the worldvolume of probe flavor branes involving Yang-Mills and Chern-Simons terms for the flavor gauge group. Taking into account that correlators of the vector and axial currents are two independent solutions to the second order differential operator in the radial coordinate z in AdS space their Wronskian is z-independent. This argument works when the CS term is neglected. On the other hand CS term itself yields the nontrivial $\langle VA\rangle$ correlator in the magnetic field which is proportional to the same Wronskian. Hence in a weak magnetic field the following relation holds [5]

$$w_T(Q^2) = \frac{N_C}{Q^2} - \frac{N_C}{f_\pi^2}\left[\Pi_A(Q^2) - \Pi_V(Q^2)\right] \tag{2}$$

where w_T is defined via the two-point correlator in the external weak electromagnetic field with the constant field strength $F_{\alpha\beta}$

$$\langle V_\mu A_\nu\rangle_{\tilde F} = \frac{1}{4\pi^2}\left[w_T(q^2)(-q^2\tilde F_{\mu\nu} + q_\nu q^\sigma \tilde F_{\mu\sigma} - q_\mu q^\sigma \tilde F_{\nu\sigma}) + w_L(q^2)\, q_\nu q^\sigma \tilde F_{\mu\sigma}\right], \tag{3}$$

V, A are the vector, $\bar q\,\mathcal{V}\gamma_\mu q$, and axial, $\bar q\,\mathcal{A}\gamma_\mu\gamma_5 q$, currents, $\tilde F$ denotes the dual field strength, $\tilde F_{\gamma\delta} = \frac{1}{2}\,\epsilon_{\gamma\delta\alpha\beta}F^{\alpha\beta}$, and Π_A , Π_V are the corresponding two-point correlators. The relation holds for all values of $Q^2 = -q^2$.

The definitions of the above correlators are

$$\begin{aligned}
\frac{1}{2}\,\mathrm{Tr}\,(\mathcal{Q}\mathcal{V}\mathcal{A})\,\langle V_\mu A_\nu\rangle_{\tilde F} &\equiv \int d^4x\, e^{iqx}\langle T\{V_\mu(x)A_\nu(0)\}\rangle_{\tilde F}\,,\\
\frac{1}{2}\,\mathrm{Tr}\,(\mathcal{V}\mathcal{V})\,\Pi_V(Q^2)(q_\mu q_\nu - g_{\mu\nu}q^2) &\equiv \int d^4x\, e^{iqx}\langle T\{V_\mu(x)V_\nu(0)\}\rangle_0\,,\\
\frac{1}{2}\,\mathrm{Tr}\,(\mathcal{A}\mathcal{A})\,\Pi_A(Q^2)(q_\mu q_\nu - g_{\mu\nu}q^2) &\equiv \int d^4x\, e^{iqx}\langle T\{A_\mu(x)A_\nu(0)\}\rangle_0\,,
\end{aligned} \tag{4}$$

where flavor dependence on matrices of vector and axial currents, $\mathcal{V}$ and $\mathcal{A}$, as well as that for the electric charge, $\mathcal{Q}$, is factored out.

It is natural to look at the matching of the SY relation with the ChPT since the holography provides the ChPT derivation from the "first principles". The SY relation is derived from the $D = 5$ equations of motion hence according the holographic dictionary it should correspond to the tree approximation in the ChPT. The condition imposed by the SY at $Q^2 = 0$ on the parameters of the chiral lagrangian has been found in [6]

$$L_{10} = -4\pi^2 C_{22} \tag{5}$$

where L_{10} corresponds to the even term in the chiral Lagrangian at the $O(p^4)$ order while C_{22} corresponds to a particular odd term at the $O(p^6)$ order. The condition is unexpected since it relates the odd and even terms in the Lagrangian. Unfortunately this relation between constants can not be used as the test of the SY relation since C_{22} is not known with the high accuracy.

To get some test of the SY relation let us focus at the terms in the correlators involving the chiral logs. Some comments are required before the looking at log terms. Naively such terms are subleading in $1/N_c$ hence in the holographic approach these should be considered as corrections to the equations of motion. On the other hand the log terms are considered as the renormalization of the constants in the chiral Lagrangian hence one should assume that the relation (5) valid at the tree level holds upon the renormalization. Therefore it is natural to look at the matching of the chiral logs.

At the right hand side of Eq. (2) the chiral log follows from the pion loop in the correlator of the vector currents

$$\Pi_V^{\rm chir}(Q^2 \to 0) = c \log Q^2 \,, \qquad c = -\frac{1}{48\pi^2} \,. \tag{6}$$

There are no logs in the correlator of the axial currents. On the other hand the chiral log in the $\langle VVA \rangle$ correlator can be traced from the particular term in the WZW term in the chiral Lagrangian responsible for the decay $\gamma^* \to 3\pi$,

$$S_{WZW}^{3\pi} = -\frac{N_c}{24\pi^2} \int {\rm Tr}\, A\big(dU^{-1}U\big)^3 \to -\frac{iN_c}{24\pi^2 f_\pi^3} \int {\rm d}^4x \, {\rm Tr}\, \tilde{F}^{\gamma\delta} \pi \, \partial_\gamma \pi \, \partial_\delta \pi \,. \tag{7}$$

Converting two pions from this vertex to the vector current and associating the remaining pion with the axial current we get the $\log Q^2$ contribution to w_T ,

$$w_T^{\rm chir} = c_1 \log Q^2 \,, \qquad c_1 = \frac{N_c}{f_\pi^2}\, c \,, \tag{8}$$

which is consistent with the SY relation.

Let us emphasize that there is no freedom in the terms in the chiral lagrangian involved into the chiral logs, hence the matching is exact although at the subleading order in $1/N_c$.

Note that similarly to the tree-level case, the SY relation implies an unexpected relation between the coefficient in front of the even kinetic term at the $O(p^2)$ order and of an odd WZW term at the $O(p^4)$ order in the chiral Lagrangian.

B. The mixed term in the chiral Lagrangian

Let us argue that nonvanishing magnetic susceptibility implies a peculiar term in the effective Lagrangian. To this aim we introduce a source term for the quark tensor current into the QCD Lagrangian

$$\delta L_1 = B_{\mu\nu}\,\bar{q}\,\mathcal{B}\,\sigma^{\mu\nu}q \equiv i\tilde{B}_{\mu\nu}\,\bar{q}\,\mathcal{B}\sigma^{\mu\nu}\gamma_5 q \tag{9}$$

where $B_{\mu\nu}$ is an external source field whose possible interpretation shall be discussed below and $\mathcal{B}$ is a diagonal flavor matrix. Accounting for the chiral features of the quark operator in Eq. (9) it simple to determine the corresponding term in the chiral Lagrangian in the linear approximation in the $B_{\mu\nu}$ field,

$$\delta L_{WZW} = -\frac{1}{2}\,\chi\,\langle\bar{q}q\rangle\,B_{\mu\nu}F^{\mu\nu}\,\mathrm{Tr}\,(U+U^{\dagger})\mathcal{B}\mathcal{Q}\,, \tag{10}$$

where χ is the magnetic susceptibility, $\langle\bar{q}q\rangle$ is the quark condensate and $U = \exp(2i\pi^a t^a/f_\pi)$ is the mesonic matrix ($f_\pi = 92\,\mathrm{MeV}$). This can be viewed as a definition of the magnetic susceptibility. Note that this term to some extent can be considered as the shift of the effective quark mass in the external fields. In the next Section we shall see that this term promoted into the holographic $D = 5$ action provides an important scalar-tensor mixing.

It it worth making a few comments concerning the implications of this effective WZW-like term in the chiral Lagrangian. First, the vacuum tensor current proportional to the chiral condensate in the magnetic field can be attributed to the stringy degrees of freedom if we identify the tensor source in (10) as NS or Ramond two-form fields. With such identification the flow of the F1 or D1 strings in the vacuum occurs in the magnetic field. On the other hands the mesons are identified holographically as the F1 strings connecting the flavor branes hence such "stringy" current corresponds in fact to a kind of mesonic vacuum current.

Secondly, there is an anomalous electromagnetic current proportional to the condensate in the external tensor field. Indeed we defined the current

$$\langle J_\nu\rangle_B = \frac{\delta S_{WZW}}{\delta A_\nu} \tag{11}$$

which gets contributions from the sources of the tensor field or from the varying pion field.

$$\langle J_\nu \rangle_B = \frac{1}{2}\, \chi \, \langle \bar{q}q \rangle \, \partial^\mu \big[B_{\mu\nu} \; \mathrm{Tr}\, (U + U^\dagger) \mathcal{B}\mathcal{Q} \big]. \tag{12}$$

For the varying pion case the anomalous current is the analogue of the Goldstone-Wilczek current. For the varying tensor field an interesting possibility emerges. Using the relation between the massive vector and tensor in four dimensions we could get the nonvanishing electromagnetic current if the vector meson gets condensed. There are some indications of such a condensation in the magnetic field both in the effective theory [12] and in the holographic framework [13]. Hence one could speculate about the nonperturbative current proportional to the product of quark and vector meson condensates.

Finally, if we expand the anomalous term in the pion field we could get the anomalous interaction of pions with the tensor current in the magnetic field. For instance, the matrix element

$$\langle 0 | \bar{q} \sigma_{\mu\nu} q | \pi^0 \pi^0 \rangle = \frac{1}{3 f_\pi^2}\, \chi \, \langle \bar{q} q \rangle F_{\mu\nu} \, . \tag{13}$$

C. On the derivation of the SY relation

The SY relation has been obtained in a slightly tricky way, hence it would be nice to get it more regularly as a kind of a Ward identity. Here we restrict ourselves by two generic remarks. Since the key observation in the derivation in [5] was the z-invariance of the Wronskian of the vector and axial currents it is reasonable to look at the radial variable z in the Hamiltonian framework. That is, following [14] we assume that it is considered as a time variable for the RG Hamiltonian evolution in the $D = 5$ gauge theory.

In the Hamiltonian framework of the gauge theories there are two natural equations involving the dependence on the boundary values of the dynamical variables. These are the gauge constraint or Gauss law and the Hamiltonian constraint or a kind of the Hamilton-Jacobi (HJ) equation. We are in a peculiar situation with the Hamiltonian constraint since the metric depends on the radial coordinate and is therefore "time-dependent".

First consider the Gauss law constraint with respect to the flavor gauge group $\mathrm{SU}_L(N_F) \times \mathrm{SU}_R(N_F)$ on the flavor branes. In the Hamiltonian approach the Gauss law reflect the gauge invariance with respect to the flavor gauge group and can be identified with the generator of the z-independent gauge transformations. Since there are $D = 5$ CS terms for the left and right gauge flavor fields the canonical momenta get modified

$$\Pi_L = E_L + A_L F_L\,, \qquad \Pi_R = E_R - A_L F_L\,, \tag{14}$$

while the canonical momenta for the scalar field are standard. Using the Hamiltonian relation

$$\Pi = \frac{\delta S}{\delta x}\,, \tag{15}$$

where S is the action, and the standard holographic relation for the $D = 4$ currents $J_\mu = \frac{\delta S}{\delta A}$ one immediately recognizes that the Gauss law constraint in the bulk theory precisely produces the anomaly equation for the axial current at the boundary including the mass term. The fact that the Gauss law is valid at any time in $D = 5$ theory gets translated into the claim that the axial anomaly is seen at all scales in the boundary $D = 4$ theory. Note that in the conventional gauge theory the Gauss law is complemented by the gauge $A_0 = 0$. In the current situation the similar equation reads as $A_z = 0$; however, one should not forget that the pion field can be identified with the holonomy of the radial component of the flavor gauge group.

In the holographic setting the HJ equation for the bulk metric has been identified as the RG equation in the boundary theory in [14]. Here we have to consider a similar HJ equation for the gauge fields and scalars. Taking into account the shift of the canonical momenta and forgetting for a moment the metric one obtains for the left gauge part of the total Hamiltonian

$$\left(\frac{\delta S}{\delta A} - AF\right)^2 + F_{ij}^2 \tag{16}$$

and similarly for the contribution of the right gauge field and scalars. The HJ-type equations are quite convenient for the derivation of Ward identities in the boundary theory since it involves the desired variational derivatives. It is important that the HJ-like equations due to the change of the canonical momenta involve the terms with the different number of the variational derivatives. Hence potentially one could hope that the additional variational derivatives of the HJ equation upon taking into account the Gauss law constraint would yield the SY relation. We did not succeed along this way of reasoning, however we plan to discuss the complete set of the Ward identities induced from the bulk theory elsewhere. In particular we plan to elaborate the constraints emerged from the dynamics of the higher rank fields induced by the color branes in the brane approach. Note that some examples of the derivation of the boundary Ward identities from the bulk HJ equation can be found in [15].

There is also some analogy with the $N = 1/2$ SUSY YM case which can be considered as $N = 1$ SYM theory in the self-dual constant graviphoton background C_1. The following term gets induced in the graviphoton field

$$\delta L = dC_1 \wedge F\bar{\lambda}\lambda \tag{17}$$

which is analogous to the anomalous term in QCD we have discussed. The analogy with QCD becomes even more close when we remind that the gluino condensate is developed in $N = 1/2$ SUSY YM like the chiral condensate in QCD. Moreover in the $N = 1/2$ theory one can consider the Ward identities reflecting the single unbroken SUSY [16]. This Ward identity amounts to a particular degeneration in the spectral densities in the $J = 1^{\pm}$ channels [17]. Since the spectral densities follow from the two-point correlators these Ward identities can be considered as some analogue of the SY relation in QCD without the anomalous three-point correlator.

III. A HOLOGRAPHIC MODEL WITH THE TENSOR FIELD

In this Section we shall consider the scalar-tensor mixing in the improved holographic model of QCD which involves the tensor field [9–11]. It is a 5-dimensional gauge theory embedded in a pure AdS geometry with an infrared hard-wall boundary:

$$ds^2 = \frac{\ell^2}{z^2}\left(-dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right), \qquad 0 \le z \le z_m\,, \tag{18}$$

where $\eta_{\mu\nu}$ is mostly negative: $\eta = \mathrm{diag}(+---)$, and ℓ is the AdS_5 radius and shall be omitted henceforward (thus rescaling the coupling constants). This model contains three types of fields: a complex scalar X, two gauge fields L_μ and R_μ, and a complex antisymmetric tensor $B_{\mu\nu}$. They are put into correspondence with the following operators of QCD:

$$\begin{aligned} &\bar{q}_{R\,\bar{f}}\, q^f_L \leftrightarrow X^f_{\bar{f}}\,, && \bar{q}_{R\,\bar{g}}\gamma_\mu q^{\bar{f}}_R \leftrightarrow R^{\bar{f}}_{\mu\,\bar{g}}\,, \\ &\bar{q}_{R\,\bar{f}}\,\sigma_{\mu\nu} q^f_L \leftrightarrow B^f_{\mu\nu\,\bar{f}}\,, && \bar{q}_{L\,g}\gamma_\mu q^f_L \leftrightarrow L^f_{\mu\,g}\,, \end{aligned} \tag{19}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$; $f, \bar{f}$ are the flavor indices of QCD with respect to the (global) $U(N_f)_L \times U(N_f)_R$ symmetry which becomes the gauge group of the five-dimensional theory.

Accordingly, the fields X and $B_{\mu\nu}$ are bifundamentals, whereas L_μ and R_μ are adjoint with respect to $U(N_f)_L$ and $U(N_f)_R$. These properties allow us to properly introduce covariant derivatives: $DX = dX - iLX + iXR$, $H = DB = dB - iL \wedge B + iB \wedge R$.

The action proposed in [11] is:

$$\mathcal{S} = \int d^5x\sqrt{-g}\ \mathrm{Tr}\Bigg\{-\frac{1}{4g_5^2}\left(F_L^2 + F_R^2\right) + g_X^2\left(|DX|^2 - m_X^2|X|^2\right) + \frac{\lambda}{2}\left(X^+F_LB + BF_RX^+ + \text{c.c.}\right) \\ - 2g_B\left(\frac{i}{6}\frac{\epsilon^{MNPQR}}{\sqrt{-g}}\left(B_{MN}H^+_{PQR} - B^+_{MN}H_{PQR}\right) + m_B|B|^2\right)\Bigg\}. \tag{20}$$

This action is a modification of a simpler hard-wall action [18] which takes into account the tensor field. The interaction term XFB on the AdS boundary is reduced to the term (10) of the chiral Lagrangian which we have discussed in the previous Section.

The constants have been fixed in previous works by comparing various correlators at large Euclidean Q^2 with OPE in QCD [11, 18–20]. The masses are fixed by requiring that the scaling properties of the fields match those of the corresponding operators in the UV: $m_X^2 = -3$, $m_B = 2$. Note that due to a non-canonical form of the kinetic term of the tensor field its physical mass is actually 1 in units of ℓ^{-1}. In this case the vacuum solution for X is

$$X(z) = \frac{1}{2}\left(mz + \frac{1}{g_X^2}\langle\bar{q}q\rangle\, z^3\right) \times \mathbf{1}_{N_f\times N_f}\,. \tag{21}$$

From now on we shall only consider the Abelian degrees of freedom, as the flavor structure of the 5D fields is trivial, since the condensates, both scalar and tensor, as well as the electromagnetic field, are diagonal in the flavor space. Hence, the equations of motion for each individual flavor q_f are the same as for the singlet component with a substitution $F^{MN}_{L,R} \to e_f F^{MN}_{L,R}$, where e_f is the electromagnetic charge of q_f. Furthermore, we shall be working in an ansatz where the axial field $(L_M - R_M)/\sqrt{2}$ is zero, which is consistent with the equations of motion. This choice is motivated by the fact that we are considering a setup with no sources and with a zero vacuum expectation value of the axial current. The same argument can be made from the point of view of the chiral perturbation theory. Moreover, introduction of a nonzero axial field would nontrivially influence the dynamics of the phase of the scalar field, i.e. the pion. We are left with only the vector field $V_M = (L_M + R_M)/\sqrt{2}$. In this case the covariant derivatives become ordinary. Let us also split the scalar and tensor fields into real and imaginary parts: $X = \dfrac{X_+ + iX_-}{2}$; $B_{MN} = \dfrac{(B_+ + iB_-)_{MN}}{\sqrt{2}}$. These new fields are dual to the following operators in QCD:

$$\bar{q}q \leftrightarrow X_+ \,, \qquad \frac{1}{\sqrt{2}}\,\bar{q}\sigma_{\mu\nu}q \leftrightarrow B_{+\mu\nu} \,,$$
$$i\bar{q}\gamma_5 q \leftrightarrow X_- \,, \qquad \frac{i}{\sqrt{2}}\,\bar{q}\gamma_5\sigma_{\mu\nu}q \leftrightarrow B_{-\mu\nu} \,,$$
$$\bar{q}\gamma_\mu q \leftrightarrow V_\mu \,. \tag{22}$$

We have noted that $B_{\mu\nu}$ is bifundamental with respect to $U(N_f)_L \times U(N_f)_R$ which guarantees its being complex-valued, its real and imaginary parts corresponding to the tensor and pseudotensor operators (22). These operators happen to be related to each other in 4D, $\bar{q}\sigma^{\mu\nu}\gamma_5 q = \frac{i}{2}\epsilon^{\mu\nu}{}_{\lambda\rho}\bar{q}\sigma^{\lambda\rho}q$, that fact is reflected in Eq. (9). From the holographic point of view, this condition is ensured by the fact that the kinetic term for $B_{\mu\nu}$ (20) is of the first order in derivatives, which leads to its complex self-duality [10, 11]. Thus, as we shall see, the "double counting" of the degrees of freedom that arises after we have introduced a complex tensor field is compensated by constraints imposed on half of them, see Eqs. (27). One may wonder whether it is possible to avoid this redundancy by dealing with a real-valued tensor field from the beginning; however, while this is not impossible, amending the model in this way while preserving holographic field-operator correspondence rules and general self-consistency appears to be quite cumbersome.

Let us now rewrite the action (20) in terms of those fields:

$$\mathcal{S} = \int d^5x\sqrt{-g} \quad \mathrm{Tr}\Bigg\{ -\frac{1}{4g_5^2}F_V^2 + \frac{g_B}{3}\frac{\epsilon^{MNPQR}}{\sqrt{-g}}\left(B_{-MN}H_{+PQR} - B_{+MN}H_{-PQR}\right)$$
$$+ \sum_{+,-}\Big[-g_B m_B B_{\pm MN}B^{\pm MN} + \frac{g_X^2}{4}\left(\partial_M X_\pm \partial^M X_\pm - m_X^2 X_\pm^2\right) + \frac{\lambda}{2}X_\pm\left(F_V\right)_{MN}B_\pm^{MN}\Big]\Bigg\} \tag{23}$$

A. Equations of motion

The action (23) yields the following first-order equations of motion for the tensor field:

$$\pm z\epsilon^{\mu\nu\lambda\rho}H_{\pm z\lambda\rho} + 2B_\mp^{\mu\nu} = \frac{\lambda}{4g_B}X_\mp F_V^{\mu\nu},$$
$$\pm\frac{z}{3}\epsilon^{\mu\lambda\rho\sigma}H_{\pm\lambda\rho\sigma} + 2B_\mp^{\mu z} = \frac{\lambda}{4g_B}X_\mp F_V^{\mu z}, \tag{24}$$

where the indices are contracted with a flat metric diag$(+----)$. They may be rewritten as second-order equations in which the real and imaginary components are disentangled and

$(F_V)_{\mu\nu}$ is assumed to be z-independent (which is a self-consistent solution in the case of a constant uniform magnetic field). Along with the equation on X we have:

$$z\partial_z\left(zH_\pm^{z\alpha\beta}\right)+B_\pm^{\alpha\beta}+z^2\partial_\mu H_\pm^{\mu\alpha\beta}=\frac{\lambda}{8g_B}\left[X_\pm F_V^{\alpha\beta}\pm z\partial_z X_\mp\tilde{F}_V^{\alpha\beta}\right],$$
$$z^2\partial_\lambda H_\pm^{\lambda\mu z}+B_\pm^{\mu z}=\pm\frac{\lambda}{2g_B}z\partial_\lambda\left(X_\mp\tilde{F}_V^{\mu\lambda}\right),$$
$$\partial_z\left(\frac{1}{z^3}\partial_z X_\pm\right)+\frac{3}{z^5}X_\pm-\frac{1}{z^3}\partial_\mu\partial^\mu X_\pm=-\frac{\lambda}{g_X^2}\frac{1}{z}(F_V)_{\mu\nu}B_\pm^{\mu\nu}. \tag{25}$$

Directing the third axis along the magnetic field so that $(F_V)_{12}=(\tilde{F}_V)_{03}=\mathbf{B}$ we get the following equations on $(B_\pm)_{12}$ and $X_\pm$:

$$\left(\partial_z^2+\frac{1}{z}\partial_z-\frac{1}{z^2}-\partial_\mu\partial^\mu\right)(B_\pm)_{12}=-\frac{\lambda}{8g_B}\frac{1}{z^2}X_\pm\,(F_V)_{12}\,,$$
$$\left(\partial_z^2-\frac{3}{z}\partial_z+\frac{3}{z^2}-\partial_\mu\partial^\mu\right)X_\pm=-\frac{2\lambda}{g_X^2}z^2\,(F_V)_{12}\,(B_\pm)_{12}\,. \tag{26}$$

From Eqs. (25) it follows that we will also have nontrivial $(B_\mp)_{03}$, $(B_\mp)_{0z}$, and $(B_\mp)_{3z}$ components, which may be expressed through $(B_\pm)_{12}$ with the use of Eqs. (24) (assuming $\epsilon^{0123}=1$):

$$(B_\mp)_{03}=\pm z\partial_z\,(B_\pm)_{12}\,,\qquad (B_\mp)_{0z}=\pm z\partial_3\,(B_\pm)_{12}\,,\qquad (B_\mp)_{3z}=\pm z\partial_0\,(B_\pm)_{12}\,. \tag{27}$$

A most general property of the equations is that the scalar and tensor degrees of freedom X_+, B_{+12} decouple from the pseudoscalar and pseudotensor X_-, B_{-12}, thus forming two independent sectors, while due to complex self-duality $B_{\mp 03},\ B_{\mp 0z},\ B_{\mp 3z}$ are admixed to the first (second) sector of the solution. Those sectors may be treated independently.

B. Solutions and boundary conditions

After we Fourier-transform the equations, solutions of the Eqs. (26, 26) assume the form:

$$X_+ + iX_- = z^2 f_X(qz)e^{ikx_3-i\omega t};$$
$$(B_+ + iB_-)_{12}=\frac{g_X}{4\sqrt{g_B}}f_B(qz)e^{ikx_3-i\omega t}, \tag{28}$$

where $f_X(qz)$ and $f_B(qz)$ are, generally speaking, superpositions of four Bessel functions:

$$f_X(qz)=C_1\mathcal{J}_1\big(\sqrt{1+\beta}qz\big)+C_2\mathcal{J}_1\big(\sqrt{1-\beta}qz\big)+C_3\mathcal{Y}_1\big(\sqrt{1+\beta}qz\big)+C_4\mathcal{Y}_1\big(\sqrt{1-\beta}qz\big);$$
$$f_B(qz)=C_1\mathcal{J}_1\big(\sqrt{1+\beta}qz\big)-C_2\mathcal{J}_1\big(\sqrt{1-\beta}qz\big)+C_3\mathcal{Y}_1\big(\sqrt{1+\beta}qz\big)-C_4\mathcal{Y}_1\big(\sqrt{1-\beta}qz\big), \tag{29}$$

where $\beta = \dfrac{|\lambda|}{2g_X\sqrt{g_B}}|\mathbf{B}/q^2|$ and $q^2 = \omega^2 - k^2$ is the Minkowski 4-momentum squared. $\mathcal{J}$, $\mathcal{Y}$ are the Bessel and Neumann functions or their analytical continuations (if we want to consider greater magnetic fields or solutions with Euclidean momenta). The Neumann functions in (29) correspond to non-normalizable modes of the AdS_5 fields (28) in the UV. As we are only interested in the mixing between vacuum expectation values without sources, $C_3 = C_4 = 0$. C_1 and C_2 are determined by the boundary conditions in the IR.

According to (27), there is a constraint that relates $(B_\mp)_{03}$ to $(B_\pm)_{12}$, which means that in order to construct a self-consistent variation principle for the tensor field one needs to take into account that half of the tensor degrees of freedom are not independent due to the tensor field's complex self-duality. Such variation principle has been proposed in [11], and it states that in our case

$$\delta_B \mathcal{S} = 2g_B \int d^4x \left(B_{+12} + B_{-03}\right) \delta\left(B_{12} - B_{-03}\right). \tag{30}$$

Note that according to (27) B_{+12} and B_{-03} have equal normalizable modes, thus contributing equally to the tensor condensate (30). Since the kinetic term of the tensor field is of the first order in derivatives, the boundary variation term in (30) contains no differentiation with respect to z. Hence it makes sense to impose on it a Dirichlet boundary condition at $z = z_m$ rather than a Neumann one. From (30) it also follows that a Dirichlet condition has to be imposed on the sum $(B_{+12} + B_{-03})$. Hence,

$$\frac{C_1}{C_2} = \frac{\mathcal{J}_1\big(\sqrt{1-\beta}\,qz_m\big) + \sqrt{1-\beta}\,qz_m\mathcal{J}_1'\big(\sqrt{1-\beta}\,qz_m\big)}{\mathcal{J}_1\big(\sqrt{1+\beta}\,qz_m\big) + \sqrt{1+\beta}\,qz_m\mathcal{J}_1'\big(\sqrt{1+\beta}qz_m\big)}. \tag{31}$$

There is no infrared boundary condition for X, so the overall value of C_i remains undetermined. Nevertheless, we can obtain the ratio of the tensor and scalar condensates (the scalar one is determined from (21), while the tensor condensate is read off of the variation of the action with respect to the tensor field (30)):

$$\langle\bar{q}\sigma_{12}q\rangle \propto 8g_B\frac{g_X}{4\sqrt{g_B}}\lim_{z\to 0}\frac{f_B(qz)}{z}; \qquad \langle\bar{q}q\rangle \propto g_X^2\lim_{z\to 0}\frac{f_X(qz)}{z},$$

hence

$$\mu(\mathbf{B};q) = \frac{\langle\bar{q}\sigma_{12}q\rangle}{\langle\bar{q}q\rangle} = \frac{2\sqrt{g_B}}{g_X}\lim_{z\to 0}\frac{f_B(qz;\mathbf{B})}{f_X(qz;\mathbf{B})}. \tag{32}$$

Setting the 4-momentum to zero we are able to obtain the magnetization $\mu(\mathbf{B})$ and the magnetic susceptibility $\chi(\mathbf{B}) = \dfrac{d}{d\mathbf{B}}\mu(\mathbf{B})$ for a uniform condensate in terms of Bessel functions of $\sqrt{\frac{|\lambda|}{2g_X\sqrt{g_B}}\mathbf{B}z_m^2}$. They are presented here on Figs. 1 and 2, respectively.

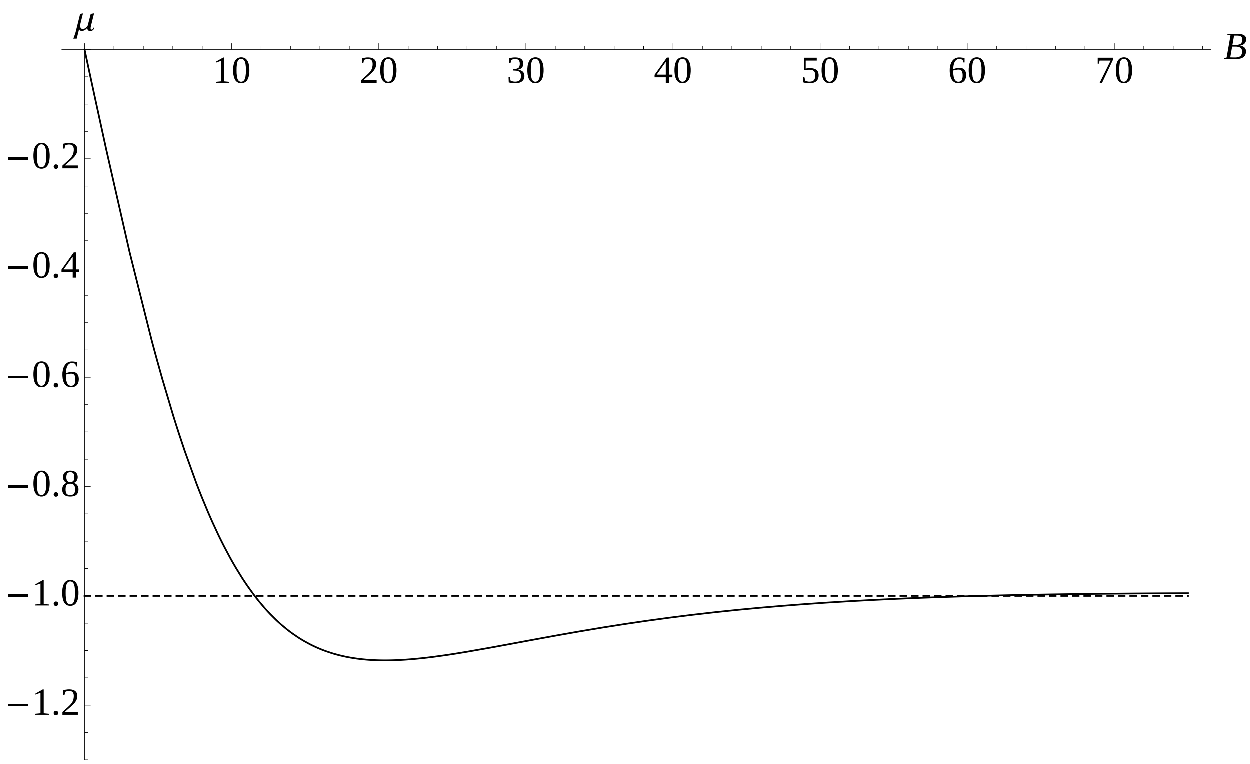

FIG. 1. Magnetization $\mu(\mathbf{B})$ in terms of $\frac{2\sqrt{g_B}}{g_X}$ (solid line) vs its strong field asymptotics (dashed line). The magnetic field is in units of $z_m^{-2} \times \frac{g_X\sqrt{g_B}}{\lambda}$

Their main properties are that

$$\lim_{\mathbf{B}\to\infty} \mu(\mathbf{B}) = -\frac{2\sqrt{g_B}}{g_X} \tag{33}$$

and

$$\chi(\mathbf{B}) = -\frac{|\lambda|}{g_X^2}\frac{z_m^2}{4}\left(1 - \frac{1}{96}\frac{\lambda^2}{g_X^2 g_B}\mathbf{B}^2 z_m^4 + \mathcal{O}\left(\mathbf{B}^4 z_m^8\right)\right), \qquad \mathbf{B}\to 0. \tag{34}$$

Note that the magnetization changes its behavior from one linear in the magnetic field to a constant at values of the magnetic field of order of $\mathbf{B} \sim z_m^{-2} \sim \Lambda_{QCD}^2$. Its constant asymptotic is a behavior to be expected. At large magnetic fields the dynamics of the theory become effectively two-dimensional and the tensor chiral condensate is kinematically reduced to a scalar one.

If we substitute the values of the constants of the model from [11] we obtain[1]

$$\mu(\infty) = 1/\sqrt{3}\,, \qquad \chi(0) = -\frac{z_m^2}{72}\,. \tag{35}$$

[1] Our constants g_B and λ are 3 times larger than those in [11] due to the fact that our variation of the action (30) with respect to the tensor field is, in a similar way, 3 times larger than the variation in [11].

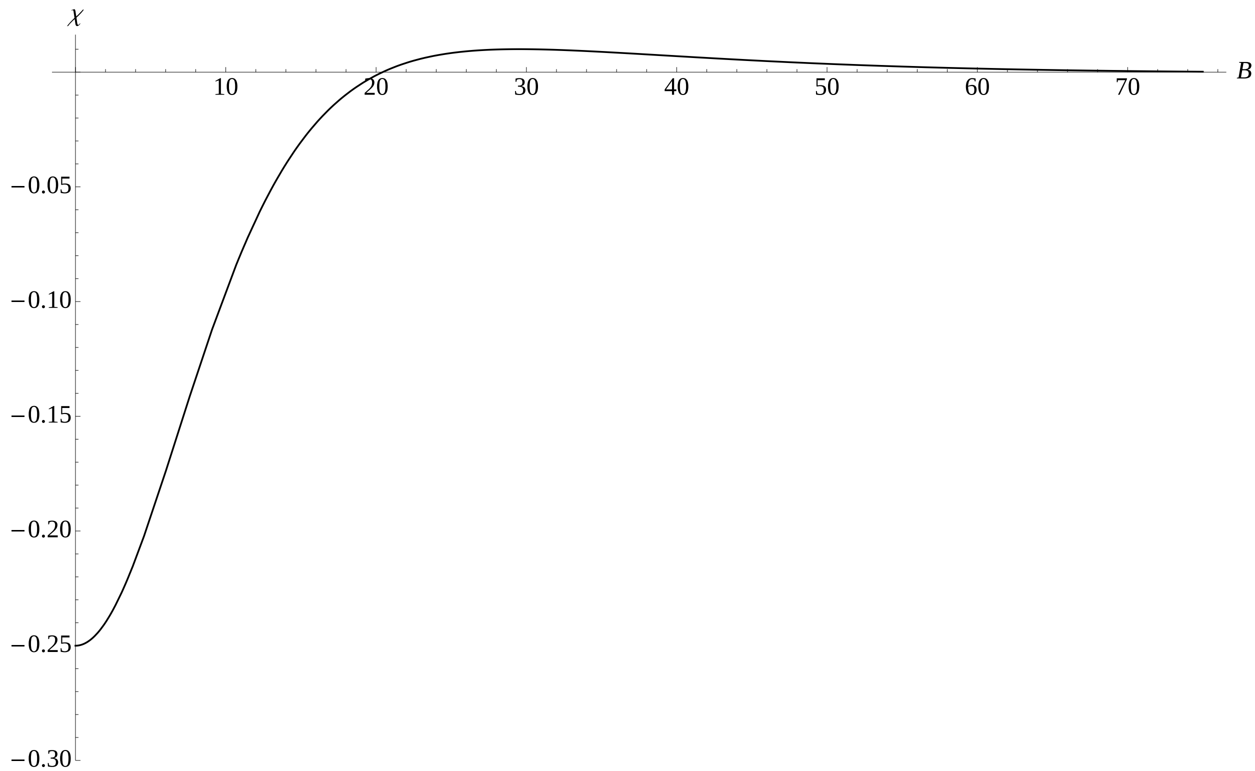

FIG. 2. Magnetic susceptibility of the quark condensate $\chi(\mathbf{B})$ in units of $z_m^2 \times \frac{\lambda}{g_X^2}$

Let us recall that z_m is fixed by the mass of the ρ-meson [18], $z_m \sim 2.4\ m_\rho^{-1}$, which means that $\chi(0) \sim -0.08\ m_\rho^{-2} \sim -0.13\ \mathrm{GeV}^{-2}$. One may compare that to the results [21], where the susceptibility has been analyzed from the point of view of sum rules and has been determined to be $\chi \sim -3.15 \pm 0.30\ \mathrm{GeV}^{-2}$, while the pion dominance and OPE for the $\langle VVA \rangle$ diagram give [2] $\chi \sim -8.9\ \mathrm{GeV}^{-2}$. (Other results include a holographic computation of the $\langle VVA \rangle$ diagram, which led to a value $\chi \sim -11.5\ \mathrm{GeV}^{-2}$ [4]; the use of vector dominance [22] gives $\chi \sim -(3.38 \div 5.67)\ \mathrm{GeV}^{-2}$.) One can see that there is a large discrepancy between the numerical results that clearly requires more investigation. However and more importantly, our results reproduce the general properties both of the susceptibility and of the magnetization – the weak-field expansion of the former and the negative constant asymptotic of the latter.

As for the particular value of the magnetization, the lattice calculation yields a different saturation value in a large field [23]: $\lim\limits_{\mathbf{B}\to\infty} \mu(\mathbf{B}) = -1$. It has also been discussed in the NJL model [24]. In theory, such a value would tell us that both condensates get contributions only from the LLL (lowest Landau level) in the strong magnetic field. On the other hand a different saturation value obtained in this paper implies that the picture is more complicated. Indeed, as it has been shown by Miransky *et al.* [25], in some problems the summation of the infinite number of higher Landau levels is needed to reproduce the correct result. It has also been argued in [25] that the LLL approximation is reliable only in the kinematic region when the momenta satisfy the condition $q_\perp^2 \gg q_\parallel^2$. In our particular case the problem is completely static ($q_i = 0$), so this condition is not fulfilled and the careful treatment of the higher Landau levels is desirable. Another simple argument concerns the derivation of the magnetization via the Dirac operator spectrum [4, 23]. An analogue of the Casher-Banks formula implies that the result obtained in [23] is based on the factorization of the product of two operators under the averaging over the gluon configuration. The lack of factorization could be the origin of disagreement with the lattice result. Anyway, this point needs further clarification.

C. Vector current

Let us now consider perturbations of the vector field about the solution $F_{V12} = \mathbf{B}$. They obey the following equations:

$$\partial_z\left(\frac{1}{z}\partial_z V_3\right) - \frac{1}{z}\partial_0^2 V_3 + \frac{1}{z}\partial_0\partial_3 V_0 = 4g_5^2\lambda\sum_{+,-}\left[-\partial_z\left(\frac{1}{z}X_\pm B_{\pm 3z}\right) - \frac{1}{z}\partial_0\left(X_\pm B_{\pm 03}\right)\right],$$

$$-\partial_z\left(\frac{1}{z}\partial_z V_0\right) - \frac{1}{z}\partial_3^2 V_0 + \frac{1}{z}\partial_0\partial_3 V_3 = 4g_5^2\lambda\sum_{+,-}\left[-\partial_z\left(\frac{1}{z}X_\pm B_{\pm 0z}\right) + \frac{1}{z}\partial_0\left(X_\pm B_{\pm 03}\right)\right]. \quad (36)$$

One may note that the longitudinal components $B_{\pm z\mu}$ (as well as $B_{\pm 03}$) become sources for the vector current and charge density. However, the r.h.s. of the Eqs. (36,36) contains products of fields from different sectors of the solution. Furthermore, if we consider small fluctuations, the vector field turns out to be a fluctuation of the second order. There are obvious similarities with Eq. (12), where $\pi = \arg(X_+ + iX_-)$.

IV. DISCUSSION

In this paper we have discussed a few issues concerning the magnetic susceptibility of the quark condensate. We have shown that the SY relation which yields the value of the susceptibility at large Q^2 is consistent with the chiral log counting at small Q^2. The nonvanishing value of the susceptibility implies a specific term in the effective lagrangian and we have analyzed the role of this term in the holographic approach. It turns out that it provides the magnetization at any value of the magnetic field. Surprisingly the saturation value in the strong magnetic field is small and disagrees with the lattice simulation. This disagreement deserves additional studies.

A satisfactory explanation of the SY relation is still absent. It implies a peculiar relation between the kinetic and topological terms. Such relation is natural from the brane viewpoint and one could expect a kind of Ward identity to stand behind it. We have not found the symmetry which would provide such a Ward identity, however more efforts could be made in this direction and we plan to return to this point elsewhere.

Which vacuum excitations are responsible for the magnetic susceptibility? This question can be rephrased as one concerning the localization of the quarks involved into the composite operator on some vacuum defects excited by the external magnetic field. The answer potentially depends on the interpretation of the background $C_{\mu\nu}$ field. There could be several interpretations. If it is the two-form field in NS or RR sectors it would mean that the F1 or D1 degrees of freedom are under the carpet. The variant with NS $B_{\mu\nu}$ field has some trouble since in this case we are dealing with noncommutative field theory and the field enters other terms in the Lagrangian. Hence it is unclear if it would be possible to separate the desired term in the effective action in a clear-cut way. If we choose the RR two-form field C_2 the product $C_2 \wedge F$ follows from the CS term immediately, however the proportionality to the chiral condensate needs an explanation.

Finally let us comment on the possible interpretation of the tensor source as the curvature of the graviphoton one-form RR field C_1: $C = dC_1$. The degrees of freedom naturally charged with respect to the graviphoton field are the D0 branes, hence one could question how D0 particles or instantons are captured by the magnetic field. The potential object which could be relevant is the dyonic instanton, that is, a blown up instanton with a string attached to it. Upon the blow up it behaves as a magnetic dipole with a topological charge.

V. ACKNOWLEDGMENTS

We would like to thank P. Buividovich and M. Shifman for the useful discussions. A.G. thanks FTPI at University of Minnesota where the part of the work has been done for the hospitality and support. The work of A.G. is partially supported by grant CRDF-RUP2-2961-MO-09. The research of A.K. is supported by the Dynasty Foundation, by the Ministry of Education and Science of the Russian Federation under contract 14.740.11.0347 and by the RFBR grant no. 10-02-0148. The research of P.N.K. is supported by the Dynasty Foundation and by the Ministry of Education and Science of the Russian Federation under contract 14.740.11.0081. The work of A.V. is supported in part by the DOE grant DE-FG02-94ER40823.

[1] B. L. Ioffe and A. V. Smilga, "Nucleon Magnetic Moments And Magnetic Properties Of Vacuum In QCD," *Nucl. Phys.* **B 232**, 109 (1984).

[2] A. Vainshtein, "Perturbative and nonperturbative renormalization of anomalous quark triangles," *Phys. Lett.* **B 569**, 187 (2003) [arXiv:hep-ph/0212231].

[3] O. Cata, "Relations between vacuum condensates and low energy parameters from a rational approach," *Phys. Rev.* **D 81**, 054011 (2010) [arXiv:0911.4736 [hep-ph]].

[4] A. Gorsky and A. Krikun, "Magnetic susceptibility of the quark condensate via holography," *Phys. Rev.* **D 79**, 086015 (2009) [arXiv:0902.1832 [hep-ph]].

[5] D. T. Son and N. Yamamoto, "Holography and Anomaly Matching for Resonances," arXiv:1010.0718 [hep-ph].

[6] M. Knecht, S. Peris and E. de Rafael, "On Anomaly Matching and Holography," *JHEP* **1110**, 048 (2011) [arXiv:1101.0706 [hep-ph]].

[7] P. Colangelo, F. De Fazio, F. Giannuzzi, S. Nicotri and J. J. Sanz-Cillero, "Anomalous AV^*V vertex function in the soft-wall holographic model of QCD," [arXiv:1108.5945 [hep-ph]].

[8] I. Iatrakis and E. Kiritsis, "Vector-axial vector correlators in weak electric field and the holographic dynamics of the chiral condensate," [arXiv:1109.1282 [hep-ph]].

[9] L. Cappiello, O. Cata and G. D'Ambrosio, "Antisymmetric tensors in holographic approaches to QCD," *Phys. Rev.* **D 82**, 095008 (2010) [arXiv:1004.2497 [hep-ph]].

[10] S. K. Domokos, J. A. Harvey and A. B. Royston, "Completing the framework of AdS/QCD: h_1/b_1 mesons and excited omega/rho's," *JHEP* **1105**, 107 (2011) [arXiv:1101.3315 [hep-th]].

[11] R. Alvares, C. Hoyos and A. Karch, "An improved model of vector mesons in holographic QCD," *Phys. Rev.* **D 84**, 095020 (2011) [arXiv:1108.1191 [hep-ph]].

[12] M. N. Chernodub, "Spontaneous electromagnetic superconductivity of vacuum in strong magnetic field: evidence from the Nambu–Jona-Lasinio model," *Phys. Rev. Lett.* **106**, 142003 (2011) [arXiv:1101.0117 [hep-ph]].

[13] M. Ammon, J. Erdmenger, P. Kerner and M. Strydom, "Black Hole Instability Induced by a Magnetic Field," *Phys. Lett.* **B 706**, 94 (2011) [arXiv:1106.4551 [hep-th]].

N. Callebaut, D. Dudal and H. Verschelde, "Holographic study of rho meson mass in an external magnetic field: Paving the road towards a magnetically induced superconducting QCD vacuum?," PoS FACESQCD , 046 (2010) [arXiv:1102.3103 [hep-ph]].

[14] J. de Boer, E. P. Verlinde and H. L. Verlinde, "On the holographic renormalization group," *JHEP* **0008**, 003 (2000) [arXiv:hep-th/9912012].

[15] S. Corley, "A note on holographic Ward identities," *Phys. Lett.* **B 484**, 141 (2000) [arXiv:hep-th/0004030].

D. Martelli and W. Mueck, "Holographic renormalization and Ward identities with the Hamilton-Jacobi method," *Nucl. Phys.* **B 654**, 248 (2003) [arXiv:hep-th/0205061].

[16] A. Imaanpur, "Comments on gluino condensates in N = 1/2 SYM theory," *JHEP* **0312**, 009 (2003) [arXiv:hep-th/0311137].

[17] A. Gorsky and M. Shifman, "Spectral degeneracy in supersymmetric gluodynamics and one-flavor QCD related to N = 1/2 SUSY," *Phys. Rev.* **D 71**, 025009 (2005) [arXiv:hep-th/0410099].

[18] J. Erlich, E. Katz, D. T. Son and M. A. Stephanov, "QCD and a Holographic Model of Hadrons," *Phys. Rev. Lett.* **95**, 261602 (2005) [arXiv:hep-ph/0501128].

[19] A. Cherman, T. D. Cohen, and E. S. Werbos, *Phys. Rev.* **C 79**, 045203 (2009), [arXiv:0804.1096].

[20] A. Krikun, *Phys. Rev.* **D 77**, 126014 (2008) [arXiv:0801.4215 [hep-th]].

[21] B. L. Ioffe, V. S. Fadin, and L. N. Lipatov, *"Quantum Chromodynamics: Perturbative and Nonperturbative Aspects"*, Cambridge University Press, 2010.

[22] I. I. Balitsky and A. V. Yung, *Phys. Lett.* **B 129**, 328 (1983).

V. M. Belyaev and Y. I. Kogan, *Yad. Fiz.* **40**, 1035 (1984).

I. I. Balitsky, A. V. Kolesnichenko and A. V. Yung, *Yad. Fiz.* **41**, 282 (1985).

P. Ball, V. M. Braun and N. Kivel, "Photon distribution amplitudes in QCD", [arXiv:hep-ph/0207307]

[23] P. V. Buividovich, M. N. Chernodub, E. V. Luschevskaya and M. I. Polikarpov, "Chiral magnetization of non-Abelian vacuum: a lattice study," *Nucl. Phys.* **B 826**, 313 (2010) [arXiv:0906.0488 [hep-lat]].

[24] M. Frasca and M. Ruggieri, "Magnetic Susceptibility of the Quark Condensate and Polarization from Chiral Models," *Phys. Rev.* **D 83**, 094024 (2011) [arXiv:1103.1194 [hep-ph]].

[25] E. V. Gorbar, M. Hashimoto and V. A. Miransky, "Nondecoupling phenomena in QED in a magnetic field and noncommutative QED," *Phys. Lett.* **B 611**, 207 (2005) [arXiv:hep-th/0501135].