Comparison of some exact and perturbative results for a supersymmetric SU(N_{c}) gauge theory

Thomas A. Ryttov and Robert Shrock

Phys. Rev. D **85**, 076009 — Published 24 April 2012

DOI: 10.1103/PhysRevD.85.076009

Comparison of Some Exact and Perturbative Results for a Supersymmetric SU(N_c) Gauge Theory

Thomas A. Ryttov[a] and Robert Shrock[b]

(a) Department of Physics, Harvard University, Cambridge, MA 02138 and
(b) C.N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, NY 11794

We consider vectorial, asymptotically free $\mathcal{N} = 1$ supersymmetric SU(N_c) gauge theories with N_f copies of massless chiral superfields in various representations and study how perturbative predictions for the lower boundary of the infrared conformal phase, as a function of N_f, compare with exact results. We make use of two-loop and three-loop calculations of the beta function and anomalous dimension of the quadratic chiral superfield operator product for this purpose. The specific chiral superfield contents that we consider are N_f copies of (i) $F + \bar{F}$, (ii) Adj, (iii) $S_2 + \bar{S}_2$, and (iv) $A_2 + \bar{A}_2$, where F, Adj, S_2, and A_2 denote, respectively, the fundamental, adjoint, and symmetric and antisymmetric rank-2 tensor representations. We find that perturbative results slightly overestimate the value of $N_{f,cr}$ relative to the respective exact results for these representations, i.e., slightly underestimate the interval in N_f for which the theory has infrared conformal behavior. Our results provide a measure of how closely perturbative calculations reproduce exact results for these theories.

I. INTRODUCTION

A longstanding question in gauge theories concerns how well the beta function, calculated to some order in a perturbation expansion in the gauge coupling, describes the properties of an asymptotically free gauge theory, in particular, its evolution from high Euclidean momentum scales μ in the ultraviolet (UV) to low μ in the infrared (IR). Part of the difficulty in answering this question stems from the fact that only the first two terms (i.e., the one-loop and two-loop terms) in the beta function are scheme-independent. Recall that the beta function effectively resums the naive perturbation expansion in defining a running gauge coupling $g(\mu)$. There is special interest in the case where the two-loop beta function has a zero away from the origin. If this zero occurs at a very small value of $\alpha(\mu) = g(\mu)^2/(4\pi)$, then one expects that the theory does not confine or produce bilinear fermion condensates and associated spontaneous chiral symmetry breaking (SχSB). In contrast, if the theory has sufficiently few fermions, then this IR zero of the beta function occurs at sufficiently large α that one expects the theory to confine and spontaneously break chiral symmetry. For ordinary non-supersymmetric SU(N_c) gauge theories, based on the calculations of the one-loop [1] and two-loop [2] terms in the beta function, there have been a number of studies of this UV to IR evolution (an early work is [3]). A particularly interesting possibility is that the IR zero of the beta function could occur at a value only slightly larger than the critical value for SχSB, so that the theory would remain quasi-conformal for a large interval in $\ln \mu$, with a large but slowly running ("walking") coupling and an associated large anomalous dimension γ_m of the bilinear fermion operator $\bar{\psi}\psi$ [4, 5]. There has been intensive recent work to study quasi-conformal behavior using lattice methods, and much progress has been made [6]. These studies have considered not only fermions in the fundamental representation of the gauge group (usually SU(2) or SU(3)), but also fermions in higher-dimensional representations. This has reflected interest in quasi-conformal behavior in gauge theories with fermions in higher-dimensional representations; for a review, see [7].

In this paper we consider vectorial, asymptotically free $\mathcal{N} = 1$ supersymmetric SU(N_c) gauge theories (at zero temperature and chemical potential) with content consisting of N_f copies of massless chiral superfields Φ_i and $\tilde{\Phi}_i$, $i = 1, ..., N_f$, which transform according to representations R and $\bar{R}$, respectively, where R denotes the representation of the gauge group. We will present various results for arbitrary R and will analyze the following specific representations: fundamental, adjoint, and rank-2 symmetric and antisymmetric tensor, denoted F, Adj, S_2, and A_2, respectively. Using two- and three-loop perturbative results, we study the evolution of the theory from the deep UV to the IR and compare these perturbative results with exact results [8]-[12]. We investigate how the IR zero of the beta function, calculated to a certain loop order, and the mass anomalous dimension, γ_m, of the composite superfield operator product that contains the component-field $\bar{\psi}\psi$, evaluated at this IR zero, calculated to the same loop order, compare with exact results. If the theory evolves from the UV to a conformally invariant IR phase (a non-Abelian Coulomb phase), there is a rigorous upper bound on γ_m, namely $\gamma_m \leq 1$ [13]-[15]. We apply this to a scheme-independent calculation of γ_m and to the two-loop and three-loop values of γ_m at the IR zeros of β, calculated to the same order, to calculate perturbative estimates of the minimal value of N_f, denoted $N_{f,cr}$, for which the IR behavior of the theory is conformal. We carry out this analysis in full detail for the case of Φ_i, $\tilde{\Phi}_i$ in the $F + \bar{F}$ representation and give briefer analyses for higher representations. By comparing these perturbative predictions for $N_{f,cr}$ with exact results, we obtain a measure of the accuracy of the perturbative analysis. A related way to estimate $N_{f,cr}$ is from an approximate solution of the

Dyson-Schwinger equations for the relevant propagators. The comparison of the resultant predictions with exact results for the fundamental representation was given in the important paper [16]. Our work complements Ref. [16], since we analyze the perturbative γ_m rather than approximate solutions to Dyson-Schwinger equations.

There are several motivations for this work. First, it is always of fundamental field-theoretic interest to compare how well a perturbative calculation reproduces an exact result. This is especially true in a quantum field theory in view of the fact that a pertubative expansion is not a Taylor series expansion, with finite radius of convergence, but instead, only asymptotic, with zero radius of convergence [17]. This is to be contrasted, for example, with high-temperature series expansions in statistical mechanics and strong-bare-coupling expansions in lattice gauge theory, which are true Taylor series with (at least) finite radii of convergence. Secondly, given the great current interest in the nature of the UV to IR evolution of asymptotically free gauge theories, as a function of their fermion content, it is valuable to have a quantitative measure of the accuracy of the (semi)perturbative approach of calculating the IR zero of the beta function and evaluating the anomalous dimension evaluated at this zero of β. We have previously carried out such a study for non-supersymmetric SU(N_c) gauge theories with various fermion contents [18] (see also [19], whose results are in agreement with those in [18]). An advantage of making this comparison in a supersymmetric gauge theory is that, in contrast to the non-supersymmetric case, one can compare the perturbative calculation with exact results on the IR properties of the theory.

This paper is organized as follows. In Sect. II we define some notation and recall the relevant coefficients of the beta function. Sect. III is devoted to a discussion of the perturbative expression for γ_m. In Sect. IV we review some exact results on the IR phase structure of the theory. In Sect. V we give a general discussion of perturbative estimates of $N_{f,cr}$. The subsequent sections VI-XIII contain our results for the various representations considered here. Our conclusions are in Sect. IX.

II. BETA FUNCTION

A. General

The beta function of the theory is denoted $\beta = dg/dt$, where $dt = d\ln\mu$. In terms of the variable

$$a \equiv \frac{g^2}{16\pi^2} = \frac{\alpha}{4\pi} \ , \tag{2.1}$$

the beta function can be written equivalently as $\beta_\alpha \equiv d\alpha/dt = g\beta/(2\pi)$, expressed as a series

$$\frac{d\alpha}{dt} = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar b_\ell \, \alpha^\ell \ , \tag{2.2}$$

where ℓ denotes the number of loops involved in the calculation of b_ℓ and $\bar b_\ell = b_\ell/(4\pi)^\ell$. The first two coefficients in the expansion (2.2), which are scheme-independent, are [20]

$$b_1 = 3C_A - 2T_f N_f \tag{2.3}$$

and [21]-[23]

$$b_2 = 6C_A^2 - 4(C_A + 2C_f)T_f N_f \ . \tag{2.4}$$

A commonly used regularization scheme for supersymmetric theories is dimensional reduction with minimal subtraction, denoted $\overline{DR}$ [24, 25] (a recent discussion is [26]). In this regularization scheme, the coefficient of the three-loop term in the beta function is [27]-[32]

$$\begin{aligned} b_3 &= 21C_A^3 + 4(-5C_A^2 - 13C_A C_f + 4C_f^2)T_f N_f \\ &+ 4(C_A + 6C_f)T_f^2 N_f^2 \ . \end{aligned} \tag{2.5}$$

Although the beta function coefficients b_ℓ with $\ell \geq 3$ are scheme-dependent, the use of three- and four-loop coefficients in the comparison of the QCD beta function with experimental data has shown the value in incorporating these higher-loop contributions [33]. The purpose in our present work is different from that for QCD; there, one wanted to obtain the most precise comparison possible with data, in order to extract the value of $\alpha_s(\mu)$. Here, we would like to compare the predictions of a particular scheme, namely DR, with the exact results concerning the infrared behavior of the theory for various ranges of N_f, in order to obtain a measure of the accuracy and reliability of the perturbative calculations as a guide to the infrared phase structure of the theory.

Before proceeding, it is appropriate to include several cautionary remarks. First, as is well-known, expansions such as (2.2) and (3.1) in powers of α are not Taylor series, but instead, only asymptotic series, with zero radius of convergence. However, a wealth of experience in particle physics has shown that if the effective expansion parameter (here, (α/π) times various group invariants) is not too large, then the first few terms can provide useful information about the physics. Second, the expansions for β and γ_m in Eqs. (2.2) and (3.1) are perturbative and do not incorporate nonperturbative properties of the physics, such as instantons. Instanton effects are absent to any order of a perturbative expansion in α, but play an important role in a non-Abelian Yang Mills gauge theory. Terms arising from instanton effects characteristically involve essential zeros of the form $\exp(-\kappa\pi/\alpha)$, where $\kappa > 0$ is a numerical constant. Indeed, instanton effects play an important role in the derivation of exact results on supersymmetric gauge theories [8],[34].

The requirement that the theory be asymptotically free means that $\beta < 0$, which, with the overall minus sign in Eq. (2.2), is true if and only if $b_1 > 0$. Note that, *a priori*, the condition $\beta > 0$ could be satisfied with $b_1 = 0$ if $b_2 > 0$, but this is actually not possible, because the value of N_f that renders $b_1 = 0$, namely [36]

$$N_{f,b1z} = \frac{3C_A}{2T_f} \equiv N_{f,max} \ , \tag{2.6}$$

yields a negative value of b_2 for any representation R, viz., $-12C_AC_f$. Hence, the requirement of asymptotic freedom implies

$$N_f < N_{f,max} \ . \tag{2.7}$$

The number $N_{f,max}$ depends on the representation R, and, where necessary for clarity, we shall indicate this by writing $N_{f,max,R}$. We shall assume N_f satisfies this upper bound in our present study. Although one could generalize the analysis to non-asymptotically free theories, the coupling $\alpha(\mu) \to 0$ as the energy scale $\mu \to 0$ in such theories, so the infrared behavior would be that of a free theory.

B. Zero of the Two-Loop Beta Function

Since only the two-loop beta function is scheme-independent, at the perturbative level, if it does not have an IR zero, then, even if such a zero were present at the level of three or more loops, it could not be reliably considered to be physical. Here we have an even more stringent criterion, based on the exact results of Ref. [8], which specify, as a function of N_c and the chiral fermion content, whether the theory evolves from the UV to a conformally invariant IR phase (a non-Abelian Coulomb phase) . These results are equivalent to having an exact beta function and knowing whether it has an exact IR fixed point of the renormalization group. If the exact analysis does not have an IR zero but the perturbative 2-loop beta function does have an IR zero, then even though the latter is scheme-independent, one would still have to reject its prediction, since it differs from the exact result. In Ref. [37], a nonperturbative IR zero of the beta function of a non-Abelian gauge theory has also been discussed.

For zero and sufficiently small N_f, the coefficients b_2 and b_3 are both positive. As N_f increases, these coefficients both decrease. The coefficient b_2 passes through zero and reverses sign from positive to negative at the value

$$N_{f,b2z} = \frac{3C_A^2}{2T_f(C_A+2C_f)} \ . \tag{2.8}$$

This value of N_f is less than $N_{f,b1z} = N_{f,max}$, as is clear from the fact that

$$N_{f,b2z} = \frac{N_{f,b1z}}{1+\frac{2C_f}{C_A}} < N_{f,b1z} \ . \tag{2.9}$$

The two-loop (2ℓ) beta function has a zero away from the origin at $a_{IR,2\ell} = -b_1/b_2$, i.e.,

$$\begin{aligned}\alpha_{IR,2\ell} &= -\frac{4\pi b_1}{b_2} \\ &= \frac{2\pi(3C_A - 2T_fN_f)}{2(C_A+2C_f)T_fN_f - 3C_A^2} \ . \end{aligned} \tag{2.10}$$

Clearly, for N_f only slightly larger than $N_{f,b2z}$, $\alpha_{IR,2\ell}$ is too large for this perturbative result to be trustworthy; a necessary condition for it to be reliable is that N_f is sufficiently far above $N_{f,b2z}$ that $\alpha_{IR,2\ell}$ is not too large. For our analysis below, it will be important whether the formal divergence in $\alpha_{IR,2\ell}$ at $b_2 = 0$, i.e., $N_f = N_{f,b2z}$, occurs above or below the lower boundary of the IR conformal phase, which is given by $N_{f,cr}$ in Eq. (4.5). The difference is

$$N_{f,b2z} - N_{f,cr} = \frac{3C_A(C_A-2C_f)}{4T_f(C_A+2C_f)} \ . \tag{2.11}$$

We find that this can be positive or negative. For example, for the fundamental representation,

$$N_{f,b2z} - N_{f,cr} = \frac{3N_c}{2(2N_c^2-1)} > 0 \quad \text{for fund. rep.} \ , \tag{2.12}$$

so that $b_2 = 0$ and $\alpha_{IR,2\ell}$ diverges within the IR conformal phase. In contrast, for the adjoint representation,

$$N_{f,b2z} - N_{f,cr} = -\frac{1}{4} \quad \text{for Adj. rep.} \ , \tag{2.13}$$

so that in this case, b_2 is nonzero (and negative) all throughout the IR conformal phase. For the symmetric and antisymmetric rank-2 tensor representations, we find

$$\begin{aligned} N_{f,b2z} - N_{f,cr} &= -\frac{3N_c(N_c^2 \pm 2N_c - 4)}{2(N_c \pm 2)(3N_c^2 \pm 2N_c - 4)} \\ & \qquad \text{for } S_2, \ A_2 \text{ rep.} \ , \end{aligned} \tag{2.14}$$

where the upper and lower signs apply for the S_2 and A_2 representations, respectively. For S_2, the numerator factor $N_c^2 + 2N_c - 4$ vanishes at the unphysical, negative value $N_c = -(1+\sqrt{5}\ )$ and at $N_c = -1+\sqrt{5} \simeq 1.236$, which is less than the minimal non-Abelian value, $N_c = 2$. Hence, for the S_2 representation, $N_{f,b2z} < N_{f,cr}$ for all non-Abelian N_c and b_2 has fixed (negative) sign throughout the IR conformal phase. For the A_2 representation, N_c is restricted to the nontrivial range $N_c \ge 3$. In this A_2 case, the numerator factor $N_c^2 - 2N_c - 4$ vanishes at $N_c = 1+\sqrt{5} \simeq 3.236$ (as well as at the negative, unphysical value $N_c = 1-\sqrt{5}$), so that $N_{f,b2z} > N_{f,cr}$ for the real interval $3 \le N_c < 1+\sqrt{5}$, while $N_{f,b2z} < N_{f,cr}$ for $N_c > 1+\sqrt{5}$, i.e., the integer values $N_c \ge 4$. Note that the A_2 representation with $N_c = 3$ is equivalent to the conjugate fundamental representation. Hence, for all representations R for which A_2 is distinct from the fundamental, b_2 has fixed (negative) sign throughout the IR conformal phase.

Given that $N_f < N_{f,max}$ to maintain the asymptotic freedom of the theory, this $\alpha_{IR,2\ell}$ is positive and hence physical if and only if N_f lies in the range $N_{f,b2z} < N_f < N_{f,b1z}$, i.e.,

$$\frac{3C_A^2}{2T_f(C_A+2C_f)} < N_f < \frac{3C_A}{2T_f} \ . \tag{2.15}$$

We will thus focus on this interval for N_f. The zero of the two-loop beta function at $\alpha = \alpha_{IR,2\ell}$ is either an approximate or exact infrared (IR) fixed point (IRFP) of the renormalization group. If the gauge interaction spontaneously breaks the global chiral symmetry of the theory via the formation of a bilinear matter (chiral) superfield condensate, then this IR zero is only approximate, since in this case the matter superfield picks up a dynamically generated mass Σ and as the scale μ decreases below Σ, one integrates out the matter superfields in defining the effective low-energy field theory. Consequently, as the theory evolves further into the infrared, the beta function becomes that of the pure supersymmetric gauge theory without these matter superfields, and hence $\alpha(\mu)$ evolves away from the approximate IR fixed point.

C. Zeros of the Three-Loop Beta Function

To three-loop order, the beta function formally has two zeros away from the origin, given by the equation $b_1 + b_2 a + b_3 a^2 = 0$, where a was given in Eq. (2.1). The solutions, in terms of $\alpha = 4\pi a$, are

$$\alpha = \frac{2\pi}{b_3}\Big[-b_2 \pm \sqrt{b_2^2 - 4b_1 b_3} \ \Big] \ . \tag{2.16}$$

Only the physical, smaller one of these two solutions will be relevant for our analysis, and we label it as $\alpha_{IR,3\ell}$. As discussed above, the requirement that the two-loop beta function has an IR zero means that N_f is in the interval (2.15) where $b_1 > 0$ and $b_2 < 0$.

III. ANOMALOUS DIMENSION γ_m

The anomalous dimension γ_m describes the scaling properties of the quadratic superfield operator product $\Phi_i \tilde{\Phi}_i$ containing the bilinear product $\psi^T C \tilde{\psi}$, or equivalently, $\bar{\psi}\psi$, of component fermion fields. If one has an input mass m for ψ, then, with our definition, $\gamma_m = -d \ln m/dt$, where $t = \ln \mu$. Since we are studying the evolution from the UV to the IR conformal phase, we do not put in such a bare mass m here, since, if we did, then as μ decreases below m, these fields would be integrated out as the theory evolved deeper into the infrared and the IR behavior would be that of a supersymmetric SU(N_c) theory with just gluons and gluinos. For notational simplicity we will often suppress the subscript m. This anomalous dimension can be expressed as a series in a or equivalently, α:

$$\gamma_m = \sum_{\ell=1}^{\infty} c_\ell \, a^\ell = \sum_{\ell=1}^{\infty} \bar{c}_\ell \, \alpha^\ell \ , \tag{3.1}$$

where $\bar{c}_\ell = c_\ell/(4\pi)^\ell$ is the ℓ-loop series coefficient. We denote $\gamma_{n\ell}$ as the n-loop value of γ_m, i.e., $\gamma_{n\ell} = \sum_{n=1}^{\ell} c_n \, a^n$.

The coefficients c_ℓ have been calculated to three-loop order. The one-loop coefficient c_1 is scheme-independent:

$$c_1 = 4C_f \ , \tag{3.2}$$

The higher-loop coefficients c_ℓ with $\ell \ge 2$ are scheme-dependent. In the $\overline{DR}$ scheme, c_2 and c_3 are [28, 32]

$$c_2 = 4C_f(-2C_f + 3C_A - 2T_f N_f) \ , \tag{3.3}$$

and

$$c_3 = 8C_f\Big[4C_f^2 + 3C_A(C_A - C_f) + T_f N_f\Big[(-8+12\zeta(3))C_f + (1-12\zeta(3))C_A\Big] - 2T_f^2 N_f^2\Big] \ , \tag{3.4}$$

where $\zeta(s)$ is the Riemann zeta function, with $\zeta(3) = 1.20205690..$ As N_f approaches $N_{f,max}$ from below, $b_1 \to 0$ with nonzero b_2 and hence $\alpha_{IR} \to 0$; since the perturbative calculation expresses γ_m in a power series in α, it follows that $\gamma_m \to 0$ as $N_f \to N_{f,max}$.

The n-loop value of γ_m at the IR zero of β, calculated to the same loop order (IR fixed point of the renormalization group), is obtained by setting $\alpha = \alpha_{IR,n\ell}$ in $\gamma_{n\ell}(\alpha)$ and is denoted

$$\gamma_{IR,n\ell} \equiv \gamma_{n\ell}(\alpha_{IR,n\ell}) \ . \tag{3.5}$$

where the dependence on the chiral superfield representation R is implicit. Thus, at the two-loop level,

$$\begin{aligned}\gamma_{IR,2\ell} &= a(c_1 + c_2 a)|_{a=a_{IR,2\ell}} \\ &= \frac{b_1(-c_1 b_2 + c_2 b_1)}{b_2^2} \ . \end{aligned} \tag{3.6}$$

Explicitly,

$$\gamma_{IR,2\ell} = \frac{C_f(3C_A - 2T_fN_f)(2T_fN_f - C_A)(2T_fN_f - 3C_A + 6C_f)}{[2(C_A + 2C_f)T_fN_f - 3C_A^2]^2} \ . \tag{3.7}$$

Thus, $\gamma_{IR,2\ell}$ has, formally, three zeros and one pole. One of the zeros occurs at $N_f = N_{f,max}$, as given in Eq. (2.6). The second zero occurs at

$$N_f = \frac{C_A}{2T_f} = \frac{N_{f,max}}{3} = \frac{2N_{f,cr}}{3} \ . \tag{3.8}$$

Because this lies below the exact $N_{f,cr}$, it is not directly relevant for our current analysis. Furthermore, for the representations of interest here, it also lies below $N_{f,b2z}$, and hence is not present where the theory has a two-loop zero in β. The third formal zero in $\gamma_{IR,2\ell}$ occurs at

$$N_f = \frac{3(C_A - 2C_f)}{2T_f} = N_{f,max} - \frac{3C_f}{T_f} \ . \tag{3.9}$$

This third zero occurs for N_f less than $N_{f,cr}$ (and, for some representations, at negative N_f), and hence also will not be relevant for our analysis. The pole in $\gamma_{IR,2\ell}$ occurs at $N_{f,b2z}$, and is a consequence of the pole in $\alpha_{IR,2\ell}$ where $b_2 = 0$. Clearly, the two-loop calculation of γ_m ceases to be reliable for N_f values less than $N_{f,b2z}$, so this pole is obviously an unphysical artifact. Thus, over the range of interest here, $\gamma_{IR,2\ell}$ increases monotonically above zero as N_f decreases below $N_{f,max}$.

In the procedure described above, one evaluates the n-loop expression for γ_m at the IR zero of β, calculated to the same n-loop order. For this procedure, one necessarily uses the β function calculated at least to the two-loop level, since an IR zero only appears at this loop level, and also the two-loop or higher-loop expressions for γ_m. Since the coefficients c_ℓ for $\ell \ge 2$ are scheme-dependent, this process necessarily entails scheme-dependence. (At an IR zero of the exact beta function, γ_m would be physical and scheme-independent, but, as noted above, we are dealing only with a perturbative expansion of β, truncated at a given loop order.) We thus also present an alternate perturbative estimate for γ_m, which has the advantage of preserving scheme-independence but the disadvantage of mixing different orders of perturbation theory. For this alternative estimate, we use only scheme-independent (SI) inputs, and hence evaluate the one-loop expression for γ_m at the two-loop IR zero of β, obtaining

$$\begin{aligned} \gamma_{IR,SI} &= c_1 a_{IR,2\ell} = \bar{c}_1 \alpha_{IR,2\ell} = -\frac{c_1 b_1}{b_2} \\ &= \frac{2C_f(3C_A - 2T_fN_f)}{2(C_A + 2C_f)T_fN_f - 3C_A^2} \ . \end{aligned} \tag{3.10}$$

IV. REVIEW OF SOME EXACT RESULTS

Since we will compare our perturbative results with certain exact results, a brief review of these is appropriate. For a vectorlike SU(N_c) gauge theory with $\mathcal{N} = 1$ supersymmetry and N_f copies of massless chiral superfields Φ_i and $\tilde{\Phi}_i$ in the fundamental and conjugate fundamental representation, respectively, exact results on the phase structure and corresponding properties of the theory in the infrared were derived by Seiberg [8]. These results were subsequently generalized to theories with gauge groups SO(N_c) and Sp(N_c) in [9] and [10] (reviews include [11]). A further generalization to arbitrary representations was given in [12]. In our present work we will focus on the comparison of perturbative estimates and exact results concerning the minimal value of N_f (for a given chiral superfield content), denoted $N_{f,cr}$ such that, for $N_f > N_{f,cr}$, the theory evolves from the UV to the IR in a chirally symmetric manner, so that the IR theory is a conformal, non-Abelian Coulomb phase. This value, $N_{f,cr}$, is often called the lower end of the conformal phase or conformal "window" (with the upper end, $N_{f,max}$, determined by the requirement of asymptotic freedom). We shall carry out this comparison at the two- and three-loop level.

We recall how, for a given R, the conformal region in N_f is determined. A crucial tool in determining $N_{f,cr}$, the lower end, as a function of N_f, of the IR conformal phase, is the existence of an exact relation between the beta function and the mass anomalous dimension, γ_m. This relation is embodied in the following form for the beta function of the theory with a vectorlike massless chiral superfield content consisting of N_f copies of the representations $R + \bar{R}$ of the gauge group [34, 35]:

$$\beta_\alpha = \frac{d\alpha}{dt} = -\frac{\alpha^2}{2\pi}\left[\frac{b_1 - 2T_fN_f\gamma_m(\alpha)}{1 - \frac{C_A\,\alpha}{2\pi}}\right] \ . \tag{4.1}$$

The IR zero of β_α is determined by the condition

$$\gamma_m = \frac{3C_A - 2T_fN_f}{2T_fN_f} = \frac{N_{f,max}}{N_f} - 1 \ . \tag{4.2}$$

Let us assume that the theory flows to an exact IR fixed point, i.e., that N_f is in the sub-interval of (2.15) in which, as the theory evolves down from the UV to the IR, no spontaneous chiral symmetry breaking takes place. Given that the theory has evolved down to an (exact) infrared fixed point,the resultant theory at this IRFP is conformally invariant.

It is a special property of a conformally invariant field theory (whether supersymmetric or not) that the full dimension of a spinless operator (other than the identity) must be larger than unity in order that the theory not contain any negative-norm states, which would violate unitarity [13–15]. Specifically, for the dimension D_m of the bilinear operator $\Phi_i\tilde{\Phi}_i$ (with no sum on i) for any $i = 1, ..., N_f$ in the present theory, this is the inequality

$$D_m \ge 1 \ . \tag{4.3}$$

In terms of its component scalar and fermion fields ϕ_i and ψ_i, the chiral superfield is expressed as $\Phi_i = \phi_i + \sqrt{2}\,\theta\psi_i + \theta\theta F_i$, where θ is a Grassmann variable and F_i is an auxiliary field. Thus (for any i), the term $\Phi_i\tilde{\Phi}_i$ yields, as the (holomorphic) term bilinear in component fermion fields, $\theta\theta\psi_i\tilde{\psi}_i$. Taking into account that the dimension of θ is $-1/2$, the free-field dimension of $\psi_i\tilde{\psi}_i$ (for any i) is 3, and using our definition of γ_m, it follows that $D_m = 2 - \gamma_m$, so that the bound (4.3) is equivalent to the following upper bound on γ_m:

$$\gamma_m \le 1 \ . \tag{4.4}$$

This may be contrasted with the situation in a non-supersymmetric SU(N_c) theory. There, the bound that the full operator dimension of $\bar{\psi}_i\psi_i$ be larger than 1 implies that $\gamma_m \le 2$, as we noted in Eq. (4.2) of [18] (equivalent to the bound from Eq. (4.1) of [18]). The more stringent upper bound (4.4) on γ_m in the supersymmetric theory is due to the fact that the fermion field ψ_i is part of a chiral superfield and the holomorphic fermion bilinear resulting from the quadratic $\Phi_i\tilde{\Phi}_i$ product carries with it a $\theta\theta$ factor.

We next assume that, in the relevant range of N_f where the theory evolves from the UV to an IR-conformal phase, γ_m evaluated at the IR fixed point, $\alpha = \alpha_{IR}$, increases monotonically as N_f decreases below $N_{f,max}$. This assumption is satisfied by γ_m as calculated in a scheme-independent manner, as will be discussed further below. The inequality (4.4) then implies that $N_{f,cr}$, the value of N_f below which the theory cannot be conformally invariant, is bounded below as $N_{f,cr} \ge 3C_A/(4T_f)$. The application of duality relations provides strong evidence that this inequality is saturated [8, 12] and hence that

$$N_{f,cr} = \frac{3C_A}{4T_f} \ . \tag{4.5}$$

We refer to this as an exact result, although, as we have indicated, there are some nonrigorous steps in its derivation. Note that

$$N_{f,cr} = \frac{N_{f,max}}{2} \ . \tag{4.6}$$

Thus, the theory evolves from the UV to an IR fixed point in the conformal phase if and only if $N_{f,cr}$ lies in the interval

$$\frac{3C_A}{4T_f} < N_f < \frac{3C_A}{2T_f} \ . \tag{4.7}$$

(The marginal value $N_f = 3C_A/(4T_f)$ itself is not in this conformal phase [8, 9].) For both Eqs. (4.5) and (4.7), it is understood that, physically, N_f must be an integer [36]. Thus, the actual values of N_f in the conformal phase are understood to be the integers that satisfy the inequality (4.7).

V. PERTURBATIVE ESTIMATES OF $N_{f,cr}$ FOR GENERAL R

As discussed above, although a perturbative calculation is not exact, one gains valuable information by comparing it with exact results. We carry out this comparison here for a general representation R, using perturbative estimates for $N_{f,cr}$, the lower boundary of the IR conformal phase. For this purpose, we utilize $\gamma_{IR,SI}$ and $\gamma_{IR,2\ell}$. With a monotonic increase in γ_m as N_f decreases below $N_{f,max}$, we can then calculate a perturbative estimate for $N_{f,cr}$ by assuming that γ_m saturates the inequality (4.4) as N_f decreases through $N_{f,cr}$. (Here, again, we are implicitly analytically continuing N_f from physical integer values to real numbers.) Setting the perturbative $\gamma_m = 1$ and solving for the value of N_f at which this happens yields the corresponding perturbative estimate of $N_{f,cr}$. Since a perturbatively calculated expression for γ_m is not, in general, equal to the exact γ_m, one does not expect these estimates to agree precisely with the exactly known values for $N_{f,cr}$ for the various representations. However, this comparison gives quantitative insight as to the accuracy of the perturbative calculations.

A. Estimate Using $\gamma_{IR,SI}$

The scheme-independent perturbative result for γ_m, $\gamma_{IR,SI}$, increases monotonically as N_f decreases from its maximal value (2.6) and reaches the rigorous upper bound as N_f decreases through the value

$$N_{f,cr,SI} = \frac{3C_A(C_A + 2C_f)}{2T_f(C_A + 4C_f)} \ . \tag{5.1}$$

This is larger than the exact value of $N_{f,cr}$, as is evident from the difference

$$N_{f,cr,SI} - N_{f,cr} = \frac{3C_A^2}{4T_f(C_A + 4C_f)} > 0 \tag{5.2}$$

or the ratio

$$\frac{N_{f,cr,SI}}{N_{f,cr}} = 2\left(\frac{C_A + 2C_f}{C_A + 4C_f}\right) > 1 \ . \tag{5.3}$$

This difference between the scheme-independent perturbative estimate of the lower boundary of the conformal phase, $N_{f,cr,SI}$, and the exact lower boundary, $N_{f,cr}$, provides one quantitative measure of the accuracy of perturbation theory. Our conclusion from this comparison is that perturbation theory slightly overestimates the value of this lower boundary and hence underestimates the size of the conformal phase as a function of N_f. Related to this, as N_f decreases below $N_{f,cr,SI}$ toward the exact lower boundary of the IR conformal phase at $N_{f,cr}$, $\gamma_{IR,SI}$ continues to increase. In this regime, its behavior is unphysical since it violates the rigorous bound (4.4). This happens for both representations where $N_{f,b2z} > N_{f,cr}$ and representations where $N_{f,b2z} < N_{f,cr}$. Formally,

$$\gamma_{IR,SI} = \frac{2C_f}{2C_f - C_A} \quad \text{at} \quad N_f = N_{f,cr} \ . \tag{5.4}$$

B. Estimate Using $\gamma_{IR,2\ell}$

Setting the two-loop result for the anomalous dimension, $\gamma_{IR,2\ell}$, equal to the rigorous upper bound, unity, we derive the corresponding two-loop perturbative prediction for $N_{f,cr}$. The equation $\gamma_{IR,2\ell} = 1$ is a cubic equation in N_f, which yields the resultant estimate for $N_{f,cr}$, together with two other roots that are not of direct relevance. Formally,

$$\gamma_{IR,2\ell} = \frac{C_f(4C_f - C_A)}{2(2C_f - C_A)^2} \quad \text{at} \quad N_f = N_{f,cr} \ , \tag{5.5}$$

but as with $\gamma_{IR,SI}$, this is only formal, since this perturbative result generically violates the upper bound (4.4). We comment below on the situation at the three-loop level. We proceed to present results for the various representations of interest here.

VI. CHIRAL SUPERFIELDS IN THE FUNDAMENTAL REPRESENTATION

A. IR Zeros of the Beta Function

1. *Two-Loop Analysis*

In this section we consider the case where the theory has N_f copies of massless chiral superfields Φ_i and $\tilde{\Phi}_i$, $i = 1, ..., N_f$, transforming according to

$$\Phi_i : \ F; \quad \tilde{\Phi}_i : \ \bar{F} \ , \quad i = 1, ..., N_f \ , \tag{6.1}$$

i.e., the fundamental plus conjugate fundamental, representation of the gauge group. The requirement of asymptotic freedom implies

$$N_f < 3N_c \ . \tag{6.2}$$

For this case, the exact result (4.5) on the value of N_f at the lower boundary of the conformal phase in the infrared is

$$N_{f,cr} = \frac{3N_c}{2} \ , \tag{6.3}$$

where it is understood that this is only formal if N_c is odd, since $N_{f,cr}$ must be an integer. Thus, the IR conformal phase is given, from Eq. (4.7), as

$$\frac{3N_c}{2} < N_f < 3N_c \ . \tag{6.4}$$

TABLE I: Values of $N_{f,b1z} = N_{f,max}$, $N_{f,b2z}$, and $N_{f,b3z}$ for the supersymmetric SU(N_c) theory with N_f chiral superfields Φ_i, $\tilde{\Phi}_i$ in the F and $\bar{F}$ representations, respectively. We also list the exact value of $N_{f,cr}$. These results are given for the illustrtive values $2 \le N_c \le 5$.

N_c	$N_{f,cr}$	$N_{f,b2z}$	$N_{f,b3z}$	$N_{f,b1z}$
2	3	3.43	3.09	6
3	4.5	4.76	4.27	9
4	6	6.19	5.55	12
5	7.5	7.65	6.85	15

Physically, N_f must be a (non-negative) integer, so the actual physical values of N_f in the IR conformal phase for $2 \le N_c \le 5$ are $N_c = 2: \quad N_f = 4, 5$; $N_c = 3: N_f = 5, 6, 7, 8$; $N_c = 4: \quad N_f = 7, 8, 9, 10, 11$;, and $N_c = 5: \quad N_f = 8, 9, 10, 11, 12, 13, 14$.

Evaluating Eq. (2.8), we find that b_2 reverses sign from positive to negative as N_f increases through the value [36]

$$N_{f,b2z} = \frac{3N_c}{2 - N_c^{-2}} \ . \tag{6.5}$$

The interval of N_f values in Eq. (2.15) where the two-loop beta function has an IR zero is therefore [36]

$$\frac{3N_c}{2 - N_c^{-2}} < N_f < 3N_c \ . \tag{6.6}$$

Numerical values of $N_{f,cr}$, $N_{f,b2z}$, $N_{f,b3z}$, and $N_{f,b1z} = N_{f,max}$ are listed in Table I for the illustrative values $2 \le N_c \le 5$ [36]. As discussed before in connection with Eq. (2.12), the value of $N_{f,cr}$ in Eq. (6.5) is greater (for all N_c) than the exactly known lower boundary of the conformal phase in Eq. (6.3). As $N_c \to \infty$, $N_{f,b2z}$ asymptotically approaches $(3/2)N_c$ from above.

Since $N_{f,b2z} > N_{f,cr}$, it follows that the two-loop beta function only has a (perturbative) infrared zero for N_f values in the interval where the theory is conformally invariant. This is different from the non-supersymmetric SU(N_c) gauge theory, in which the two-loop beta function may have an IR zero for values of N_f less than the estimate, from the Dyson-Schwinger equation, of $N_{f,cr}$, i.e., in the phase where the theory has spontaneous chiral symmetry breaking. (Because of this SχSB, this IR zero is only approximate.)

For our supersymmetric theory with the $F + \bar{F}$ chiral superfield content of Eq. (6.1), the general formula (2.10) for the IR zero of the two-loop beta function reduces to

$$\alpha_{IR,2\ell} = \frac{2\pi(3N_c - N_f)}{(2N_c - N_c^{-1})N_f - 3N_c^2} \tag{6.7}$$

This decreases monotonically from arbitrarily large values (where, of course, the perturbative beta function does not apply reliably) to zero as N_f increases throughout the interval (6.6). Numerical values of $\alpha_{IR,2\ell}$ are listed in Table II for the illustrative cases $2 \le N_c \le 5$.

TABLE II: Values of the IR zero of the beta function in the supersymmetric SU(N_c) gauge theory with N_f pairs of chiral superfields in Φ_i, $\tilde{\Phi}_i$ in the fundamental and conjugate fundamental represention, respectively, calculated at n-loop order, and denoted as $\alpha_{IR,n\ell}$. Results are given for the illustrative values $2 \le N_c \le 5$. For each N_c, we only give results for the integral N_f values in the interval (2.15) where the theory is asymptotically free and the two-loop beta function has an infrared zero.

N_c	N_f	$\alpha_{IR,2\ell}$	$\alpha_{IR,3\ell}$
2	4	6.28	2.65
2	5	1.14	0.898
3	5	18.85	3.05
3	6	2.69	1.40
3	7	0.992	0.734
3	8	0.343	0.308
4	7	5.03	1.64
4	8	1.795	0.984
4	9	0.867	0.615
4	10	0.426	0.357
4	11	0.169	0.158
5	8	12.94	1.90
5	9	2.86	1.13
5	10	1.37	0.765
5	11	0.766	0.528
5	12	0.442	0.353
5	13	0.240	0.212
5	14	0.101	0.0963

It is often of interest to consider the 't Hooft limit $N_c \to \infty$ with $g^2 N_c$ fixed and finite. For the present $F + \bar{F}$ superfield content it is also natural to consider taking $N_f \to \infty$ with the ratio

$$r \equiv \frac{N_f}{N_c} \tag{6.8}$$

fixed and finite (sometimes called the Veneziano limit). In this limit, the relevant interval for r where the two-loop beta function has an IR zero is thus

$$\frac{3}{2} < r < 3 \ . \tag{6.9}$$

Here,

$$\alpha_{IR,2\ell} N_c = \frac{2\pi(3-r)}{2r-3} \ , \tag{6.10}$$

which decreases monotonically to 0 as r increases through the interval $3/2 < r < 3$.

2. Three-Loop Analysis

For the present case, Eq. (6.1), the general result in Eq. (2.5) for the three-loop coefficient b_3 takes the form

$$b_3 \ = \ 21N_c^3 + (9 - 21N_c^2 + 2N_c^{-2})N_f + \ (4N_c - 3N_c^{-1})N_f^2 \ . \tag{6.11}$$

For small N_f, b_3 is positive. As N_f increases, b_3 passes through zero and reverses sign from positive to negative. To investigate this, one solves the equation $b_3 = 0$ for N_f. Since b_3 is a quadratic function of N_f, there are formally two solutions to this equation, namely

$$N_{f,b3z,\pm} \ = \ [2N_c(4N_c^2-3)]^{-1}\Big[21N_c^4 - 9N_c^2 - 2 \pm \ \sqrt{105N_c^8 - 126N_c^6 - 3N_c^4 + 36N_c^2 + 4}\ \Big] \ . \tag{6.12}$$

For $N_c = 2$, this gives $N_{f,b3z,-} = 3.09$ (to the indicated accuracy), slightly above $(3/2)N_c = 3$, while for $N_c \ge 3$, we find that $N_{f,b3z} < (3/2)N_c$. As examples, for $N_c = 3$, $N_{f,b3z,-} = 4.27 < (3/2)N_c = 4.5$, while for $N_c = 4$, $N_{f,b3z,-} = 5.55 < (3/2)N_c = 6$, and so forth for higher values of N_c. For large N_c,

$$\frac{N_{f,b3z,-}}{(3/2)N_c} = \frac{1}{12}\Big[21 - \sqrt{105}\ \Big] + O\bigg(\frac{1}{N_c^2}\bigg) \ . \tag{6.13}$$

The numerical value of the first term is approximately 0.896. As N_f increases past the larger value $N_{f,b3z,+}$, b_3 vanishes and reverses sign again, becoming positive. However, this larger zero is not relevant to our analysis, since for all $N_c \ge 2$, $N_{f,b3z,+} > 3N_c$, i.e., this occurs for N_f beyond the upper limit imposed by the constraint of asymptotic freedom. For example, for $N_c = 2$, $N_{f,b3z,+} \simeq 8.38$, which is greater than $N_{f,max} = 6$, and so forth for larger values of N_c. For large N_c, we have

$$\frac{N_{f,b3z,+}}{3N_c} = \frac{1}{24}\Big[21 + \sqrt{105}\ \Big] + O\bigg(\frac{1}{N_c^2}\bigg) \ . \tag{6.14}$$

The numerical value of the first term is 1.30.

Since we restrict here to the interval (2.15) where the two-loop beta function has an IR zero, we observe that for physical, integral values of N_f in this interval, b_3 is always negative. Note that for $N_c = 2$, b_3 vanishes and changes sign from positive to negative as N_f increases through $N_f = 3.09$, but this value of N_f is less than the value $N_{f,b2z} = 3.43$ where the beta function first has an IR zero for this N_c. Hence, for all $N_c \ge 2$ and for integral values of N_f in the interval (2.15) where there is an IR zero at the two-loop level, $b_3 < 0$.

We list values of $\alpha_{IR,3\ell}$ for $2 \le N_c \le 5$ in Table II, together with the values of $\alpha_{IR,2\ell}$ already given. As is evident from this table,

$$\alpha_{IR,3\ell} < \alpha_{IR,2\ell} \ . \tag{6.15}$$

This is the same trend that we found in [18] for a non-supersymmetric SU(N_c) with N_f copies of massless fermions in the fundamental representation.

In the large-N_c, large-N_f limit with $N_f = rN_c$, we calculate

$$\alpha_{IR,3\ell}N_c = \frac{4\pi\Big[3-2r+\sqrt{4r^3-29r^2+72r-54}\ \Big]}{21(r-1)-4r^2} \ . \tag{6.16}$$

The right-hand side of Eq. (6.16) decreases monotonically from $4\pi \simeq 12.57$ to 0 as r increases from 3/2 to 3. Note that the denominator in Eq. (6.16), $21(r-1)-4r^2$, is positive-definite in the relevant interval $3/2 < r < 3$; it has zeros at $r = (1/8)(21 \pm \sqrt{105})$, i.e., approximately 3.906 and 1.344. In the numerator, the first part, $3-2r$, is negative-definite in this interval, but is smaller than the square root.

B. Values of γ_m at IR Zero of β

1. Coefficients

Evaluating Eqs. (3.2)-(3.4) for our case (6.1), we have

$$c_1 = \frac{2(N_c^2-1)}{N_c} \ , \tag{6.17}$$

$$c_2 = 2(N_c^2-1)N_c^{-2}(2N_c^2+1-N_cN_f) \ , \tag{6.18}$$

and

$$c_3 = 2(N_c^2-1)N_c^{-3}\Big[5N_c^4-N_c^2+2+N_c(-3N_c^2+4)N_f-6\zeta(3)N_c(N_c^2+1)N_f-N_c^2N_f^2\Big] \ , \tag{6.19}$$

where $\zeta(s)$ is the Riemann zeta function. As N_f approaches $N_{f,max}$ from below, $b_1 \to 0$ with nonzero b_2 and hence $\alpha_{IR} \to 0$; since the perturbative calculation expresses γ_m in a power series in α, it follows that as $\gamma_m \to 0$ as $N_f \to N_{f,max}$.

To get an analytic understanding of the behavior of γ_m, we study the signs of the coefficients c_ℓ with $\ell = 1,\ 2,\ 3$. Since the one-loop coefficient c_1 is positive, γ_m increases from zero as N_f decreases just below $N_{f,max}$. In the conformal phase the two-loop coefficient c_2 may be either positive or negative, depending on N_f. This coefficient c_2 vanishes at

$$N_f = 2N_c + N_c^{-1} \equiv N_{f,c2z} \ . \tag{6.20}$$

and

$$\begin{aligned} c_2 &> 0 \quad \text{for} \quad N_f < N_{f,c2z} \ , \\ c_2 &< 0 \quad \text{for} \quad N_{f,c2z} < N_f < N_{f,max} = 3N_c \ . \end{aligned} \tag{6.21}$$

The three-loop coefficient c_3 is a quadratic function of N_f and vanishes at two values of $N_f = 0$, namely

$$N_{f,c3z,fund,\pm} = \frac{4-3N_c^2-6(N_c^2+1)\zeta(3) \pm \sqrt{R_{c3z}}}{2N_c} \tag{6.22}$$

where

$$\begin{aligned} R_{c3z} &= 29N_c^4-28N_c^2+24+12(N_c^2+1)(3N_c^2-4)\zeta(3) \\ &+ [6(N_c^2+1)]^2\zeta(3)^2 \ . \end{aligned} \tag{6.23}$$

For $N_c = 2$, $N_{f,c3z,fund,-}$ is equal to -22.88 and hence is unphysical, while $N_{f,c3z,fund,+} = 0.8522$, so that in the physical range, c_3 is negative for all positive integral values of N_f. For $N_c = 3$, $N_{f,c3z,fund,-}$ is equal to -33.05 and hence is again unphysical, while $N_{f,c3z,fund,+} = 1.338$, so that in the physical range, c_3 is positive for $N_f = 1$, but negative for $N_f \geq 2$, including all of the interval of interest here. This qualitative behavior continues to hold for higher values of N_c, as is evident from the Taylor series expansion

$$\begin{aligned} \frac{N_{f,c3z,fund,+}}{N_c} &= \frac{1}{2}\Big[-3(1+2\zeta(3)) \\ &+ \sqrt{36\zeta(3)(\zeta(3)+1)+29}\ \Big] + O\Big(\frac{1}{N_c^2}\Big) \ . \end{aligned} \tag{6.24}$$

The numerical value of the constant term is 0.468 to the indicated accuracy. Thus, as $N_c \to \infty$, $N_{f,c3z,fund,+} \sim 0.468N_c$, which is less than the exact value of the lower boundary of the conformal phase, $N_{f,cr} = (3/2)N_c$, given in Eq. (6.3). Thus, a general characterization of the three-loop γ_m is that c_1 is positive for any representation R, and for $R = F$; c_2 is positive in the lower part of the conformal phase but negative in the upper part, as specified by Eq. (6.21), and c_3 is negative throughout all of the N_f interval of interest, including the conformal phase.

2. $\gamma_{IR,SI}$

For the chiral fermion content (6.1), we calculate

$$\gamma_{IR,SI} = \frac{(N_c^2-1)(3N_c-N_f)}{(2N_c^2-1)N_f-3N_c^3} \ . \tag{6.25}$$

As N_f decreases below its maximal value, $3N_c$, $\gamma_{IR,SI}$ increases monotonically. We list values of $\gamma_{IR,SI}$ in Table III for the illustrative values $2 \leq N_c \leq 5$. For each N_c,

TABLE III: Values of the anomalous dimension γ_m in the SU(N_c) supersymmetric gauge theory with N_f copies of massless chiral superfields Φ_i, $\tilde{\Phi}_i$ in the F and $\bar{F}$ representations, calculated to the n-loop order in perturbation theory and evaluated at the IR zero of the beta function calculated to this order, $\alpha_{IR,n\ell}$, for $\ell = 2, 3$. We denote these as $\gamma_{IR,n\ell} \equiv \gamma_{n\ell}(\alpha_{IR,n\ell})$. Results are given for the illustrative values $2 \le N_c \le 5$. For sufficiently small $N_f > N_{f,b2z}$ for each N_c, $\alpha_{IR,2\ell}$ is so large that the formal values of $\gamma_{IR,2\ell}$ and/or $\gamma_{IR,3\ell}$ are either larger than unity or negative and hence are unphysical. We indicate this by placing these values in parentheses.

N_c	N_f	$\gamma_{IR,SI}$	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$
2	4	(1.500)	(1.875)	(−1.68)
2	5	0.273	0.260	0.0802
3	6	(1.14)	(1.22)	(−0.730)
3	7	0.421	0.399	0.0584
3	8	0.145	0.139	0.104
4	8	(1.07)	(1.11)	(−0.546)
4	9	0.517	0.490	0.0219
4	10	0.254	0.239	0.127
4	11	0.101	0.0970	0.0835
5	10	(1.04)	(1.07)	(−0.475)
5	11	0.585	0.557	(−0.0135)
5	12	0.338	0.317	0.120
5	13	0.183	0.173	0.121
5	14	0.0772	0.0748	0.0680

we omit values in the lower range of N_f that strongly violate the bound (4.4). This violation is due to both the inexactness of the perturbative calculation of γ_m and the fact that the two-loop IR zero of the beta function, $\alpha_{IR,2\ell}$ gets arbitrarily large as N_f decreases toward $N_{f,b2z}$. In Figs. 1-3 we show plots of $\gamma_{IR,SI}$ as a function of N_f for the approximate respective subintervals of the conformal phase where $\gamma_{IR,SI}$ satisfies the upper bound (4.4). We will discuss below the other curves on these plots.

Specializing Eq. (5.1) to the case of the fundamental representation, we obtain the scheme-independent perturbative estimate of the lower boundary of the conformal phase,

$$N_{f,cr,SI} = \frac{3N_c(2N_c^2-1)}{3N_c^2-2} \ . \qquad (6.26)$$

As noted before, this estimate lies above the actual exact lower boundary, which, in the present case, occurs at $N_{f,cr} = (3/2)N_c$:

$$\frac{N_{f,cr,SI}}{N_{f,cr}} = \frac{2(2N_c^2-1)}{3N_c^2-2} \ . \qquad (6.27)$$

This ratio decreases from the value $7/5 = 1.40$ at $N_c = 2$ to $4/3$ as $N_c \to \infty$ and has the Taylor series expansion

$$\frac{N_{f,cr,SI}}{N_{f,cr}} = \frac{4}{3} + \frac{2}{9N_c^2} + O\left(\frac{1}{N_c^4}\right) \quad \text{as} \quad N_c \to \infty \ . \quad (6.28)$$

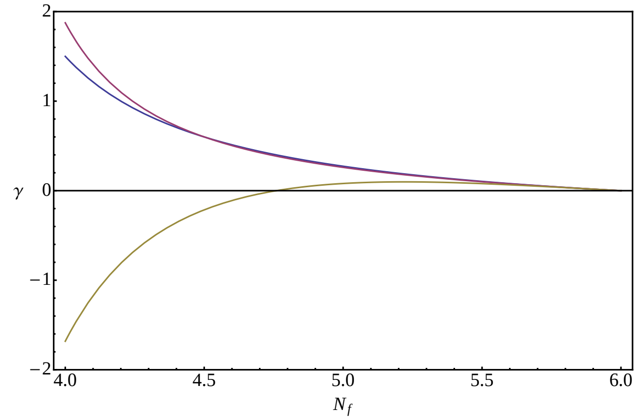

FIG. 1: Plot of the n-loop fermion anomalous dimension, $\gamma_{n\ell}$, evaluated at the respective n-loop value of the IR zero of β, $\alpha_{IR,n\ell}$, and denoted as $\gamma_{IR,n\ell}$, for two-loop and three-loop order, in the case of N_f chiral superfields Φ_i, $\tilde{\Phi}_i$ in the F and $\bar{F}$ representation of SU(N_c) for $N_c = 2$. We use the generic label γ for the vertical axis. At the lower end of the plot, from top to bottom, the curves are for (i) $\gamma_{IR,2\ell}$, (ii) $\gamma_{IR,SI}$, and (iii) $\gamma_{IR,3\ell}$. The curves involve an implicit analytic continuation of N_f from integer values to real values; of course, only the integer values are physical. We only show the region in N_f where $\gamma_{IR,2\ell}$ and $\gamma_{IR,SI}$ approximately satisfy the upper bound (4.4).

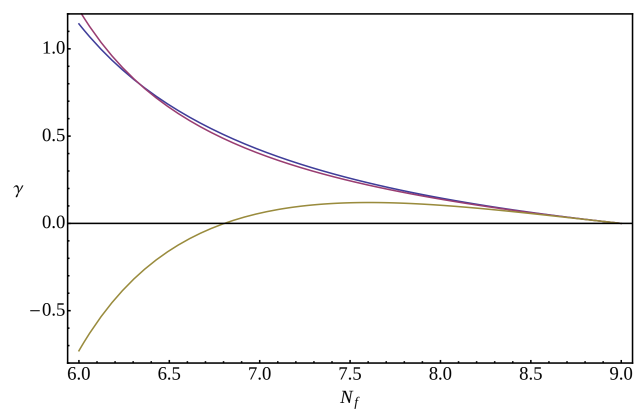

FIG. 2: Same as Fig. 1 for $N_c = 3$.

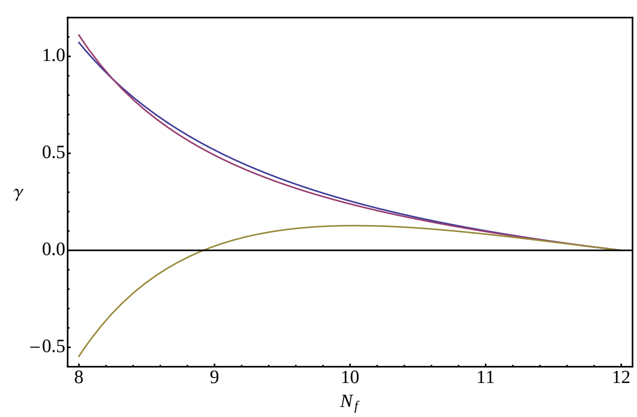

FIG. 3: Same as Fig. 1 for $N_c = 4$.

We next proceed to two-loop and three-loop analyses. As we have remarked above, these have the advantage of using the same n-loop orders in calculating γ_m and $\alpha_{IR,n\ell}$, but are subject to the standard caution that they involve scheme-dependence.

3. Two-Loop Analysis

Evaluating γ_m calculated to two-loop order at the zero of the beta function calculated to the same order for $R = F$, we obtain

$$\gamma_{IR,2\ell} = \frac{(N_c^2-1)(3N_c-N_f)(N_f-N_c)(N_cN_f-3)}{2(-3N_c^3+2N_c^2N_f-N_f)^2} \ . \tag{6.29}$$

We list values of $\gamma_{IR,2\ell}$ in Table III and plot curves of $\gamma_{IR,2\ell}$ as a function of N_f (analytically continued from integer to real values) in Figs. 1-3. One sees from these results that over the range where the calculations are reliable, $\gamma_{IR,2\ell}$ is rather close to $\gamma_{IR,SI}$. The anomalous dimension $\gamma_{IR,2\ell}$ increases monotonically as N_f decreases from $N_{f,max} = 3N_c$ in the interval (2.15). As N_f decreases toward $N_{f,b2z}$ (given in Eq. (6.5), $\alpha_{IR,2\ell}$ gets arbitrarily large, and this perturbative expression obviously ceases to be reliable. As N_f increases toward $N_{f,max,fund} = 3N_c$, $\alpha_{IR,2\ell} \to 0$, and hence also $\gamma_{IR,2\ell} \to 0$.

The condition that $\gamma_{IR,2\ell} = 1$ is a cubic equation in N_f, namely

$$N_c(N_c^2-1)N_f^3 + (4N_c^4-7N_c^2+5)N_f^2 + 3N_c(-7N_c^4+7N_c^2-4)N_f + 9N_c^2(2N_c^4-N_c^2+1) = 0 \ . \tag{6.30}$$

As N_f decreases from $N_{f,max,fund} = 3N_c$ toward $N_{f,b2z,fund}$, γ_m exceeds the rigorous upper bound $\gamma_m < 1$ at a value of N_f which is given as the relevant one among the three roots of Eq. (6.30). For example, for $N_c = 2$, this equation has the physical root $N_f = 4.242$, together with two other roots which are not relevant, namely $N_f = 2.929$ and $N_f = -14.00$ (to four significant figures). The first of these other roots is irrelevant since it is below the lower boundary of the IR conformal phase, $N_{f,cr}$ given in Eq. (6.3), and the second is irrelevant since it is negative and hence unphysical. Thus, insofar as one can compare a perturbative 2-loop calculation of γ_m with an upper bound on the exact γ_m, one finds that this 2-loop prediction for the lower end of the conformal phase in N_f for $N_c = 2$ is $N_f = 4.242$. The ratio of this to the exact result $N_{f,cr}$, which is 3 for $N_c = 2$, is approximately 1.414. Similarly, for $N_c = 3$, the physical root of Eq. (6.30) is $N_f = 6.150$. (The other two roots are 3.983, and -21.22, which are again irrelevant). The ratio of this to the (formal, half-integral) exact result $N_{f,cr} = 9/2$ is 1.367. As $N_c \to \infty$, we find that this ratio of the physical root of the cubic for $N_{f,cr}$ divided by the exact value $N_{f,cr} = (3/2)N_c$ approaches the same value, 4/3, as was true of the ratio of $N_{f,cr,SI}$ divided by this exact value, as given in Eqs. (6.27) and (6.28) above:

$$\frac{N_{f,cr,\gamma_{IR,2\ell}}}{N_{f,cr}} = \frac{4}{3} + O\left(\frac{1}{N_c^2}\right) \ . \tag{6.31}$$

We thus find that a 2-loop perturbative analysis of β and γ_m overestimates the value of $N_{f,cr}$ somewhat and hence underestimates the size of the conformal phase in N_f for this case (6.1). This is qualitatively the same as we found for the estimate of $N_{f,cr}$ using $\gamma_{IR,SI}$.

It is interesting to compare these estimates for $N_{f,cr}$ from the scheme-independent $\gamma_{IR,SI}$ and the two-loop $\gamma_{IR,2\ell}$ with the results obtained via the different method of equating $\alpha_{IR,2\ell}$ with the critical value α_{cr} for dynamical mass generation calculated from the Dyson-Schwinger equation for the fermion propagator in Ref. [16]. This mass generation, associated with fermion condensation, was found to occur at [16]

$$N_{f,cr,DS} = \frac{3N_c(3N_c^2-2)}{4N_c^2-3} \ . \tag{6.32}$$

The ratio of this to the exact value for $N_{f,cr}$ in Eq. (6.3) is

$$\frac{N_{f,cr,DS}}{N_{f,cr}} = \frac{3}{2}\left(\frac{1-(2/3)N_c^{-2}}{1-(3/4)N_c^{-2}}\right) \ , \tag{6.33}$$

This ratio is equal to 1.54 and 1.515 for $N_c = 2$ and $N_c = 3$ and decreases monotonically to 3/2 as $N_c \to \infty$. Thus, for a given N_c, our estimates of $N_{f,cr}$ obtained from equating $\gamma_{IR,SI}$ and $\gamma_{IR,2\ell}$ to the upper bound of unity are slightly closer to the exact result (6.3) than the estimate from the analysis of the Dyson-Schwinger equation, and all agree qualitatively, i.e., all are slight overestimates of $N_{f,cr}$.

In the large-N_c, large-N_f limit, with $N_f = rN_c$, $\gamma_{IR,2\ell}$ has the Taylor series expansion

$$\gamma_{IR,2\ell} = \frac{r(r-1)(3-r)}{2(3-2r)^2} - \frac{3(r-1)(3-r)^2}{2(3-2r)^3N_c^2} + O\left(\frac{1}{N_c^4}\right) \ . \tag{6.34}$$

In this limit, the value of r, and hence N_f, where $\gamma_{IR,2\ell} = 1$ is given as the relevant one among the three roots of the cubic equation $(r-2)(r^2-6r-9) = 0$, namely $r = 2$. The other two roots are $-3+3\sqrt{2}$ and $-3-3\sqrt{2}$; the first of these has the value 1.243 and hence is below the value $r = 3/2$ (cf. Eq. (6.5)) where, for increasing r, an IR zero of the beta function first appears (which coincides with the exact value of the lower boundary of the IR conformal phase in this limit), and the second is negative and hence obviously unphysical. Thus, in this large-N_c limit, with $r = N_f/N_c$ fixed and finite, the 2-loop analysis predicts that $N_{f,cr} = 2N_c$, which is 4/3 times the exact result of Eq. (6.3), in agreement with our analysis in Eq. (6.31).

4. Three-Loop Analysis

In the same manner, we evaluate the three-loop expression for γ_m at the 3-loop value of the IR zero of the beta function $\alpha_{IR,3\ell}$. Since the analytic formulas are somewhat complicated, we will restrict ourselves to giving numerical results for $N_c = 2$ through $N_c = 4$ in Table III and Figs. 1-3. In contrast to $\gamma_{IR,SI}$ and $\gamma_{IR,2\ell}$, we find that $\gamma_{IR,3\ell}$ does not increase monotonically as N_f decreases below $N_{f,max}$. Instead, it reaches a maximum well below unity in the interior of the conformal phase and then decreases, vanishing and becoming negative. Because of this behavior, we cannot use our procedure of setting the perturbative expression for γ_m equal to 1 and then solving for N_f for $\gamma_{IR,3\ell}$. Since this behavior of $\gamma_{IR,3\ell}$ clearly differs from the behavior of the scheme-independent $\gamma_{IR,SI}$, it may reflect scheme-dependence. It also shows that as N_f decreases toward the lower end of the IR conformal phase and α_{IR} increases to larger values, a perturbative calculation becomes less reliable. In general, a necessary condition for these perturbative calculations to be reliable is that inclusion of the next higher-loop order term should not drastically change the qualitative behavior. Imposing this condition for the comparison of $\gamma_{IR,2\ell}$ and $\gamma_{IR,3\ell}$, we may obtain an estimate of the interval in N_f where the calculation could be reasonably reliable. For $N_c = 2$, we find that this interval plausibly includes $N_f = 5$ but does not include $N_f = 4$. For $N_c = 3$, this interval includes $N_f = 8$ but not lower values of N_f. In view of this behavior of $\gamma_{IR,3\ell}$, one must view the three-loop results with appropriate caution, recognizing that perturbative calculations become less reliable as the coupling becomes stronger.

VII. SUPERFIELDS IN THE ADJOINT REPRESENTATION

A. Beta Function

The adjoint representation is self-conjugate, so here a theory with N_f copies of a massless chiral superfield content consisting of Φ_i and $\tilde{\Phi}_i$, $i = 1, ..., N_f$, is equivalent to a theory with $N'_f = 2N_f$ copies of Φ_i. We shall thus consider half-integral values of N_f as physical here. The beta function coefficients are

$$b_1 = N_c(3 - 2N_f) \ , \tag{7.1}$$

$$b_2 = -6N_c^2(2N_f - 1) \tag{7.2}$$

and, in the $\overline{DR}$ scheme,

$$b_3 = -7N_c^3(2N_f - 1)(3 - 2N_f) \ . \tag{7.3}$$

Note that b_3 vanishes at the same (formal, non-integral) value of N_f at which b_1 vanishes, namely $N_f = 3/2$. The condition that the theory be asymptotically free, i.e., that $b_1 > 0$, implies the upper bound $N_f < 3/2$.

$$N_f < \frac{3}{2} = N_{f,max} \tag{7.4}$$

For the Φ_i, $\tilde{\Phi}_i$ content, this only allows the choice $N_f = 1$, while for the reduced content consisting only of Φ_i, this allows $N'_f = 1$ and $N'_f = 2$. (Note that the $N'_f = 1$ theory has been solved exactly [38].)

From Eq. (4.5), the lower boundary of the IR conformal phase is given formally by

$$N_{f,cr} = \frac{3}{4} \tag{7.5}$$

or equivalently, $N'_{f,cr} = 3/2$. Since neither of these is an integer, they must be regarded only as quantities defined via a requisite analytic continuation of the theory in N_f or N'_f away from the integers to the real numbers. With this understanding, the IR conformal phase is thus given by

$$\frac{3}{4} < N_f < \frac{3}{2} \ . \tag{7.6}$$

Hence, with the Φ_i, $\tilde{\Phi}_i$ superfield content, the only integer value of N_f allowed by the requirement of asymptotic freedom, namely $N_f = 1$ yields an IR conformal phase. For the theory with just the Φ_i superfield, Eq. (7.6) reads $3/2 < N'_f < 3$, so for $N'_f = 2$ ($N'_f = 1$) the theory evolves into the infrared in a conformal (nonconformal) manner, respectively.

In the theory with Φ, $\tilde{\Phi}$ superfield content, the two-loop β function coefficient b_2 is negative for the only relevant value of N_f, namely $N_f = 1$. In the reduced theory, $b_2 = -6N_c^2(N'_f - 1)$, which is zero for $N'_f = 1$ and negative for $N'_f = 2$. Thus, at the two-loop level, the IR zero of the β functions occurs at

$$\alpha_{IR,2\ell} = \frac{2\pi(3 - 2N_f)}{3N_c(2N_f - 1)} \ . \tag{7.7}$$

At the three-loop level, β has two zeros away from the origin, at

$$\frac{\alpha}{4\pi} = \frac{-3 \pm \sqrt{\frac{2(14N_f^2 - 33N_f + 27)}{2N_f - 1}}}{7N_c(3 - 2N_f)} \ . \tag{7.8}$$

The $-$ sign choice yields an unphysical, negative result, so the physical three-loop IR zero of the beta function is given by Eq. (7.8) with the $+$ sign. We denote this as $\alpha_{IR,3\ell}$ (suppressing the Adj for simplicity). This $\alpha_{IR,3\ell}$ exhibits unphysical behavior, vanishing at $N_f = 9/8$ and becoming negative in the range $9/8 < N_f < N_{f,max} = 3/2$. One could take the point of view that this precludes a reliable three-loop perturbative analysis of this case. However, we will at least give results for the one case for which the theory has an IR fixed point, namely $N_f = 1$. For $N_f = 1$, we have

$$\alpha_{IR,2\ell,N_f=1} = \frac{2\pi}{3N_c} \tag{7.9}$$

and

$$\alpha_{IR,3\ell,N_f=1} = \frac{4\pi}{7N_c} \ , \tag{7.10}$$

so that, for this value of N_f,

$$\frac{\alpha_{IR,3\ell}}{\alpha_{IR,2\ell}} = \frac{6}{7} \ , \tag{7.11}$$

independent of N_c. This is the same trend that we found for the case of matter superfields in the $F + \bar{F}$ representation, i.e., the value of the IR fixed point calculated to three-loop order is somewhat smaller than the value calculated to two-loop order.

B. Anomalous Dimension

For this theory with chiral superfields in the adjoint representation, the coefficients in Eq. (3.1) are

$$c_1 = 4N_c \ , \tag{7.12}$$

$$c_2 = -4N_c^2(2N_f - 1) \ , \tag{7.13}$$

and

$$c_3 = -8N_c^3(N_f + 4)(2N_f - 1) \ . \tag{7.14}$$

From our general result (3.10), we calculate the scheme-independent anomalous dimension

$$\gamma_{IR,SI} = \frac{2(3 - 2N_f)}{3(2N_f - 1)} \ . \tag{7.15}$$

This increases monotonically as N_f decreases from its maximal to its minimal value in the IR conformal phase. $\gamma_{IR,SI}$ increases through its upper limit of unity as N_f decreases through the value $N_{f,cr,SI}$. Evaluating our general formula in Eq. (5.1) for the present case of the adjoint representation, we obtain

$$N_{f,cr,SI} = \frac{9}{10} \ . \tag{7.16}$$

As is true for general R, this is larger than the exact result, which in the present case is $N_{f,cr} = 3/4$. (This exact value is only formal, since it is non-integral.) The ratio (5.3) here is $6/5 = 1.2$. As N_f decreases from 9/10 to $N_{f,cr} = 3/4$, $\gamma_{IR,SI}$ increases from 1 to 2, exhibiting unphysical behavior.

Evaluating Eq. (3.6) for the present case of the adjoint representation, we find

$$\gamma_{IR,2\ell} = \frac{(3 - 2N_f)(2N_f + 3)}{9(2N_f - 1)} \ . \tag{7.17}$$

Setting this equal to 1, we derive another perturbative estimate of $N_{f,cr}$, which is the positive root of a quadratic equation,

$$N_f = \frac{3(-3 + \sqrt{17}\ )}{4} \simeq 0.8423 \ . \tag{7.18}$$

(The other root is negative). This is again slightly larger than the formal exact $N_{f,cr} = 3/4$.

For the only physical value where there is an IR fixed point, $N_f = 1$, we thus have

$$\begin{aligned}
\gamma_{IR,SI} &= \frac{2}{3} = 0.6666... \\
\gamma_{IR,2\ell} &= \frac{5}{9} = 0.5555... \\
\gamma_{IR,3\ell} &= \frac{2^7}{7^3} = 0.37317... \quad \text{for} \quad N_f = 1 \ .
\end{aligned} \tag{7.19}$$

Again, we find the same trend as for Φ_i, $\tilde{\Phi}_i$ in $F + \bar{F}$, namely that the value of the anomalous dimension γ_m evaluated at the IR fixed point decreases somewhat when one goes from two-loop order (or the scheme-independent result) to three-loop order.

VIII. CHIRAL SUPERFIELDS IN THE SYMMETRIC OR ANTISYMMETRIC RANK-2 TENSOR REPRESENTATION

In this section we analyze the UV to IR evolution of the supersymmetric SU(N_c) theory with Φ_i, $\tilde{\Phi}_i$ in the R and $\bar{R}$ representation, where R is a symmetric or antisymmetric rank-2 tensor representation, denoted S_2, A_2, respectively. Since many formulas are closely related to each other, it is convenient to treat these two cases together, as the T_2 representation. In each of the combined formulas involving a $\pm$ or $\mp$ sign, the upper and lower signs apply to the S_2 and A_2 representations, respectively. For $N_c = 2$, the S_2 representation is the adjoint representation, which has already been discussed. Thus, for the S_2 representation, the distinct cases begin with $N_c \geq 3$. For the A_2 case, N_c is implicitly taken to be $N_c \geq 3$, since this representation is the singlet if $N_c = 2$. Further, note that the A_2 representation with $N_c = 3$ is equivalent to the conjugate fundamental representation so, with our vectorlike content of chiral superfields, this reduces to the case Φ_i, $\tilde{\Phi}_i$ in the $F + \bar{F}$ representation already covered above. Thus, the A_2 cases that are distinct have $N_c \geq 4$. We focus here on $\gamma_{IR,SI}$ and $\gamma_{IR,2\ell}$.

A. β Function

We have

$$b_1 = 3N_c - (N_c \pm 2)N_f \ , \tag{8.1}$$

and

$$b_2 = 2\Big[3N_c^2(1-N_f) \mp 8(N_c - N_c^{-1})N_f\Big] \ . \qquad (8.2)$$

The expression for b_3 is similarly obtained in a straightforward manner from the general result (2.5). The requirement of asymptotic freedom requires $b_1 > 0$, i.e.,

$$N_f < N_{f,b1z} = N_{f,max} = \frac{3N_c}{N_c \pm 2} \ . \qquad (8.3)$$

For the S_2 representation, $N_{f,max}$ increases monotonically from 3/2 for $N_c = 2$ to 3 as $N_c \to \infty$, while for the A_2 representation, $N_{f,max}$ decreases monotonically from 9 for $N_c = 3$ to 3 as $N_c \to \infty$.

The exact result for the lower boundary of the IR conformal phase is

$$N_{f,cr} = \frac{N_{f,max}}{2} = \frac{3N_c}{2(N_c \pm 2)} \ . \qquad (8.4)$$

For the S_2 representation, $N_{f,cr}$ increases monotonically from 3/4 for $N_c = 2$ to 3/2 as $N_c \to \infty$, while for the A_2 representation, $N_{f,cr}$ decreases monotonically from 9/2 for $N_c = 3$ to 3/2 as $N_c \to \infty$. Thus, the IR conformal phase exists for

$$\frac{3N_c}{2(N_c \pm 2)} < N_f < \frac{3N_c}{N_c \pm 2} \ . \qquad (8.5)$$

The coefficient $b_2 = 0$ for

$$N_f = N_{f,b2z} = \frac{3N_c^2}{3N_c^2 \pm 8(N_c - N_c^{-1})} \ . \qquad (8.6)$$

This is always smaller than $N_{f,cr}$ for the S_2 representation, so that b_2 has fixed (negative) sign in the IR conformal phase in this case. For the A_2 representation, if $N_f < 1 + \sqrt{5} = 3.236..$, then $N_{f,b2z} > N_{f,cr}$, while if $3.22 < N_f < N_{f,max}$, then $N_{f,b2z} < N_{f,cr}$. Hence, the only physical case where $N_{b2z} > N_{f,cr}$ is for the integer value $N_c = 3$, where the $A_2 + \bar{A}_2$ representation is equivalent to the $F + \bar{F}$ representation.

At the two-loop level, the IR zero of β occurs at $a_{IR,2\ell} = -b_1/b2$, i.e.,

$$\alpha_{IR,2\ell} = \frac{2\pi[3N_c - (N_c \pm 2)N_f]}{3N_c^2(N_f - 1) \pm 8(N_c - N_c^{-1})N_f} \ . \qquad (8.7)$$

B. γ_m

For this T_2 case,

$$c_1 = \frac{4(N_c \pm 2)(N_c \mp 1)}{N_c} \ , \qquad (8.8)$$

and

$$c_2 = \frac{4(N_c \pm 2)(N_c \mp 1)[N_c^2 \mp 2N_c - 4 + N_c(N_c \pm 2)N_f]}{N_c^2} \ . \qquad (8.9)$$

The expression for c_3 is similarly obtained from the general result (3.4).

Hence,

$$\gamma_{IR,SI} = \frac{2(N_c \pm 2)(N_c \mp 1)[3N_c - (N_c \pm 2)N_f]}{3N_c^3(N_f - 1) \pm 8(N_c^2 - 1)N_c} \ . \qquad (8.10)$$

and

$$\gamma_{IR,2\ell} = \frac{(N_c \pm 2)(N_c \mp 1)[3N_c - (N_c \pm 2)N_f][N_c^2(3 + N_f) \pm 2N_c(3 + N_f) - 12][N_c(N_f - 1) \pm 2N_f]}{[3N_c^3(N_f - 1) \pm 8(N_c^2 - 1)N_f]^2} \ . \qquad (8.11)$$

One perturbative estimate of $N_{f,cr}$ is obtained by setting $\gamma_{IR,SI} = 1$ and solving for N_f. This gives

$$N_{f,cr,SI} = \frac{3N_c(3N_c^2 \pm 2N_c - 4)}{(N_c \pm 2)(5N_c^2 \pm 4N_c - 8)} \ . \qquad (8.12)$$

Comparing these with the respective exact expressions for $N_{f,cr}$ for S_2 and A_2, we find

$$N_{f,cr,SI} - N_{f,cr} = \frac{3N_c^3}{(N_c \pm 2)(5N_c^2 \pm 4N_c - 8)} \ . \qquad (8.13)$$

This difference is positive for all N_c for both the S_2 and A_2 cases. Thus, as with the fundamental and adjoint representations, for these rank-2 tensor representations, this perturbative approach overestimates $N_{f,cr}$ and hence underestimates the size of the IR conformal phase. A second perturbative estimate of $N_{f,cr}$ is obtained by setting $\gamma_{IR,2\ell} = 1$ and solving for N_f . The condition $\gamma_{IR,2\ell} = 1$ is a cubic equation in N_f, from which we extract the physically relevant root. This second method yields estimates of $N_{f,cr}$ that are qualitatively similar to those obtained with the first method with $\gamma_{IR,SI}$. This qualitative agreement between these two perturbative methods for these rank-2 tensor representations is the same as what we found for the fundamental and adjoint representations.

IX. DISCUSSION AND COMPARISON WITH NON-SUPERSYMMETRIC SU(N_c) GAUGE THEORY

From our calculations on an SU(N_c) gauge theory with $\mathcal{N}=1$ supersymmetry in this paper, we have found several general results. Our most detailed analyses here were for the cases $R=F$ and $R=Adj$, with briefer studies of the S_2 and A_2 cases. It is useful to compare our results with what we found in [18] (see also [19], whose results were in agreement with those in [18]) for a non-supersymmetric SU(N_c) gauge theory with N_f copies of massless fermions in various representations. We believe that our findings for the supersymmetric gauge theory, besides being of interest in their own right, provide further insight into the results that we obtained previously for the non-supersymmetric theory.

First, for the IR zero of β, we find that $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$. This is the same type of shift that we showed earlier for the non-supersymmetric theory with the same R and suggests that the lowest-order (two-loop) perturbative calculation of the IR fixed point gives a larger value than the true value. Second, we find that when one goes from the two-loop anomalous dimension evaluated at the two-loop IR zero of β, $\gamma_{IR,2\ell}$ or the scheme-independent $\gamma_{IR,SI}$, to the three-loop result $\gamma_{IR,3\ell}$, the value decreases. Again, this is the same trend that we found for the corresponding non-supersymmetric theory in [18]. Thus, as with the IR zero, this suggests that for both the non-supersymmetric and the supersymmetric theory with corresponding matter field representation content, the lowest-order calculation of the value of γ_m at the IR fixed point gives a larger value than the true value. The exact value of $N_{f,cr}$ is not known for the non-supersymmetric theory, and an intensive research program has been underway for several years, especially using lattice measurements, to determine $N_{f,cr}$ for a given N_c and R. Here we have taken advantage of the fact that $N_{f,cr}$ is known exactly (at least with the level or rigor that is usual in physics) for the supersymmetric SU(N_c) theory. We have used one method for obtaining a perturbative estimate of $N_{f,cr}$ here, namely to set $\gamma_m=1$ and solve for the value of N_f where this occurs. With this method, we have found that, for a given N_c and matter superfield content, the perturbative calculation yields a slight overestimate of $N_{f,cr}$ as compared with the exactly known value. This result agrees with and complements the different analysis in Ref. [16], which was based on an analysis of an approximate solution to the Dyson-Schwinger equation for the fermion matter field propagator. Our calculation of γ_m at the IR zero of the beta function provides some insight into this. Since, at least at the scheme-independent and two-loop level, γ_m increases throughout the IR conformal phase as N_f decreases from $N_{f,max}$, and since the lowest-order calculations yield a larger value of α_{IR} and γ_m than the true value, it would follow that setting $\gamma_m=1$ to determine the lower boundary of the IR conformal phase would yield a value of $N_{f,cr}$ that is larger than the true value. One must, however, add the caveat that at the four-loop level for the non-supersymmetric theory and at the three-loop level for the corresponding supersymmetric theory, we have found that γ_m does not increase monotonically as N_f decreases from $N_{f,max}$, which complicates the intepretation of the results. The deviation of $\gamma_{IR,4\ell}$ from $\gamma_{IR,3\ell}$ in the non-supersymmetric theory was relatively small throughout much of the N_f interval of interest, but here we have found that the deviation of $\gamma_{IR,3\ell}$ from $\gamma_{IR,2\ell}$ (or $\gamma_{IR,SI}$) is significant in the N_f region of interest, which limits what one can infer from calculations of $\gamma_{IR,3\ell}$.

X. CONCLUSIONS

In this paper we have considered an asymptotically free vectorial SU(N_c) gauge theory with $\mathcal{N}=1$ supersymmetry and N_f pairs of chiral superfields Φ_i, $\tilde{\Phi}_i$, $i=1,...,N_f$, transforming according to the representations R and $\bar{R}$, respectively, where R includes fundamental, adjoint, and symmetric and antisymmetric rank-2tensor representations. We have studied the evolution of this theory from the ultraviolet to the infrared, taking account of higher-loop corrections to the β function and the anomalous dimension γ_m. We have compared the results obtained from the two- and three-loop calculations of the beta function and anomalous dimension (in the $\overline{DR}$ scheme) with exact results. In particular, we have calculated perturbative estimates for the lower boundary of the conformal phase, $N_{f,cr}$, by setting the scheme-independent and two-loop perturbative expressions for γ_m equal to the rigorous upper bound (4.4), taken to be saturated at $N_{f,cr}$. We find that this perturbative calculation somewhat overestimates $N_{f,cr}$ as compared with the exact results, and thus underestimates the size of the IR conformal phase. Keeping in mind the caution that perturbative calculations become less reliable as the infrared fixed point α_{IR} gets larger, our results provide a measure of how closely perturbative calculations reproduce exact results for these theories.

Acknowledgments

This research was partially supported by the *Sapere Aude* Award (TA) and the grant NSF-PHY-09-69739 (RS).

[1] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973); H. D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973); G. 't Hooft, unpublished.

[2] W. E. Caswell, Phys. Rev. Lett. **33**, 244 (1974); D. R. T. Jones, Nucl. Phys. B **75**, 531 (1974).

[3] T. Banks and A. Zaks, Nucl. Phys. B **196**, 189 (1982).

[4] T. Appelquist, D. Karabali, and L. C. R. Wijewardhana, Phys. Rev. Lett. **57**, 957 (1986); T. Appelquist and L. C. R. Wijewardhana, Phys. Rev. D **35**, 774 (1987); Phys. Rev. D **36**, 568 (1987); T. Appelquist, J. Terning, and L. C. R. Wijewardhana, Phys. Rev. Lett. **77**, 1214 (1996); T. Appelquist, K. Lane, and U. Mahanta, Phys. Rev. Lett. **61**, 1553 (1998).

[5] Besides [4], early discussions of walking and a possible large γ_m include B. Holdom, Phys. Lett. B **150**, 301 (1985); K. Yamawaki, M. Bando, and K. Matumoto, Phys. Rev. Lett. **56**, 1335 (1986).

[6] For recent reviews and references to the considerable literature, see, e.g., T. Appelquist, Int. J. Mod. Phys. A **25**, 5114 (2010); E. Neil, in Lattice 2011, https://latt11.llnl.gov and talks at the workshop Lattice Meets Experiment 2011: Beyond the Standard Model, Fermilab, http://lqcd.fnal.gov/ eneil/lat-exp-2011.

[7] F. Sannino, Acta Phys. Polon. B **40**, 3533 (2009).

[8] N. Seiberg, Phys. Rev. D **49**. 6857 (1994); Nucl. Phys. B **435**, 129 (1995) [hep-th/9411149].

[9] K. A. Intriligator and N. Seiberg, Nucl. Phys. B **444**, 125 (1995) [hep-th/9503179].

[10] K. A. Intriligator, P. Pouliot, Phys. Lett. B **353**, 471 (1995) [hep-th/9505006].

[11] Two reviews are M. Shifman, hep-th/9704114 and J. Terning, *Modern Supersymmetry: Dynamics and Duality* (Oxford Clarendon Press, Oxford, UK, 2006).

[12] T. A. Ryttov, F. Sannino, Phys. Rev. D **76**, 105004 (2007) [arXiv:0707.3166]. For a different approach, see M. Frandsen, T. Pickup, and M. Teper, Phys. Lett. B **695**, 231 (2011).

[13] G. Mack, Commun. Math. Phys. **55**, 1 (1977).

[14] M. Flato and C. Fronsdal, Lett. Math. Phys. **8**, 159 (1984).

[15] V. K. Dobrev and V. B. Petkova, Phys. Lett. B **162**, 127 (1985).

[16] T. Appelquist, A. Nyffeler, and S. B. Selipsky, Phys. Lett. B **425**, 300 (1998).

[17] We recall that the lack of analyticity at $\alpha = 0$ is clear, for example, from the fact that if the expressions were analytic at $\alpha = 0$, then one could analytically continue them from positive to negative α, but this would lead to an imaginary gauge coupling, which would render the theory non-unitary.

[18] T. A. Ryttov and R. Shrock, Phys. Rev. D **83**, 056011 (2011).

[19] C. Pica and F. Sannino, Phys. Rev. D **83**, 035013 (2011).

[20] D. R. T. Jones, Nucl. Phys. B **87**, 127 (1975).

[21] M. Machacek and M. Vaughn, Nucl. Phys. B **222**, 83 (1983).

[22] A. J. Parkes and P. C. West, Phys. Lett. B **138**, 99 (1984); Nucl. Phys. B **256**, 340 (1985).

[23] D. R. T. Jones and L. Mezincescu, Phys. Lett. B **136**, 242 (1984); Phys. Lett. B **138**, 293 (1984).

[24] W. Siegel, Phys. Lett. B **84**, 193 (1979); W. Siegel, Phys. Lett. B **94**, 37 (1980); D. M. Capper, D. R. T. Jones, and P. van Nieuwenhuizen, Nucl Phys. B **167**, 479 (1980).

[25] S. J. Gates, Jr.. M. T. Grisaru, M. Roček, and W. Siegel, *Superspace, or 1001 Lessons in Supersymmetry* (Benjamin, New York, 1983); online copy: arXiv:hep-th/0108200.

[26] W. Stöckinger, JHEP 0503, 076 (2005).

[27] I. Jack, D. R. T. Jones, and C. G. North, Phys. Lett. B **386**, 138 (1996).

[28] I. Jack, D. R. T. Jones, and C. G. North, Nucl. Phys. B **473**, 308 (1996).

[29] I. Jack, D. R. T. Jones, and C. G. North, Nucl. Phys. B **486**, 479 (1996).

[30] A. G. M. Pickering, J. A. Gracey, and D. R. T. Jones, Phys. Lett. B **510**, 347 (2001).

[31] R. V. Harlander, D. R. T. Jones, P. Kant, L. Mihaila, and M. Steinhauser, JHEP 0612, 024 (2006).

[32] R. Harlander, L. Mihaila, and M. Steinhauser (HMS), Eur. Phys. J. C **63**, 383 (2009). Since HMS define β function as $-\alpha^2 \sum_{n=0}^{\infty} \beta_n (\alpha/\pi)^n$, while we define ours as in Eq. (2.2), the respective coefficients are related according to $b_\ell = 4^\ell \beta_{\ell-1}$ for $\ell \ge 1$. Similarly, since HMS define $\gamma_m = (1/2) d \ln m/dt = -\sum_{n=0}^{\infty} \gamma_n (\alpha/\pi)^n$, our coefficients c_ℓ in Eq. (3.1) are related to the coefficients γ_n according to $c_\ell = 2(4^\ell)\gamma_{\ell-1}$.

[33] S. Bethke, Eur. Phys. J. C **64**, 689 (2009).

[34] Novikov, Shifman, Vainshtein, and Zakharov, Nucl. Phys. B **229**, 381 (1983); Phys. Lett. B **166**, 329 (1986); Nucl. Phys. B **277**, 426 (1986).

[35] We note in passing that the form (4.1) motivated a related ansatz for non-supersymmetric gauge theories in T. A. Ryttov and F. Sannino, Phys. Rev. D **78**, 065001 (2008). See also C. Pica and F. Sannino, Phys. Rev. D **83**, 116001 (2011).

[36] Here and elsewhere, when expressions are given for N_f that evaluate to non-integral real values, it is understood that they are formal and are to be interpreted via an analytic continuation of N_f from physical integer values to real values.

[37] S. J. Brodsky and G. F. de Téramond, in M. Harada, M. Tanabashi, and K. Yamawaki, eds., *The Origin of Mass and Strong Coupling Gauge Theories, SCGT2006* (World Scientific, Singapore, 2008), p. 31; S. J. Brodsky and R. Shrock, Phys. Lett. B **666**, 95 (2008); S. J. Brodsky, G. F. de Téramond, and A. Deur, Phys. Rev. D **81**, 096010 (2010). +

[38] N. Seiberg and E. Witten, Nucl. Phys. B **426**, 19 (1994); Err. ibid. B **430**, 485 (1994); N. Seiberg and E. Witten, Nucl. Phys. B **431**, 484 (1994).