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# Small Orbits

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We study both the "large" and "small" U-duality charge orbits of extremal black holes appearing in $D = 5$ and $D = 4$ Maxwell-Einstein supergravity theories with symmetric scalar manifolds. We exploit a formalism based on cubic Jordan algebras and their associated Freudenthal triple systems, in order to derive the minimal charge representatives, their stabilizers and the associated "moduli spaces". After recalling $\mathcal{N} = 8$ maximal supergravity, we consider $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories coupled to an arbitrary number of vector multiplets, as well as $\mathcal{N} = 2$ magic, $STU$, $ST^2$ and $T^3$ models. While the $STU$ model may be considered as part of the general $\mathcal{N} = 2$ sequence, albeit with an additional triality symmetry, the $ST^2$ and $T^3$ models demand a separate treatment, since their representative Jordan algebras are Euclidean or only admit non-zero elements of rank 3, respectively. Finally, we also consider *minimally coupled* $\mathcal{N} = 2$, matter coupled $\mathcal{N} = 3$, and "pure" $\mathcal{N} = 5$ theories.

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## I. INTRODUCTION

### A. Background

A concerted effort has been made to understand the physically distinct black hole (BH) solutions appearing in various 4-dimensional supergravity theories. The extremal solutions typically carry electromagnetic charges transforming linearly under $G_4$, the $D = 4$ U-duality group[1]. BHs with charges lying in different orbits of $G_4$ therefore correspond to distinct solutions. Moreover, thanks to the attractor mechanism [3–7] the entropy of the extremal BH solutions loses all memory of the scalars at infinity and is a function of only the charges. Consequently, the Bekenstein-Hawking [8, 9] entropy is given by a U-duality invariant quartic in the electromagnetic charges. Hence, the classification of the U-duality charge orbits captures many significant features of the possible BH solutions, which in turn have provided a range of important string or M-theoretic insights.

We focus on those theories in which the scalars live in a symmetric coset $G_4/H_4$. The orbits of the 4-dimensional $\mathcal{N} = 8$ [1] and the exceptional octonionic "magic" $\mathcal{N} = 2$ [10] supergravities were obtained in [11] for both "large" and "small" BHs, which have non-vanishing or vanishing classical entropy, respectively. The large orbits of the $\mathcal{N} = 2$ Maxwell-Einstein supergravities coupled to $n_V$ vector multiplets, which also include the three non-exceptional magic examples, were analysed in [11, 12]. The small orbits of the $STU$ model [13–19], which exhibits a discrete triality, exchanging the roles of $S$, $T$ and $U$, over and above the continuous U-duality group, were found in [20]. Meanwhile, for the infinite sequence of $\mathcal{N} = 4, 2$ theories coupled to $n_V$ vector multiplets the U-duality invariant charge constraints defining the distinct orbits and their supersymmetry preserving properties, for both large and small cases, were obtained in [21, 22], and further discussed in [23, 24].

In the present work, we aim at essentially completing this story in $D = 4$. In particular, we obtain the small orbits for the $\mathcal{N} = 2$ $\mathbb{R}, \mathbb{C}, \mathbb{H}$ magic supergravities, $\mathcal{N} = 2, 4$ supergravity coupled to an arbitrary number of vector multiplets including the special cases of the $STU$, $ST^2$ and $T^3$ models, as well as the *minimally coupled* $\mathcal{N} = 2$, matter coupled $\mathcal{N} = 3$, and "pure" $\mathcal{N} = 5$ theories.

We begin by repeating the $\mathcal{N} = 8$ theory as it provides an instructive example, setting the stage for all the other cases. We then study both the "large" and "small" U-duality BH charge orbits of the $D = 4$, $\mathcal{N} = 4$ and $\mathcal{N} = 2$

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[1] We work in the classical regime for which the electromagnetic charges are real valued. Here U-duality $G_4$ is referred to as the "continuous" symmetries of [1]. Their discrete versions are the non-perturbative U-duality string theory symmetries described in [2].

Maxwell-Einstein supergravity theories coupled to an arbitrary number $n_V$ of vector multiplets, including the magic theories. The $\mathcal{N}=2$ *STU* model is retreated as part of the generic sequence ($n_V=3$), revealing additional subtleties which were previously obscured by the triality symmetry. Its degeneration into the $ST^2$ and $T^3$ models is also treated. A formalism based on cubic Jordan algebras and their associated Freudenthal triple systems (FTS) is used to derive the minimal charge orbit representatives, their stabilizers and the associated "moduli spaces" of attractor solutions. In particular, we make use of [25] and [26, 27]. While the *STU* model may be considered as part of the general $\mathcal{N}=2$ sequence, albeit with an additional triality symmetry, the $ST^2$ and $T^3$ models demand a separate treatment. This is due to their representative Jordan algebras being, in some sense, degenerate: the $ST^2$ Jordan algebra is Euclidean, as opposed to the Lorentzian nature of the general sequence, while the $T^3$ Jordan algebra only contains non-zero elements of rank 3. Finally, in section III G, section III H and section III I, we respectively include the analogous treatment of the *minimally coupled* $\mathcal{N}=2$, matter coupled $\mathcal{N}=3$, and "pure" $\mathcal{N}=5$ theories, which cannot all be uplifted to $D=5$ space-time dimensions.

Physically speaking, the FTS makes the symmetries of the parent $D=5$ theory manifest. This allows us to make extensive use of the orbits and their minimal charge representatives of the $D=5$ theories, which are simpler to derive and already appeared in the literature. In particular, we exploit the analysis of [11, 22, 24, 28–30]. Note, one may also use the *integral* FTS to address the orbit classification of the discrete stringy U-duality groups [2], as was done for the maximally supersymmetric $D=6,5,4$ theories in [31, 32]. Moreover, for $D=4, \mathcal{N}=8$ it has recently been observed that some of the orbits of $E_{7(7)}(\mathbb{Z})$ should play an important role in counting microstates of this theory [33, 34]. The importance of discrete invariants and orbits to the dyon spectrum of string theory has been the subject of much investigation [34–41].

### B. Summary

We summarise the key results here. For each of the theories considered (aside from the $\mathcal{N}=2$ minimally coupled, $\mathcal{N}=3$ and $\mathcal{N}=5$ theories), the electromagnetic BH charges may be regarded as elements of a Freudenthal triple system

$$\mathfrak{F}(\mathfrak{J}_3) := \mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{J}_3 \oplus \mathfrak{J}_3, \tag{1}$$

defined over a cubic Jordan algebra $\mathfrak{J}_3$. The electric (magnetic) BH (black string - BS -) charges of the parent $D=5$ theory may be regarded as elements of $\mathfrak{J}_3$. The FTS comes equipped with three maps: (i) a bilinear antisymmetric form $\{\bullet,\bullet\} : \mathfrak{F}\times\mathfrak{F}\to\mathbb{R}$, which encodes the symplectic structure of the charge representations (see for example [42], and Refs. therein); (ii) a quartic norm $\Delta : \mathfrak{F}\to\mathbb{R}$; (iii) a triple product $T : \mathfrak{F}\times\mathfrak{F}\times\mathfrak{F}\to\mathfrak{F}$. A brief summary may be found in section III A. Full details can be found in [25] and Refs. therein. The automorphism group $\mathrm{Aut}(\mathfrak{F}) \cong \mathrm{Conf}(\mathfrak{J}_3)$ is the set of invertible $\mathbb{R}$-linear transformations preserving the quartic norm and bilinear form. It coincides with the $D=4$ U-duality group: $\mathrm{Aut}(\mathfrak{F}) = G_4$. Hence, the unique quartic $G_4$-invariant, denoted $I_4$, is given by $\Delta$. The Bekenstein-Hawking entropy therefore reads

$$S_{\mathrm{BH}} = \pi\sqrt{|\Delta|} = \pi\sqrt{|I_4|}. \tag{2}$$

Let us briefly review some of the analogous features of cubic Jordan algebras and the BHs (BSs) in $D=5$, which we will make extensive use of throughout. A cubic Jordan algebra $\mathfrak{J}_3$ is a vector space equipped with an admissible cubic norm $N : \mathfrak{J}_3\to\mathbb{R}$ and an element $c\in\mathfrak{J}_3$, referred to as a *base point*, satisfying $N(c)=1$. The cubic norm defines the Jordan product, $-\circ- : \mathfrak{J}_3\times\mathfrak{J}_3\to\mathfrak{J}_3$, satisfying,

$$X^2\circ(X\circ Y) = X\circ(X^2\circ Y), \qquad \forall X, Y\in\mathfrak{J}_3. \tag{3}$$

A brief summary may be found in section III A. Full details can be found in [25] and Refs. therein. For each of the theories considered in the present investigation (but the $\mathcal{N}=2$ minimally coupled, $\mathcal{N}=3$ and $\mathcal{N}=5$ theories), the electromagnetic BH charges may be regarded as elements of some cubic Jordan algebra $\mathfrak{J}_3$. The automorphism group $\mathrm{Aut}(\mathfrak{J}_3)$ is the set of invertible $\mathbb{R}$-linear transformations preserving the Jordan product. The reduced structure group $\mathrm{Str}_0(\mathfrak{J}_3)$ is the set of invertible $\mathbb{R}$-linear transformations preserving the cubic norm $N$ [25]. $\mathrm{Str}_0(\mathfrak{J}_3)$ is the $D=5$ U-duality group, $\mathrm{Str}_0(\mathfrak{J}_3) = G_5$. Hence, the unique cubic $G_5$-invariant, denoted $I_3$, is given by $N$. The Bekenstein-Hawking BH (BS) entropy is therefore

$$S_{\mathrm{BH}} = \pi\sqrt{|N|}. \tag{4}$$

The models we consider are itemized here:

- $\mathcal{N}=8$: $28+28$ electric/magnetic BH charges belong to $\mathfrak{F}^{\mathbb{O}^s}:=\mathfrak{F}(\mathfrak{J}_3^{\mathbb{O}^s})$, where $\mathfrak{J}_3^{\mathbb{O}^s}$ is the cubic Jordan algebra of $3\times 3$ Hermitian matrices defined over the split-octonions. The 56 charges transform linearly as the fundamental $\mathbf{56}$ of $\mathrm{Aut}(\mathfrak{F}^{\mathbb{O}^s})=E_{7(7)}\cong \mathrm{Conf}\left(\mathfrak{J}_3^{\mathbb{O}_s}\right)$, the maximally non-compact (split) real form of $E_7(\mathbb{C})$. The scalar manifold is given by (apart from discrete factors, see *e.g.* [43])

$$\frac{E_{7(7)}}{\mathrm{SU}(8)}. \tag{5}$$

- Magic $\mathcal{N}=2$ theories: Given by $\mathcal{N}=2$ supergravity coupled to $(3+3\dim\mathbb{A})$ vector multiplets, where $\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$. The $(4+3\dim\mathbb{A})+(4+3\dim\mathbb{A})$ electric/magnetic BH charges belong to $\mathfrak{F}^{\mathbb{A}}:=\mathfrak{F}(\mathfrak{J}_3^{\mathbb{A}})$, where $\mathfrak{J}_3^{\mathbb{A}}$ is the cubic Jordan algebra of $3\times 3$ Hermitian matrices defined over one of the four division algebras $\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$. The $(8+6\dim\mathbb{A})$ charges transform linearly as the threefold antisymmetric traceless tensor $\mathbf{14'}$, the threefold antisymmetric self-dual tensor $\mathbf{20}$, the chiral spinor $\mathbf{32}$ and the fundamental $\mathbf{56}$ of $\mathrm{Aut}(\mathfrak{F}^{\mathbb{A}})\cong \mathrm{Conf}\left(\mathfrak{J}_3^{\mathbb{A}}\right)=\mathrm{Sp}(6,\mathbb{R}),\mathrm{SU}(3,3),\mathrm{SO}^\star(12),E_{7(-25)}$ for $\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$, respectively. The scalar manifolds are given by (apart from discrete factors, see *e.g.* [43])

$$\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{U}(3)},\quad \frac{\mathrm{SU}(3,3)}{\mathrm{U}(1)\times\mathrm{SU}(3)\times\mathrm{SU}(3)},\quad \frac{\mathrm{SO}^\star(12)}{\mathrm{U}(6)},\quad \frac{E_{7(-25)}}{\mathrm{U}(1)\times E_{6(-78)}}. \tag{6}$$

- $\mathcal{N}=4$ supergravity (6 graviphotons) coupled to $n=n_V$ vector multiplets: the $(n_V+6)+(n_V+6)$ electric/magnetic BH charges belong to $\mathfrak{F}^{6,n}:=\mathfrak{F}(\mathfrak{J}_{5,n-1})$, where $\mathfrak{J}_{5,n-1}\cong\mathbb{R}\oplus\mathbf{\Gamma}_{5,n-1}$ is the cubic Jordan algebra of pseudo-Euclidean spin factors [44] (see also [25]). In general, $\mathbf{\Gamma}_{m,n}$ is a Jordan algebra with a quadratic form of pseudo-Euclidean signature $(m,n)$, *i.e.* the Clifford algebra of $O(m,n)$ [45]. The $2(n_V+6)$ charges transform linearly as the $(\mathbf{2},\mathbf{6}+\mathbf{n_V})$ of $\mathrm{Aut}(\mathfrak{F}^{6,n})\cong\mathrm{Conf}(\mathfrak{J}_{5,n-1})=\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n_V)$. The scalar manifolds are given by the infinite sequence of globally symmetric Riemannian manifolds

$$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}\times\frac{\mathrm{SO}(6,n_V)}{\mathrm{SO}(6)\times\mathrm{SO}(n_V)},\ n_V\geqslant 0. \tag{7}$$

- $\mathcal{N}=2$ supergravity (1 graviphoton) coupled to $n_V$ vector multiplets: the $(n_V+1)+(n_V+1)$ electric/magnetic BH charges belong to $\mathfrak{F}^{2,n}:=\mathfrak{F}(\mathfrak{J}_{1,n-1})$, where $\mathfrak{J}_{1,n-1}\cong\mathbb{R}\oplus\mathbf{\Gamma}_{1,n-1}$ is the cubic Jordan algebra of Lorentzian spin factors [44] (see also [25]), and $n=n_V-1$. The $2(n_V+1)$ charges transform linearly as the $(\mathbf{2},\mathbf{1}+\mathbf{n_V})$ of $\mathrm{Aut}(\mathfrak{F}^{2,n})\cong\mathrm{Conf}(\mathfrak{J}_{1,n-1})=\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)$. The scalar manifolds are given by the infinite sequence of globally symmetric special Kähler manifolds

$$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}\times\frac{\mathrm{SO}(2,n_V-1)}{\mathrm{SO}(2)\times\mathrm{SO}(n_V-1)},\ n_V\geqslant 2. \tag{8}$$

- $\mathcal{N}=2$ *STU* model: it is nothing but $n_V=3$ element of the Jordan symmetric sequence (8), but we single it out for two reasons. First, over and above the continuous U-duality group it has a discrete *triality symmetry* which swaps the roles of the three complex moduli $S,T,U$ [14], and is manifested in the structure of the duality orbits. Second, it may be considered as the common sector of all $D=4$ Maxwell-Einstein supergravity theories with a rank-3 symmetric vector multiplets' scalar manifold and related to Jordan algebras (which we will dub *"symmetric"* supergravities). Furthermore, it also provides a link to the degenerate cases described below. The $4+4$ electric/magnetic BH charges belong to $\mathfrak{F}_{STU}:=\mathfrak{F}(\mathfrak{J}_{STU})$, where $\mathfrak{J}_{STU}=\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$ is isomorphic to the Lorentzian spin factor $\mathfrak{J}_{1,1}$ [25, 44]. The 8 charges transform linearly as the $(\mathbf{2},\mathbf{2},\mathbf{2})$ of $\mathrm{Aut}(\mathfrak{F}_{STU})\cong\mathrm{Conf}(\mathfrak{J}_{STU})=\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$. This symmetry is made manifest by organising the charges into a $2\times2\times2$ hypermatrix $a_{ABC}$, where $A,B,C=0,1$, transforming under $\mathrm{SL}_A(2,\mathbb{R})\times\mathrm{SL}_B(2,\mathbb{R})\times\mathrm{SL}_C(2,\mathbb{R})$ [46]. The scalar manifold is given by

$$\left[\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}\right]^3. \tag{9}$$

It is worth noting that, by using U-duality, the charge vectors of the *symmetric* supergravity theories described above may be reduced to a subsector living in $\mathfrak{F}_{STU}$. Hence, the *STU* charges are common to all the above theories which, indeed, may all be consistently truncated to the *STU* model. Moreover, the special Kähler geometry characterising the completely factorised rank-3 symmetric manifold (9) is defined by the triality-symmetric prepotential

$$F=STU. \tag{10}$$

See, for example, [3, 47–49] for the details of special geometry. By identifying $T = U$ and $S = T = U$ in (10) we obtain the $ST^2$ and $T^3$ models, respectively (see *e.g.* [18] for the consistent exploitation of such a degeneration/reduction procedure). In this sense, the $STU$ model is the linchpin of all the theories considered here.

- $\mathcal{N} = 2$ $ST^2$ model: coupled to two vector multiplets. The $3 + 3$ electric/magnetic BH charges belong to $\mathfrak{F}_{ST^2} := \mathfrak{F}(\mathfrak{J}_{ST^2})$, where $\mathfrak{J}_{ST^2} = \mathbb{R} \oplus \mathbb{R}$ is isomorphic to the Euclidean spin factor $\mathfrak{J}_1$ [25, 44]. The 6 charges transform linearly as the $(\mathbf{2}, \mathbf{3})$ of $\mathrm{Aut}(\mathfrak{F}_{ST^2}) = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$. This symmetry is made manifest by organising the charges into a partially symmetrised hypermatrix $a_{A(B_1B_2)}$, where $A, B_1, B_2 = 0, 1$, transforming under $\mathrm{SL}_A(2, \mathbb{R}) \times \mathrm{SL}_B(2, \mathbb{R})$ [18]. The scalar manifold is given by

$$\left[\frac{\mathrm{SL}\,(2,\mathbb{R})}{\mathrm{SO}\,(2)}\right]^2. \tag{11}$$

- $\mathcal{N} = 2$ $T^3$ model: this is a *non-generic* irreducible model, coupled to a single vector multiplet. May be obtained as a circle compactification of minimal supergravity in five dimensions. The $2+2$ electric/magnetic BH charges belong to $\mathfrak{F}_{T^3} := \mathfrak{F}(\mathfrak{J}_{T^3})$, where $\mathfrak{J}_{T^3} = \mathbb{R}$. The 4 charges transform linearly as the $\mathbf{4}$ (spin $s = 3/2$) of $\mathrm{Aut}(\mathfrak{F}_{T^3}) \cong \mathrm{Conf}\,(\mathfrak{J}_{T^3}) = \mathrm{SL}(2, \mathbb{R})$. This symmetry is made manifest by organising the charges into a totally symmetrised hypermatrix $a_{(A_1A_2A_3)}$, where $A_1, A_2, A_3 = 0, 1$, transforming under $\mathrm{SL}_A(2, \mathbb{R})$ [18] (see also e.g. [50], as well as the recent discussion in [51]). The scalar manifold is given by the special Kähler manifold (with scalar curvature $R = -2/3$ [52])

$$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}. \tag{12}$$

In all aforementioned cases, excluding the $T^3$ model, the charge orbits are split into four classes first identified in [11]. There are three small classes with vanishing Bekenstein-Hawking entropy: doubly critical, critical and light-like. There is one large class with non-zero Bekenstein-Hawking entropy, which actually is a one-parameter ($I_4$) family of orbits. The $T^3$ model is the exception in that the doubly critical and critical classes collapse into a single orbit. This is precisely due to the fact that the underlying cubic Jordan algebra $\mathfrak{J}_{T^3}$ only admits non-zero elements of rank 3, as opposed to the other examples, which all possess elements of rank 1,2 and 3 (including the $ST^2$ model). From a physical perspective, this is equivalent to the fact that there is only one gauge potential (namely, only one Abelian vector multiplet) outside the gravity multiplet to support both the doubly critical and critical orbits.

These four classes are coded in the "rank" of the FTS element: ranks 1, 2, 3 and 4 imply doubly critical, critical, light-like and large, respectively. For the $\mathcal{N} = 8$ (maximal supersymmetry) theory the ranks are sufficient to capture all the orbit details, *i.e.* there is precisely one orbit per rank. The only subtlety is that the large BHs are supported by a 1/8-BPS or a non-BPS orbit, according as $I_4 > 0$ or $I_4 < 0$, respectively [11]. For theories of gravity with non-maximal local supersymmetry, this identification between rank and orbit generally becomes more subtle: while rank 1 (doubly critical) elements lie in a single orbit, higher ranks split into two or more orbits. Moreover, BHs with $I_4 > 0$ may also be non-BPS; in contrast, all BHs with $I_4 < 0$ are non-BPS. In every case, there is only one $I_4 < 0$ orbit.

We summarise the key features of this orbit splitting here, while laying out the organisation of the letter.

First, let us mention that the technical aspects of Jordan algebras, the FTS and the proofs of the associated theorems used here may be found in [25] and in Refs. therein. We begin in section II with a summary of the $D = 5$ parent theories: their Jordan algebras, minimal charge orbit representatives, cosets and *moduli spaces*. This lays the foundations for the $D = 4$ analysis. In section III the details of $D = 4$ minimal charge orbit representatives, cosets and *moduli spaces* are presented for each of the aforementioned theories. The $\mathcal{N} = 8$ treatment, while having been well understood for sometime now [11, 32], is given first as the simplest example (only one orbit per rank of FTS element), with ranks 1, 2, 3 corresponding to 1/2-, 1/4- and 1/8-BPS states, respectively. As mentioned, the unique subtlety is that the rank 4 large orbit is 1/8-BPS or non-BPS orbit according as $I_4 > 0$ or $I_4 < 0$. The orbits and their representatives are given in Table V and Theorem 5, respectively. Also, notice that the supersymmetry BPS-preserving features are not sufficient to uniquely characterise the charge orbits; indeed, there are two 1/8-BPS orbits, one large (rank 4) and one small lightlike (rank 3). All subsequent treatments may be seen as a fine-graining of the treatment of $\mathcal{N} = 8$ orbits. Only the rank 1 (doubly critical) and the rank 4 ($I_4 < 0$) cases do not split, remaining as a single 1/2-BPS and non-BPS orbit, respectively, for all non-maximally supersymmetric theories. The next simplest cases are the magic $\mathcal{N} = 2$ supergravities. Here the rank 2, 3 and 4 ($I_4 > 0$) orbits split into one 1/2-BPS and non-BPS orbit each. The non-BPS large ($I_4 > 0$) orbit has vanishing central charge at the unique BH event horizon. The orbits and their representatives are given in Table VI and Theorem 6, respectively. The exceptional octonionic case is given as a detailed example in section A 1, which thus provides an alternative derivation of the result obtained in [11]. Next, comes $\mathcal{N}=4$ Maxwell-Einstein supergravity. The major difference is that the corresponding FTS is *reducible*. As a consequence, as proved in [25], an extra rank 2 orbit is introduced, making a total of three: 1/2-BPS, 1/4-BPS and non-BPS. Rank 3 has one 1/4-BPS and one non-BPS, as does rank 4 ($I_4>0$). The orbits and their representatives are given in Table VII and Theorem 7, respectively. Finally, we consider $\mathcal{N}=2$ Maxwell-Einstein supergravity based on the Jordan symmetric sequence (8), which has the most intricate orbit structure. However, it may be derived directly from the $\mathcal{N}=4$ case by splitting each 1/4-BPS orbit into one 1/2-BPS and one non-BPS (with vanishing central charge at the horizon); see section III D. We conclude with the "degenerate" cases of $ST^2$ (non-generic reducible) and $T^3$ (non-generic irreducible) $\mathcal{N}=2$, $D=4$ supergravity models in section III F.

Finally, we consider the remaining D=4 theories with symmetric scalar manifolds, which cannot be uplifted to D=5, namely:

- $\mathcal{N}=2$ supergravity *minimally coupled* to $n$ vector multiplets [53] (in section III G). It has a quadratic U-invariant polynomial, and it does *not* enjoy a Jordan algebraic formulation.
- $\mathcal{N}=3$ matter coupled supergravity [54] (in section III H). It has a quadratic U-invariant polynomial, and it does *not* enjoy a Jordan algebraic formulation.
- $\mathcal{N}=5$ *"pure"* supergravity [55] (in section III I). It enjoys a formulation in terms of $M_{2,1}(\mathbb{O})$, the Jordan triple system generated by the $2\times 1$ vector over the octonions $\mathbb{O}$ [10, 56]. Among the symmetric supergravities with *quartic* U-invariant polynomial, it stands on a special footing, because its U-invariant polynomial is a *perfect square* when written in terms of the scalar-dependent skew-eigenvalues of the $5\times 5$ complex antisymmetric central charge matrix $Z_{AB}$. This property, discussed in [57], drastically simplifies the case study of charge orbits.

For the convenience of the reader we summarize here our main original results together with where they appear in the text:

(1) In section III C the small (rank 3,2,1) orbits and moduli spaces of the magic $D=4,\mathcal{N}=2$ models based on degree-3 quaternionic, complex, real Jordan algebras are derived. The results are presented in the three $\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H}$ sub-blocks of Table VI. The $\mathbb{A}=\mathbb{O}$ orbits as well as the large $\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H}$ orbits appearing in Table VI were previously obtained in [11]. In section III C 1 the $\mathcal{N}=2, D=4$ magic quaternionic case is compared to its "twin" $\mathcal{N}=6$ theory [12, 57, 112] and the supersymmetry analysis of twin black hole charge orbits is carried out and presented in (65).

(2) In section III D the small (rank 3,2,1) orbits and moduli spaces of the infinite sequences of $D=4,\mathcal{N}=4$ and $D=4,\mathcal{N}=2$ Maxwell-Einstein theories are derived. The results are presented in Table VII and Table VIII, respectively. The the large orbits appearing in Table VII and Table VIII were previously obtained in [11, 12, 22, 114]. In section III F 1 it is observed that for the triality-symmetric $\mathcal{N}=2$ $STU$ model each of the rank 3 and rank 2 orbits split into two *isomorphic yet physically distinct* (BPS vs. non-BPS) orbits.

(4) In section III F 2 and section III F 3 the small orbits and moduli spaces of the $ST^2$ and $T^3$ models are derived. For the $ST^2$ model the small orbits may be obtained from Table VIII by setting $n=1$ (when this is still well defined - when it is not, the orbit is not present). The $T^3$ orbits are presented in Table IX. It is established that while the BPS large orbit of the $T^3$ model (which one could think as the simplest example of BPS-supporting charge orbit in $D=4,\mathcal{N}=2$ Maxwell-Einstein supergravity) has no continuous stabilizer it does in fact have a $\mathbb{Z}_3$ stabilizer.

(5) In section III G, section III H and section III I the unique small orbits and moduli spaces of the $\mathcal{N}=2$ minimally coupled, $\mathcal{N}=3$ matter coupled and $\mathcal{N}=5$ pure supergravities are obtained, respectively.

## II. BH CHARGE ORBITS IN $D=5$ SYMMETRIC SUPERGRAVITIES

### A. Cubic Jordan Algebras

A Jordan algebra $\mathfrak{J}$ is a vector space defined over a ground field $\mathbb{F}$ equipped with a bilinear product satisfying

$$X\circ Y=Y\circ X,\quad X^2\circ(X\circ Y)=X\circ(X^2\circ Y),\quad\forall\ X,Y\in\mathfrak{J}. \tag{13}$$

The class of *cubic* Jordan algebras is constructed as follows [44]. Let $V$ be a vector space equipped with a cubic norm, *i.e.* an homogeneous map of degree three,

$$N:V\to\mathbb{F},\quad\text{where}\quad N(\lambda X)=\lambda^3N(X),\forall\lambda\in\mathbb{F},X\in V,$$

such that

$$N(X,Y,Z) := \frac{1}{6}[N(X+Y+Z) - N(X+Y) - N(X+Z) - N(Y+Z) + N(X) + N(Y) + N(Z)] \tag{14}$$

is trilinear. If $V$ further contains a base point $N(c) = 1, c \in V$ one may define the following three maps,

$$\begin{aligned} \mathrm{Tr}: V \to \mathbb{F}; &\quad X \mapsto 3N(c,c,X), \\ S: V \times V \to \mathbb{F}; &\quad (X,Y) \mapsto 6N(X,Y,c), \\ \mathrm{Tr}: V \times V \to \mathbb{F}; &\quad (X,Y) \mapsto \mathrm{Tr}(X)\,\mathrm{Tr}(Y) - S(X,Y). \end{aligned} \tag{15}$$

A cubic Jordan algebra $\mathfrak{J}$, with multiplicative identity $\mathbb{1} = c$, may be derived from any such vector space if $N$ is *Jordan cubic*. That is: if (i) the trace bilinear form (15) is non-degenerate, and if (ii) the quadratic adjoint map

$$\sharp : \mathfrak{J} \to \mathfrak{J}, \tag{16}$$

uniquely defined by

$$\mathrm{Tr}(X^\sharp, Y) = 3N(X,X,Y), \tag{17}$$

satisfies $(X^\sharp)^\sharp = N(X)X$, $\forall X \in \mathfrak{J}$. The Jordan product can then be implemented as follows:

$$X \circ Y = \tfrac{1}{2}\big(X \times Y + \mathrm{Tr}(X)Y + \mathrm{Tr}(Y)X - S(X,Y)\mathbb{1}\big), \tag{18}$$

where, $X \times Y$ is the linearisation of the quadratic adjoint: $X \times Y := (X+Y)^\sharp - X^\sharp - Y^\sharp$.

The *degree* of a cubic Jordan algebra is defined as the number of linearly independent *irreducible idempotents*:

$$E \circ E = E, \quad \mathrm{Tr}(E) = 1, \quad E \in \mathfrak{J}.$$

Two important symmetry groups, $\mathrm{Aut}(\mathfrak{J})$ and $\mathrm{Str}_0(\mathfrak{J})$, are given by the set of $\mathbb{F}$-linear transformations preserving the Jordan product and cubic norm, respectively. In particular, $\mathrm{Str}_0(\mathfrak{J})$ is the U-duality group $G_5$ of the corresponding $D = 5$ supergravity, and the corresponding vector multiplets' scalar manifold is given by

$$\frac{\mathrm{Str}_0\,(\mathfrak{J})}{\mathrm{Aut}\,(\mathfrak{J})}, \tag{19}$$

which is isomorphic to the BPS rank 3 orbit in the symmetries theories with 8 supersymmetries - related to Jordan algebras - in which $\mathrm{Aut}\,(\mathfrak{J})$ is the maximal compact subgroup (*mcs*) of $\mathrm{Str}_0\,(\mathfrak{J})$, as well.

The conventional concept of matrix rank may be generalised to a cubic Jordan algebra in a natural and $\mathrm{Str}_0(\mathfrak{J})$ invariant manner. The rank of an arbitrary element $X \in \mathfrak{J}$ is uniquely defined by [58]:

$$\begin{aligned} &\mathrm{Rank}X = 1 \Leftrightarrow X^\sharp = 0; \\ &\mathrm{Rank}X = 2 \Leftrightarrow N(X) = 0,\ X^\sharp \neq 0; \\ &\mathrm{Rank}X = 3 \Leftrightarrow N(X) \neq 0. \end{aligned} \tag{20}$$

### B. $\mathcal{N} = 8$

The $27 = 3 + 3\dim_{\mathbb{R}}\mathbb{O}^s$ electric BH charges may be represented as elements

$$Q = \begin{pmatrix} q_1 & Q_s & \overline{Q_c} \\ \overline{Q_s} & q_2 & Q_v \\ Q_c & \overline{Q_v} & q_3 \end{pmatrix}, \quad \text{where} \quad q_1, q_2, q_3 \in \mathbb{R} \quad \text{and} \quad Q_{v,s,c} \in \mathbb{O}^s \tag{21}$$

of the 27-dimensional Jordan algebra $\mathfrak{J}_3^{\mathbb{O}^s}$ of $3 \times 3$ Hermitian matrices over the split-octonions $\mathbb{O}^s$. The cubic norm is defined as,

$$N(Q) = q_1q_2q_3 - q_1Q_v\overline{Q_v} - q_2Q_c\overline{Q_c} - q_3Q_s\overline{Q_s} + (Q_vQ_c)Q_s + \overline{Q_s}(\overline{Q_cQ_v}). \tag{22}$$

One finds that the quadratic adjoint (16) is given by

$$Q^\sharp = \begin{pmatrix} q_2 q_3 - |Q_v|^2 & \overline{Q_c Q_v} - q_3 Q_s & Q_s Q_v - q_2 \overline{Q_c} \\ \overline{Q_v Q_c - q_3 \overline{Q_s}} & q_1 q_3 - |Q_c|^2 & \overline{Q_s Q_c} - q_1 Q_v \\ \overline{Q_v Q_s} - q_2 Q_c & Q_c Q_s - q_1 \overline{Q_v} & q_1 q_2 - |Q_s|^2 \end{pmatrix}, \tag{23}$$

from which it is derived that $Q \circ P = \frac{1}{2}(QP + PQ)$. The cubic Jordan algebra $\mathfrak{J}_3^{\mathbb{O}^s}$ has irreducible idempotents given by

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{24}$$

The $D = 5, \mathcal{N} = 8$ U-duality group is given by the reduced structure group $\mathrm{Str}_0(\mathfrak{J}_3^{\mathbb{O}^s}) = E_{6(6)}$, which is the maximally non-compact (split) form of $E_6(\mathbb{C})$ under which $Q \in \mathfrak{J}_3^{\mathbb{O}^s}$ transforms as the fundamental $\mathbf{27}$. The BH entropy is then given by (recall Eq. (4)

$$S_{D=5,\mathrm{BH}} = \pi\sqrt{|I_3(Q)|} = \pi\sqrt{|N(Q)|}. \tag{25}$$

The U-duality charge orbits are classified according to the $E_{6(6)}$-invariant Jordan rank of the charge vector, as defined in (20). This precisely reproduces the classification originally obtained in [11, 59]. The maximally split form of the U-duality group, which corresponds to the use of the split-octonions[2], is the most powerful in the sense that for each rank there is a *unique* canonical form to which all elements may be transformed. More precisely, we have the following

**Theorem 1.** [11, 60] *Every BH charge vector $Q \in \mathfrak{J}_3^{\mathbb{O}^s}$ of a given rank is $E_{6(6)}$ related to one of the following canonical forms:*

1. *Rank 1*
   (a) $Q_1 = (1, 0, 0) = E_1$
2. *Rank 2*
   (a) $Q_2 = (1, 1, 0) = E_1 + E_2$
3. *Rank 3*
   (a) $Q_3 = (1, 1, k) = E_1 + E_2 + kE_3$

The orbit stabilizers are summarized in Table I. We will see that the orbit structure of theories with less supersymmetry is a progressive splitting of this exceptionally simple case [].

TABLE I. Charge orbits, corresponding *moduli spaces* and the number # of "non-flat" scalar directions of $D = 5, \mathcal{N} = 8$ supergravity defined over $\mathfrak{J}_3^{\mathbb{O}^s}$ [11].

| $\mathfrak{J}_3^{\mathbb{O}^s}$, $M = E_{6(6)}/\,\mathrm{Usp}(8)$ | | | | | |
|---|---|---|---|---|---|
| Rank | BH | Susy | Charge orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # |
| 1 | small critical | 1/2 | $\frac{E_{6(6)}}{\mathrm{SO}(5,5)\ltimes\mathbb{R}^{16}}$ | $\frac{\mathrm{SO}(5,5)}{\mathrm{SO}(5)\times\mathrm{SO}(5)}\ltimes\mathbb{R}^{16}$ | 1 |
| 2 | small light-like | 1/4 | $\frac{E_{6(6)}}{\mathrm{SO}(5,4)\ltimes\mathbb{R}^{16}}$ | $\frac{\mathrm{SO}(5,4)}{\mathrm{SO}(5)\times\mathrm{SO}(4)}\ltimes\mathbb{R}^{16}$ | 6 |
| 3 | large | 1/8 | $\frac{E_{6(6)}}{F_{4(4)}}$ | $\frac{F_{4(4)}}{\mathrm{Usp}(6)\times\mathrm{SU}(2)}$ | 14 |

[2] The split-octonions are not division, but are composition: $|ab| = |a||b|$.

### C. $\mathcal{N}=2$ Magic

The $3+3\dim\mathbb{A}$ electric BH charges may be represented as elements

$$Q=\begin{pmatrix} q_1 & Q_s & \overline{Q_c}\\ \overline{Q_s} & q_2 & Q_v\\ Q_c & \overline{Q_v} & q_3\end{pmatrix},\quad \text{where}\quad q_1,q_2,q_3\in\mathbb{R}\quad\text{and}\quad Q_{v,s,c}\in\mathbb{A} \tag{26}$$

of the $(3+3\dim\mathbb{A})$-dimensional Jordan algebra $\mathfrak{J}_3^{\mathbb{A}}$ of $3\times 3$ Hermitian matrices over the division algebra $\mathbb{A}$ [56]. The irreducible idempotents, quadratic adjoint and cubic norm are as in section II B. The magic $D=5$, $\mathcal{N}=2$ U-duality groups $G_5^{\mathbb{A}}$ are given by the reduced structure group $\mathrm{Str}_0(\mathfrak{J}_3^{\mathbb{A}})$. For $\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$ the U-duality $G_5^{\mathbb{A}}$ is $\mathrm{SL}(3,\mathbb{R}),\mathrm{SL}(3,\mathbb{C}),\mathrm{SU}^\star(6),E_{6(-26)}$ under which $Q\in\mathfrak{J}_3^{\mathbb{A}}$ transforms as a $\mathbf{6},\mathbf{9},\mathbf{15},\mathbf{27}$, respectively. The BH entropy is given by Eq. (25). Once again, the U-duality charge orbits are classified according to the $G_5^{\mathbb{A}}$-invariant Jordan rank of the charge vector. More precisely, we have the following

**Theorem 2.** [11, 27] *Every BH charge vector $Q\in\mathfrak{J}_3^{\mathbb{A}}$ of a given rank is $G_5^{\mathbb{A}}$ related to one of the following canonical forms:*

1. *Rank 1*
   - *(a)* $Q_{1a}=(1,0,0)=E_1$
   - *(b)* $Q_{1b}=(-1,0,0)=-E_1$
2. *Rank 2*
   - *(a)* $Q_{2a}=(1,1,0)=E_1+E_2$
   - *(b)* $Q_{2b}=(-1,1,0)=-E_1+E_2$
   - *(c)* $Q_{2c}=(-1,-1,0)=-E_1-E_2$
3. *Rank 3*
   - *(a)* $Q_{3a}=(1,1,k)=E_1+E_2+kE_3$
   - *(b)* $Q_{3b}=(-1,-1,k)=-E_1-E_2+kE_3$

Note, the orbits generated by the conical forms $Q_{1a}$ and $Q_{1b}$ are isomorphic, as are those generated by $Q_{2a}$ and $Q_{2c}$. The light-like 1/4-BPS orbit of the $\mathcal{N}=8$ splits into one 1/2-BPS and one non-BPS orbit, as does the large 1/8-BPS orbit. Note, the critical 1/2-BPS orbit remains intact [30]. The orbits are summarized in Table II (the exceptional - octonionic - case was firstly derived in [11]). Note that the $\mathcal{N}=2$ $\mathfrak{J}_3^{\mathbb{H}}$ theory has a "dual" interpretation as $\mathcal{N}=6$ supergravity, as described in [30].

### D. The $\mathcal{N}=4$ and $\mathcal{N}=2$ Reducible Jordan Symmetric Sequences

#### 1. $\mathcal{N}=4$

For $\mathcal{N}=4$ supergravity coupled to $n_V$ vector multiplets, the $n+5$ electric BH charges may be represented as elements ($\mu:=0,I$, where $I=1,...,n+3$)

$$Q=(q;q_\mu),\quad \text{where}\quad q\in\mathbb{R},\ q_\mu\in\mathbb{R}^{5,n-1}, \tag{27}$$

of the $(n+5)$-dimensional reducible cubic Jordan algebra $\mathfrak{J}_{5,n-1}$ (note that the index 0 pertains to one of the 5 graviphotons). Note, we have adopted the $(5,n-1)$ convention to emphasize the relation to the corresponding $D=4$ theory, whereas in [30] the $(5,n_V)$ convention was used, *i.e.* $n=n_V+1$. The cubic norm is defined as

$$N(Q)=qq_\mu q^\mu, \tag{28}$$

where the index $\mu$ has been raised with the $(+^5,-^{n-1})$ signature metric $\eta^{\mu\nu}$; the positive signature pertains to the 5 graviphotons of the theory, whereas the negative one pertains to the $n-1$ Abelian matter (vector) supermultiplets coupled to the gravity multiplet. The reduced structure group reads

$$G_5=\mathrm{Str}_0(\mathfrak{J}_{5,n-1})=\mathrm{SO}(1,1)\times\mathrm{SO}(5,n-1). \tag{29}$$

TABLE II. Charge orbits, corresponding *moduli spaces*, and number # of ”non-flat” scalar directions of the magic $D = 5, \mathcal{N} = 2$ supergravities defined over $\mathfrak{J}_3^{\mathbb{A}}$, $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ [28, 30].

| | | | | | |
|---|---|---|---|---|---|
| | | $\mathfrak{J}_3^{\mathbb{O}},\ n_V = 26,\ M = E_{6(-26)}/F_{4(-52)}$ | | | |
| Rank | BH | Susy | Charge orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # |
| 1 | small critical | 1/2 | $\frac{E_{6(-26)}}{\mathrm{SO}(1,9)\ltimes\mathbb{R}^{16}}$ | $\frac{\mathrm{SO}(1,9)}{\mathrm{SO}(9)}\ltimes\mathbb{R}^{16}$ | 1 |
| $2a$ | small light-like | 0 | $\frac{E_{6(-26)}}{\mathrm{SO}(1,8)\ltimes\mathbb{R}^{16}}$ | $\frac{\mathrm{SO}(1,8)}{\mathrm{SO}(8)}\ltimes\mathbb{R}^{16}$ | 2 |
| $2b$ | small light-like | 1/2 | $\frac{E_{6(-26)}}{\mathrm{SO}(9)\ltimes\mathbb{R}^{16}}$ | $\mathbb{R}^{16}$ | 10 |
| $3a(k>0)$ | large | 1/2 | $\frac{E_{6(-26)}}{F_{4(-52)}} = M$ | – | 26 |
| $3b(k>0)$ | large | $0\ (Z_H\neq 0)$ | $\frac{E_{6(-26)}}{F_{4(-20)}}$ | $\frac{F_{4(-20)}}{\mathrm{SO}(9)}$ | 10 |
| | | $\mathfrak{J}_3^{\mathbb{H}},\ n_V = 14,\ M = \mathrm{SU}^\star(6)/\mathrm{Usp}(6)$ | | | |
| Rank | BH | Susy | Charge orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # |
| 1 | small critical | 1/2 | $\frac{\mathrm{SU}^\star(6)}{[\mathrm{SO}(1,5)\times\mathrm{SO}(3)]\ltimes\mathbb{R}^{(4,2)}}$ | $\frac{\mathrm{SO}(1,5)}{\mathrm{SO}(5)}\ltimes\mathbb{R}^{(4,2)}$ | 1 |
| $2a$ | small light-like | 0 | $\frac{\mathrm{SU}^\star(6)}{[\mathrm{SO}(1,4)\times\mathrm{SO}(3)]\ltimes\mathbb{R}^{(4,2)}}$ | $\frac{\mathrm{SO}(1,4)}{\mathrm{SO}(4)}\ltimes\mathbb{R}^{(4,2)}$ | 2 |
| $2b$ | small light-like | 1/2 | $\frac{\mathrm{SU}^\star(6)}{[\mathrm{SO}(5)\times\mathrm{SO}(3)]\ltimes\mathbb{R}^{(4,2)}}$ | $\mathbb{R}^{(4,2)}$ | 6 |
| $3a(k>0)$ | large | 1/2 | $\frac{\mathrm{SU}^\star(6)}{\mathrm{Usp}(6)} = M$ | – | 14 |
| $3b(k>0)$ | large | $0\ (Z_H\neq 0)$ | $\frac{\mathrm{SU}^\star(6)}{\mathrm{Usp}(2,4)}$ | $\frac{\mathrm{Usp}(2,4)}{\mathrm{Usp}(2)\times\mathrm{Usp}(4)}$ | 6 |
| | | $\mathfrak{J}_3^{\mathbb{C}},\ n_V = 8,\ M = \mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$ | | | |
| Rank | BH | Susy | Charge orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # |
| 1 | small critical | 1/2 | $\frac{\mathrm{SL}(3,\mathbb{C})}{[\mathrm{SO}(1,3)\times\mathrm{SO}(2)]\ltimes\mathbb{R}^{(2,2)}}$ | $\frac{\mathrm{SO}(1,3)}{\mathrm{SO}(3)}\ltimes\mathbb{R}^{(2,2)}$ | 1 |
| $2a$ | small light-like | 0 | $\frac{\mathrm{SL}(3,\mathbb{C})}{[\mathrm{SO}(1,2)\times\mathrm{SO}(2)]\ltimes\mathbb{R}^{(2,2)}}$ | $\frac{\mathrm{SO}(1,2)}{\mathrm{SO}(2)}\ltimes\mathbb{R}^{(2,2)}$ | 2 |
| $2b$ | small light-like | 1/2 | $\frac{\mathrm{SL}(3,\mathbb{C})}{[\mathrm{SO}(3)\times\mathrm{SO}(2)]\ltimes\mathbb{R}^{(2,2)}}$ | $\mathbb{R}^{(2,2)}$ | 4 |
| $3a(k>0)$ | large | 1/2 | $\frac{\mathrm{SL}(3,\mathbb{C})}{\mathrm{SU}(3)} = M$ | – | 8 |
| $3b(k>0)$ | large | $0\ (Z_H\neq 0)$ | $\frac{\mathrm{SL}(3,\mathbb{C})}{\mathrm{SU}(1,2)}$ | $\frac{\mathrm{SU}(1,2)}{\mathrm{U}(1)\times\mathrm{SU}(2)}$ | 4 |
| | | $\mathfrak{J}_3^{\mathbb{R}},\ n_V = 5,\ M = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ | | | |
| Rank | BH | Susy | Charge orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # |
| 1 | small critical | 1/2 | $\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(1,2)\ltimes\mathbb{R}^2}$ | $\frac{\mathrm{SO}(1,2)}{\mathrm{SO}(2)}\ltimes\mathbb{R}^2$ | 1 |
| $2a$ | small light-like | 0 | $\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(1,1)\ltimes\mathbb{R}^2}$ | $\mathrm{SO}(1,1)\ltimes\mathbb{R}^2$ | 2 |
| $2b$ | small light-like | 1/2 | $\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(2)\ltimes\mathbb{R}^2}$ | $\mathbb{R}^2$ | 3 |
| $3a(k>0)$ | large | 1/2 | $\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(3)} = M$ | – | 5 |
| $3b(k>0)$ | large | $0\ (Z_H\neq 0)$ | $\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(1,2)}$ | $\frac{\mathrm{SO}(1,2)}{\mathrm{SO}(2)}$ | 3 |

For $\lambda \in \mathbb{R}, \Lambda \in \mathrm{SO}(5, n-1)$, its action on the charge vector reads

$$(q; q_\mu) \mapsto (e^{2\lambda} q; e^{-\lambda} \Lambda_\mu{}^\nu q_\nu). \tag{30}$$

One finds that the quadratic adjoint (16) is given by,

$$Q^\sharp = (q_\mu q^\mu; q q_0, -q q_I), \tag{31}$$

from which it is derived that[3]

$$Q \circ P = (qp; q_0 p_0 - q_J p^J, q_0 p_I + p_0 q_I), \tag{32}$$

where the index $I$ has been raised with the $(+^4, -^{n-1})$ signature metric $\eta^{nm}$. Consequently, the automorphism group is given by

$$\mathrm{Aut}(\mathfrak{J}_{5,n-1}) = \mathrm{SO}(4, n-1). \tag{33}$$

Three irreducible idempotents are given by

$$E_1 = (1; 0); \quad E_2 = (0; \tfrac{1}{2}, 0, 0, 0, 0, \tfrac{1}{2}, 0, \ldots); \quad E_3 = (0; \tfrac{1}{2}, 0, 0, 0, 0, -\tfrac{1}{2}, 0, \ldots). \tag{34}$$

The U-duality charge orbits are classified according to the $\mathrm{SO}(1,1) \times \mathrm{SO}(5, n-1)$ invariant Jordan *rank* of the charge vector. More precisely, the following theorem [25] holds.

**Theorem 3.** *Every BH charge vector* $Q = (q; q_\mu) \in \mathfrak{J}_{5,n-1}$ *of a given rank is* $\mathrm{SO}(1,1) \times \mathrm{SO}(5, n-1)$ *related one of the following canonical forms:*

1. *Rank 1*
   - *(a)* $Q_{1a} = E_1$
   - *(b)* $Q_{1b} = -E_1$
   - *(c)* $Q_{1c} = E_2$
2. *Rank 2*
   - *(a)* $Q_{2a} = E_2 + E_3$
   - *(b)* $Q_{2b} = E_2 - E_3$
   - *(c)* $Q_{2c} = E_1 + E_2$
   - *(d)* $Q_{2d} = -E_1 - E_2$
3. *Rank 3*
   - *(a)* $Q_{3a} = E_1 + E_2 + kE_3$
   - *(b)* $Q_{3b} = -E_1 + E_2 + kE_3$

Note, the orbits $1a$ and $1b$ are physically equivalent, and have isomorphic cosets. The same applies to $2c$ and $2d$. The orbits are summarized in Table III [30].

TABLE III. Charge orbits, corresponding *moduli spaces* and number # of "non-flat" scalar directions of the reducible $D = 5, \mathcal{N} = 4$ supergravities defined over $\mathfrak{J}_{5,n-1} = \mathbb{R} \oplus \Gamma_{5,n-1}$ [30]. The scalar manifold reads $M = [\mathrm{SO}(1,1) \times \mathrm{SO}(5, n-1]/[\mathrm{SO}(5) \times \mathrm{SO}(n-1)]$, with $\dim_{\mathbb{R}} = 5n - 4$

| Rank | BH | Susy | Charge orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # |
|---|---|---|---|---|---|
| $1a$ | small critical | 1/2 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(5,n-1)}{\mathrm{SO}(5,n-1)}$ | $\frac{\mathrm{SO}(5,n-1)}{\mathrm{SO}(5)\times\mathrm{SO}(n-1)}$ | 1 |
| $1c$ | | 1/2 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(5,n-1)}{\mathrm{SO}(1,1)\times\mathrm{SO}(4,n-2)\ltimes\mathbb{R}^{4,n-2}}$ | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(4,n-2)}{\mathrm{SO}(4)\times\mathrm{SO}(n-2)} \ltimes \mathbb{R}^{4,n-2}$ | 2 |
| $2a$ | small light-like | 1/2 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(5,n-1)}{\mathrm{SO}(4,n-1)}$ | $\frac{\mathrm{SO}(4,n-1)}{\mathrm{SO}(4)\times\mathrm{SO}(n-1)}$ | $n$ |
| $2b$ | | 0 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(5,n-1)}{\mathrm{SO}(5,n-2)}$ | $\frac{\mathrm{SO}(5,n-2)}{\mathrm{SO}(5)\times\mathrm{SO}(n-2)}$ | 6 |
| $2c$ | | 1/4 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(5,n-1)}{\mathrm{SO}(4,n-2)\ltimes\mathbb{R}^{4,n-2}}$ | $\frac{\mathrm{SO}(4,n-2)}{\mathrm{SO}(4)\times\mathrm{SO}(n-2)} \ltimes \mathbb{R}^{4,n-2}$ | 2 |
| $3ab(k > 0)$ | large | 1/4 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(5,n-1)}{\mathrm{SO}(4,n-1)}$ | $\frac{\mathrm{SO}(4,n-1)}{\mathrm{SO}(4)\times\mathrm{SO}(n-1)}$ | $n$ |
| $3b(k < 0)$ | | 0 $(\hat{Z}_{AB,H}=0)$ | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(5,n-1)}{\mathrm{SO}(5,n-2)}$ | $\frac{\mathrm{SO}(5,n-2)}{\mathrm{SO}(5)\times\mathrm{SO}(n-2)}$ | 6 |

[3] Note, this construction appears to be undemocratic in the sense that it picks out one of the graviphotons $q_0$ as special. This is due to the undemocratic choice of base point $c = (1; 1, 0)$ we have used. This choice was made for convenience, but one could have equally used a "democratic" base point, valid for any signature $\mathfrak{J}_{p,q}$ with $p \geq 1$, $c = (p^{-1}; 1, 1, \ldots, 1, 0, 0, \ldots, 0)$, which for $p = 5$ treats all five graviphotons on the same footing. Of course, this is just a matter of conventions and the results are unaffected.

### 2. $\mathcal{N}=2$

For $\mathcal{N}=2$ theories coupled to $n_V$ vector multiplets, whose scalar manifolds belong to the so-called Jordan symmetric sequence of the real special geometry, the $n+1$ electric BH charges may be represented as elements ($\mu := 0, I$, where $I = 1, ..., n-1$)

$$Q = (q; q_\mu), \quad \text{where} \quad q \in \mathbb{R},\ q_\mu \in \mathbb{R}^{1,n-1}, \tag{35}$$

of the $(n+1)$-dimensional reducible cubic Jordan algebra $\mathfrak{J}_{1,n-1}$. Once again, let us note that we have adopted the $(1, n-1)$ convention, in order to to emphasize the relation to the corresponding $D=4$ theory, whereas in [30] the $(1, n_V)$ convention was used, *i.e.* $n = n_V + 1$. The set-up and analysis is essentially as for the $\mathcal{N}=4$ case. The principle difference is that the 1/4-BPS orbits split into one 1/2-BPS and one non-BPS orbit. This is captured in the connectedness of the charge orbits [30], as we will discuss below. This may be seen as a consequence of the Lorentzian nature of $\mathfrak{J}_{1,n-1}$, contrasted to the genuine pseudo-Euclidean nature of $\mathfrak{J}_{5,n-1}$. As for $\mathcal{N}=4$, the cubic norm is defined by (28), but now the index $\mu$ is raised with the $(+, -^{n-1})$ signature metric $\eta^{\mu\nu}$. The reduced structure group is therefore

$$G_5 = \mathrm{Str}_0(\mathfrak{J}_{1,n-1}) = \mathrm{SO}(1,1) \times \mathrm{SO}(1, n-1). \tag{36}$$

For $\lambda \in \mathbb{R}, \Lambda \in \mathrm{SO}(1, n-1)$, its action on the charge vector is given by Eq. (30). Then, one finds that the quadratic adjoint (16) is given by

$$Q^\sharp = (q_\mu q^\mu; q q^\mu), \tag{37}$$

from which Eq. (32) can be derived. Consequently, the automorphism group is given by

$$\mathrm{Aut}(\mathfrak{J}_{1,n-1}) = \mathrm{SO}(n-1) = mcs\left(\mathrm{Str}_0\left(\mathfrak{J}_{1,n-1}\right)\right). \tag{38}$$

Three irreducible idempotents are given by

$$E_1 = (1; 0); \quad E_2 = (0; \tfrac{1}{2}, \tfrac{1}{2}, 0, \ldots); \quad E_3 = (0; \tfrac{1}{2}, -\tfrac{1}{2}, 0, \ldots). \tag{39}$$

The U-duality charge orbits are classified according to the $\mathrm{SO}(1,1) \times \mathrm{SO}(1, n-1)$ invariant Jordan *rank* of the charge vector. More precisely, the following theorem [25] holds.

**Theorem 4.** *Every BH charge vector $Q = (q; q_\mu) \in \mathfrak{J}_{1,n-1}$ of a given rank is $\mathrm{SO}(1,1) \times \mathrm{SO}(1, n-1)$ related to one of the following canonical forms:*

1. *Rank 1*
   - *(a)* $Q_{1a} = E_1$
   - *(b)* $Q_{1b} = -E_1$
   - *(c)* $Q_{1c} = E_2$
2. *Rank 2*
   - *(a)* $Q_{2a} = E_2 + E_3$
   - *(b)* $Q_{2b} = E_2 - E_3$
   - *(c)* $Q_{2c} = E_1 + E_2$
   - *(d)* $Q_{2d} = -E_1 - E_2$
3. *Rank 3*
   - *(a)* $Q_{3a} = E_1 + E_2 + kE_3$
   - *(b)* $Q_{3b} = -E_1 + E_2 + kE_3$

*Note, if one restricts to the identity-connected component of $\mathrm{SO}(1, n-1)$, each of the orbits $Q_{1c}$, $Q_{2c}$ and $Q_{2d}$ splits into two cases, $Q^{\pm}_{1c}$, $Q^{\pm}_{2c}$ and $Q^{\pm}_{2d}$, corresponding to the future and past light cones. Similarly, $Q_{2a}$ splits into two disconnected components, $Q^{\pm}_{2a}$, corresponding to the future and past hyperboloids. For $k > 0$ the orbits $Q_{3a}$ and $Q_{3b}$ also split into disconnected future and past hyperboloids, $Q^{\pm}_{3a}$ and $Q^{\pm}_{3b}$.*

TABLE IV. Charge orbits, corresponding *moduli spaces*, and number # of "non-flat" scalar directions of the reducible $D = 5, \mathcal{N} = 2$ supergravities defined over $\mathfrak{J}_{1,n-1} = \mathbb{R} \oplus \Gamma_{1,n-1}$ [30]. The scalar manifold reads $M = [\mathrm{SO}(1,1) \times \mathrm{SO}(1,n-1]/\mathrm{SO}(n-1)$, with $\dim_{\mathbb{R}} M = n$.

| Rank | BH | Susy | Charge orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # |
|---|---|---|---|---|---|
| $1a$ | small critical | 1/2 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(1,n-1)}$ | $\frac{\mathrm{SO}(1,n-1)}{\mathrm{SO}(n-1)}$ | 1 |
| $1c$ | | 1/2 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(1,1)\times\mathrm{SO}(n-2)\ltimes\mathbb{R}^{n-2}}$ | $\mathrm{SO}(1,1)\times\mathbb{R}^{n-2}$ | 2 |
| $2a$ | | 1/2 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(n-1)}$ | – | $n$ |
| $2b$ | small light-like | 0 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(1,n-2)}$ | $\frac{\mathrm{SO}(1,n-2)}{\mathrm{SO}(n-2)}$ | 2 |
| $2c^+$ | | 1/2 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(n-2)\ltimes\mathbb{R}^{n-2}}$ | $\mathbb{R}^{n-2}$ | 2 |
| $2c^-$ | | 0 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(n-2)\ltimes\mathbb{R}^{n-2}}$ | $\mathbb{R}^{n-2}$ | 2 |
| $2d^-$ | | 1/2 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(n-2)\ltimes\mathbb{R}^{n-2}}$ | $\mathbb{R}^{n-2}$ | 2 |
| $2d^+$ | | 0 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(n-2)\ltimes\mathbb{R}^{n-2}}$ | $\mathbb{R}^{n-2}$ | 2 |
| $3a^+(k>0)$ | | 1/2 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(n-1)}$ | – | $n$ |
| $3a^-(k>0)$ | large | 0 $(Z_H\neq 0)$ | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(n-1)}$ | – | $n$ |
| $3b^-(k>0)$ | | 1/2 | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(n-1)}$ | – | $n$ |
| $3b^+(k>0)$ | | 0 $(Z_H\neq 0)$ | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(n-1)}$ | – | $n$ |
| $3ab(k<0)$ | | 0 $(Z_H\neq 0)$ | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(1,n-2)}$ | $\frac{\mathrm{SO}(1,n-2)}{\mathrm{SO}(n-2)}$ | 2 |

The orbits are summarized in Table IV. As described in [30], the orbits $Q^{\pm}_{2c}$, $Q^{\pm}_{2d}$, $Q^{\pm}_{3a}$ and $Q^{\pm}_{3b}$ are BPS or non-BPS according as the sign $+/-$ of $q$ is correlated or anti-correlated, respectively, with the future/past branch on which the orbit is defined.

The *non-Jordan symmetric* sequence [61]

$$M_{nJ,5,n} \equiv \frac{SO(1,n)}{SO(n)},\ n = n_V \in \mathbb{N}, \tag{40}$$

($n_V$ being the number of Abelian vector supermultiplets coupled to the $\mathcal{N}=2$, $D=5$ supergravity one) is the only (sequence of) symmetric *real special geometry* which is *not* related to a cubic Jordan algebra. It is usually denoted by $L(-1,n-1)$ in the classification of homogeneous Riemannian $d$-spaces (see *e.g.* [62], and Refs. therein).

As discussed in [61], the isometries of the symmetric real special space (40) are not all contained in the invariance group of the corresponding supergravity theory, despite the fact that the latter group still acts transitively on the space. By using the parametrization introduced in the last Sec. of [63] and comparing *e.g.* Eq. (5.1) of [62] to Eq. (7) of [61], it is immediate to conclude that the $D=5$, $\mathcal{N}=2$ Maxwell-Einstein supergravity theory whose scalar manifold is given by (40) can be uplifted to a $D=6$, $(1,0)$ supergravity theory with $n-1$ vector multiplets, but *no tensor multiplets at all* ($n_T = 0$). Thus, in absence of matter fields charged under a non-trivial gauge group, the gravitational anomaly-free condition implies that [64, 65] $n_H = 272 + n$ hypermultiplets must be coupled to the theory. On the other hand, this theory is known not to satisfy the condition of conservation of the gauge vector current (required by the consistency of the gauge invariance [66–70]); therefore, it seemingly has a $D=6$ uplift to $(1,0)$ chiral supergravity which is *not* anomaly-free, unless it is embedded in a model where a non-trivial gauge group is present, with charged matter (see *e.g.* [71, 72]).

We will not further considered this theory in the present investigation, because it does not correspond to symmetric spaces in $D=4$ [61].

## III. BH CHARGE ORBITS IN $D=4$ SYMMETRIC SUPERGRAVITIES

### A. The Freudenthal Triple System

Given a cubic Jordan algebra $\mathfrak{J}$ defined over a field $\mathbb{F}$, one is able to construct a FTS by defining the vector space $\mathfrak{F}(\mathfrak{J})(:=\mathfrak{F})$,

$$\mathfrak{F}(\mathfrak{J})=\mathbb{F}\oplus\mathbb{F}\oplus\mathfrak{J}\oplus\mathfrak{J}. \tag{41}$$

An arbitrary element $x\in\mathfrak{F}(\mathfrak{J})$ may be written as a formal "$2\times 2$ matrix",

$$x=\begin{pmatrix}\alpha & X\\ Y & \beta\end{pmatrix}\quad\text{where } \alpha,\beta\in\mathbb{F}\quad\text{and}\quad X,Y\in\mathfrak{J}. \tag{42}$$

The FTS comes equipped with a non-degenerate bilinear antisymmetric quadratic form, a quartic form and a trilinear triple product [73, 74]:

1. Quadratic form $\{x,y\}$: $\mathfrak{F}\times\mathfrak{F}\to\mathbb{F}$

$$\{x,y\}=\alpha\delta-\beta\gamma+\mathrm{Tr}(X,W)-\mathrm{Tr}(Y,Z),\quad\text{where}\quad x=\begin{pmatrix}\alpha & X\\ Y & \beta\end{pmatrix},\ y=\begin{pmatrix}\gamma & Z\\ W & \delta\end{pmatrix}. \tag{43a}$$

2. Quartic form $q:\mathfrak{F}\to\mathbb{F}$

$$q(x)=-2[\alpha\beta-\mathrm{Tr}(X,Y)]^2-8[\alpha N(X)+\beta N(Y)-\mathrm{Tr}(X^\sharp,Y^\sharp)]. \tag{43b}$$

3. Triple product $T:\mathfrak{F}\times\mathfrak{F}\times\mathfrak{F}\to\mathfrak{F}$ which is uniquely defined by

$$\{T(x,y,w),z\}=q(x,y,w,z) \tag{43c}$$

where $q(x,y,w,z)$ is the full linearisation of $q(x)$ such that $q(x,x,x,x)=q(x)$.

The *automorphism* group is given by the set of invertible $\mathbb{F}$-linear transformations preserving the quadratic and quartic forms [73, 74],

$$\mathrm{Aut}(\mathfrak{F}):=\{\sigma\in\mathrm{Iso}_{\mathbb{F}}(\mathfrak{F})|q(\sigma x)=q(x),\{\sigma x,\sigma y\}=\{x,y\},\ \forall x,y\in\mathfrak{F}\}=\mathrm{Conf}\,(\mathfrak{J})\,. \tag{44}$$

Generally, the automorphism group corresponds to the U-duality group of corresponding 4-dimensional supergravities (see for example [12, 32, 75, 76], and Refs. therein). The conventional concept of matrix rank may be generalised to Freudenthal triple systems in a natural and $\mathrm{Aut}(\mathfrak{F})$ invariant manner. The rank of an arbitrary element $x\in\mathfrak{F}$ is uniquely defined by [26, 77]:

$$\begin{aligned}
&\mathrm{Rank}x=1\Leftrightarrow 3T(x,x,y)+x\{x,y\}x=0\ \forall y;\\
&\mathrm{Rank}x=2\Leftrightarrow \exists y\text{ s.t. }3T(x,x,y)+x\{x,y\}x\neq 0,\ T(x,x,x)=0;\\
&\mathrm{Rank}x=3\Leftrightarrow T(x,x,x)\neq 0,\ q(x)=0;\\
&\mathrm{Rank}x=4\Leftrightarrow q(x)\neq 0.
\end{aligned} \tag{45}$$

### B. $\mathcal{N}=8$

The $(1+27)+(1+27)$ electric+magnetic BH charges may be represented as elements

$$x=\begin{pmatrix}-q_0 & P\\ Q & p^0\end{pmatrix},\quad\text{where}\quad p^0,q^0\in\mathbb{R}\quad\text{and}\quad Q,P\in\mathfrak{J}_3^{\mathbb{O}^s} \tag{46}$$

of the Freudenthal triple system $\mathfrak{F}^{\mathbb{A}}:=\mathfrak{F}(\mathfrak{J}_3^{\mathbb{O}^s})$. The details may be found in section III A of [25], and in Refs. therein. The automorphism group $\mathrm{Aut}(\mathfrak{F}^{\mathbb{O}^s})\cong\mathrm{Conf}\left(\mathfrak{J}_3^{\mathbb{O}^s}\right)=E_{7(7)}$ is the $D=4$, $\mathcal{N}=8$ U-duality group, where $x\in\mathfrak{F}^{\mathbb{A}}$ transforms as the fundamental **56**. The BH entropy is given by Eq. (2), where $I_4(x)=\Delta(x)=\frac{1}{2}q(x)$ is Cartan's unique quartic invariant polynomial of $E_{7(7)}$ [78]. The U-duality charge orbits are classified according to the $E_{7(7)}$-invariant FTS *rank* of the charge vector, as defined in (45). This reproduces the classification originally obtained in [11, 59]. More precisely, we have the following

**Theorem 5.** [11, 26, 50] *Every BH charge vector $x \in \mathfrak{F}^{\mathbb{O}^s}$ of a given rank is $E_{7(7)}$ related one of the following canonical forms:*

1. *Rank 1*

   (a) $x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

2. *Rank 2*

   (a) $x_2 = \begin{pmatrix} 1 & (1,0,0) \\ 0 & 0 \end{pmatrix}$

3. *Rank 3*

   (a) $x_3 = \begin{pmatrix} 1 & (1,1,0) \\ 0 & 0 \end{pmatrix}$

4. *Rank 4*

   (a) $x_{4a} = k \begin{pmatrix} 1 & (-1,-1,-1) \\ 0 & 0 \end{pmatrix}$

   (b) $x_{4b} = k \begin{pmatrix} 1 & (1,1,1) \\ 0 & 0 \end{pmatrix}$

   *where* $k > 0$.

As anticipated, there is one orbit per rank, but with rank 4 splitting into $4a$ ($\Delta > 0$) 1/8-BPS and $4b$ ($\Delta < 0$) non-BPS. The orbits are summarized in Table V.

TABLE V. Charge orbits, *moduli spaces*, and number # of "non-flat" scalar directions of $D = 4, \mathcal{N} = 8$ supergravity defined over $\mathfrak{F}^{\mathbb{O}^s}$. $M = E_{7(7)}/\,\mathrm{SU}(8)$, $\dim_{\mathbb{R}} = 70$ [11].

| Rank | BH | Susy | Charge orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # |
|---|---|---|---|---|---|
| 1 | doubly critical | 1/2 | $\frac{E_{7(7)}}{E_{6(6)} \ltimes \mathbb{R}^{27}}$ | $\frac{E_{6(6)}}{\mathrm{Usp}(8)} \ltimes \mathbb{R}^{27}$ | 1 |
| 2 | critical | 1/4 | $\frac{E_{7(7)}}{\mathrm{SO}(6,5) \ltimes \mathbb{R}^{32} \times \mathbb{R}}$ | $\frac{\mathrm{SO}(6,5)}{\mathrm{SO}(6)\times\mathrm{SO}(5)} \ltimes \mathbb{R}^{32} \times \mathbb{R}$ | 7 |
| 3 | light-like | 1/8 | $\frac{E_{7(7)}}{F_{4(4)} \ltimes \mathbb{R}^{26}}$ | $\frac{F_{4(4)}}{\mathrm{Usp}(6)\times\mathrm{SU}(2)} \ltimes \mathbb{R}^{26}$ | 16 |
| $4(\Delta > 0)$ | large | 1/8 | $\frac{E_{7(7)}}{E_{6(2)}}$ | $\frac{E_{6(2)}}{\mathrm{SU}(6)\times\mathrm{SU}(2)}$ | 30 |
| $4(\Delta < 0)$ | | 0 | $\frac{E_{7(7)}}{E_{6(6)}}$ | $\frac{E_{6(6)}}{\mathrm{Usp}(8)}$ | 28 |

### C. $\mathcal{N} = 2$ Magic

The $(4 + 3\dim\mathbb{A}) + (4 + 3\dim\mathbb{A})$ electric+magnetic BH charges may be represented as elements

$$x = \begin{pmatrix} -q_0 & P \\ Q & p^0 \end{pmatrix}, \quad \text{where} \quad p^0, q^0 \in \mathbb{R} \quad \text{and} \quad Q, P \in \mathfrak{J}_3^{\mathbb{A}} \tag{47}$$

of the Freudenthal triple system $\mathfrak{F}^{\mathbb{A}} := \mathfrak{F}(\mathfrak{J}_3^{\mathbb{A}})$. The details may be found in section III A, Ref. [25], and in Refs. therein. The magic $D = 4$, $\mathcal{N} = 2$ U-duality groups $G_4^{\mathbb{A}}$ are given by the automorphism group $\mathrm{Aut}(\mathfrak{F}^{\mathbb{A}}) \cong \mathrm{Conf}\left(\mathfrak{J}_3^{\mathbb{A}}\right)$. For $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ the U-duality group $G_4^{\mathbb{A}}$ is $\mathrm{Sp}(6,\mathbb{R}), \mathrm{SU}(3,3), \mathrm{SO}^\star(12), E_{7(-25)}$. The $(8 + 6\dim\mathbb{A})$ charges transform linearly as the threefold antisymmetric traceless tensor $\mathbf{14}'$, the threefold antisymmetric self-dual tensor $\mathbf{20}$, the chiral spinor $\mathbf{32}$ and the fundamental $\mathbf{56}$ of $\mathrm{Sp}(6,\mathbb{R})$, $\mathrm{SU}(3,3)$, $\mathrm{SO}^\star(12)$ and $E_{7(-25)}$, respectively.

The BH entropy is given by Eq. (2), where $I_4(x) = \Delta(x) = \frac{1}{2}q(x)$ is the unique quartic invariant polynomial of $G_4^{\mathbb{A}}$. The U-duality charge orbits are classified according to the $G_4^{\mathbb{A}}$-invariant FTS *rank* of the charge vector, as defined in (45). More precisely, we have the following

**Theorem 6.** [11, 27] *Every BH charge vector $x \in \mathfrak{F}^{\mathbb{A}}$ of a given rank is $G_4^{\mathbb{A}}$ related one of the following canonical forms:*

1. *Rank 1*

   (a) $x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

2. *Rank 2*

   (a) $x_{2a} = \begin{pmatrix} 1 & (1,0,0) \\ 0 & 0 \end{pmatrix}$

   (b) $x_{2b} = \begin{pmatrix} 1 & (-1,0,0) \\ 0 & 0 \end{pmatrix}$

3. *Rank 3*

   (a) $x_{3a} = \begin{pmatrix} 1 & (1,1,0) \\ 0 & 0 \end{pmatrix}$

   (b) $x_{3b} = \begin{pmatrix} 1 & (-1,-1,0) \\ 0 & 0 \end{pmatrix}$

4. *Rank 4*

   (a) $x_{4a} = k \begin{pmatrix} 1 & (-1,-1,-1) \\ 0 & 0 \end{pmatrix}$

   (b) $x_{4b} = k \begin{pmatrix} 1 & (1,1,-1) \\ 0 & 0 \end{pmatrix}$

   (c) $x_{4c} = k \begin{pmatrix} 1 & (1,1,1) \\ 0 & 0 \end{pmatrix}$

   *where $k > 0$.*

Here, we see that the rank 2 and 3 orbits of the $\mathcal{N} = 8$ theory split in to one 1/2-BPS orbit and one non-BPS orbit each. The splitting of the large BHs is a little more subtle [12]. There is, as always for $\mathcal{N} = 2$, one 1/2-BPS ($I_4 > 0$) orbit, which we label $4a$. However, there is also one non-BPS orbit for $I_4 > 0$, which has vanishing central charge at the horizon $Z_H = 0$. Finally, there is the universal non-BPS $I_4 < 0$, which has non-vanishing central charge at the horizon. The orbit stabilizers are summarized in Table VI. The exceptional octonionic case is given as a detailed example in section A 1, which thus provides an alternative derivation of the result obtained in [11].

### 1. $\mathcal{N} = 2$ *Magic Quaternionic versus* $\mathcal{N} = 6$

As is well known [12, 57, 112], $\mathcal{N} = 2$ magic quaternionic and $\mathcal{N} = 6$ supergravity share the very same bosonic sector; they are both related to the simple, rank-3 Jordan algebra $\mathfrak{J}_3^{\mathbb{H}}$ over the quaternions, and their scalar manifold is the rank-3 symmetric coset $\frac{SO^*(12)}{U(6)}$.

It should also be noticed that the two real, non-compact forms of $E_7$ given by $E_{7(7)}$ and $E_{7(-25)}$ contain $SO^*(12) \times SU(2)$ as a maximal subgroup, and indeed both manifolds $\frac{E_{7(-25)}}{E_6 \times U(1)}$ (rank-3 special Kähler, with $\dim_{\mathbb{C}} = 27$) and $\frac{E_{7(7)}}{SU(8)}$ (rank-7, with $\dim_{\mathbb{R}} = 70$) contain the coset space $\frac{SO^*(12)}{U(6)}$ as a submanifold. Such an observation reveals the *dual* role of the manifold $\frac{SO^*(12)}{U(6)}$: it is at the same time the $\sigma$-model scalar manifold of $\mathcal{N} = 6$ supergravity and of $\mathcal{N} = 2$ magic quaternionic Maxwell-Einstein supergravity.

Starting from $\mathcal{N} = 8$, the supersymmetry truncation down to $\mathcal{N} = 6$ goes as follows:

$$\begin{array}{c} \mathcal{N} = 8 : \left[ (2)\,,\ 8\left(\frac{3}{2}\right)\,,\ 28\,(1)\,,\ 56\left(\frac{1}{2}\right)\,,\ 70\,(0) \right] \text{ gravity mult.} \\ \downarrow \\ \mathcal{N} = 6 : \left\{ \begin{array}{l} \left[ (2)\,,\ 6\left(\frac{3}{2}\right)\,,\ 16\,(1)\,,\ 26\left(\frac{1}{2}\right)\,,\ 30\,(0) \right] \text{ gravity mult.} \\ \\ 2\ \left[ \left(\frac{3}{2}\right)\,,\ 6\,(1)\,,\ 15\left(\frac{1}{2}\right)\,,\ 20\,(0)\,, \right] \text{ gravitino mults.} \end{array} \right. \end{array} \tag{48}$$

In order to truncate the two $\mathcal{N}=6$ gravitino multiplets away, one has to consider the $U$-duality branching for vectors reads

$$\begin{aligned} E_{7(7)} &\supset SO^{*}(12)\times SU(2)\,; \\ \mathbf{56} &= (\mathbf{32},\mathbf{1})+(\mathbf{12},\mathbf{2})\,, \end{aligned} \tag{49}$$

implying the truncation condition

$$SO^{*}(12)\times SU(2):(\mathbf{12},\mathbf{2})=0, \tag{50}$$

as well as the $\mathcal{R}$-symmetry branching (omitting $U(1)$ charges)

$$\begin{aligned} \overset{\mathcal{N}=8\ \mathcal{R}\text{-symmetry}}{SU(8)} &\supset \overset{\mathcal{N}=6\ \mathcal{R}\text{-symmetry}}{U(6)} \times SU(2)\,; \\ \mathbf{8} &= (\mathbf{6},\mathbf{1})+(\mathbf{1},\mathbf{2})\,; \\ \mathbf{28} &= (\mathbf{15},\mathbf{1})+(\mathbf{1},\mathbf{1})+(\mathbf{6},\mathbf{2})\,; \\ \mathbf{56} &= (\mathbf{20},\mathbf{1})+(\mathbf{6},\mathbf{1})+(\mathbf{15},\mathbf{2})\,; \\ \mathbf{70} &= (\mathbf{15},\mathbf{1})+(\mathbf{15},\mathbf{1})+(\mathbf{20},\mathbf{2})\,, \end{aligned} \tag{51}$$

implying the truncation conditions

$$U(6)\times\ SU(2):(\mathbf{1},\mathbf{2})=(\mathbf{6},\mathbf{2})=(\mathbf{15},\mathbf{2})=(\mathbf{20},\mathbf{2})=0. \tag{52}$$

Note that the commuting $SU(2)$ factor in (51) may be regarded as the "extra" $\mathcal{R}$-symmetry truncated away in the supersymmetry reduction $\mathcal{N}=8\rightarrow\mathcal{N}=6$ obtained by imposing (50) and (52), which corresponds to the following scalar manifold embedding:

$$\frac{E_{7(7)}}{SU(8)}\supset\frac{SO^{*}(12)}{U(6)}. \tag{53}$$

On the other hand, the supersymmetry truncation $\mathcal{N}=8\rightarrow\mathcal{N}=2$ goes as follows:

$$\begin{gathered} \mathcal{N}=8:\left[(2)\,,\ 8\left(\frac{3}{2}\right),\ 28\,(1)\,,\ 56\left(\frac{1}{2}\right),\ 70\,(0)\right]\ \text{gravity mult.} \\ \downarrow \\ \mathcal{N}=2:\left\{\begin{array}{l} \left[(2)\,,\ 2\left(\frac{3}{2}\right),\ (1)\right]\ \text{gravity mult.} \\[6pt] 6\ \left[\left(\frac{3}{2}\right),\ 2\,(1)\,,\ \left(\frac{1}{2}\right)\right]\ \text{gravitino mults.} \\[6pt] 15\ \left[(1)\,,\ 2\left(\frac{1}{2}\right),\ 2\,(0)\right]\ \text{vector mults.} \\[6pt] 10\ \left[2\left(\frac{1}{2}\right),\ 4\,(0)\right]\ \text{hypermults.} \end{array}\right. \end{gathered} \tag{54}$$

In order to truncate the six $\mathcal{N}=2$ gravitino multiplets away, the same condition (50) on $U$-irreps. has to be imposed. On the other hand, by reconsidering (51) with the the different interpretation of $\mathcal{R}$-symmetry branching $\mathcal{N}=8\rightarrow\mathcal{N}=2$ (the commuting $SU(6)$ factor in (51) now refers to the "extra" $\mathcal{R}$-symmetry truncated away), the following truncation conditions, different from (52), are obtained:

$$U(6)\times\ SU(2):(\mathbf{6},\mathbf{1})=(\mathbf{6},\mathbf{2})=0. \tag{55}$$

Thus, by imposing (50) and (55), one achieves a consistent truncation of $\mathcal{N}=8$ down to $\mathcal{N}=2$ magic octonionic supergravity coupled to 15 vector multiplets and 10 hypermultiplets, which at the level of the scalar manifold reads:

$$\frac{E_{7(7)}}{SU(8)}\supset\frac{SO^{*}(12)}{U(6)}\times\frac{E_{6(2)}}{SU(6)\times SU(2)}. \tag{56}$$

The $\mathcal{N}=2$ hyper sector can be consistently truncated away, by further imposing

$$U(6)\times\ SU(2):(\mathbf{20},\mathbf{1})=(\mathbf{20},\mathbf{2})=0, \tag{57}$$

thus yielding (53).

On the other hand, starting from the $\mathcal{N}=2$ exceptional magic supergravity with no hypermultiplets, the truncation down to its $\mathcal{N}=2$ magic quaternionic sub-theory is dictated by the following branchings ($H$ is the local symmetry group of the scalar manifold, up to a $U(1)$ factor):

$$U\text{-duality}:\left\{\begin{array}{l} E_{7(-25)}\supset SO^{*}(12)\times SU(2), \\ \mathbf{56}=(\mathbf{32},\mathbf{1})+(\mathbf{12},\mathbf{2}); \end{array}\right. \tag{58}$$

$$H\text{-symmetry}:\left\{\begin{array}{l} E_{6(-78)}\supset SU(6)\times SU(2), \\ \mathbf{27}=\left(\overline{\mathbf{6}},\mathbf{2}\right)+(\mathbf{15},\mathbf{1}), \end{array}\right. \tag{59}$$

implying the truncation conditions

$$SO^{*}(12)\times SU(2):(\mathbf{12},\mathbf{2})=0; \tag{60}$$

$$SU(6)\times SU(2):\left(\overline{\mathbf{6}},\mathbf{2}\right)=0. \tag{61}$$

Under such positions, one achieves a consistent truncation of $\mathcal{N}=2$ exceptional Maxwell-Einstein supergravity down to its $\mathcal{N}=2$ magic quaternionic sub-theory which at the level of the scalar manifold reads:

$$\frac{E_{7(-25)}}{E_{6(-78)}\times U(1)}\supset\frac{SO^{*}(12)}{U(6)}. \tag{62}$$

Once their origin as truncation has been clarified, it is thus evident that $\mathcal{N}=2$ quaternionic and $\mathcal{N}=6$, $D=4$ supergravities exhibit *indistinguishable* bosonic sectors, and therefore their charge orbits are the same, and their attractor equations [12] have the same solutions.

In order to elucidate the different supersymmetry properties of the charge orbits, by recalling the spin content of the $\mathcal{N}=6$ gravity multiplet, it should be noticed that its 16 vector fields decompose as $\mathbf{15}+\mathbf{1}$ with respect to the $\mathcal{N}=6$ $\mathcal{R}$-symmetry (as well as the 26 gauginos and the 30 scalar fields decompose as $\mathbf{20}+\mathbf{6}$ and $\mathbf{15}+\overline{\mathbf{15}}$, respectively). Thus, the $\mathcal{N}=6$ dyonic charge vector $\mathcal{Q}$ splits as

$$\mathcal{N}=6:\mathcal{Q}=\left(X,\ Z_{AB},\ \overline{Z}^{AB},\ \overline{X}\right), \tag{63}$$

where $X$ is a *complex* $SU(6)$-singlet, and $Z_{AB}$ ($A=1,...,6$) is the complex $6\times 6$ antisymmetric central charge matrix. The intertwining supersymmetry-preserving properties for the *"twin"* theories $\mathcal{N}=2$ magic quaternionic *versus* "pure" $\mathcal{N}=6$ can be obtained by noticing that the $\mathcal{N}=2$ counterpart of (63) is given by

$$\mathcal{N}=2:\mathcal{Q}=\left(Z,\ Z_{i},\ \overline{Z}_{\overline{i}},\ \overline{Z}\right), \tag{64}$$

where $Z_i\equiv D_iZ$ are the so-called *matter charges* (namely, the Kähler-covariant derivatives of the $\mathcal{N}=2$ central charge $Z$). As summarized in Table 9 of [12], (63) and (64) imply that the role of "large" BPS orbits and non-BPS orbits with (all) central charge(s) vanishing is *flipped* under the *exchange* $\mathcal{N}=2\longleftrightarrow\mathcal{N}=6$; as mentioned, such a kind of *"cross-symmetry"* is easily understood when noticing that the $\mathcal{N}=2$ central charge $Z$ corresponds to the $SU(6)$-singlet component $X$ of $\mathcal{Q}$ (63), and that the 15 complex $\mathcal{N}=2$ matter charges $Z_i$ correspond to the 15 independent complex elements of the $6\times 6$ antisymmetric $\mathcal{N}=6$ central charge matrix $Z_{AB}$.

These considerations can be extended to "small" charge orbits, by observing that orbits with representatives having $Z=0$ necessarily are non-BPS orbits (because they cannot saturate any BPS bound) and, in light of the above reasoning, they correspond to $\mathcal{N}=6$ orbits with $X=0$ representative. These simple arguments, combined with the nilpotent orbits' analysis summarized in Table V of [79], allows one to determine the intertwining supersymmetry-preserving properties related to the charge orbits, listed in the Table below (we use the orbit nomenclature reported in Table VI, and for small orbits the representatives are reported in brackets):

$$\begin{array}{lll}
\mathcal{O} & \mathcal{N}=2,\ J_3^{\mathbb{H}} & \mathcal{N}=6,\ J_3^{\mathbb{H}} \\
 & & \\
4a & 1/2\text{-BPS} & \text{nBPS}:\ X_H\neq 0, Z_{AB,H}=0 \\
4b & \text{nBPS}:\ Z_H=0 & 1/6\text{-BPS}: X_H=0, Z_{AB,H}\neq 0 \\
4c & \text{nBPS}:\ Z_H\neq 0 & \text{nBPS}:\ X_H\neq 0, Z_{AB,H}\neq 0 \\
3a & \text{nBPS}\ (Z=0) & 1/6\text{-BPS}\ (X=0) \\
3b & 1/2\text{-BPS}\ (Z\neq 0) & \text{nBPS}\ (X\neq 0) \\
2a & \text{nBPS}\ (Z=0) & 1/3\text{-BPS}\ (X=0) \\
2b & 1/2\text{-BPS}\ (Z\neq 0) & 1/6\text{-BPS}\ (X\neq 0) \\
1 & 1/2\text{-BPS}\ (Z\neq 0) & 1/2\text{-BPS}\ (X\neq 0)
\end{array} \tag{65}$$

For analogue treatment in $D=5$, see [30].

TABLE VI. Charge orbits, *moduli spaces*, and number # of "non-flat" scalar directions of the magic $D = 4, \mathcal{N} = 2$ supergravities defined over $\mathfrak{F}^{\mathbb{A}}, \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. $M = \mathrm{Aut}(\mathfrak{F}^{\mathbb{A}})/mcs(\mathfrak{J}_3^{\mathbb{A}})$. $\dim_{\mathbb{R}} M = 6 + 6 \dim \mathbb{A}$ [11].

| Rank | BH | Susy | $\mathfrak{F}^{\mathbb{O}}$, $n_V = 27$, $M = E_{7(-25)}/[\mathrm{U}(1) \times E_{6(-78)}]$: Orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # | $\mathfrak{F}^{\mathbb{H}}$, $n_V = 15$, $M = \mathrm{SO}^\star(12)/\mathrm{U}(6)$: Orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # |
|---|---|---|---|---|---|---|---|---|
| 1 | small d. critical | 1/2 | $\frac{E_{7(-25)}}{E_{6(-26)} \ltimes \mathbb{R}^{27}}$ | $\frac{E_{6(-26)}}{F_{4(-52)}} \ltimes \mathbb{R}^{27}$ | 1 | $\frac{\mathrm{SO}^\star(12)}{\mathrm{SU}^\star(6) \ltimes \mathbb{R}^{15}}$ | $\frac{\mathrm{SU}^\star(6)}{\mathrm{Usp}(6)} \ltimes \mathbb{R}^{15}$ | 1 |
| $2a$ | small critical | 0 | $\frac{E_{7(-25)}}{\mathrm{SO}(2,9) \ltimes \mathbb{R}^{32} \oplus \mathbb{R}}$ | $\frac{\mathrm{SO}(2,9)}{\mathrm{SO}(2) \times \mathrm{SO}(9)} \ltimes \mathbb{R}^{32} \oplus \mathbb{R}$ | 3 | $\frac{\mathrm{SO}^\star(12)}{[\mathrm{SO}(2,5) \times \mathrm{SO}(3)] \ltimes \mathbb{R}^{(8,2)} \oplus \mathbb{R}}$ | $\frac{\mathrm{SO}(2,5)}{\mathrm{SO}(2) \times \mathrm{SO}(5)} \ltimes \mathbb{R}^8 \oplus \mathbb{R}^8 \oplus \mathbb{R}$ | 3 |
| $2b$ | small critical | 1/2 | $\frac{E_{7(-25)}}{\mathrm{SO}(1,10) \ltimes \mathbb{R}^{32} \oplus \mathbb{R}}$ | $\frac{\mathrm{SO}(1,10)}{\mathrm{SO}(10)} \ltimes \mathbb{R}^{32} \oplus \mathbb{R}$ | 11 | $\frac{\mathrm{SO}^\star(12)}{[\mathrm{SO}(1,6) \times \mathrm{SO}(3)] \ltimes \mathbb{R}^{(8,2)} \oplus \mathbb{R}}$ | $\frac{\mathrm{SO}(1,6)}{\mathrm{SO}(6)} \ltimes \mathbb{R}^8 \oplus \mathbb{R}^8 \oplus \mathbb{R}$ | 7 |
| $3a$ | small light-like | 0 | $\frac{E_{7(-25)}}{F_{4(-20)} \ltimes \mathbb{R}^{26}}$ | $\frac{F_{4(-20)}}{\mathrm{SO}(9)} \ltimes \mathbb{R}^{26}$ | 12 | $\frac{\mathrm{SO}^\star(12)}{\mathrm{Usp}(2,4) \ltimes \mathbb{R}^{14}}$ | $\frac{\mathrm{Usp}(2,4)}{\mathrm{Usp}(2) \times \mathrm{Usp}(4)} \ltimes \mathbb{R}^{14}$ | 8 |
| $3b$ | small light-like | 1/2 | $\frac{E_{7(-25)}}{F_{4(-52)} \ltimes \mathbb{R}^{26}}$ | $\mathbb{R}^{26}$ | 28 | $\frac{\mathrm{SO}^\star(12)}{\mathrm{Usp}(6) \ltimes \mathbb{R}^{14}}$ | $\mathbb{R}^{14}$ | 16 |
| $4a$ | large time-like | 1/2 | $\frac{E_{7(-25)}}{E_{6(-78)}}$ | – | 54 | $\frac{\mathrm{SO}^\star(12)}{\mathrm{SU}(6)}$ | – | 30 |
| $4b$ | large time-like | 0 $(Z_H{=}0)$ | $\frac{E_{7(-25)}}{E_{6(-14)}}$ | $\frac{E_{6(-14)}}{\mathrm{SO}(10) \times \mathrm{SO}(2)}$ | 22 | $\frac{\mathrm{SO}^\star(12)}{\mathrm{SU}(4,2)}$ | $\frac{\mathrm{SU}(4,2)}{\mathrm{SU}(4) \times \mathrm{SU}(2)}$ | 13 |
| $4c$ | large space-like | 0 $(Z_H{\neq}0)$ | $\frac{E_{7(-25)}}{E_{6(-26)}}$ | $\frac{E_{6(-26)}}{F_{4(-52)}}$ | 28 | $\frac{\mathrm{SO}^\star(12)}{\mathrm{SU}^\star(6)}$ | $\frac{\mathrm{SU}^\star(6)}{\mathrm{Usp}(6)}$ | 16 |

| Rank | BH | Susy | $\mathfrak{F}^{\mathbb{C}}$, $n_V = 9$, $M = \mathrm{SU}(3,3)/[\mathrm{U}(1) \times \mathrm{SU}(3) \times \mathrm{SU}(3)]$: Orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # | $\mathfrak{F}^{\mathbb{R}}$, $n_V = 6$, $M = \mathrm{Sp}(6,\mathbb{R})/\mathrm{U}(3)$: Orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # |
|---|---|---|---|---|---|---|---|---|
| 1 | small d. critical | 1/2 | $\frac{\mathrm{SU}(3,3)}{\mathrm{SL}(3,\mathbb{C}) \ltimes \mathbb{R}^9}$ | $\frac{\mathrm{SL}(3,\mathbb{C})}{\mathrm{SU}(3)} \ltimes \mathbb{R}^9$ | 1 | $\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{SL}(3,\mathbb{R}) \ltimes \mathbb{R}^6}$ | $\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(3)} \ltimes \mathbb{R}^6$ | 1 |
| $2a$ | small critical | 0 | $\frac{\mathrm{SU}(3,3)}{[\mathrm{SO}(2,3) \times \mathrm{SO}(2)] \ltimes \mathbb{R}^{(4,2)} \oplus \mathbb{R}}$ | $\frac{\mathrm{SO}(2,3)}{\mathrm{SO}(2) \times \mathrm{SO}(3)} \ltimes \mathbb{R}^4 \oplus \mathbb{R}^4 \oplus \mathbb{R}$ | 3 | $\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{SO}(2,2) \ltimes \mathbb{R}^4 \oplus \mathbb{R}}$ | $\frac{\mathrm{SO}(2,2)}{\mathrm{SO}(2) \times \mathrm{SO}(2)} \ltimes \mathbb{R}^4 \oplus \mathbb{R}$ | 3 |
| $2b$ | small critical | 1/2 | $\frac{\mathrm{SU}(3,3)}{[\mathrm{SO}(1,4) \times \mathrm{SO}(2)] \ltimes \mathbb{R}^{(4,2)} \oplus \mathbb{R}}$ | $\frac{\mathrm{SO}(1,4)}{\mathrm{SO}(4)} \ltimes \mathbb{R}^4 \oplus \mathbb{R}^4 \oplus \mathbb{R}$ | 5 | $\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{SO}(1,3) \ltimes \mathbb{R}^4 \oplus \mathbb{R}}$ | $\frac{\mathrm{SO}(1,3)}{\mathrm{SO}(3)} \ltimes \mathbb{R}^4 \oplus \mathbb{R}$ | 4 |
| $3a$ | small light-like | 0 | $\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(1,2) \ltimes \mathbb{R}^8}$ | $\frac{\mathrm{SU}(1,2)}{\mathrm{U}(1) \times \mathrm{SU}(2)} \ltimes \mathbb{R}^8$ | 6 | $\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{SU}(1,1) \ltimes \mathbb{R}^5}$ | $\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1) \times \mathrm{U}(1)} \ltimes \mathbb{R}^5$ | 6 |
| $3b$ | small light-like | 1/2 | $\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3) \ltimes \mathbb{R}^8}$ | $\mathbb{R}^8$ | 10 | $\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{SU}(2) \ltimes \mathbb{R}^5}$ | $\mathbb{R}^5$ | 7 |
| $4a$ | large time-like | 1/2 | $\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3) \times \mathrm{SU}(3)}$ | – | 18 | $\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{SU}(3)}$ | – | 12 |
| $4b$ | large time-like | 0 $(Z_H{=}0)$ | $\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(1,2) \times \mathrm{SU}(1,2)}$ | $\frac{\mathrm{SU}(1,2) \times \mathrm{SU}(1,2)}{[\mathrm{U}(1) \times \mathrm{SU}(2)]^2}$ | 9 | $\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{SU}(1,2)}$ | $\frac{\mathrm{SU}(1,2)}{\mathrm{U}(1) \times \mathrm{SU}(2)}$ | 8 |
| $4c$ | large space-like | 0 $(Z_H{\neq}0)$ | $\frac{\mathrm{SU}(3,3)}{\mathrm{SL}(3,\mathbb{C})}$ | $\frac{\mathrm{SL}(3,\mathbb{C})}{\mathrm{SU}(3)}$ | 10 | $\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{SL}(3,\mathbb{R})}$ | $\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SO}(3)}$ | 7 |

### D. The $\mathcal{N}=4$ and $\mathcal{N}=2$ Reducible Jordan Symmetric Sequences

#### 1. $\mathcal{N}=4$

For $\mathcal{N}=4$ supergravity coupled to $n_V$ vector multiplets, the $(n+6)+(n+6)$ electric+magnetic BH charges (where $n=n_V\geqslant 0$) may be represented as elements

$$x=\begin{pmatrix}-q_0 & P\\ Q & p^0\end{pmatrix},\quad \text{where}\quad p^0,q^0\in\mathbb{R}\quad\text{and}\quad Q,P\in\mathfrak{J}_{5,n-1} \tag{66}$$

of the Freudenthal triple system $\mathfrak{F}^{6,n}:=\mathfrak{F}(\mathfrak{J}_{5,n-1})$. The details may be found in section III A of [25], and in Refs. therein. The *reducible* $D=4$, $\mathcal{N}=4$ U-duality group is given by the automorphism group $\mathrm{Aut}(\mathfrak{F}^{6,n})=\mathrm{Conf}(\mathfrak{J}_{5,n-1})=\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)$ under which $x\in\mathfrak{F}^{6,n}$ transforms as a $(\mathbf{2},\mathbf{6}+\mathbf{n})$. The BH entropy is once again given by Eq. (2), where $I_4(x)=\Delta(x)=\frac{1}{2}q(x)$ is the unique quartic invariant polynomial of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)$. The U-duality charge orbits are classified according to the $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)$-invariant FTS *rank* of the charge vector. More precisely, we have the following theorem [25].

**Theorem 7.** *Every BH charge vector $x\in\mathfrak{F}^{6,n}$ of a given rank is $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)$ related one of the following canonical forms:*

1. *Rank 1*

   (a) $x_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}$

2. *Rank 2*

   (a) $x_{2a}=\begin{pmatrix}1&E_1\\0&0\end{pmatrix}$

   (b) $x_{2b}=\begin{pmatrix}1&-E_1\\0&0\end{pmatrix}$

   (c) $x_{2c}=\begin{pmatrix}1&E_2\\0&0\end{pmatrix}$

3. *Rank 3*

   (a) $x_{3a}=\begin{pmatrix}1&E_2+E_3\\0&0\end{pmatrix}$

   (b) $x_{3b}=\begin{pmatrix}1&E_2-E_3\\0&0\end{pmatrix}$

4. *Rank 4*

   (a) $x_{4a}=k\begin{pmatrix}1&-E_1+E_2+E_3\\0&0\end{pmatrix}$

   (b) $x_{4b}=k\begin{pmatrix}1&E_1+E_2-E_3\\0&0\end{pmatrix}$

   (c) $x_{4c}=k\begin{pmatrix}1&-E_1+E_2-E_3\\0&0\end{pmatrix}$

*where $k>0$ and the $E_i$ are as given in* (34).

The orbit stabilizers are summarized in Table VII.

TABLE VII. Charge orbits, *moduli spaces*, the number # of "non-flat" scalar directions of the reducible $D=4, \mathcal{N}=4$ supergravities defined over $\mathfrak{F}^{6,n} := \mathfrak{F}(\mathfrak{J}_{5,n-1})$. $M = [\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(6,n)]/[\mathrm{SO}(2) \times \mathrm{SO}(6) \times \mathrm{SO}(n)]$. $\dim_{\mathbb{R}}(M) = 6n+2$. For comparison we have included the orbit labeling used in [22], and then in [23] and [24]. The table is split according as the BHs are small or large.

| Rank | BH | Susy | Charge orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # |
|---|---|---|---|---|---|
| $1/\mathbf{A.3}$ | d. critical | 1/2 | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)}{[\mathrm{SO}(1,1)\times\mathrm{SO}(5,n-1)]\ltimes(\mathbb{R}\times\mathbb{R}^{5,n-1})}$ | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(5,n-1)}{\mathrm{SO}(5)\times\mathrm{SO}(n-1)} \ltimes \mathbb{R}\times\mathbb{R}^{5,n-1}$ | 1 |
| $2a/\mathbf{A.2}$ | critical | 0 | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)}{\mathrm{SO}(6,n-1)\times\mathbb{R}}$ | $\frac{\mathrm{SO}(6,n-1)}{\mathrm{SO}(6)\times\mathrm{SO}(n-1)} \ltimes \mathbb{R}$ | 7 |
| $2b/\mathbf{A.1}$ | critical | 1/2 | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)}{\mathrm{SO}(5,n)\times\mathbb{R}}$ | $\frac{\mathrm{SO}(5,n)}{\mathrm{SO}(5)\times\mathrm{SO}(n)} \ltimes \mathbb{R}$ | $2n+2$ |
| $2c/\mathbf{B}$ | critical | 1/4 | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)}{[\mathrm{SO}(2,1)\ltimes\mathbb{R}]\times[\mathrm{SO}(4,n-2)\ltimes(\mathbb{R}^{4,n-2}\oplus\mathbb{R}^{4,n-2})]}$ | $\frac{\mathrm{SO}(2,1)\times\mathrm{SO}(4,n-2)}{\mathrm{SO}(2)\times\mathrm{SO}(4)\times\mathrm{SO}(n-2)} \ltimes \mathbb{R}\times[\mathbb{R}^{4,n-2}\oplus\mathbb{R}^{4,n-2}]$ | 4 |
| $3a/\mathbf{C.1}$ | light-like | 1/4 | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)}{[\mathrm{SO}(4,n-1)\ltimes\mathbb{R}^{4,n-1}]\times\mathbb{R}}$ | $\frac{\mathrm{SO}(4,n-1)}{\mathrm{SO}(4)\times\mathrm{SO}(n-1)} \ltimes \mathbb{R}\times\mathbb{R}^{4,n-1}$ | $n$ |
| $3b/\mathbf{C.2}$ | light-like | 0 | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)}{[\mathrm{SO}(5,n-2)\ltimes\mathbb{R}^{5,n-2}]\times\mathbb{R}}$ | $\frac{\mathrm{SO}(5,n-2)}{\mathrm{SO}(5)\times\mathrm{SO}(n-2)} \ltimes \mathbb{R}\times\mathbb{R}^{5,n-2}$ | 8 |
| $4a/\alpha$ | time-like | 1/4 | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)}{\mathrm{SO}(2)\times\mathrm{SO}(4,n)}$ | $\frac{\mathrm{SO}(4,n)}{\mathrm{SO}(4)\times\mathrm{SO}(n)}$ | $2n+2$ |
| $4b/\gamma$ | time-like | $0$ $(\hat{Z}_{AB,H}=0)$ | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)}{\mathrm{SO}(2)\times\mathrm{SO}(6,n-2)}$ | $\frac{\mathrm{SO}(6,n-2)}{\mathrm{SO}(6)\times\mathrm{SO}(n-2)}$ | 14 |
| $4c/\beta$ | space-like | $0$ $(\hat{Z}_{AB,H}\neq 0)$ | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,n)}{\mathrm{SO}(1,1)\times\mathrm{SO}(5,n-1)}$ | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(5,n-1)}{\mathrm{SO}(5)\times\mathrm{SO}(n-1)}$ | $n+6$ |

### 2. $\mathcal{N}=2$

For $\mathcal{N}=2$ supergravity theories coupled to $n_V$ vector multiplets whose scalar manifolds belong to the so-called Jordan symmetric sequence of special Kähler geometry, the $(n+2)+(n+2)$ electric+magnetic BH charges (where $n=n_V-1\geqslant 1$) may be represented as elements

$$x=\begin{pmatrix}-q_0 & P\\ Q & p^0\end{pmatrix},\quad \text{where}\quad p^0,q^0\in\mathbb{R}\quad\text{and}\quad Q,P\in\mathfrak{J}_{1,n-1} \tag{67}$$

of the Freudenthal triple system $\mathfrak{F}^{2,n}:=\mathfrak{F}(\mathfrak{J}_{1,n-1})$. The details may be found in section III A of [25], as well as in Refs. therein. The *reducible* $D=4$, $\mathcal{N}=2$ U-duality group is given by the automorphism group $\mathrm{Aut}(\mathfrak{F}^{2,n})\cong\mathrm{Conf}(\mathfrak{J}_{1,n-1})=\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)$ under which $x\in\mathfrak{F}^{2,n}$ transforms as a $(\mathbf{2},\mathbf{2}+\mathbf{n})$. The BH entropy is once again given by Eq. (2), where $I_4(x)=\Delta(x)=\frac{1}{2}q(x)$ is the unique quartic invariant polynomial of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)$. The U-duality charge orbits are classified according to the $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)$-invariant FTS *rank* of the charge vector. The orbit representatives are as in Theorem 7 [25]. However, physically each 1/4-BPS orbits of Table VII splits into one 1/2-BPS orbit and one non-BPS orbit, see Table VIII. This splitting is determined by the sign of the quantity [12]

$$\mathcal{I}_2=|Z|^2-|D_SZ|^2. \tag{68}$$

Here, $Z$ is the central charge and $D_SZ$ is the axion-dilaton *matter charge*, where $D_S$ is the Kähler covariant derivative on the scalar manifold along the axion-dilaton direction; this is a "privileged" scalar direction, because the scalar manifold is factorized. In fact, noting that the $\mathcal{N}=4$, $D=4$ 1/4-BPS canonical forms all have a Jordan algebra element that has two disconnected components under $\mathrm{Str}_0(\mathfrak{J}_{1,n-1})$, the sign condition on (68) can be rephrased in terms of the charges.

## E. Interpretation of $\sharp_{\frac{1}{2}-BPS,\text{rank-1}}=1$

As reported in the Tables, all *symmetric* $D=4$ theories share the same result, namely:

$$\sharp_{\frac{1}{2}-BPS,\text{rank-1}}=1. \tag{69}$$

Note that the rank-1, doubly critical orbit is always unique, corresponding to the maximum weight vector in the relevant representation space. Up to U-duality all rank-1 $D=4$ black holes may be regarded as a pure KK state of the 5-dimensional parent theory. All along the $\frac{1}{2}$-BPS rank-1 scalar flow [23], there is only one "non-flat" scalar degree of freedom.

This can be easily interpreted by recalling that the first-order superpotential of the $\mathcal{N}=2$ BPS flows is nothing but $\mathcal{W}=|Z|$, where $Z$ is the $\mathcal{N}=2$ central charge [80]. Thus, by considering the general expression of $Z$ in a generic $d$-special Kähler geometry (given by Eq. (4.9) of [29]) for the relevant representative 1-charge configuration in which the dependence on only one scalar field is manifest (which turns out to be $\{q_0\}$), one obtains:

$$\mathcal{W}_{\frac{1}{2}-BPS,\text{rank-1}}=|Z|_{\{q_0\}}=\frac{|q_0|}{2\sqrt{2}}\mathcal{V}^{-1/2}, \tag{70}$$

where $\mathcal{V}\equiv r_{KK}^3$, $r_{KK}$ denoting the KK radius in the KK reduction $D=5\longrightarrow D=4$ [29].

In the cases $\mathcal{N}=8$ and $\mathcal{N}=4$, similar results can be obtained from the treatment given in [81] and [22]. Analogous explanations can be given for the result (69) for $D=5$ charge orbits, as reported in the relevant Tables.

## F. The $\mathcal{N}=2$ $STU$, $ST^2$ and $T^3$ Models

### 1. $STU$

The $STU$ model is $\mathcal{N}=2$ supergravity coupled to three vector multiplets. However, it has an additional discrete triality, which exchanges the roles of the three complex moduli. This triality has a stringy explanation first identified in [14]. It is essentially a remnant of the $D=6$ equivalence between the heterotic string on $T^4$, the Type IIA string on $K3$ and the Type IIB string on its mirror. The $STU$ model is thus a noteworthy element ($n=2$) of the $\mathcal{N}=2$, $D=4$ Jordan symmetric sequence discussed above.

TABLE VIII. Charge orbits, *moduli spaces*, and number # of "non-flat" scalar directions of the reducible $D=4,\mathcal{N}=2$ supergravities defined over $\mathfrak{F}^{2,n}:=\mathfrak{F}(\mathfrak{J}_{1,n-1})$. $M=[\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)]/[\mathrm{SO}(2)^2\times\mathrm{SO}(n)]$. $\dim_{\mathbb{R}}(M)=2n+2$. For comparison, we have included the orbit labelling used in [22], and then in [23] and [24]. The table is split according as the BHs are small or large.

| Rank | [22] | BH | Susy | Charge orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # |
|---|---|---|---|---|---|---|
| 1 | **A.3** | d. critical | 1/2 | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)}{[\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)]\ltimes(\mathbb{R}\times\mathbb{R}^{1,n-1})}$ | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(n-1)}\ltimes\mathbb{R}\times\mathbb{R}^{1,n-1}$ | 1 |
| $2a$ | **A.2** | critical | 0 | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)}{\mathrm{SO}(2,n-1)\times\mathbb{R}}$ | $\frac{\mathrm{SO}(2,n-1)}{\mathrm{SO}(2)\times\mathrm{SO}(n-1)}\ltimes\mathbb{R}$ | 3 |
| $2b$ | **A.1** | critical | 1/2 | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)}{\mathrm{SO}(1,n)\times\mathbb{R}}$ | $\frac{\mathrm{SO}(1,n)}{\mathrm{SO}(n)}\ltimes\mathbb{R}$ | $n+1$ |
| $2c^+$ | **B** | critical | 1/2 $\mathcal{I}_2>0$ | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)}{[\mathrm{SO}(2,1)\ltimes\mathbb{R}]\times[\mathrm{SO}(n-2)\ltimes(\mathbb{R}^{n-2}\oplus\mathbb{R}^{n-2})]}$ | $\frac{\mathrm{SO}(2,1)}{\mathrm{SO}(2)}\ltimes\mathbb{R}\times[\mathbb{R}^{n-2}\oplus\mathbb{R}^{n-2}]$ | 3 |
| $2c^-$ | **B** | critical | 0 $\mathcal{I}_2<0$ | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)}{[\mathrm{SO}(2,1)\ltimes\mathbb{R}]\times[\mathrm{SO}(n-2)\ltimes(\mathbb{R}^{n-2}\oplus\mathbb{R}^{n-2})]}$ | $\frac{\mathrm{SO}(2,1)}{\mathrm{SO}(2)}\ltimes\mathbb{R}\times[\mathbb{R}^{n-2}\oplus\mathbb{R}^{n-2}]$ | 3 |
| $3a^+$ | **C.1** | light-like | 1/2 $\mathcal{I}_2>0$ | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)}{[\mathrm{SO}(n-1)\ltimes\mathbb{R}^{n-1}]\times\mathbb{R}}$ | $\mathbb{R}\times\mathbb{R}^{n-1}$ | $n+2$ |
| $3a^-$ | **C.1** | light-like | 0 $\mathcal{I}_2<0$ | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)}{[\mathrm{SO}(n-1)\ltimes\mathbb{R}^{n-1}]\times\mathbb{R}}$ | $\mathbb{R}\times\mathbb{R}^{n-1}$ | $n+2$ |
| $3b$ | **C.2** | light-like | 0 | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)}{[\mathrm{SO}(1,n-2)\ltimes\mathbb{R}^{n-1}]\times\mathbb{R}}$ | $\frac{\mathrm{SO}(1,n-2)}{\mathrm{SO}(n-2)}\ltimes\mathbb{R}^{n-1}\times\mathbb{R}$ | 4 |
| $4a^+$ | $\alpha$ | time-like | 1/2 $\mathcal{I}_2>0$ | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)}{\mathrm{SO}(2)\times\mathrm{SO}(n)}$ | – | $2n+2$ |
| $4a^-$ | $\alpha$ | time-like | 0 $\mathcal{I}_2<0$ | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)}{\mathrm{SO}(2)\times\mathrm{SO}(n)}$ | – | $2n+2$ |
| $4b$ | $\gamma$ | time-like | 0 $Z_H=0$ | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)}{\mathrm{SO}(2)\times\mathrm{SO}(2,n-2)}$ | $\frac{\mathrm{SO}(2,n-2)}{\mathrm{SO}(2)\times\mathrm{SO}(n-2)}$ | 8 |
| $4c$ | $\beta$ | space-like | 0 $Z_H\neq0$ | $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(2,n)}{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}$ | $\frac{\mathrm{SO}(1,1)\times\mathrm{SO}(1,n-1)}{\mathrm{SO}(n-1)}$ | $n+2$ |

The $(1+3)+(1+3)$ electromagnetic charges may be represented as elements

$$x = \begin{pmatrix} -q_0 & (p;p^\mu) \\ (q;q_\nu) & p^0 \end{pmatrix}, \quad \text{where} \quad p^0, q^0 \in \mathbb{R} \quad \text{and} \quad (q;q_\nu),(p;p^\mu) \in \mathfrak{J}_{1,1} \tag{71}$$

of the Freudenthal triple system $\mathfrak{F}^{2,2} := \mathfrak{F}(\mathfrak{J}_{1,1})$.
The U-duality group $\text{Aut}(\mathfrak{F}_{STU}) \cong \text{Conf}\,(\mathfrak{J}_{1,1} = \mathbb{R} \oplus \mathbf{\Gamma}_{1,1} = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}) = \text{SL}(2,\mathbb{R}) \times \text{SO}(2,2)$ may be recast in a form reflecting this triality symmetry using the isomorphism $\text{SO}(2,2) \cong \text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R})$. From the the heterotic string perspective this corresponds to an $\text{SL}(2,\mathbb{Z})_S$ strong/weak coupling duality and an $\text{SL}(2,\mathbb{Z})_T \times \text{SL}(2,\mathbb{Z})_U$ target space duality acting on the dilaton/axion, complex Kähler form and the complex structure fields $S,T,U$ respectively. At the level of the FTS [20, 50, 82], this is realised by the Jordan algebra isomorphism $\mathfrak{J}_{1,1} = \mathbb{R} \oplus \Gamma_{1,1} \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} = \mathfrak{J}_{STU}$ which, for $(q_1,q_2,q_3) \in \mathfrak{J}_{STU}$ and $(q;q_\nu) \in \mathfrak{J}_{1,1}$ is given by,

$$q_1 = q, \quad q_2 = q_0 + q_1, \quad q_3 = q_0 - q_1, \tag{72}$$

so that the $STU$ cubic norm becomes

$$N(Q) = q_1 q_2 q_3. \tag{73}$$

By renaming

$$\begin{pmatrix} -q_0 & (p_1,p_2,p_3) \\ (q_1,q_2,q_3) & p^0 \end{pmatrix} \mapsto \begin{pmatrix} a_{000} & (a_{011},a_{101},a_{110}) \\ (a_{100},a_{010},a_{001}) & a_{111} \end{pmatrix}, \tag{74}$$

the charges may be arranged into a $2\times2\times2$ hypermatrix $a_{ABC}$, where $A,B,C = 0,1$, transform as a $(\mathbf{2},\mathbf{2},\mathbf{2})$ under $\text{SL}_A(2,\mathbb{R}) \times \text{SL}_B(2,\mathbb{R}) \times \text{SL}_C(2,\mathbb{R})$. In such a way, the quartic norm is given by Cayley's hyperdeterminant $\text{Det}\, a_{ABC}$ [46, 83],

$$\Delta = -\,\text{Det}\, a = \frac{1}{2}\epsilon^{A_1A_2}\epsilon^{B_1B_2}\epsilon^{C_1C_3}\epsilon^{A_3A_4}\epsilon^{B_3B_4}\epsilon^{C_2C_4} a_{A_1B_1C_1}a_{A_2B_2C_2}a_{A_3B_3C_3}a_{A_4B_4C_4} \tag{75}$$

and

$$S_{D=4,\text{BH}} = \pi\sqrt{|\,\text{Det}\, a|}. \tag{76}$$

This observation lies at the origin of the "black-hole/qubit correspondence" [50, 51, 82, 84–96]. The hyperdeterminant is manifestly invariant under the triality $A \leftrightarrow B \leftrightarrow C$. The role of more general hyperdeterminants in M-theory can be found in [97, 98].

The implication of this triality for the structure of the orbits is that what are distinct cosets for generic $n_V$ become isomorphic for the $STU$ case. In particular, we find that for the $STU$ model [20]

$$\mathcal{O}_{2a} \cong \mathcal{O}_{2b} \cong \mathcal{O}_{2c}, \qquad \mathcal{O}_{3a} \cong \mathcal{O}_{3b} \tag{77}$$

as can be seen immediately from Table VIII setting $n = 2$. However, while the cosets are isomorphic the distinct physical properties of each orbit are preserved, so that the $STU$ model can really be included in the generic sequence.

### 2. $ST^2$

On the other hand, the orbit structure of the $ST^2$ model, which can be seen as the first $(n = 1)$ element of the Jordan symmetric sequence, $\mathcal{N} = 2$ coupled to two vector multiplets, does depart from the one discussed so far. The $(1+2)+(1+2)$ electromagnetic charges may be represented as elements

$$x = \begin{pmatrix} -q_0 & (p^1,p^2) \\ (q_1,q_2) & p^0 \end{pmatrix}, \quad \text{where} \quad p^0, q^0 \in \mathbb{R} \quad \text{and} \quad (p^1,p^2),(q_1,q_2) \in \mathbb{R} \oplus \mathbb{R} \tag{78}$$

of the Freudenthal triple system $\mathfrak{F}^{2,1} := \mathfrak{F}(\mathfrak{J}_1)$. Here, $\mathfrak{J}_1 = \mathbb{R} \oplus \Gamma_1 = \mathbb{R} \oplus \mathbb{R}$ now has an "Euclidean" cubic norm

$$N(Q) = q_1(q_2)^2, \quad Q \in \mathfrak{J}_{ST^2}, \tag{79}$$

which implies there is only one rank 2 $Q \in \mathfrak{J}_{ST^2}$ up to $\text{Str}_0(\mathfrak{J}_{ST^2}) = \text{SO}(1,1)$, which is now pure dilatation. Consequently, the third rank 2 orbit (in the FTS) of the generic sequence $(n_V \geq 3)$ vanishes [25].

The U-duality group is $\mathrm{Aut}(\mathfrak{F}_{ST^2}) \cong \mathrm{Conf}(\mathbb{R} \oplus \mathbb{R}) = \mathrm{SL}_A(2, \mathbb{R}) \times \mathrm{SL}_B(2, \mathbb{R})$ under which the charges transform as a $(\mathbf{2}, \mathbf{3})$. Again, this symmetry is made manifest by writing the charges as a hypermatrix

$$Q = a_{A(B_1 B_2)}. \tag{80}$$

The BH entropy is given by Eq. (76), with the hyperdeterminant now being the "$ST^2$ degeneration" of the expression holding for the $STU$ model (see e.g. [18] for further details). The canonical forms are presented in Theorem 8 [25]. The orbits may be obtained from Table VIII by setting $n = 1$ (when this is still well defined - when it is not, the orbit is not present).

**Theorem 8.** [25] *Every element* $x \in \mathfrak{F}_{ST^2}$ *of a given rank is* $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ *related to one of the following canonical forms:*

1. *Rank 1*

   (a) $x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

2. *Rank 2*

   (a) $x_{2a} = \begin{pmatrix} 1 & (1; 0) \\ 0 & 0 \end{pmatrix}$

   (b) $x_{2b} = \begin{pmatrix} 1 & (-1; 0) \\ 0 & 0 \end{pmatrix}$

3. *Rank 3*

   (a) $x_{3a} = \begin{pmatrix} 1 & (0; 1) \\ 0 & 0 \end{pmatrix}$

4. *Rank 4*

   (a) $x_{4a} = k \begin{pmatrix} 1 & (-1; 1) \\ 0 & 0 \end{pmatrix}$

   (b) $x_{4b} = k \begin{pmatrix} 1 & (1; 1) \\ 0 & 0 \end{pmatrix}$

### *3.* $T^3$

Finally, we come to the $T^3$ model. Unlike all the other cases treated here, the $T^3$ has a cubic Jordan algebra, $\mathfrak{J}_{T^3} = \mathbb{R}$, with a single non-zero rank. The cubic norm is given by

$$N(Q) = q^3, \quad q \in \mathbb{R}. \tag{81}$$

Hence, there is *only a single* rank given by $N(Q) \neq 0$: all non-zero elements are rank 3. Consequently, the rank 2, where we now mean in the FTS $\mathfrak{F}(\mathfrak{J}_{T^3})$, orbit disappears entirely [25]. That is, if a small BH is critical, then it is doubly critical.

The U-duality group is $\mathrm{Aut}(\mathfrak{F}_{T^3}) \cong \mathrm{Conf}(\mathbb{R}) = \mathrm{SL}_A(2, \mathbb{R})$ under which the charges transform as a $\mathbf{4}$ (spin $s = 3/2$). Again, this symmetry is made manifest by writing the charges as a hypermatrix

$$Q = a_{(A_1 A_1 A_2)}. \tag{82}$$

The BH entropy is given by Eq. (76), with the hyperdeterminant now being the "$T^3$ degeneration" of the expression holding for the $STU$ model (see e.g. [18] for further details).

Accounting for the vanishing rank 2 case, the remaining $\mathrm{SL}_A(2, \mathbb{R})$-orbits are given in Theorem 9.

**Theorem 9.** [25] *Every element* $x \in \mathfrak{F}_{T^3}$ *of s given rank is* $\mathrm{SL}(2, \mathbb{R})$ *related to one of the following canonical forms:*

1. *Rank 1*

   (a) $x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

*2. Rank 3*

(a) $x_{3a} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

*3. Rank 4*

(a) $x_{4a} = k \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

(b) $x_{4b} = k \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

There are now just four orbits: small doubly critical (rank 1) 1/2-BPS, small light-like (rank 3) 1/2-BPS, large (rank 4) 1/2-BPS and non-BPS. This is consistent with the analysis of [99–101], which relies on the theory of nilpotent orbits. The BPS nature of both "small" (rank 3 and rank 1) charge orbits of this model can also be easily understood by recalling the result derived in Sec. 5.5 of [23], namely that the "small" limit of the first-order ("fake") superpotentials of both BPS and non-BPS attractor scalar flows yields nothing but the absolute value $|Z|$ of the $\mathcal{N} = 2$ central charge.

Performing a time-like reduction (since we are interested in stationary solutions) the resulting 3-dimensional $T^3$ model has $G_{2(2)}$ U-duality, with scalars parametrising the pseudo-Riemannian coset,

$$\frac{G_{2(2)}}{\mathrm{SO}_0(2,2)}. \tag{83}$$

The nilpotent $\mathrm{SO}_0(2,2)$-orbits of $\mathfrak{g}_{2(2)}$ correspond to six static (*i.e.* single or non-interacting centre) extremal solutions [99]. However, only four of these orbits, labeled $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_{3K}, \mathcal{O}_{4K'}$ in [99], correspond to physically acceptable static solutions [99]. From our perspective the unphysical orbits cannot be seen and it can be checked that the four orbits we describe correspond precisely to the four physical orbits of [99–101]. Explicitly, where we use the labeling in Theorem 9,

$$\begin{array}{lcll} \mathcal{O}_1 & \longleftrightarrow & \mathcal{O}_{x_1} & \text{small doubly critical (rank 1) 1/2-BPS,} \\ \mathcal{O}_2 & \longleftrightarrow & \mathcal{O}_{x_3} & \text{small light-like (rank 3) 1/2-BPS,} \\ \mathcal{O}_{3K} & \longleftrightarrow & \mathcal{O}_{x_{4a}} & \text{large (rank 4) 1/2-BPS,} \\ \mathcal{O}_{4K'} & \longleftrightarrow & \mathcal{O}_{x_{4b}} & \text{large (rank 4) non-BPS.} \end{array} \tag{84}$$

The orbit stabilizers are summarized in Table IX. Note, the two large (1/2-BPS and non-BPS) orbits have no continuous stabilizers. However, the 1/2-BPS case does have a discrete $\mathbb{Z}_3$ stabilizer generated by

$$M = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \tag{85}$$

where $M \in \mathrm{SL}(2,\mathbb{R})$. Note, this is a finite subgroup of the $\mathrm{SL}(2,\mathbb{R})$ U-duality and should not be misconstrued as a sub-group the $STU$ triality symmetry, which collapses upon identifying the moduli. The origin of $\mathbb{Z}_3$ is easily understood in terms of the "parent" 1/2-BPS rank-4 $STU$ orbit stabilizer $\mathrm{SO}(2) \times \mathrm{SO}(2)$. Recall, the Lie algebra of the automorphism group $\mathfrak{Aut}(\mathfrak{F}(\mathfrak{J}))$ decomposes under the reduced structure group $\mathrm{Str}_0(\mathfrak{J})$ according as

$$\mathfrak{Aut}(\mathfrak{F}(\mathfrak{J})) = \mathfrak{Str}_0(\mathfrak{J}) \oplus \mathfrak{J} \oplus \mathfrak{J} \oplus \mathbb{R}. \tag{86}$$

The 1/2-BPS rank-4 $STU$ stability group is conjugate to[4] an $\mathrm{SO}(2) \times \mathrm{SO}(2)$ generated by (using the notation introduced in appendix A) $\Phi = (0, X, -X, 0)$, $\Phi \in \mathfrak{Str}_0(\mathfrak{J}) \oplus \mathfrak{J} \oplus \mathfrak{J} \oplus \mathbb{R}$, such that $\mathrm{Tr}(X) = 0$. One possible parametrization of $\mathrm{SO}(2) \times \mathrm{SO}(2) \subset \mathrm{SL}_A(2,\mathbb{R}) \times \mathrm{SL}_B(2,\mathbb{R}) \times \mathrm{SL}_C(2,\mathbb{R})$, obtained by exponentiating $\Phi$, is given by,

$$\begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \otimes \begin{pmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{pmatrix} \otimes \begin{pmatrix} \cos(\phi+\psi) & \sin(\phi+\psi) \\ -\sin(\phi+\psi) & \cos(\phi+\psi) \end{pmatrix}. \tag{87}$$

Symmetrizing down from the $STU$ model to the $T^3$ model implies identifying the three factors appearing in the above parametrization. This gives (85) and its powers, hence picking out a $\mathbb{Z}_3$ finite subgroup. Alternatively, this may be checked directly using the totally symmetrized hypermatrix, which transforms as

$$a_{(A_1A_2A_3)} \mapsto \tilde{a}_{(A_1A_2A_3)} = M_{A_1}{}^{A'_1} M_{A_2}{}^{A'_2} M_{A_3}{}^{A'_3} a_{(A'_1A'_2A'_3)}, \tag{88}$$

[4] In fact, for our orbit representative, equal to.

under $\mathrm{SL}(2,\mathbb{R})$. Solving $\tilde{a}^{4a}_{(A_1A_2A_3)} = a^{4a}_{(A_1A_2A_3)}$, where $a^{4a}_{(A_1A_2A_3)}$ is the orbit representative appearing in Theorem 9, yields the same conclusion. Since this $\mathbb{Z}_3$ forms a finite sub-group of a *compact* stabilizer there should be no corresponding "discrete" moduli space.

By considering its embedding in the $STU$ model it is also particularly easy to see why there is no discrete stabilizer in the unique $\Delta < 0$ non-BPS orbit. The $\Delta < 0$ non-BPS $STU$ orbit stabilizer is conjugate to an $\mathrm{SO}(1,1)\times\mathrm{SO}(1,1)$ generated by $\Phi = (\phi, 0, 0, 0)$, $\phi \in \mathfrak{Str}_0(\mathfrak{J})$. Equivalently, there is a U-duality frame in which only the two graviphoton charges are turned on. Since the graviphotons are singlets under the $D = 5$ U-duality group the stabilizer is precisely $\mathrm{Str}_0(\mathfrak{J})$. This is true for all $D = 4$ theories based on cubic Jordan algebras, explaining this common feature of the $\Delta < 0$ non-BPS orbits. However, for the $T^3$ model $\mathrm{Str}_0(\mathfrak{J})$ contains only the identity, hence there can be no discrete stabilizer. This expectation is borne out by explicit computation. Note, since the presence of only graviphoton charges implies $\Delta < 0$, this charge configuration is only possible for $\Delta < 0$ non-BPS states.

TABLE IX. Charge orbits, *moduli spaces*, and number # of "non-flat" scalar directions of the $D = 4, T^3$ model. $M = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, $\dim_{\mathbb{R}} = 2$. $L_+$ is the generator of $\mathrm{SL}(2,\mathbb{R})$ with positive grading with respect to its maximal subgroup $\mathrm{SO}(1,1)$.

| Rank | BH | Susy | Charge orbit $\mathcal{O}$ | Moduli space $\mathcal{M}$ | # |
|---|---|---|---|---|---|
| 1 | doubly critical | 1/2 | $\frac{\mathrm{SL}(2,\mathbb{R})}{L_+}$ | $\mathbb{R}$ | 1 |
| 3 | light-like | 1/2 | $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathbb{1}}$ | – | 2 |
| $4(\Delta > 0)$ | large | 1/2 | $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathbb{Z}_3}$ | – | 2 |
| $4(\Delta < 0)$ | | 0 | $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathbb{1}}$ | – | 2 |

### G. $\mathcal{N} = 2$ *Minimally Coupled*

We now consider $\mathcal{N} = 2$, $d = 4$ ungauged supergravity *minimally coupled (mc)* [53] to $n_V$ Abelian vector multiplets, whose scalar manifold is given by the sequence of homogeneous symmetric *rank*-1 special Kähler manifolds

$$\mathcal{M}_{\mathcal{N}=2,mc,n} = \mathbb{CP}^n \equiv \frac{G_{\mathcal{N}=2,mc,n}}{H_{\mathcal{N}=2,mc,n}} = \frac{\mathrm{U}(1,n)}{\mathrm{U}(n)\times\mathrm{U}(1)},\ \dim_{\mathbb{R}} = 2n,\ n = n_V \in \mathbb{N}. \tag{89}$$

This theory cannot be uplifted to $D = 5$, and it does not enjoy an interpretation in terms of Jordan algebras. The $1+n$ vector field strengths and their duals, as well as their asymptotical fluxes, sit in the *fundamental* $\mathbf{1+n}$ representation of the U-duality group $G_{\mathcal{N}=2,mc,n} = \mathrm{U}(1,n)$, in turn embedded in the symplectic group $Sp(2+2n,\mathbb{R})$. The unique algebraically independent invariant polynomial in the $\mathbf{1+n}$ of $\mathrm{U}(1,n)$ is quadratic:

$$\mathcal{I}_2 = \frac{1}{2}\left[q_0^2 - q_i^2 + \left(p^0\right)^2 - \left(p^i\right)^2\right] = |Z|^2 - Z_i\overline{Z}^i. \tag{90}$$

The general analysis of the Attractor Equations, BH charge orbits, attractor *moduli spaces* and split attractor of such a theory has been performed in [12, 57, 102, 103]; here we recall it briefly, and further consider the "small" charge orbit of such models.

1. the "large" (rank-2) BPS charge orbit reads [12]

$$\mathcal{O}_{BPS,rank-2} = \frac{\mathrm{U}(1,n)}{\mathrm{U}(n)},\ \dim_{\mathbb{R}} = 2n+1,\ \mathcal{I}_2 > 0. \tag{91}$$

   Thus, as for all "large" BPS charge orbits [7], there is no associated attractor *moduli space* or, equivalently, the number of "non-flat" scalar directions along the flow is $\# = 2n$.

2. the "large" (rank-2) non-BPS charge orbit (with $Z_H = 0$) reads [12]

$$\mathcal{O}_{nBPS,rank-2} = \frac{\mathrm{U}(1,n)}{\mathrm{U}(1,n-1)},\ \dim_{\mathbb{R}} = 2n+1,\ \mathcal{I}_2 < 0. \tag{92}$$

Thus, the associated attractor *moduli space* reads

$$\mathcal{M}_{nBPS,rank-2} = \mathbb{CP}^{n-1},\ \# = 2. \tag{93}$$

3. the unique "small" (rank-1) BPS charge orbit reads

$$\mathcal{O}_{BPS,rank-1} = \frac{\mathrm{U}(1,n)}{\mathrm{U}(n-1)\times\mathrm{U}(1)\ltimes\mathbb{C}_n^{n-1}},\ \dim_{\mathbb{R}} = 2n+1,\ \mathcal{I}_2 = 0, \tag{94}$$

where the subscript denotes charge with respect to the U(1) commuting factor of the stabilizer. Thus, the associated attractor *moduli space* reads

$$\mathcal{M}_{BPS,rank-1} = \mathbb{C}^{n-1},\ \# = 2. \tag{95}$$

It is worth of notice that (non-compact forms of) $\mathbb{CP}^n$ spaces as moduli spaces of string compactifications have appeared in the literature, either as particular subspaces of complex structure deformations of certain Calabi-Yau manifold [104, 105] or as moduli spaces of some asymmetric orbifolds of Type II superstrings [106–109], or of orientifolds [110].

## H. $\mathcal{N} = 3$

The (Kähler) scalar manifold is [54]

$$\mathcal{M}_{\mathcal{N}=3,n} = \frac{G_{\mathcal{N}=3,n}}{H_{\mathcal{N}=3,n}} = \frac{\mathrm{U}\left(3,n\right)}{\mathrm{SU}\left(3\right)\times\mathrm{U}\left(n\right)\times\mathrm{U}\left(1\right)},\ \dim_{\mathbb{R}} = 6n. \tag{96}$$

This theory cannot be uplifted to $D = 5$, and it does not enjoy an interpretation in terms of Jordan algebras.

The $3+n$ vector field strengths and their duals, as well as their asymptotical fluxes, sit in the *fundamental* $\mathbf{3}+\mathbf{n}$ representation of the U-duality group $G_{\mathcal{N}=3,n} = \mathrm{U}\left(3,n\right)$, in turn embedded in the symplectic group $Sp\left(6+2n,\mathbb{R}\right)$. The unique algebraically independent invariant polynomial in the $\mathbf{3}+\mathbf{n}$ of $\mathrm{U}\left(3,n\right)$ is quadratic, and it reads ($A = 1,2,3,\ I = 1,...,n$) [57]:

$$\mathcal{I}_2 = \frac{1}{2}\left[q_A^2 - q_i^2 + \left(p^A\right)^2 - \left(p^i\right)^2\right] = \frac{1}{2}Z_{AB}\overline{Z}^{AB} - Z_I\overline{Z}^I, \tag{97}$$

The general analysis of the Attractor Equations, BH charge orbits, attractor *moduli spaces* and split attractor of such a theory has been performed in [57, 102, 103]; here we recall it briefly, and further consider the "small" charge orbit of this theory (the results are also consistent with the $D = 3$ analysis of [79]).

1. the "large" (rank-2) $\frac{1}{3}$-BPS charge orbit reads [111]

$$\mathcal{O}_{\frac{1}{3}-BPS,rank-2} = \frac{\mathrm{U}(3,n)}{\mathrm{U}(2,n)},\ \dim_{\mathbb{R}} = 2n+5,\ \mathcal{I}_2 > 0. \tag{98}$$

The associated attractor *moduli space*, as all the $\frac{1}{\mathcal{N}}$-BPS attractor moduli spaces of $\mathcal{N} \geqslant 3$-extended, $D = 4$ supergravity theories [112], is a quaternionic symmetric space (recall Eq. (93)):

$$\mathcal{M}_{\frac{1}{3}-BPS,rank-2} = \frac{\mathrm{SU}(2,n)}{\mathrm{SU}(2)\times\mathrm{SU}(n)\times\mathrm{U}(1)} = c\left(\mathbb{CP}^{n-1}\right) = c\left(\mathcal{M}_{\mathcal{N}=2,mc,nBPS,rank-2}\right),\ \# = 2n, \tag{99}$$

where "$c$" denotes the $c$-map [113].

2. the "large" (rank-2) non-BPS charge orbit (with $Z_{AB,H} = 0$) reads [111]

$$\mathcal{O}_{nBPS,rank-2} = \frac{\mathrm{U}(3,n)}{\mathrm{U}(3,n-1)},\ \dim_{\mathbb{R}} = 2n+5,\ \mathcal{I}_2 < 0. \tag{100}$$

Thus, the associated attractor *moduli space* reads

$$\mathcal{M}_{nBPS,rank-2} = \frac{\mathrm{U}\left(3,n-1\right)}{\mathrm{SU}\left(3\right)\times\mathrm{U}\left(n-1\right)\times\mathrm{U}\left(1\right)} = \mathcal{M}_{\mathcal{N}=3,n-1},\ \# = 6. \tag{101}$$

3. the unique "small" (rank-1) $\frac{2}{3}$-BPS charge orbit reads

$$\mathcal{O}_{\frac{2}{3}-BPS,rank-1}=\frac{\mathrm{U}(3,n)}{\mathrm{U}(2,n-1)\times\mathrm{U}(1)\ltimes\mathbb{C}^{2,n-1}_{n+2}},\ \dim_{\mathbb{R}}=2n+5,\ \mathcal{I}_2=0, \tag{102}$$

where the subscript denotes charge with respect to the $\mathrm{U}(1)$ commuting factor of the stabilizer. Thus, the associated attractor *moduli space* reads (recall Eq. (93))

$$\begin{aligned}\mathcal{M}_{\frac{2}{3}-BPS,rank-1}&=\frac{\mathrm{SU}(2,n-1)}{\mathrm{SU}(2)\times\mathrm{SU}(n-1)\times\mathrm{U}(1)}\\&=c\left(\mathbb{CP}^{n-2}\right)=c\left(\mathcal{M}_{\mathcal{N}=2,mc,nBPS,rank-2}|_{n\to n-1}\right),\ \#=2.\end{aligned} \tag{103}$$

## I. $\mathcal{N}=5$

The (special Kähler) *scalar manifold* is [55]

$$\mathcal{M}_{\mathcal{N}=5}=\frac{G_{\mathcal{N}=5}}{H_{\mathcal{N}=5}}=\frac{\mathrm{SU}(1,5)}{\mathrm{SU}(5)\times\mathrm{U}(1)},\ \dim_{\mathbb{R}}=10. \tag{104}$$

*No* matter coupling is allowed (*pure* supergravity). This theory cannot be uplifted to $D=5$, but it is associated to the Jordan triple system $M_{2,1}(\mathbb{O})$ generated by the $2\times1$ vectors over $\mathbb{O}$ [10, 56].

The 10 vector field strengths and their duals, as well as their asymptotical fluxes, sit in the *three-fold antisymmetric* irrepr. **20** of the U-duality group $G_{\mathcal{N}=5}=\mathrm{SU}(1,5)$. As discussed in [57], unique algebraically independent invariant polynomial in the **20** of $\mathrm{SU}(1,5)$ is quartic in the bare charges (see *e.g.* the treatment of [57]), but is a *perfect square* of a quadratic expression when written in terms of the scalar-dependent *skew-eigenvalues* $\mathcal{Z}_1$ and $\mathcal{Z}_2$ of the central charge matrix $Z_{AB}$ ($A=1,...,5$):

$$\mathcal{I}_4(p,q)\equiv Z_{AB}\overline{Z}^{BC}Z_{CD}\overline{Z}^{DA}-\frac{1}{4}\left(Z_{AB}\overline{Z}^{AB}\right)^2=\left(\mathcal{Z}_1^2-\mathcal{Z}_2^2\right)^2. \tag{105}$$

This property distinguishes the $\mathcal{N}=5$ "pure" theory from the previously treated $\mathcal{N}=2$, $D=4$ magic Maxwell-Einstein theory associated to $\mathfrak{J}_3^{\mathbb{C}}$, whose U-duality group $\mathrm{SU}(3,3)$ is a different non-compact from of $\mathrm{SU}(6)$, and makes the discussion of charge orbits much simpler.

The general analysis of the Attractor Equations, BH charge orbits and attractor *moduli spaces* of such a theory has been performed in [57, 114]; here we recall it briefly, and further consider the "small" charge orbit of this theory (the results are also consistent with the $D=3$ analysis of [79]).

1. the "large" (rank-2) $\frac{1}{5}$-BPS charge orbit reads [111]

$$\mathcal{O}_{\frac{1}{5}-BPS,rank-2}=\frac{\mathrm{SU}(1,5)}{\mathrm{SU}(3)\times\mathrm{SU}(2,1)},\ \dim_{\mathbb{R}}=19,\ \mathcal{I}_4>0. \tag{106}$$

The associated attractor *moduli space*, as all the $\frac{1}{\mathcal{N}}$-BPS attractor moduli spaces of $\mathcal{N}\geqslant3$-extended, $D=4$ supergravity theories [112], is a quaternionic symmetric space, namely the *universal hypermultiplet* space:

$$\mathcal{M}_{\frac{1}{5}-BPS,rank-2}=\frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2)\times\mathrm{U}(1)}=\mathbb{CP}^2,\ \#=6. \tag{107}$$

2. the unique "small" (rank-1) $\frac{2}{5}$-BPS charge orbit reads

$$\mathcal{O}_{\frac{2}{5}-BPS,rank-1}=\frac{\mathrm{SU}(1,5)}{\mathrm{SU}(3)\ltimes\mathbb{R}^8},\ \dim_{\mathbb{R}}=19,\ \mathcal{I}_4=0\Leftrightarrow\mathcal{Z}_1=\mathcal{Z}_2. \tag{108}$$

Thus, the associated attractor *moduli space* reads

$$\mathcal{M}_{\frac{2}{5}-BPS,rank-1}=\mathbb{R}^8,\ \#=2. \tag{109}$$

Note that the stabilizer of $\mathcal{O}_{\frac{2}{5}-BPS,rank-1}$ is the same as the stabilizer of the rank-3 $\frac{1}{2}$-BPS orbit of the $\mathcal{N}=2$ magic theory associated to $\mathfrak{J}_3^{\mathbb{C}}$.

By comparing Eqs. (95), (103) and (109), it follows that the $\mathcal{N}=2$ *minimally coupled*, $\mathcal{N}=3$ matter-coupled and $\mathcal{N}=5$ "pure" theories, besides the fact that they cannot be uplifted to $D=5$, all share the property that the number of "non-flat" directions supported by the unique rank-1 charge orbit is 2.

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## Appendix A: Orbit Stabilizers

In order to determine the stabilizers of the orbits we will use the infinitesimal Lie action of $\mathrm{Aut}(\mathfrak{F}) \cong \mathrm{Conf}(\mathfrak{J})$ acting on the corresponding representative canonical forms. Hence, one needs to define the action of the Lie algebra $\mathfrak{Aut}(\mathfrak{F}(\mathfrak{J}))$ in the $\mathrm{Str}_0(\mathfrak{J})$-covariant basis. To this end, one can introduce the *Freudenthal product*, $\wedge : \mathfrak{F} \times \mathfrak{F} \to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{F})$, which for $x = (\alpha, \beta, A, B),\ y = (\delta, \gamma, C, D)$ is defined by

$$x \wedge y = \Phi(\phi, X, Y, \nu), \quad \text{where} \quad \begin{cases} \phi &= -(A \vee D + B \vee C) \\ X &= -\frac{1}{2}(B \times D - \alpha C - \delta A) \\ Y &= \frac{1}{2}(A \times C - \beta D - \gamma B) \\ \nu &= \frac{1}{4}(\mathrm{Tr}(A, D) + \mathrm{Tr}(C, B) - 3(\alpha\gamma + \beta\delta)) \end{cases} \tag{A1}$$

and $A \vee B \in \mathfrak{Str}_0(\mathfrak{J})$ is defined by $(A \vee B)C = \frac{1}{2}\mathrm{Tr}(B, C)A + \frac{1}{6}\mathrm{Tr}(A, B)C - \frac{1}{2}B \times (A \times C)$. The action of $\Phi : \mathfrak{F} \to \mathfrak{F}$ is given by

$$\Phi(\phi, X, Y, \nu)\begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} = \begin{pmatrix} \alpha\nu + (Y, B) & \phi A - \frac{1}{3}\nu A + 2Y \times B + \beta X \\ -{}^t\phi B + \frac{1}{3}\nu B + 2X \times A + \alpha Y & -\beta\nu + (X, A) \end{pmatrix}. \tag{A2}$$

The maps $\Phi \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{F})$ are in fact Lie algebra elements. Moreover, every Lie algebra element is given by some $\Phi$. More precisely we have the following theorem [43].

**Theorem 10** (Imai and Yokota, 1980)**.**

$$\mathfrak{Aut}(\mathfrak{F}) = \{\Phi(\phi, X, Y, \nu) \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{F}) | \phi \in \mathfrak{Str}_0(\mathfrak{J}), X, Y \in \mathfrak{J}, \nu \in \mathbb{R}\}. \tag{A3}$$

*where the Lie bracket*

$$[\Phi(\phi_1, X_1, Y_1, \nu_1), \Phi(\phi_2, X_2, Y_2, \nu_2)] = \Phi(\phi, X, Y, \nu) \tag{A4}$$

*is given by*

$$\begin{aligned} \phi &= [\phi_1, \phi_2] + 2(X_1 \vee Y_2 - X_2 \vee Y_1) \\ X &= (\phi_1 + \frac{2}{3}\nu_1)X_2 - (\phi_2 + \frac{2}{3}\nu_2)X_1 \\ Y &= (\phi_2 + \frac{2}{3}\nu_2)Y_1 - ({}^t\phi_1 + \frac{2}{3}\nu_1)Y_2 \\ \nu &= \mathrm{Tr}(X_1, Y_2) - \mathrm{Tr}(Y_1, X_2). \end{aligned} \tag{A5}$$

We will frequently consider (see also [25]) the Lie algebra elements of the form

$$\widehat{\Phi}(X, Y) := \Phi(0, X, Y, 0). \tag{A6}$$

The Hermitian conjugate is defined by

$$\widehat{\Phi}^\dagger(X, Y) = \widehat{\Phi}(Y, X). \tag{A7}$$

Hermitian (resp. anti-Hermitian) generators are non-compact (resp. compact) [12].

### 1. An Example : The Exceptional Magic Theory

As an example, which may be quite simply generalised to all models treated here, we examine the case of $\mathfrak{F}(\mathfrak{J}_3^{\mathbb{O}})$. In order to determine the stabilizers of the the orbits, we will use the infinitesimal Lie algebra action (A2) to fix the Lie sub-algebras annihilating the the canonical forms presented in Theorem 6 [27]. Note, in this specific case the construction of the Lie algebra elements $\Phi(\phi, X, Y, \nu)$ corresponds to the decomposition,

$$\begin{gathered} E_{7(-25)} \supset E_{6(-26)} \\ \mathbf{133} \to \mathbf{1} + \mathbf{27} + \mathbf{27}' + \mathbf{78} \end{gathered} \tag{A8}$$

where $\phi, X, Y$, and $\nu$ sit in the $\mathbf{78}, \mathbf{27}, \mathbf{27}'$ and $\mathbf{1}$, respectively.

For all canonical forms one obtains

$$\Phi(x_{\text{can}}) = \begin{pmatrix} \nu & \phi A_{\text{can}} - \frac{1}{3}\nu A_{\text{can}} \\ X \times A_{\text{can}} + Y & \operatorname{Tr}(Y, A_{\text{can}}) \end{pmatrix}, \quad \text{where} \quad x_{\text{can}} = \begin{pmatrix} 1 & A_{\text{can}} \\ 0 & 0 \end{pmatrix}, \tag{A9}$$

so we may set the dilatation generator $\nu$ to zero throughout.

*a. Rank 1:* $A_{\text{can}} = 0$

$$\Phi(x_1) = \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} \tag{A10}$$

$\Rightarrow Y = 0$ while $X$ and $\phi$ are unconstrained. Hence, the stability group is

$$H_1 = E_{6(-26)} \ltimes \mathbb{R}^{27}, \tag{A11}$$

where $E_{6(-26)}$ is generated by $\phi$ and the 27 translations are generated by $X$.

*b. Rank 2a:* $A_{\text{can}} = (1, 0, 0)$

$$\Phi(x_{2a}) = \begin{pmatrix} 0 & \phi A_{\text{can}} \\ X \times A_{\text{can}} + Y & \operatorname{Tr}(Y, A_{\text{can}}) \end{pmatrix} \tag{A12}$$

From the $D = 5$ analysis [27] we know that the Lie sub-algebra of $\mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$ satisfying $\phi A_{\text{can}} = 0$ has 36 compact, 9 non-compact semi-simple generators and 16 translational generators giving $\mathfrak{so}(1, 9) \oplus \mathbb{R}^{16}$. For the remaining $27 + 27$ generators we obtain the following constraints:

1.

$$\operatorname{Tr}(Y, A_{\text{can}}) = 0 \Rightarrow y_{11} = 0. \tag{A13}$$

2.

$$X \times A_{\text{can}} + Y = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_{33} & -x_{23} \\ 0 & -\overline{x}_{23} & x_{22} \end{pmatrix} = \begin{pmatrix} 0 & -y_{12} & -\overline{y}_{13} \\ -\overline{y}_{12} & -y_{22} & -y_{23} \\ -y_{13} & -\overline{y}_{23} & -y_{33} \end{pmatrix} \tag{A14}$$

This gives 1 compact and 9 non-compact semi-simple generators

$$\widehat{\Phi}(\tilde{X}, \tilde{Y}), \tag{A15}$$

where, writing $x_{22} = x + y$ and $x_{33} = x - y$,

$$\tilde{X} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x + y & x_{23} \\ 0 & \overline{x}_{23} & x - y \end{pmatrix}, \quad \tilde{Y} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -x + y & x_{23} \\ 0 & \overline{x}_{23} & -x - y \end{pmatrix}. \tag{A16}$$

These, together with the 36 compact and 9 non-compact generators from $\mathfrak{so}(1, 9) \subset \mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$, give a total of 37 compact generators and 18 non-compact semi-simple generators producing $\mathfrak{so}(2, 9)$, where we have used the fact that $\mathrm{SO}(m, n)$ has $[m(m-1) + n(n-1)]/2$ compact and $mn$ non-compact generators.

The other $1 + 16$ components of $X$ generate translations,

$$X' = \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X'' = \begin{pmatrix} 0 & x_{12} & \overline{x}_{13} \\ \overline{x}_{12} & 0 & 0 \\ x_{13} & 0 & 0 \end{pmatrix}, \tag{A17}$$

where $X'$ commutes with $\mathfrak{so}(2, 9)$. The remaining $16 + 16$ translational generators transform as the spinor of $\mathfrak{so}(2, 9)$. Hence, the stability group is

$$H_{2a} = \mathrm{SO}(2, 9) \ltimes \mathbb{R}^{32} \times \mathbb{R}. \tag{A18}$$

*c. Rank 2b:* $A_{\text{can}} = (-1, 0, 0)$

$$\Phi(x_1) = \begin{pmatrix} 0 & \phi A_{\text{can}} \\ X \times A_{\text{can}} - Y & \operatorname{Tr}(Y, A_{\text{can}}) \end{pmatrix} \tag{A19}$$

The analysis goes through as above but with the sign of $\tilde{Y}$ flipped. This gives a total of 45 compact and 10 non-compact semi-simple generators giving $\mathfrak{so}(1, 10)$. Hence, the stability group is

$$H_{2b} = \mathrm{SO}(1, 10) \ltimes \mathbb{R}^{32} \times \mathbb{R}. \tag{A20}$$

*d. Rank 3a:* $A_{\text{can}} = (1, 1, 0)$

$$\Phi(x_{3a}) = \begin{pmatrix} 0 & \phi A_{\text{can}} \\ X \times A_{\text{can}} + Y & \operatorname{Tr}(Y, A_{\text{can}}) \end{pmatrix} \tag{A21}$$

From the $D = 5$ analysis [27], we know that the Lie sub-algebra of $\mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$ satisfying $\phi A_{\text{can}} = 0$ has 36 compact semi-simple generators and 16 translational generators, yielding $\mathfrak{so}(9) \oplus \mathbb{R}^{16}$. For the remaining $27 + 27$ generators, we obtain the following constraints:

1.

$$\operatorname{Tr}(Y, A_{\text{can}}) = 0 \Rightarrow y_{11} = -y_{22}. \tag{A22}$$

2.

$$\begin{aligned} X \times A_{\text{can}} + Y = 0 \Rightarrow \begin{pmatrix} x_{33} & 0 & -\overline{x}_{13} \\ 0 & x_{33} & -x_{23} \\ -x_{13} & -\overline{x}_{23} & x_{11} + x_{22} \end{pmatrix} &= \begin{pmatrix} -y_{11} & -y_{12} & -\overline{y}_{13} \\ -\overline{y}_{12} & y_{11} & -y_{23} \\ -y_{13} & -\overline{y}_{23} & -y_{33} \end{pmatrix} \\ \Rightarrow x_{33} = y_{11} = 0. & \end{aligned} \tag{A23}$$

This gives 16 non-compact semi-simple generators,

$$\widehat{\Phi}(\tilde{X}, \tilde{Y}), \tag{A24}$$

where,

$$\tilde{X} = \tilde{Y} = \begin{pmatrix} 0 & 0 & \overline{x}_{13} \\ 0 & 0 & x_{23} \\ x_{13} & \overline{x}_{23} & 0 \end{pmatrix}. \tag{A25}$$

These, together with the 36 semi-simple generators from $\mathfrak{so}(9) \subset \mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$, give a total of 36 compact generators and 16 non-compact generators producing $F_{4(-20)}$, which is a non-compact form of $\operatorname{Aut}(\mathfrak{J}_3^{\mathbb{O}})$.

The remaining 10 components of $X$ generate translations which, together with the 16 preserved translational generators of $\mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$, transform as the fundamental **26** of $F_{4(-20)}$.

Hence, the stability group is

$$H_{3a} = F_{4(-20)} \ltimes \mathbb{R}^{26}. \tag{A26}$$

*e. Rank 3b:* $A_{\text{can}} = (-1, -1, 0)$

$$\Phi(R_1) = \begin{pmatrix} 0 & \phi A_{\text{can}} \\ X \times A_{\text{can}} - Y & \operatorname{Tr}(Y, A_{\text{can}}) \end{pmatrix} \tag{A27}$$

The analysis goes through as above, but with the sign of $\tilde{Y}$ flipped so that the 16 previously non-compact semi-simple generators become compact giving the compact form $F_{4(-52)} = \operatorname{Aut}(\mathfrak{J}_3^{\mathbb{O}})$. Hence, the stability group is

$$H_{3a} = F_{4(-52)} \ltimes \mathbb{R}^{26}. \tag{A28}$$

*f. Rank 4a:* $A_{\rm can} = (-1,-1,-1)$

$$\Phi(x_{4a}) = \begin{pmatrix} 0 & \phi A_{\rm can} \\ X\times A_{\rm can} + Y & \mathrm{Tr}(Y, A_{\rm can}) \end{pmatrix} \tag{A29}$$

From the $D = 5$ analysis we know that the Lie sub-algebra of $\mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$ satisfying $\phi A_{\rm can} = 0$ has 52 compact semi-simple generators giving $F_{4(-52)}$. For the remaining $27 + 27$ generators, we obtain the following constraints:

1.
$$\mathrm{Tr}(Y, A_{\rm can}) = 0 \Rightarrow y_{11} + y_{22} + y_{33} = 0. \tag{A30}$$

2.
$$X\times A_{\rm can} + Y = 0 \Rightarrow \begin{pmatrix} x_{11} & x_{12} & \overline{x}_{13} \\ \overline{x}_{12} & x_{22} & x_{23} \\ x_{13} & \overline{x}_{23} & -(x_{11}+x_{22}) \end{pmatrix} = \begin{pmatrix} -y_{11} & -y_{12} & -\overline{y}_{13} \\ -\overline{y}_{12} & -y_{22} & -y_{23} \\ -y_{13} & -\overline{y}_{23} & (y_{11}+y_{22}) \end{pmatrix}, \tag{A31}$$

where we have abused the notation by use the same symbols for $X, Y$ after imposing the condition $\mathrm{Tr}(Y) = 0$. We have also used the identity $X \times (-\mathbb{1}) = X - \mathrm{Tr}(X)\mathbb{1}$ so that $X \times A_{\rm can} + Y = 0$ implies $\mathrm{Tr}(X) = 0$, therefore giving the implication in (A31).

This gives 26 compact semi-simple generators,

$$\widehat{\Phi}(\tilde{X}, \tilde{Y}), \tag{A32}$$

where

$$\tilde{X} = \begin{pmatrix} x_{11} & x_{12} & \overline{x}_{13} \\ \overline{x}_{12} & x_{22} & x_{23} \\ x_{13} & \overline{x}_{23} & -(x_{11}+x_{22}) \end{pmatrix}, \quad \tilde{Y} = \begin{pmatrix} -x_{11} & -x_{12} & -\overline{x}_{13} \\ -\overline{x}_{12} & -x_{22} & -x_{23} \\ -x_{13} & -\overline{x}_{23} & (x_{11}+x_{22}) \end{pmatrix}. \tag{A33}$$

These, together with the 52 compact semi-simple generators from $F_{4(-52)}$, give a total of 78 compact generators producing $E_{6(-78)}$.

Hence, the stability group is

$$H_{4a} = E_{6(-78)}. \tag{A34}$$

*g. Rank 4b:* $A_{\rm can} = (1, 1, -1)$

$$\Phi(x_{4b}) = \begin{pmatrix} 0 & \phi A_{\rm can} \\ X\times A_{\rm can} + Y & \mathrm{Tr}(Y, A_{\rm can}) \end{pmatrix} \tag{A35}$$

From the $D = 5$ analysis [27], we know that the Lie sub-algebra of $\mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$ satisfying $\phi A_{\rm can} = 0$ has 36 compact and 16 non-compact semi-simple generators giving $F_{4(-20)}$. For the remaining $27+27$ generators, we obtain the following constraints:

1.
$$\mathrm{Tr}(Y, A_{\rm can}) = 0 \Rightarrow y_{11} + y_{22} = y_{33}. \tag{A36}$$

2.
$$X\times A_{\rm can} + Y = 0 \Rightarrow \begin{pmatrix} x_{11} & x_{12} & -\overline{x}_{13} \\ \overline{x}_{12} & x_{22} & -x_{23} \\ -x_{13} & -\overline{x}_{23} & x_{11}+x_{22} \end{pmatrix} = \begin{pmatrix} -y_{11} & -y_{12} & -\overline{y}_{13} \\ -\overline{y}_{12} & -y_{22} & -y_{23} \\ -y_{13} & -\overline{y}_{23} & -(y_{11}+y_{22}) \end{pmatrix}. \tag{A37}$$

This gives 10 compact and 16 non-compact semi-simple generators,

$$\widehat{\Phi}(\tilde{X}, \tilde{Y}), \tag{A38}$$

where

$$\tilde{X} = \begin{pmatrix} x_{11} & x_{12} & \overline{x}_{13} \\ \overline{x}_{12} & x_{22} & x_{23} \\ x_{13} & \overline{x}_{23} & x_{11}+x_{22} \end{pmatrix}, \quad \tilde{Y} = \begin{pmatrix} -x_{11} & -x_{12} & \overline{x}_{13} \\ -\overline{x}_{12} & -x_{22} & x_{23} \\ x_{13} & \overline{x}_{23} & -(x_{11}+x_{22}) \end{pmatrix}. \tag{A39}$$

These, together with the 36 compact and 16 non-compact semi-simple generators from $F_{4(-20)}$, give a total of 46 compact generators and 32 non-compact generators producing $E_{6(-14)}$.

Hence, the stability group is

$$H_{4b} = E_{6(-14)}. \tag{A40}$$

*h. Rank 4c:* $A_{\mathrm{can}} = (1,1,1)$

$$\Phi(x_{4c}) = \begin{pmatrix} 0 & \phi A_{\mathrm{can}} \\ X \times A_{\mathrm{can}} + Y & \mathrm{Tr}(Y, A_{\mathrm{can}}) \end{pmatrix} \tag{A41}$$

From the $D = 5$ analysis [27], we know that the Lie sub-algebra of $\mathfrak{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$ satisfying $\phi A_{\mathrm{can}} = 0$ has 52 compact semi-simple generators giving $F_{4(-52)} = \mathrm{Aut}(\mathfrak{J}_3^{\mathbb{O}})$. For the remaining $27 + 27$ generators, we obtain the following constraints:

1.

$$\mathrm{Tr}(Y, A_{\mathrm{can}}) = 0 \Rightarrow y_{11} + y_{22} + y_{33} = 0. \tag{A42}$$

2.

$$X \times A_{\mathrm{can}} + Y = 0 \Rightarrow \begin{pmatrix} -x_{11} & -x_{12} & -\overline{x}_{13} \\ -\overline{x}_{12} & -x_{22} & -x_{23} \\ -x_{13} & -\overline{x}_{23} & x_{11} + x_{22} \end{pmatrix} = \begin{pmatrix} -y_{11} & -y_{12} & -\overline{y}_{13} \\ -\overline{y}_{12} & -y_{22} & -y_{23} \\ -y_{13} & -\overline{y}_{23} & y_{11} + y_{22} \end{pmatrix}. \tag{A43}$$

This gives 26 non-compact semi-simple generators,

$$\widehat{\Phi}(\tilde{X}, \tilde{Y}), \tag{A44}$$

where

$$\tilde{X} = \tilde{Y} = \begin{pmatrix} x_{11} & x_{12} & \overline{x}_{13} \\ \overline{x}_{12} & x_{22} & x_{23} \\ x_{13} & \overline{x}_{23} & -(x_{11} + x_{22}) \end{pmatrix}. \tag{A45}$$

These, together with the 52 compact semi-simple generators from $F_{4(-52)}$, give a total of 52 compact generators and 26 non-compact generators producing $E_{6(-26)} = \mathrm{Str}_0(\mathfrak{J}_3^{\mathbb{O}})$.

Hence, the stability group is

$$H_{4c} = E_{6(-26)}. \tag{A46}$$

This procedure can be repeated for all magical theories, yielding the results reported in Table 6, as well as for all $\mathcal{N} = 2$, $D = 4$ *symmetric* supergravity theories with a Jordan algebraic interpretation (see also the treatment of [25]). For the $D = 5$ treatment, see [27].


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