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Confinement in a Higgs Model on $R^3 \times S^1$

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We determine the phase structure of an $SU(2)$ gauge theory with an adjoint scalar on $R^3 \times S^1$ using semiclassical methods. There are two global symmetries: a $Z(2)_H$ symmetry associated with the Higgs field and a $Z(2)_C$ center symmetry associated with the Polyakov loop in the compact direction. The order of the deconfining phase transition can be either second-order or first-order for $SU(2)$, depending on the deformation used. After finding order parameters for the global symmetries, we show that there are four distinct phases: a deconfined phase, a confined phase, a Higgs phase, and a mixed confined phase. The mixed confined phase occurs where one might expect a phase in which there is both confinement and the Higgs mechanism, but the behavior of the order parameters distinguishes the two phases. In the mixed confined phase, the $Z(2)_C \times Z(2)_H$ global symmetry breaks spontaneously to a $Z(2)$ subgroup that acts non-trivially on both the scalar field and the Polyakov loop. We find explicitly the BPS and KK monopole solutions of the Euclidean field equations in the BPS limit; these monopoles are extensions of similar pure gauge theory solutions, where they are constituents of instantons. In the mixed phase, a linear combination of the Higgs field ϕ and A_4 , the component of the gauge field in the compact direction, enters into the monopole solutions. In all four phases, Wilson loops orthogonal to the compact direction are expected to show area-law behavior. We show that this confining behavior can be attributed to a dilute monopole gas in a broad region that includes portions of all four phases. The dilute monopole gas picture breaks down when the action of a BPS monopole is zero. A duality argument similar to that applied recently [1] to the Seiberg-Witten model on $R^3 \times S^1$ shows that the monopole gas picture, arrived at using Euclidean instanton methods, can be interpreted as a gas of finite-energy dyons.

I. INTRODUCTION

One of the most fundamental questions we can ask about a gauge theory is its phase diagram. In the standard model, we have seen three fundamentally different types of behavior: the familiar Coulomb behavior associated with the massless photon, the Higgs mechanism and confinement of quarks and gluons. These properties are characteristics of different phases: QCD is in a confined phase at zero temperature and density, while the electroweak sector of the standard model combines Coulomb and Higgs phases.

As shown by 't Hooft [2, 3], there is a fundamental conflict between the Higgs mechanism and confinement. There is a simple picture of this conflict based on the dual superconductor picture of confinement. In a type II superconductor, magnetic monopoles would be confined by magnetic flux tubes, which we interpret as the Higgs mechanism leading to the confinement of magnetic charges. If the confined phase of a gauge theory can be interpreted as a dual condensate of magnetic monopoles, then confinement of non-Abelian electric charge would follow.

We will study below the phase structure of an $SU(2)$ adjoint Higgs model on $R^3 \times S^1$. Together with the scalar potential, a deformation term added to the model will allow us to explore what turns out to be a very rich phase structure. The use of $R^3 \times S^1$ with a small circumference, as opposed to R^4 , makes the gauge coupling small. One-loop perturbation theory shows that the deformation term can be used to move between confined and deconfined phases. This in turn allows the study of the interplay between confinement and the Higgs mechanism using semiclassical methods. This model extends recent work on gauge theories that are confining on $R^3 \times S^1$ for small circumference L [4, 5]. Typically, we associate this geometry with finite temperature, and L with the inverse temperature β , and we would expect a high-temperature, deconfined phase for small β . Recently, methods have been found to change this result: Gauge models have been found where confinement can be understood analytically at small L using semiclassical methods. The starting point is typically a gauge theory on $R^3 \times S^1$; a small circumference L for the compact direction implies a small coupling constant $g(L)$ provided $LA \ll 1$, where Λ is the characteristic renormalization group-invariant mass scale of the theory. Such a gauge theory is generally found in the deconfined phase for small L , so it is necessary to modify the gauge action in order to obtain a confined phase. In previous work on deformed gauge theories without

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fundamental scalars, there has been good evidence that the confined and deconfined phases on $R^3 \times S^1$ for small L are continuously connected to the same phases at large L [4].

The center symmetry associated with the gauge field, which is a $Z(N)_C$ symmetry for $SU(N)$, is crucial to our modern understanding of confinement and deconfinement. The Polyakov loop operator $P(\vec{x})$ will be central to our analysis. It is defined as a Wilson loop traversing a topologically non-trivial path in the compact direction given by

$$P(\vec{x}) = \mathcal{P} \exp \left[ig \int_0^L dx_4 A_4(\vec{x}, x_4) \right] \quad (1)$$

where \mathcal{P} indicates path ordering and A_μ is the gauge field. It transforms as $P(\vec{x}) \rightarrow g(\vec{x}, 0) P(\vec{x}) g^\dagger(\vec{x}, 0)$ under a gauge transformation $g(\vec{x}, x_4)$ so that $\text{Tr}_R P^n(\vec{x})$ is gauge-invariant for any representation R and any integer n . The Polyakov loop transforms non-trivially under center symmetry. For $SU(N)$, this is a transformation that takes $\text{Tr}_R P^n(\vec{x})$ into $z^n \text{Tr}_R P^n(\vec{x})$ where $z \in Z(N)_C$. In a pure gauge theory at small L , the one-loop effective potential for P is reliable, and indicates that $Z(N)_C$ symmetry is spontaneously broken: The deconfined phase is preferred in this region. In order to restore the confined, $Z(N)_C$ -symmetric phase at small L , additional contributions to the effective potential must be present. Two methods are known, one using adjoint fermions, and the other a deformation of the gauge action. The addition of adjoint representation fermions to $SU(N)$ gauge theories preserves the global $Z(N)_C$ symmetry of the action. With normal antiperiodic boundary conditions for the fermions, the perturbative effective action for the Polyakov loop shows that the deconfined phase remains favored at high temperature, as in the pure gauge case. With periodic boundary conditions for the fermions, however, this class of field theories can avoid the transition to the deconfined phase found in the pure gauge theory for sufficiently light fermion mass and small L [6–8]. If the number of adjoint Dirac fermion flavors N_f is less than $11/2$, these systems are asymptotically free at small L , and therefore the effective potential for P is calculable using perturbation theory. An alternative approach which is closely related is to add to the gauge action deformation terms which are local in the non-compact directions, but non-local in the compact direction [4, 9]. Because these terms must respect center symmetry, they are often referred to as double-trace deformations, reflecting the fact that $\text{Tr}_A P = \text{Tr}_F P^\dagger \text{Tr}_F P - 1$. A minimal choice for the deformation term S_d , which is adequate for $SU(2)$ and $SU(3)$, takes the form

$$S_d = L \int d^3x \frac{h_1}{L^4} |\text{Tr}_F P(\vec{x})|^2 \quad (2)$$

which favors the confined phase with $\text{Tr}_F P = 0$ for $h_1 > 0$. For $N \geq 4$, it is necessary to include additional terms to avoid partially confined phases, as in the case of $SU(4)$ where $Z(4)$ can break spontaneously to $Z(2)$. In this more general case, the deformation may be taken to be

$$S_d = L \int d^3x \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{h_k}{L^4} |\text{Tr}_F P^k(\vec{x})|^2 \quad (3)$$

with the confined phase regained at small L if all the h_k 's are sufficiently positive.

The change of the action away from that of a pure gauge theory restores center symmetry in the compact direction in such a way that perturbation theory can be used to calculate fundamental quantities associated with the Polyakov loops such as string tensions. In contrast, the maintenance of center symmetry in the non-compact directions, which holds for all values of L , is non-perturbative. String tensions are measured by Wilson loops in planes orthogonal to the compact direction. The mechanism by which the Wilson loop string tension arises is monopole condensation, via a mechanism first discussed by Polyakov [10] in the context of a $d = 3$ Higgs model. Once the center symmetry is restored by either method discussed above, the gauge field in the compact direction, A_4 automatically acquires a non-zero vacuum expectation value in an appropriately chosen gauge. A_4 then behaves like a scalar field in the usual Higgs mechanism, where $SU(N)$ spontaneously breaks down to $U(1)^{N-1}$. Unlike the case of conventional scalar fields, there are N monopoles in this case; $N - 1$ BPS monopoles and one additional monopole, called Kaluza-Klein monopole due to the fact that the forth direction is compactified [11–13]. Following the work by Polyakov, Unsal and Yaffe were able to analytically calculate the string tension for the case of $SU(2)$ using the dilute gas approximation of monopoles [9]. A similar result holds for $SU(3)$, although unfortunately the more general case of $SU(N)$ is not as tractable [8].

With confinement in the pure gauge theory on $R^3 \times S^1$ under analytic control, we can now introduce an adjoint Higgs field into this setting. The addition of an adjoint scalar field to such a theory allows us to examine the interplay of confinement and the Higgs mechanism. For a quartic scalar field potential $V(\phi)$, there is a $Z(2)_H$ global symmetry given by $\phi \rightarrow -\phi$. Unlike the gauge coupling, the quartic interaction λ of such a scalar is not asymptotically free. However, we are free to set the running coupling $\lambda(\mu)$ so that it is small at the scale $\mu = 1/L$, and semiclassical

methods, including perturbation theory, are valid. An $SU(N)$ adjoint scalar Higgs model on $R^3 \times S^1$ has a natural global symmetry group $Z(N)_C \times Z(2)_H$. We will focus in what follows on the case $N = 2$. Not only is it the simplest case, but for $N \geq 3$, the gauge theory on $R^3 \times S^1$ has additional phases intermediate between the confined and deconfined phases, complicating the analysis [4].

The supersymmetric analog of this model is the Seiberg-Witten model [14], which is an $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group $SU(2)$. Seiberg and Witten found that in this model the addition of an $\mathcal{N} = 1$ mass perturbation leads to confinement by magnetic monopoles. Recently, Poppitz and Unsal have examined the behavior of this model on $R^3 \times S^1$, and concluded that the confined phase seen for small compactification circumference on $R^3 \times S^1$ is connected to the confining phase at infinite compactification circumference. In their work, Euclidean monopoles in which a linear combination of A_4 and ϕ plays the role of the scalar field appear prominently, in a very similar fashion to the non-supersymmetric model [15].

The scalar field ϕ is not gauge invariant, and cannot serve as an order parameter for the breaking of the $Z(2)_H$ symmetry associated with ϕ when gauge interactions are present. This is an old problem, a consequence of Elitzur's theorem [16]. Higgs models with scalar fields in the fundamental and adjoint representations behave differently. For Higgs models with scalar fields in the fundamental representation, the confined and Higgs phases are connected [17], in a manner similar to the connection between liquid and gas phases. In this case, the $Z(N)_C$ center symmetry is explicitly broken, and large Wilson loops do not have area-law behavior due to screening by the scalars. In the adjoint case, $Z(N)_C$ center symmetry is preserved by the action, and there is a distinct phase transition between the confined and Higgs phases. In the $R^3 \times S^1$ model we consider, we will show that there are combinations of ϕ and P , such as $Tr_F \phi P$ that can serve as gauge-invariant order parameters for the symmetries of the model.

Section II describes in detail the effective potential for the Polyakov loop and deformations added to it that restore confinement at small L , focusing on the case of $SU(2)$. We will discuss a number of possible deformation terms and their effect on the order of the deconfining phase transition. We will show that a particularly useful deformation can be obtained by considering the embedding of two-dimensional fermions into the four-dimensional theory. This deformation leads to a simple treatment of the perturbatively-determined phase diagram in section III, although our overall conclusions regarding the phase structure are general. In section III we determine the phase structure of the $SU(2)$ model using the effective potential U_{eff} , evaluated at one loop. The evaluation at finite L of the functional determinants representing one-loop contributions to U_{eff} will be the same as those needed at finite temperature. With the inclusion of a deformation term, we will show that there are four different phases in perturbation theory, corresponding to different patterns of symmetry breaking: a deconfined phase, a confined phase, a Higgs phase, and a phase which appears to exhibit both the Higgs mechanism and confinement. In section IV we show that the phase that apparently combines confinement and the Higgs mechanism is in fact a mixed confined phase, where the $Z(2)_C \times Z(2)_H$ global symmetry breaks spontaneously to the $Z(2)$ subgroup that acts non-trivially on both the scalar field and the Polyakov loop. We show that three gauge-invariant order parameters $Tr_F P$, $Tr_F \phi P$ and $Tr_F \phi P^2$ are sufficient to resolve the phase structure, and characterize all four phases in terms of their global symmetries. Section V finds in the BPS limit the classical solutions of the Euclidean equations of motion that appear as constituents of instantons in the pure gauge case. These solutions are not identical to Minkowski-space monopoles, which also occur in this model. The most interesting and general case is the mixed confined phase, where both ϕ and A_4 have expected values in an appropriately chosen gauge. In the mixed confined phase, a linear combination of ϕ and A_4 plays the same role that A_4 plays in the analysis of pure gauge theories on $R^3 \times S^1$, in line with the breaking $Z(2)_C \times Z(2)_H \rightarrow Z(2)$. The behavior of the monopole solutions in the other three phases appear as special cases of the mixed confined phase. Section VI discusses the effects of these Euclidean monopoles on the dynamics of the model. In previous studies of the confined phase in $SU(N)$ gauge theories on $R^3 \times S^1$, it has been shown that Euclidean-space monopoles play a key role in the area-law behavior of Wilson loops in planes orthogonal to the compact direction [5, 9, 18]. It is therefore no surprise to find that monopoles play an important role when an adjoint scalar field is present. However, there is great subtlety and variety in the analysis. Nevertheless, we show that a dilute monopole gas gives rise to confining behavior for Wilson loops over a broad region that includes part of all four phases. A final section gives our conclusions.

II. ROLE OF THE DEFORMATION

As explained in the introduction, the one-loop gauge boson effective potential V_g favors the deconfined phase. In the case of $SU(2)$, where the Polyakov loop can be parametrized as $Tr_F P = 2 \cos(\theta)$, V_g can be written as [19–22]

$$V_g = -\frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{Tr_A P^n}{n^4} \quad (4)$$

or equivalently

$$V_g = -\frac{\pi^2}{15L^4} + \frac{4}{3\pi^2 L^4} \theta^2 (\theta - \pi)^2 \quad (5)$$

which is minimized at $\theta = 0$ or π corresponding to $Tr_F P = \pm 2$. In order to realize the confined phase for small L , we will add a double-trace deformation term S_d to the action. This term will be a $Z(N)_C$ -invariant function of P , and therefore will be non-local in the compact variable x_4 . Many forms of S_d may be used, such that the confined phase is favored for some range of parameters. In the case of $SU(2)$, there is an interesting issue concerning the order of the transition from the confined to the deconfined phase. Although the pure gauge theory is clearly related to three-dimensional $Z(2)$ spin systems, this does not ensure a second-order transition in the Ising model universality class because the transition may be first order. This issue can easily be understood from the point of view of Landau-Ginzburg theory. Consider a general theory with a real scalar order parameter ρ and a Landau-Ginzburg free energy density $f[\rho]$ which is $Z(2)$ invariant. It may be expanded as

$$f[\rho] = \frac{1}{2} r \rho^2 + \frac{1}{4!} \lambda \rho^4 + \frac{1}{6!} \kappa \rho^6 \quad (6)$$

with $\kappa > 0$ for global stability. As long as $\lambda > 0$, the transition will be second-order, occurring at $r = 0$. However, if $\lambda < 0$, it is easy to see that the transition may be first-order [23]. Because we have a high degree of freedom in choosing our deformation in $SU(2)$, we can also choose the order of the transition.

A minimal choice for S_d takes the form

$$S_d = L \int d^3x \frac{h_1}{L^4} |Tr_F P(\vec{x})|^2 \quad (7)$$

which favors the confined phase with $Tr_F P = 0$ for $h_1 > 0$. From a Landau-Ginzburg point of view, changing h_1 is a change to r . However, the transition between the confined and deconfined phases is first-order when S_d is added to the one-loop effective action of the gauge theory. It is instructive to consider a slightly generalized form

$$V_d = h_1 L^{-4} (Tr_F P)^2 + h_2 L^{-4} (Tr_F P)^4 \quad (8)$$

where we define the potential V_d via

$$S_d = L \int d^3x V_d. \quad (9)$$

For sufficiently large $h_1 > 0$, the symmetry will be restored. Expanding the gluon potential with this deformed potential around the symmetric point, $\theta = \pi/2$, we get a potential of Landau-Ginzburg type

$$\begin{aligned} V_g + V_d \simeq & \frac{\pi^2}{60L^4} + \frac{4}{L^4} \left(h_1 - \frac{1}{6} \right) \left(\theta - \frac{\pi}{2} \right)^2 + \frac{4}{3L^4} \left(-h_1 + 12h_2 + \frac{1}{\pi^2} \right) \left(\theta - \frac{\pi}{2} \right)^4 \\ & + \frac{8}{45L^4} (h_1 - 60h_2) \left(\theta - \frac{\pi}{2} \right)^6 \end{aligned} \quad (10)$$

displaying explicitly the variation of the low-order terms in the expansion. If the phase transition is second-order, it must occur at $h_1 = 1/6$. However, if the coefficient of the quartic term is negative, the confined phase at $\theta = \pi/2$ will be unstable when $h_1 = 1/6$. This tells us that the transition is first-order for sufficiently small h_2 . On the other hand, when h_2 is large, we can ignore terms past quartic because θ is bounded, and the transition is second order. The tricritical point where the transition changes from first- to second-order, lies somewhere on the line of $h_1 = 1/6$, but it must be located numerically. We plot the phase diagram of the deformed $SU(2)$ as shown in Figure 1.

Another possibility is to choose a form for V_d which is proportional to the one-loop expression for the gauge boson contribution to the effective action, but with opposite sign [4]:

$$V_d = \frac{2h}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{|Tr_F P^n|^2}{n^4}. \quad (11)$$

The action will cancel the leading-order $1/L^4$ contribution of the gauge bosons to the effective action when $h = 1$, and the confined phase will be favored at small L when $h > 1$. This choice for V_d leads to a very strong first-order transition as h is varied between a confined phase where $Tr_F P = 0$ and a deconfined phase where $Tr_F P = \pm 2$, the largest possible value. This form for the deformation can be approximately implemented by a local addition to the

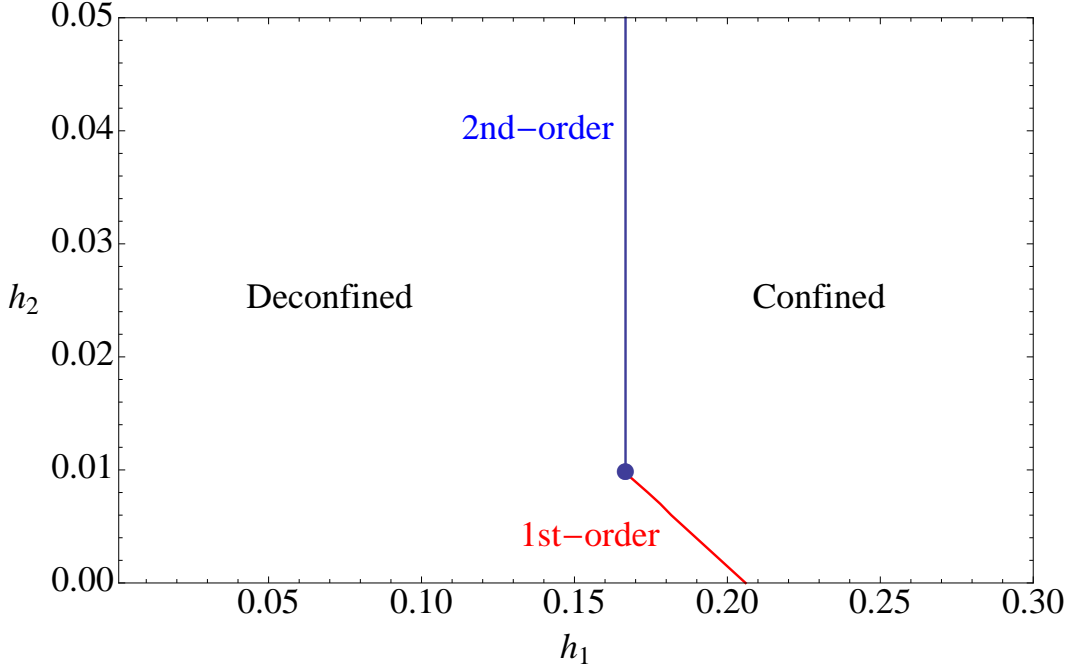


Figure 1: Phase diagram of an $SU(2)$ gauge theory as a function of the h_1 and h_2 deformation parameters.

action, corresponding to N_f flavors of adjoint Dirac fermions of mass M with periodic boundary conditions in the compact direction. In general, the potential for such adjoint fermions in $(d+1)$ -dimension is

$$4N_f s \left(\frac{M}{2\pi L} \right)^{(d+1)/2} \sum_{n=1}^{\infty} \frac{K_{(d+1)/2}(nML) \text{Tr}_A P^n}{n^{(d+1)/2}} \quad (12)$$

where $K_{(d+1)/2}$ is the modified Bessel function and s accounts for spin degeneracy [22]. In the limit $M \rightarrow 0$ with $d = 3$ spatial dimensions, the adjoint fermions will make a one-loop contribution to the effective potential of the form given above with the identification $h = sN_f$, up to a term independent of $\text{Tr}_F P$ because $\text{Tr}_A P^n = |\text{Tr}_F P^n|^2 - 1$. The transition between phases is first-order for all M .

The most analytically tractable choice we have found that yields a second-order transition is based on the one-loop potential for N_f adjoint Dirac fermions with periodic boundary conditions in two dimensions instead of four, i.e., $d = 1$ in equation 12, yielding in two dimensions

$$\frac{2MLN_f}{\pi L^2} \sum_{n=1}^{\infty} \frac{K_1(nML) \text{Tr}_A P^n}{n}. \quad (13)$$

These sheets of two-dimensional fermions can be embedded in four dimensions with a density $1/a^2$ in the plane orthogonal to the plane of the fermions. Then V_d is given by

$$V_d = \frac{2MLN_f}{\pi a^2 L^2} \sum_{n=1}^{\infty} \frac{K_1(nML) \text{Tr}_A P^n}{n}. \quad (14)$$

In a lattice implementation, we would identify a as the lattice spacing and an overall coefficient of order one would depend on the lattice fermion implementation. Using the relation $L = N_4 a$, where N_4 is the number of lattice sites in the compact direction, we would have

$$V_d = \frac{2MLN_f N_4^2}{\pi L^4} \sum_{n=1}^{\infty} \frac{K_1(nML) \text{Tr}_A P^n}{n}. \quad (15)$$

The infinite series can be summed exactly in the limit when the mass goes to zero,

$$\lim_{M \rightarrow 0} V_d = \frac{2N_f N_4^2}{\pi L^4} \sum_{n=1}^{\infty} \frac{\text{Tr}_A P^n}{n^2} = \frac{4N_f N_4^2}{\pi L^4} (\theta - \pi/2)^2 \quad (16)$$

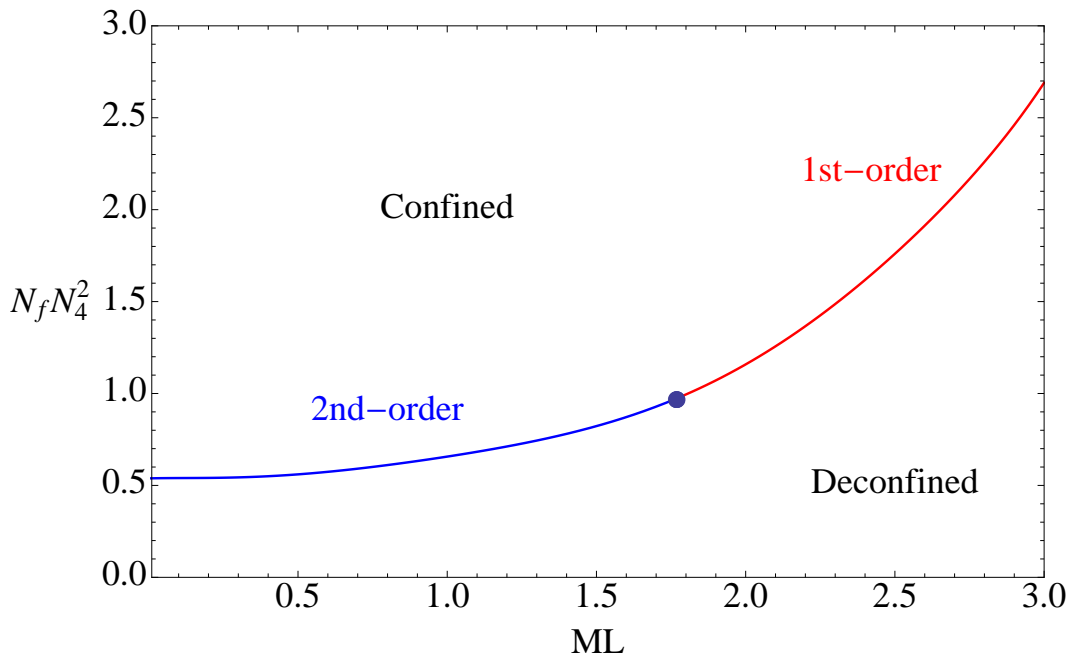


Figure 2: The phase diagram of an $SU(2)$ gauge theory with a deformation inspired by N_f two-dimensional fermions of mass M as a function of ML and $N_f N_4^2$.

where $0 \leq \theta \leq \pi$. This deformation leads to a second-order phase transition at some N_f for sufficiently small M . We stress that although the form of the deformation term was motivated by the connection with adjoint fermions, it is in fact a deformation term with no additional dynamical degrees of freedom. Because we treat this term as a deformation, we can identify the compactification circumference as an inverse temperature $L = \beta$; this would not be legitimate for periodic adjoint fermions, because spectral positivity in the compact direction would fail. We minimize the effective potential of gluons with this deformation numerically by changing the two dimensionless parameters, $N_f N_4^2$ and ML , and construct the phase diagram as shown in Figure 2. As we increase ML , the contribution from adjoint fermions is suppressed, so a larger number of flavors is needed to retain confinement. However, the transition becomes first-order for sufficiently high ML as we change $N_f N_4^2$. The tricritical point lies on $(ML, N_f N_4^2)_c \simeq (1.771, 0.955)$. We will use the $M = 0$ form in what follows, thereby obtaining a second-order deconfinement transition.

III. THE EFFECTIVE POTENTIAL

The phase diagram of our $SU(2)$ model will be calculated from an approximate form of the one-loop effective potential, including the deformation term. The effective potential will be calculated in background field gauge [24], with the background fields consisting of a scalar expectation value for ϕ and a constant value for A_4 ; the latter gives rise to a non-trivial Polyakov loop background. For a general Higgs theory, the classical Euclidean action can be written as

$$S_c = \int d^4x \left[\frac{1}{4} (F_{\mu\nu}^a)^2 + \frac{1}{2} (D_\mu \phi)^T \cdot D_\mu \phi + V(\phi) \right] \quad (17)$$

where the field ϕ is in an arbitrary real representation R of the gauge group G of dimension n , in general reducible. The index a runs over the number of generators of the group, $N^2 - 1$ for $SU(N)$.

The potential $V(\phi)$ we use is given by

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda (\phi^2)^2 \quad (18)$$

where $\phi^2 = \phi^T \phi$. The covariant derivative acts on ϕ as

$$D_\mu(A) \phi = \partial_\mu \phi - ig A_\mu \phi \quad (19)$$

where the gauge field is written as an $n \times n$ matrix using $A_\mu = A_\mu^a T^a$ where the T^a are the generators of the group in the representation R . The field strength tensor in a matrix notation is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \quad (20)$$

or

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c \quad (21)$$

in terms of components.

The classical contribution to the effective potential is the sum of the scalar potential $V(\phi)$ and a contribution from the kinetic term:

$$V_c(\phi) = -\frac{1}{2}g^2 (A_\mu \phi)^T \cdot A_\mu \phi + V(\phi). \quad (22)$$

The contribution from the kinetic term is positive-definite, despite appearances. In a real representation, the Hermitian generators T^a are purely imaginary, so $T^{a*} = -T^a$. This in turn implies $T^{aT} = -T^a$, and thus $A_\mu^T = -A_\mu$. In the case of the adjoint representation, this term can be written in matrix notation as

$$-g^2 \text{Tr}_F [A_4, \phi]^2 \quad (23)$$

where $[A_4, \phi]$ is clearly anti-Hermitian. The positivity of this term for the adjoint representation implies that the effective potential will be minimized if $[A_4, \phi] = 0$.

The calculation of the effective potential in the presence of a background Polyakov loop is similar to the case of finite temperature and density [25], because a chemical potential is an imaginary $U(1)$ background A_4 expected value. The one-loop effective action Γ for the Higgs model without the deformation term is given by the classical action S plus contributions from the functional determinants of the gauge, scalar and ghost fields. The computation is only simple in R_ξ gauge with $\xi = 1$. For the adjoint scalar model, the result is

$$\Gamma = S + \text{Tr}_A \log \left[(-\bar{D}_\mu^2)^{ac} + (M_g^2)^{ac} \right] + \frac{1}{2} \text{Tr}_R \log \left[(-\bar{D}_\mu^2) + M_s^2 \right] \quad (24)$$

where the functional traces are taken over space-time as well as the internal symmetry group and \bar{D}_μ is the covariant derivative with respect to the background field. We denote the background fields by $\bar{\phi}$ and \bar{A}_μ . The first trace represents the net contribution of the gauge and ghost fields, while the second term is the contribution of the scalar field. The mass matrices depend on the background field configuration and are given by

$$(M_g^2)^{ac} = g^2 \bar{\phi}^T T^a T^c \bar{\phi} \quad (25)$$

for the gauge fields and

$$M_s^2 = m^2 + \lambda \bar{\phi}^2 + 2\lambda \bar{\phi} \bar{\phi}^T + g^2 T^a \bar{\phi} \bar{\phi}^T T^a \quad (26)$$

for the scalar fields. For static background fields we have

$$\Gamma = \int d^4x U_{eff}. \quad (27)$$

The contribution to the effective potential from the functional determinants may be separated into a contribution independent of L , analogous to $T = 0$, of the form

$$V_{1l}^\infty = 2 \frac{1}{64\pi^2} \text{Tr}_A \left[(M_g^2)^2 \log (M_g^2/\Lambda^2) \right] + \frac{1}{64\pi^2} \text{Tr}_R \left[(M_s^2)^2 \log (M_s^2/\Lambda^2) \right] \quad (28)$$

where the traces are taken over representations of the gauge group, with A denoting the adjoint representation. Λ is the usual scale-setting parameter with dimensions of mass required by renormalization. There is also an L -dependent contribution, corresponding to $T \neq 0$, of the form $V_{1l}^L = V_{1lg}^L + V_{1l\phi}^L$ where

$$V_{1lg}^L = \frac{2}{L} \text{Tr}_A \int \frac{d^3p}{(2\pi)^3} \log \left[1 - P \exp \left(-L \sqrt{p^2 + M_g^2} \right) \right] \quad (29)$$

and

$$V_{1l\phi}^L = \frac{1}{L} Tr_R \int \frac{d^3 p}{(2\pi)^3} \log \left[1 - P \exp \left(-L \sqrt{p^2 + M_s^2} \right) \right] \quad (30)$$

where P is simply $\exp(igL\bar{A}_4)$. We have assumed in these expressions that the mass matrices are diagonal, and so commute with the Polyakov loop, as is the case if ϕ is in the adjoint representation of $SU(N)$.

We now specialize to the case of adjoint $SU(2)$ where we take $\bar{\phi} = (0, 0, v)$ and $P = \text{diag}[\exp(i\theta), \exp(-i\theta)]$ in the fundamental representation. It is easy to check that the gauge boson mass matrix has the form

$$M_g^2 = \begin{pmatrix} g^2 v^2 & & \\ & g^2 v^2 & \\ & & 0 \end{pmatrix} \quad (31)$$

and the scalar mass matrix M_s^2 is

$$M_s^2 = \begin{pmatrix} m^2 + \lambda v^2 + g^2 v^2 & & \\ & m^2 + \lambda v^2 + g^2 v^2 & \\ & & m^2 + 3\lambda v^2 \end{pmatrix}. \quad (32)$$

The complete one-loop effective potential for the scalar-gauge system is then

$$\begin{aligned} V_{eff} = & \frac{1}{2} m^2 v^2 + \frac{1}{4} \lambda v^4 + \frac{2 \cdot 2}{64\pi^2} g^4 v^4 \log(g^2 v^2 / \Lambda^2) + \frac{2}{64\pi^2} (m^2 + \lambda v^2 + g^2 v^2)^2 \log[(m^2 + \lambda v^2 + g^2 v^2) / \Lambda^2] \\ & + \frac{1}{64\pi^2} (m^2 + 3\lambda v^2)^2 \log[(m^2 + 3\lambda v^2) / \Lambda^2] + 2 \frac{1}{L} \int \frac{d^3 p}{(2\pi)^3} \log \left[1 - e^{2i\theta} e^{-L \sqrt{p^2 + g^2 v^2}} \right] \\ & + 2 \frac{1}{L} \int \frac{d^3 p}{(2\pi)^3} \log \left[1 - e^{-2i\theta} e^{-L \sqrt{p^2 + g^2 v^2}} \right] + 2 \frac{1}{L} \int \frac{d^3 p}{(2\pi)^3} \log \left[1 - e^{-L|p|} \right] \\ & + \frac{1}{L} \int \frac{d^3 p}{(2\pi)^3} \log \left[1 - e^{2i\theta} e^{-L \sqrt{p^2 + m^2 + \lambda v^2 + g^2 v^2}} \right] \\ & + \frac{1}{L} \int \frac{d^3 p}{(2\pi)^3} \log \left[1 - e^{-2i\theta} e^{-L \sqrt{p^2 + m^2 + \lambda v^2 + g^2 v^2}} \right] \\ & + \frac{1}{L} \int \frac{d^3 p}{(2\pi)^3} \log \left[1 - e^{-L \sqrt{p^2 + m^2 + 3\lambda v^2}} \right]. \end{aligned} \quad (33)$$

As explained in the introduction, the extra term S_d added to the action is used to offset one-loop terms in Γ that favor the deconfined phase. These one-loop terms are $O(1)$ in the loop expansion, whereas the classical action S_c is $O(\hbar^{-1})$. It is thus consistent to take S_d to be $O(1)$ in the loop expansion. This occurs naturally when the added term S_d represents fermions in the adjoint representation, but in the case of a deformation it is essentially a choice we make in defining what our perturbation theory is. The total one-loop effective potential is

$$U_{eff} = V_{eff} + V_d \quad (34)$$

which is a function of the expected values $\bar{\phi}$ and \bar{A}_4 and depends on the parameters g , m^2 , λ and L , as well as any additional parameters in V_d . We use the form of V_d given in section II:

$$V_d = \frac{4N_f N_4^2}{\pi L^4} (\theta - \pi/2)^2. \quad (35)$$

We now make use of an approximate form for the integrals [22]

$$\begin{aligned} V_B = & \frac{1}{L} \int \frac{d^3 p}{(2\pi)^3} \log \left[1 - e^{i\theta} e^{-L \sqrt{p^2 + M^2}} \right] + \frac{1}{L} \int \frac{d^3 p}{(2\pi)^3} \log \left[1 - e^{-i\theta} e^{-L \sqrt{p^2 + M^2}} \right] \\ \simeq & -\frac{2}{\pi^2 L^4} \left[\frac{\pi^4}{90} - \frac{1}{48} \theta_+^4 + \frac{\pi}{12} \theta_+^3 - \frac{\pi^2}{12} \theta_+^2 \right] + \frac{M^2}{2\pi^2 L^2} \left[\frac{1}{4} \theta_+^2 - \frac{\pi}{2} \theta_+ + \frac{\pi^2}{6} \right] \\ & - \frac{M^4}{16\pi^2} \left[\ln \left(\frac{LM}{4\pi} \right) + \gamma - \frac{3}{4} \right] \end{aligned} \quad (36)$$

where θ_+ means θ made periodic over the range 0 to 2π and M is a mass term. Because we work with particles in the adjoint representation, we must make the replacements $\theta_+ \rightarrow 2\theta$, and the range of θ must be taken as $[0, \pi]$. Applying this approximation to our complete expression for V_{eff} , we have

$$\begin{aligned}
U_{eff} = & \frac{1}{2}m^2v^2 + \frac{1}{4}\lambda v^4 \\
& - \frac{4}{\pi^2 L^4} \left[\frac{\pi^4}{90} - \frac{1}{48} (2\theta)_+^4 + \frac{\pi}{12} (2\theta)_+^3 - \frac{\pi^2}{12} (2\theta)_+^2 \right] \\
& + \frac{g^2 v^2}{\pi^2 L^2} \left[\frac{1}{4} (2\theta)_+^2 - \frac{\pi}{2} (2\theta)_+ + \frac{\pi^2}{6} \right] - \frac{g^4 v^4}{16\pi^2} \left[\ln \left(\frac{L^2 \Lambda^2}{16\pi^2} \right) + 2\gamma - \frac{3}{2} \right] \\
& - \frac{2}{\pi^2 L^4} \left[\frac{\pi^4}{90} \right] - \frac{2}{\pi^2 L^4} \left[\frac{\pi^4}{90} - \frac{1}{48} (2\theta)_+^4 + \frac{\pi}{12} (2\theta)_+^3 - \frac{\pi^2}{12} (2\theta)_+^2 \right] \\
& + \frac{(m^2 + \lambda v^2 + g^2 v^2)}{2\pi^2 L^2} \left[\frac{1}{4} (2\theta)_+^2 - \frac{\pi}{2} (2\theta)_+ + \frac{\pi^2}{6} \right] \\
& - \frac{(m^2 + \lambda v^2 + g^2 v^2)^2}{32\pi^2} \left[\ln \left(\frac{L^2 \Lambda^2}{16\pi^2} \right) + 2\gamma - \frac{3}{2} \right] \\
& - \frac{1}{\pi^2 L^4} \left[\frac{\pi^4}{90} \right] + \frac{m^2 + 3\lambda v^2}{4\pi^2 L^2} \left[\frac{\pi^2}{6} \right] - \frac{(m^2 + 3\lambda v^2)^2}{64\pi^2} \left[\ln \left(\frac{L^2 \Lambda^2}{16\pi^2} \right) + 2\gamma - \frac{3}{2} \right] \\
& + \frac{4N_f N_4^2}{\pi L^4} (\theta - \pi/2)^2.
\end{aligned} \tag{37}$$

Note that the logarithmic dependence on the mass matrix disappears in this small- L expansion. Rearranging the leading order terms, we have

$$\begin{aligned}
U_{eff} = & \frac{1}{2}m^2v^2 + \frac{1}{4}\lambda v^4 \\
& - \frac{6}{\pi^2 L^4} \left[\frac{\pi^4}{90} - \frac{1}{48} (2\theta)_+^4 + \frac{\pi}{12} (2\theta)_+^3 - \frac{\pi^2}{12} (2\theta)_+^2 \right] - \frac{3}{\pi^2 L^4} \left[\frac{\pi^4}{90} \right] \\
& + \frac{(m^2 + \lambda v^2 + g^2 v^2)}{2\pi^2 L^2} \left[\frac{1}{4} (2\theta)_+^2 - \frac{\pi}{2} (2\theta)_+ + \frac{\pi^2}{6} \right] \\
& + \frac{2g^2 v^2}{2\pi^2 L^2} \left[\frac{1}{4} (2\theta)_+^2 - \frac{\pi}{2} (2\theta)_+ + \frac{\pi^2}{6} \right] + \frac{m^2 + 3\lambda v^2}{4\pi^2 L^2} \left[\frac{\pi^2}{6} \right] \\
& - \frac{(m^2 + 3\lambda v^2)^2}{64\pi^2} \left[\ln \left(\frac{L^2 \Lambda^2}{16\pi^2} \right) + 2\gamma - \frac{3}{2} \right] - \frac{2g^4 v^4}{32\pi^2} \left[\ln \left(\frac{L^2 \Lambda^2}{16\pi^2} \right) + 2\gamma - \frac{3}{2} \right] \\
& - \frac{(m^2 + \lambda v^2 + g^2 v^2)^2}{32\pi^2} \left[\ln \left(\frac{L^2 \Lambda^2}{16\pi^2} \right) + 2\gamma - \frac{3}{2} \right] + \frac{4N_f N_4^2}{\pi L^4} (\theta - \pi/2)^2.
\end{aligned} \tag{38}$$

We now drop all the terms independent of v and θ from U_{eff} .

$$\frac{1}{2}m^2(L)v^2 \tag{39}$$

and

$$\frac{1}{L^4} \left[\frac{4N_f N_4^2}{\pi} - 1 \right] \left(\theta - \frac{\pi}{2} \right)^2. \tag{40}$$

Additionally, we define running couplings $m^2(L)$ and $\lambda(L)$ in such a way that all one-loop contributions are included in the running couplings when $\theta = \pi/2$

$$\begin{aligned}
U_{eff} = & \frac{1}{2}m^2(L)v^2 + \frac{1}{4}\lambda(L)v^4 \\
& + \frac{1}{\pi^2 L^4} \left[2 \left(\theta - \frac{\pi}{2} \right)^4 - \pi^2 \left(\theta - \frac{\pi}{2} \right)^2 \right] + \frac{4N_f N_4^2}{\pi L^4} (\theta - \pi/2)^2 \\
& + \frac{(m^2 + \lambda v^2 + g^2 v^2)}{2\pi^2 L^2} (\theta - \pi/2)^2 + \frac{2g^2 v^2}{2\pi^2 L^2} (\theta - \pi/2)^2.
\end{aligned} \tag{41}$$

In order for us to take the phase diagram predicted by our one-loop effective potential seriously, both the gauge coupling $g(L)$ and the scalar coupling $\lambda(L)$ must be small. The gauge coupling is naturally small at a scale where $\Lambda L \ll 1$ as a consequence of asymptotic freedom, but the scalar coupling must be tuned to make $\lambda(L)$ small.

Naively, the phase diagram is controlled in perturbation theory by the two quadratic terms

$$\frac{1}{2}m^2(L)v^2 \quad (42)$$

and

$$\frac{1}{L^4} \left[\frac{4N_f N_4^2}{\pi} - 1 \right] \left(\theta - \frac{\pi}{2} \right)^2. \quad (43)$$

The potential also has a quartic coupling that couples together the two order parameters in a way that generally can produce either four second-order transition lines meeting at a tetracritical point or two second-order lines and one first-order line meeting at a bicritical point [23]. In the case at hand, the tetracritical phase diagram is obtained, as we now show. We define the parameter

$$a \equiv \frac{4N_f N_4^2}{\pi} - 1. \quad (44)$$

It is easy to see that there are at least two second-order phase transition lines that meet at $(a = 0, m^2(L) = 0)$: one line is along $a = 0$ for $m^2(L) > 0$, and the other is along $m^2(L) = 0$ for $a > 0$. Note that when $m^2(L) = 0$, the Lagrangian parameter m^2 is negative and $\mathcal{O}(\lambda/L^2, g^2/L^2)$. It is easy to see that the critical line for θ is determined by the $\mathcal{O}(1/L^4)$ terms in U_{eff} , implying the critical line is given by $a = 0$ up to a term which is of order $m^2(L)L^2$, which is of order λ or g^2 or less in the vicinity of the tetracritical point. Thus to leading order in perturbation, the critical line associated with θ is given by $a = 0$. As a moves from $a = 0$ to negative values, θ decreases from $\pi/2$, reaching $\theta = 0$ at $a = -1$. A given value of a will determine the value of θ , which in turn determines the coefficient of a contribution to U_{eff} of the form

$$\frac{(\lambda + g^2)}{2\pi^2 L^2} (\theta - \pi/2)^2 v^2. \quad (45)$$

We can absorb this contribution into our definition of $m^2(L)$. This has the effect of straightening out what would have been a curved segment in the critical line associated with ϕ in the region $-1 < a < 0$; the critical line is straight in any case for $a > 0$, where $\theta = \pi/2$, and for $a < -1$, where $\theta = 0$. Henceforth, we will write $m^2(L)$ as simply m^2 for notational simplicity.

We now see that the one-loop effective potential predicts two second-order phase transitions. They appear to be essentially independent: When $m^2 < 0$, the scalar expectation value v is non-zero; for $m^2 > 0$, it is 0. If $a > 0$, the angle θ associated with the Polyakov loop has the value $\pi/2$, and $Z(2)_C$ center symmetry holds. For $a < 0$, center symmetry is broken. Thus there are four distinct phases. As we have seen, the order of the deconfinement transition is non-universal, depending on the deformation. The detailed structure of the phase diagram will depend on the precise model. For example, if four-dimensional adjoint fermions are used, a coupling of the form $Tr[\bar{\psi}\phi\psi]$ must be considered. A large value for v gives rise to a large fermion mass terms, which in turn reduces the ability of the adjoint fermions to restore confinement [26]. However, the basic phase structure will be the same for all models. Three of the phases are familiar: a Higgs phase, a confined phase and a deconfined phase. However, the fourth phase, where center symmetry is unbroken and $v \neq 0$ is novel, and appears to have some of the properties of both the confined phase ($Tr_F P = 0$) and the Higgs phase ($v \neq 0$). In the next three sections, we will explore this phenomenon, first in terms of symmetries using the perturbative effective action, and then non-perturbatively.

IV. SYMMETRIES AND ORDER PARAMETERS

An understanding of the overall phase structure can be based on the global symmetries of this class of models. The action is invariant under two global $Z(2)$ symmetries. The first symmetry, $Z(2)_H$ is the invariance of the action under a transformation of the scalar field $\phi \rightarrow -\phi$. Because ϕ transforms under $SO(3)$, the adjoint representation of $SU(2)$, this transformation is not a gauge transformation, but a global symmetry. The other global symmetry, $Z(2)_C$, is associated with the center symmetry of the $SU(2)$ gauge group, and is present because all fields have 0 N -ality. Under this global symmetry, the action is invariant, but the Polyakov loop P transforms as $P \rightarrow -P$. It is useful to consider three distinct gauge-invariant order parameters associated with the $Z(2)_C \times Z(2)_H$ symmetry. Although these order

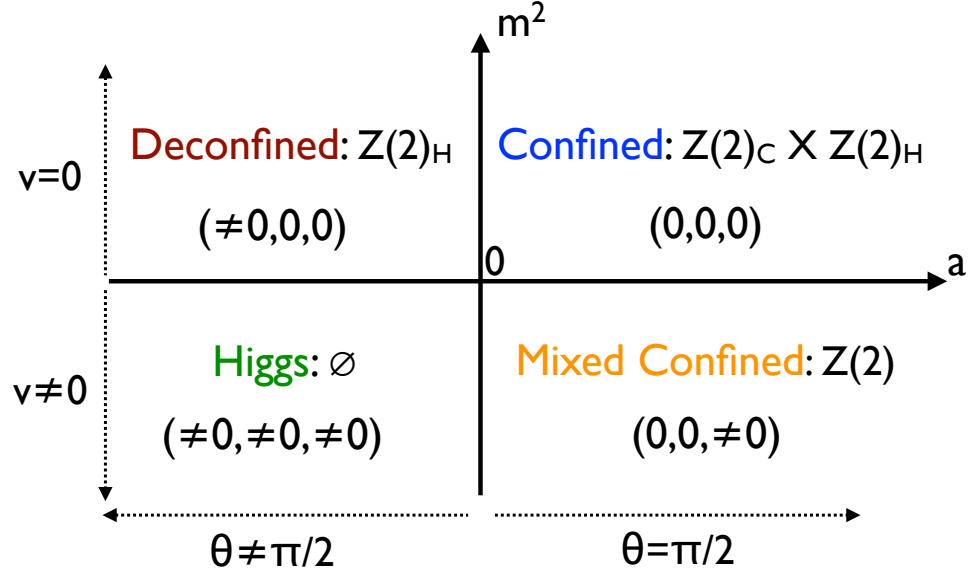


Figure 3: Phase diagram of $SU(2)$ Higgs model as a function of a and m^2 . The values of the order parameters are shown in parenthesis as $(\langle Tr_F P \rangle, \langle Tr_F [P^2 \phi] \rangle, \langle Tr_F [P \phi] \rangle)$.

parameters are non-local in the compact direction, they are local in the three non-compact directions. The first of these is the trace in the fundamental representation of the Polyakov loop P itself, $\langle Tr_F P(x) \rangle$, which is independent of x_4 . It transforms non-trivially under $Z(2)_C$ but is invariant under $Z(2)_H$. The second is $\langle Tr_F [P^2(x) \phi(x)] \rangle$ which is invariant under $Z(2)_C$, but transforms non-trivially under $Z(2)_H$. Finally, there is $\langle Tr_F [P(x) \phi(x)] \rangle$, which transforms non-trivially under both groups.

In the deconfined phase, there is spontaneous breaking of $Z(2)_C$, indicated by $\langle Tr_F P(x) \rangle \neq 0$. The Higgs phase is associated with the spontaneous breaking of $Z(2)_H$, indicated by $\langle Tr_F [P^2(x) \phi(x)] \rangle \neq 0$. It appears that five distinct phases might be possible: a confined phase, where $Z(2)_C \times Z(2)_H$ is unbroken; a deconfined phase, where $Z(2)_C$ is spontaneously broken but $Z(2)_H$ is unbroken; a Higgs phase, where both $Z(2)_C$ and $Z(2)_H$ are spontaneously broken; a phase where $Z(2)_H$ is broken but $Z(2)_C$ is unbroken; and finally a phase where $Z(2)_C \times Z(2)_H$ spontaneously breaks to $Z(2)$. This last phase is only invariant under a simultaneous transformation of P and ϕ . We will refer to this phase as the mixed confined phase. The mixed confined phase in some sense takes the place of a phase where $Z(2)_H$ is broken but $Z(2)_C$ is unbroken, which would be a phase where both the Higgs mechanism and confinement hold.

The minimum of the perturbative effective potential is specified by the expected values θ and v . They are not themselves gauge-invariant, but they can be used reliably to calculate gauge-invariant order parameters for small L . We have simply

$$\langle Tr_F P(x) \rangle = 2 \cos(\theta) \quad (46)$$

$$\langle Tr_F [P^2(x) \phi(x)] \rangle = 2iv \sin(2\theta) \quad (47)$$

$$\langle Tr_F [P(x) \phi(x)] \rangle = 2iv \sin(\theta). \quad (48)$$

The second and third expectation values are imaginary, but can be made real if desired by forming the appropriate Hermitian operator. The key technical point is that $\langle Tr_F [P^2(x) \phi(x)] \rangle$ is a gauge-invariant proxy for ϕ as long as $\sin(2\theta) \neq 0$. This restriction implies that the case of maximal center symmetry breaking where $\theta \rightarrow 0$ or $\theta \rightarrow \pi$ must be treated as a limiting case. Although one-loop perturbation theory does indicate maximal center symmetry breaking at high temperatures, lattice simulations suggest that such temperatures are not reached until well beyond the deconfinement transition. We assume that in each phase where center symmetry is broken there is a region where it is not maximally broken. It is easy to check that for our choice of deformation this is the case.

It is now easy to work out the phase diagram and the properties of the phases, as shown in Figure 3 and Table I. Naively, the phase that is both a confined and a Higgs phase occurs when $a > 0$ and $m^2 < 0$. This would be a phase where $Z(2)_H$ is broken but $Z(2)_C$ is unbroken, in the sense that $\langle Tr_F P(x) \rangle = 0$ and $\langle Tr_F [P(x) \phi(x)] \rangle = 0$ due to unbroken center symmetry, but $\langle Tr_F [P^2(x) \phi(x)] \rangle \neq 0$ as in the Higgs phase. This behavior is not possible in perturbation theory because $\langle Tr_F [P^2(x) \phi(x)] \rangle = 0$ if $\langle Tr_F P(x) \rangle = 0$. The phase that replaces it is a confining

Parameters	$\langle Tr_F P \rangle$	$\langle Tr_F [P^2 \phi] \rangle$	$\langle Tr_F [P \phi] \rangle$	Phase	Residual Symmetry
$a > 0, m^2 > 0$	0	0	0	Confined	$Z(2)_C \times Z(2)_H$
$a < 0, m^2 > 0$	$\neq 0$	0	0	Deconfined	$Z(2)_H$
$a < 0, m^2 < 0$	$\neq 0$	$\neq 0$	$\neq 0$	Higgs	\emptyset
$a > 0, m^2 < 0$	0	0	$\neq 0$	Mixed Confined	$Z(2)$
Absent	0	$\neq 0$	0	Confined & Higgs	$Z(2)_C$

Table I: Properties of the four possible phases, along with the Confined & Higgs phase, which does not occur.

phase because the Polyakov loop is zero, but center symmetry has become entwined with the global symmetry of the Higgs field. We will show in the next section that the non-perturbative dynamics of the model shows the effects of this mixing in a direct and dramatic way.

V. CLASSICAL MONOPOLE SOLUTIONS

The non-perturbative dynamics of gauge theories on $R^3 \times S^1$ are all based on Polyakov's analysis of the Georgi-Glashow model in three dimensions [10]. This is an $SU(2)$ gauge model coupled to an adjoint Higgs scalar. The model we are considering thus differs by the addition of a fourth compact dimension and a suitable deformation added to the action. The four-dimensional Georgi-Glashow model is the standard example of a gauge theory with classical monopole solutions when the Higgs expectation value is non-zero. They are topologically stable because $\Pi_2(SU(2)/U(1)) = \Pi_1(U(1)) = \mathbb{Z}$, and make a non-perturbative contribution to the partition function Z . In three dimensions, these monopoles are instantons. Polyakov showed that a gas of such three-dimensional monopoles gives rise to non-perturbative confinement in three dimensions, even though the theory appears to be in a Higgs phase perturbatively.

Because L is small in our four-dimensional theory, the three-dimensional effective theory describing the behavior of Wilson loops in the non-compact directions will have many features in common with the three-dimensional theory first discussed by Polyakov. In the four-dimensional theory, monopole solutions with short worldline trajectories in the compact direction exist, and behave as three-dimensional instantons in the effective theory. It is useful to recall the analysis of the small- L confined phase in the case of a gauge theory without scalars [4, 9]. In this theory, the role of the three-dimensional scalar field is played by the fourth component of the gauge field A_4 , which has a vacuum expected value induced by the perturbative effective potential. However, there is another way to understand the presence of monopoles in this phase, based on studies of instantons in pure gauge theories at finite temperature [11–13]. If the Polyakov loop has a non-trivial expectation value, finite-temperature instantons in $SU(N)$ may be decomposed into N monopoles, and the locations of the monopoles become parameters of the moduli space of the instanton. In the case of $SU(2)$, an instanton may be decomposed into a conventional BPS monopole and a so-called KK (Kaluza-Klein) monopole. The presence of the KK monopole solution differentiates the case of a gauge field at finite temperature from the case of an adjoint scalar breaking $SU(N)$ to $U(1)^{N-1}$, in which case there are $N - 1$ fundamental monopoles.

If a scalar field is added to the model, the coupling of A_4 to the R^3 gauge field \vec{A} is identical to the coupling of ϕ to \vec{A} , and non-zero expectation values for either or both lead to topologically non-trivial field configurations. For simplicity, we will continue to refer to these solutions as monopoles, although they are instantons, in the sense that they are solutions of the Euclidean field equations, and generally dyons in the sense that A_4 has non-trivial behavior. In the general case, both A_4 and ϕ play roles in the monopole solutions. This behavior is similar to that found in Higgs models with more than one scalar [27]. However, there is a significant difference. When an adjoint Higgs model spontaneously breaks $SU(N)$ down to $U(1)^{N-1}$, there are $N - 1$ fundamental monopoles. When A_4 is responsible for the breaking of $SU(N)$ down to $U(1)^{N-1}$, there is an additional monopole for a total of N fundamental monopoles. The solutions for all these monopoles can be found explicitly in the BPS limit; when A_4 is non-trivial, the $N - 1$ BPS monopoles are joined by a KK monopole [11–13]. In what follows, it will be useful to differentiate between solutions which saturate the Bogomolny bounds, versus solutions with the same topological properties and reduce to the solutions saturating the Bogomolny bounds in an appropriate limit. Thus we will distinguish between BPS solutions and monopoles of BPS type, meaning monopoles that reduce to BPS solutions in the appropriate limit. We will similarly distinguish between KK monopoles and KK monopole solutions.

We will now show how the monopole solutions in the general case are found. The monopole solutions in each of the four phases may be obtained as special cases. We begin with the BPS-type solution where all fields are independent of x_4 . This construction is very similar to the case of models with two Higgs fields [27]. The Euclidean Lagrangian \mathcal{L}

is given by

$$\mathcal{L} = \frac{1}{4} (F_{\mu\nu})^2 + \frac{1}{2} (D_\mu \phi)^2 + U_{eff}(\phi, A_4) \quad (49)$$

which includes potential term for both ϕ and A_4 . We assume that A_4 commutes with ϕ so that \mathcal{L} may be reduced to

$$\mathcal{L} = \frac{1}{2} (D_j A_4)^2 + \frac{1}{2} (B_j)^2 + \frac{1}{2} (D_j \phi)^2 + U_{eff}(\phi, A_4). \quad (50)$$

We can associate with \mathcal{L} an energy defined by

$$E = \int d^3x \left[\frac{1}{2} (B_j)^2 + \frac{1}{2} (D_j A_4)^2 + \frac{1}{2} (D_j \phi)^2 + U_{eff}(\phi, A_4) \right] \quad (51)$$

as well as an action $S = LE$. We will concern ourselves for now with the solutions in the BPS limit, in which the effective potential U_{eff} is neglected, but the boundary conditions on ϕ and A_4 at infinity imposed by the potential are retained.

We introduce two new fields

$$b = \cos \alpha A_4 + \sin \alpha \phi \quad (52)$$

$$c = -\sin \alpha A_4 + \cos \alpha \phi \quad (53)$$

which are orthogonal linear combinations of ϕ and A_4 , depending on an arbitrary angle α . We can write the energy as

$$\begin{aligned} E &= \int d^3x \left[\frac{1}{2} B_j^2 + \frac{1}{2} (D_j b)^2 + \frac{1}{2} (D_j c)^2 \right] \\ &= \int d^3x \left[\frac{1}{2} (B_j \pm D_j b)^2 + \frac{1}{2} (D_j c)^2 \mp B_j D_j b \right]. \end{aligned} \quad (54)$$

This expression is a sum of squares plus a term which can be converted to a surface integral, giving rise to the BPS inequality

$$E \geq \mp \int dS_j B_j b. \quad (55)$$

The BPS inequality is saturated if the following equalities hold:

$$\begin{aligned} B_j &= \mp D_j b \\ D_j c &= 0. \end{aligned} \quad (56)$$

For the case of a single monopole at the origin, we require the fields at spatial infinity to behave as

$$\begin{aligned} \lim_{r \rightarrow \infty} \phi^a &= v \frac{x^a}{r} \\ \lim_{r \rightarrow \infty} A_4^a &= w \frac{x^a}{r} \\ \lim_{r \rightarrow \infty} A_i^a &= \epsilon^{aij} \frac{x_j}{gr^2}. \end{aligned} \quad (57)$$

Note that w is related to the eigenvalues of P at large distances by $w = 2\theta/gL$. The first two terms are the usual hedgehog fields. A_i^a is chosen such that covariant terms vanish at infinity: $(D_i \phi)^a = 0$ and $(D_i A_4)^a = 0$. With the 't Hooft-Polyakov ansatz, the general expressions for the fields become

$$\begin{aligned} \phi^a &= v f(r) \frac{x^a}{r} \\ A_4^a &= w h(r) \frac{x^a}{r} \\ A_i^a &= a(r) \epsilon^{aij} \frac{x_j}{gr^2} \end{aligned} \quad (58)$$

where we define $v, w > 0$ and require $f(\infty) = 1$ or -1 , $h(\infty) = 1$ or -1 , and $a(\infty) = 1$ to obtain the correct asymptotic behavior. We must also have $f = h = a = 0$ at $r = 0$ to have well-defined functions at the origin. The equation $D_j c = 0$ gives $f = h$ everywhere. Substituting the ansatz into the expression for the energy, we obtain

$$E_{BPS} = \mp \int dS_j B_j^a (\pm) \left(\frac{x^a}{r} w \cos \alpha + \frac{x^a}{r} v \sin \alpha \right) \quad (59)$$

where the $+$ sign in parenthesis corresponds to the case $f(\infty) = 1$ and $-$ corresponds to $f(\infty) = -1$. We identify a magnetic flux

$$\Phi = (\pm) \int dS_j B_j^a \frac{x^a}{r} = (\mp) \frac{4\pi}{g} \quad (60)$$

and so the energy of the BPS monopole can be written as

$$E_{BPS} = \mp \Phi (w \cos \alpha + v \sin \alpha). \quad (61)$$

Minimizing the energy as a function of α , we obtain

$$E_{BPS} = \mp \Phi \sqrt{w^2 + v^2}. \quad (62)$$

By definition, Φ is negative for monopoles and positive for antimonopoles. Thus the upper sign corresponds to monopole with $f(\infty) = 1$ and the lower sign to antimonopoles with $f(\infty) = -1$.

In addition to the BPS monopole, there is another, topologically distinct monopole which occurs at finite temperature when A_4 is treated as a Higgs field. Starting from a static monopole solution where $|A_4| = w$ at spatial infinity, we apply a special gauge transformation

$$U_{special} = \exp \left[-\frac{i\pi x_4}{L} \tau^3 \right] \quad (63)$$

where τ^i is the Pauli matrix. Although $U_{special}$ is not periodic in x^4 , it transforms the scalar field as

$$\phi \rightarrow \exp \left[-\frac{i\pi x_4}{L} \tau^3 \right] \phi \exp \left[+\frac{i\pi x_4}{L} \tau^3 \right] \quad (64)$$

so that ϕ remains periodic: $\phi(\vec{x}, x_4 = 0) = \phi(\vec{x}, x_4 = L)$. However, A_μ transforms in such a way that the value of A_4 at spatial infinity is shifted: $w \rightarrow w - 2\pi/gL$. If we instead start from a static monopole solution such that $A_4 = 2\pi/gL - w$ at spatial infinity, then the action of $U_{special}$ gives a monopole solution with $A_4 = -w$ at spatial infinity. A final constant gauge transformation $U_{const} = \exp[i\pi\tau^2/2]$ yields a new monopole solution with $A_4 = w$ at spatial infinity. The distinction between the BPS solution, which is independent of x_4 , and the KK solution is made clear by consideration of the topological charge. The action of $U_{special}$ followed by U_{const} increases the topological charge by 1 and changes the sign of the monopole charge. Thus the KK solution is topologically distinct from the BPS solution because it carries instanton number 1. This is consistent with the KvBLL decomposition of instantons in the pure gauge theory with non-trivial Polyakov loop behavior, where $SU(2)$ instantons can be decomposed into a BPS monopole and a KK monopole. Our picture of the confined and mixed confined phases is one where instantons and anti-instantons have “melted” into their constituent monopoles and anti-monopoles, which effectively form a three-dimensional gas of magnetic monopoles. In the BPS limit, both the magnetic and scalar interactions are long-ranged; this behavior appears prominently, for example, in the construction of N -monopole solutions in the BPS limit.

We thus find that the BPS solution has energy

$$E_{BPS} = \frac{4\pi}{g} \sqrt{w^2 + v^2} \quad (65)$$

corresponding to an action

$$S_{BPS} = \frac{4\pi}{g} L \sqrt{w^2 + v^2} = \frac{4\pi}{g^2} \sqrt{4\theta^2 + g^2 L^2 v^2}. \quad (66)$$

For the KK solution, we have instead

$$S_{KK} = \frac{4\pi}{g^2} \sqrt{(2\pi - gLw)^2 + g^2 L^2 v^2} = \frac{4\pi}{g^2} \sqrt{(2\pi - 2\theta)^2 + g^2 L^2 v^2}. \quad (67)$$

Note that the action of a BPS monopole S_{BPS} can be written in the form ML , with M independent of L . With L regarded as the inverse temperature β , this might suggest an interpretation as a finite-energy solution of the Minkowski-space field equations. However, the explicit presence of θ has no obvious Minkowski-space interpretation. Furthermore, S_{KK} cannot be written in the form of a mass times L in any case. This indicates that these monopole solutions of the Euclidean field equations have no obvious continuation to Minkowski, a point we shall reconsider in section VI.

Although we used the BPS construction to exhibit the existence and some properties of the monopole solutions of our system, we must move away from the BPS limit to ensure that magnetic interaction dominate at large distances, *i.e.*, that the three-dimensional scalar interactions associated with A_4 and ϕ are not long-ranged. This behavior is natural in the confined and mixed confined phases, where the characteristic scale of the Debye (electric) screening mass associated with A_4 is large, on the order of g/L . It is well known that the BPS bound for the monopole mass holds as an equality only when the scalar potential is taken to zero. As mentioned above, in the case under consideration the scalar coupling λ must be very small for perturbation theory to be valid, but the potential for A_4 is not small. However, the two combined potential can be written together as a quartic potential in terms of the rotated fields b and c with some quartic coupling λ' for b . Numerical studies [28] have shown that the monopole action is given in general for $SU(2)$ as

$$LE_{BPS}C(\epsilon) \quad (68)$$

where $\epsilon = \sqrt{\lambda'}/g$. The function $C(\epsilon)$ varies monotonically from $C(0) = 1$ in the BPS limit to $C(\infty) = 1.787$ with the limiting behaviors

$$C = 1 + \frac{\epsilon}{2} \quad (69)$$

and

$$C = 1.787 - \frac{2.228}{\epsilon} + O(\epsilon^{-2}) \quad (70)$$

for small and large ϵ , respectively. Thus corrections to the BPS result for the monopole mass and action due to the potential terms are less than a factor of two. We will henceforth use the exact results for the actions in the BPS limit, neglecting corrections from U_{eff} for the sake of simplicity of notation. It is useful to note that the $SU(2)$ construction of the mixed phase monopoles extends to $SU(N)$ in the standard way, via the embedding of $SU(2)$ subgroups in $SU(N)$.

VI. TOPOLOGICAL EFFECTS IN THE FOUR PHASES

We can now discuss the topological content of each of the four phases we have found. It is important to understand that in all four phases, Wilson loops in planes orthogonal to the compact direction should show area law behavior. This is an old observation about the deconfined phase [29, 30] which is very clearly observed in lattice simulations of $SU(2)$ and $SU(3)$ at temperatures above the deconfinement transition [31, 32]. At first sight, this seems to directly conflict with the association of deconfinement with the loss of area-law behavior for Wilson loops. However, the introduction of a compact direction, as in the case of finite temperature, explicitly breaks space-time symmetry. In the case of finite temperature, Wilson loops measuring electric flux have perimeter behavior in the deconfined phase; Wilson loops measuring magnetic flux still obey an area law. This asymmetry in behavior can be understood on the basis of center symmetry. The full center symmetry of an $SU(N)$ gauge theory on a d -dimensional hypertorus T^d is $Z(N)^d$. While the $Z(N)$ symmetry may break spontaneously in the short compact direction, the other $Z(N)$ symmetries are unbroken, and thus the associated Wilson loops obey an area law. Given the known role of monopoles in the confined phase of $R^3 \times S^1$ [9], it is in some sense unsurprising that monopoles might play a role in the area law for Wilson loops in all four phases.

In order to understand the effects of monopoles play in the four phases we have identified, we must analyze their interactions. We begin with a discussion of quantum fluctuations around the monopole solutions. The contribution to the partition function of a single BPS monopole at finite temperature was considered by Zarembo [33], and is given formally by

$$Z_a = \int d\mu^a \exp[-S_a] \exp[-S_d] \det'[-\bar{D}_\mu^2 + M_g^2]_a^{-1} \det'[-\bar{D}_\mu^2 + M_s^2]_a^{-1/2} \quad (71)$$

where a denotes the type of monopoles, $a = \{BPS, KK, \overline{BPS}, \overline{KK}\}$, and the determinants are written with a prime to indicate that zero modes are omitted. The measure factor $d\mu^a$ associated with the collective coordinates (moduli)

of the monopole solution, including the Jacobians from the zero modes is given by [34]

$$\int d\mu^a = \mu^4 \int \frac{d^3x}{(2\pi)^{3/2}} J_x \int_0^{2\pi} \frac{d\phi}{(2\pi)^{1/2}} J_\phi \quad (72)$$

where x is the position and ϕ the $U(1)$ phase of the monopole and μ is a Pauli-Villars regulator. The corresponding Jacobians are

$$J_x = S_a^{3/2}, \quad J_\phi = NL S_a^{1/2}. \quad (73)$$

Each of the four zero modes contributes a factor of μ . We are interested in the behavior of the model in the case where the eigenvalues of M_s^2 and M_g^2 are much smaller than either μ^2 or L^{-2} . For the functional determinants, this limiting case is similar to the BPS limit, and the μ dependence of the functional determinant is given by [33]

$$\det' [-\bar{D}_\mu^2 + M_s^2]_a \approx \det' [-\bar{D}_\mu^2]_a \sim (NL\mu)^{1/3} \quad (74)$$

for the scalar determinant and similarly for the gauge field determinant. Collecting all the terms, each monopole carries a factor

$$\begin{aligned} Z_a &= c\mu^{7/2} (NL)^{1/2} S_a^2 \exp[-S_a + \mathcal{O}(1)] \int d^3x \\ &= \xi_a \exp[-S_a] \int d^3x \end{aligned} \quad (75)$$

in its contribution to Z . The factor ξ_a is $c\mu^{7/2} (NL)^{1/2} S_a^2$ where c is a numerical constant and the factor of d^3x represents the integration over the location of the monopole. From the construction of the KK monopole, we see that we have $\xi_{KK}(\theta) = \xi_{BPS}(\pi - \theta)$.

A. The Confined Phase

The renormalization of the functional determinant arising from quantum fluctuations around the monopole solution is particularly simple in the confined phase, as first observed by Davies *et al.* in the corresponding supersymmetric model [35]. The dependence on the Pauli-Villars regulator is removed, as usual, by coupling constant renormalization. We begin by reviewing the previously studied cases of a pure gauge theory with a deformation or with periodic adjoint fermions. The relation at one loop of the bare coupling and the regulator mass μ to a renormalization-group invariant scale Λ is

$$\Lambda^{b_0} = \mu^{b_0} e^{-8\pi^2/g^2 N} \quad (76)$$

where b_0 is the first coefficient of the β function divided by N :

$$b_0 = \frac{11}{3} - \frac{4}{3} \cdot \frac{n_f C(R_f)}{N} - \frac{1}{6} \cdot \frac{n_b C(R_b)}{N} \quad (77)$$

where n_f is the number of flavors of Dirac fermions in a representation R_f , n_b is the number of flavors of real scalars in a representation R_b , and $C(R)$ is obtained from $\text{Tr}_R(T^a T^b) = C(R)\delta^{ab}$. For the case of a pure gauge theory with a deformation, there are four collective coordinates and this gives a factor of μ^4 . The functional integral over gauge degrees of freedom gives rise to a factor $\det' [-D^2]^{-1} \propto \mu^{-1/3}$ and the action contributes a factor $\exp(-8\pi^2/g^2 N)$ in the confined phase. Thus the contribution of a single monopole to the partition function gives a factor

$$\mu^{4-\frac{1}{3}} e^{-8\pi^2/g^2 N} = \mu^{11/3} e^{-8\pi^2/g^2 N} = \Lambda^{11/3}. \quad (78)$$

A detailed calculation confirms what we know on dimensional grounds: the contribution $\xi_a e^{-8\pi^2/g^2 N} \propto L^{-3} (\Lambda L)^{11/3}$. Note that the eliminations of renormalization-dependent quantities by renormalization-independent quantities depends crucially on the coefficient of $1/g^2$ in the action. For the case of n_f Dirac fermions in the adjoint representation, we have a factor of $4 - 2n_f$ from the zero modes:

$$\mu^{(4-2n_f)-\frac{1}{3}+2n_f\frac{1}{3}} e^{-8\pi^2/g^2 N} = \mu^{11/3-4n_f/3} e^{-8\pi^2/g^2 N} = \Lambda^{11/3-4n_f/3} \quad (79)$$

for n_f Dirac fermions which is again renormalization group invariant.

For a gauge theory with n_b adjoint scalars plus a deformation, we have similarly that

$$\mu^{11/3-n_b/6} e^{-8\pi^2/g^2 N} = \Lambda^{11/3-n_b/6}. \quad (80)$$

This implies that for $n_b = 1$ the complete functional determinant prefactor depends on Λ and L as $L^{-3}(\Lambda L)^{7/2}$. As we have seen, the action of both the BPS and the KK monopole in the gauge plus scalar model will exactly equal $8\pi^2/g^2 N$ only in the confined phase, so this result is special to that phase.

The interaction of the monopoles is essentially the one described by Polyakov in his original treatment of the Georgi-Glashow model in three dimensions [10], slightly generalized to include both the BPS and KK monopoles. Let us consider, say, a BPS-type monopole and KK-type monopole located at \vec{x}_1 and \vec{x}_2 in the non-compact directions, with static worldlines in the compact direction. The interaction energy due to magnetic charge of such a pair is

$$E_{BPS-KK} = - \left(\frac{4\pi}{g} \right)^2 \frac{1}{4\pi |\vec{x}_1 - \vec{x}_2|} \quad (81)$$

and the associated action is approximately $S_{BPS} + S_{KK} + L E_{BPS-KK}$. As discussed above, this will be larger than the value obtained from the Bogomolny bound, but of the same order of magnitude. There is an elegant way to capture the dynamics of the monopole plasma, using an Abelian scalar field σ dual to the magnetic field. Assuming that the Abelian magnetic gauge field is three-dimensional for small L , we may write

$$L \int d^3x \frac{1}{2} B_k^2 = \int d^3x \frac{g^2}{32\pi^2 L} (\partial_k \sigma)^2 \quad (82)$$

where the normalization of σ is chosen to simplify the form of the interaction terms. The three-dimensional effective action is given by

$$L_{eff} = \frac{g^2}{32\pi^2 L} (\partial_j \sigma)^2 - \sum_a \xi_a e^{-S_a + i q_a \sigma} \quad (83)$$

where the sum is over the set $\{BPS, KK, \overline{BPS}, \overline{KK}\}$. Each species of monopole has its own magnetic charge sign $q_a = \pm$ as well as its own action S_a . The coefficients ξ_a represent the functional determinant associated with each kind of monopole, but the combination $\xi_a \exp(-S_a)$ may be usefully regarded as a monopole activity in terms of the statistical mechanics of a gas of magnetic charges. The generating functional

$$Z_\sigma = \int [d\sigma] \exp \left[- \int d^3x L_{eff} \right] \quad (84)$$

is precisely equivalent to the generating function of the monopole gas. This equivalence is a generalization of the equivalence of a sine-Gordon model to a Coulomb gas, and may be proved by expanding Z_σ in a power series in the ξ_a 's, and doing the functional integral over σ for each term of the expansion.

It is well known that the magnetic monopole plasma leads to confinement in three dimensions. For our effective three-dimensional theory, any Wilson loop in a hyperplane of fixed x_4 , for example a Wilson loop in the $x_1 - x_2$ plane, will show an area law. The original procedure of Polyakov [10] may be used to calculate the string tension, where the presence of a large planar Wilson loop causes the dual field σ to have a discontinuity on the surface associated with the loop and a half-kink profile on both sides. However, an alternative procedure is simpler where the discontinuity in the gauge field strength induced by the Wilson loop is moved to infinity so that the string tension is obtained from the kink solution connecting the two vacua of the dual field σ [9].

In the confined phase, the action and functional determinant factors for all four types of monopoles are the same, so we denote them by S_M and ξ_M . The potential term in the mixed and confined phases then reduces to

$$- \sum_a \xi_a e^{-S_a + i q_a \sigma} \rightarrow 4 \xi_M e^{-S_M} [1 - \cos(\sigma)] \quad (85)$$

which has minima at $\sigma = 0$ and $\sigma = 2\pi$; we have added a constant for convenience such that the potential is positive everywhere and zero at the minima. A one-dimensional soliton solution $\sigma_s(z)$ connects the two vacua, and the string tension σ_{3d} for Wilson loops in the three non-compact directions is given by

$$\sigma_{3d} = \int_{-\infty}^{+\infty} dz L_{eff}(\sigma_s(z)) \quad (86)$$

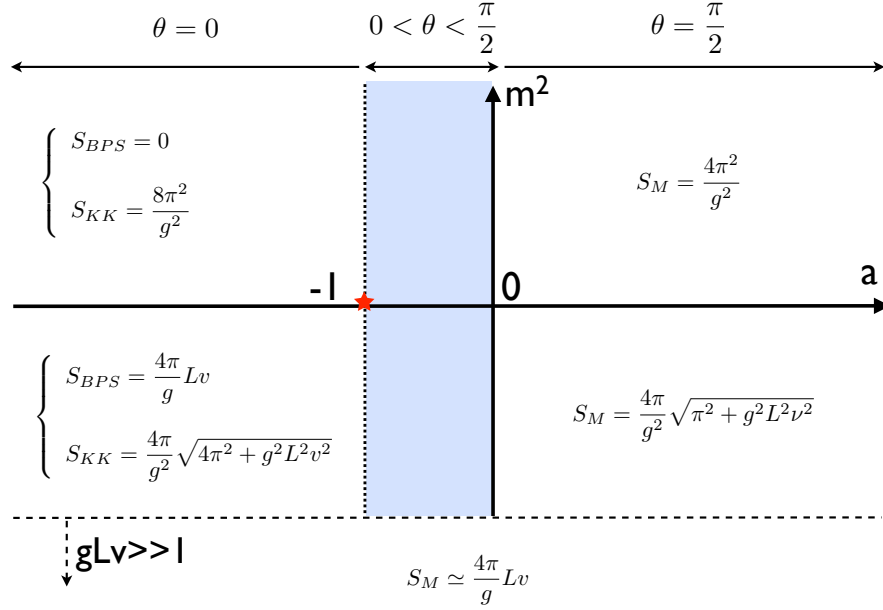


Figure 4: The phase diagram with the regions where various limiting cases for S_{BPS} and S_{KK} hold; the shaded region is a cross-over region where $2 > Tr_{FP} > 0$. Cross-over effects are negligible for $gLv \gg 1$. The star marks the point on the $m^2 = 0$ line where the deformation term is identically zero.

which can be calculated via yet another Bogomolny inequality to be

$$\sigma_{3d} = \frac{4g}{\pi} \sqrt{\frac{\xi_M}{L}} e^{-S_M}. \quad (87)$$

It is notable that in the confined phase σ_{3d} can be written in a form independent of the renormalization group scale.

B. Generalization to Other Phases

The naive generalization of the above results for the confined phase to the other three phases is straightforward. Writing explicitly the θ dependence, we have in general that $S_{BPS}(\theta)$ is not the same as $S_{KK}(\theta)$, and for arbitrary θ , $\xi_{BPS}(\theta) \neq \xi_{KK}(\theta)$. However, it is generally true that $\xi_{BPS}(\theta) = \xi_{\overline{BPS}}(\theta)$ and $\xi_{KK}(\theta) = \xi_{\overline{KK}}(\theta)$; furthermore, the explicit construction of the KK monopole from the BPS monopole shows that $\xi_{BPS}(\theta) = \xi_{KK}(\pi - \theta)$. The limiting cases of S_{BPS} and S_{KK} for $\theta = 0$ and $\pi/2$ and for $v = 0$ and large v are shown in Figure 4.

The construction of the sine-Gordon dual Lagrangian proceeds in a familiar way. Essentially, we must make the replacement

$$\xi_{BPS}(\pi/2) e^{-S_{BPS}(\pi/2)} \rightarrow \frac{1}{2} \left(\xi_{BPS}(\theta) e^{-S_{BPS}(\theta)} + \xi_{KK}(\theta) e^{-S_{KK}(\theta)} \right). \quad (88)$$

Repeating the calculation of the string tension leads to

$$\sigma_{3d} = \frac{4g}{\pi} \sqrt{\frac{1}{2L} \left(\xi_{BPS}(\theta) e^{-S_{BPS}(\theta)} + \xi_{KK}(\theta) e^{-S_{KK}(\theta)} \right)}. \quad (89)$$

However, there are two issues raised by this generalization. The first is the validity of the dilute monopole gas approximation. The assumption that the monopoles can be treated as well-separated objects will hold when $\xi_a \exp(-S_a) \ll 1$ for all monopole species. We will examine this point in detail below for all three remaining phases.

The second issue is technical: the renormalization group-dependence of the final result for σ_{3d} . As we have seen, in the confined phase σ_{3d} can be written in terms of L and Λ , with no dependence on the regulator μ . When $\theta \neq \pi/2$, the explicit cancellation of the μ dependence between ξ_a and $\exp(-S_a)$ does not occur: the μ dependence of $\xi_a(\theta)$ does

not depend on θ , but the coefficient of $1/g^2(\mu)$ in S_a is θ -dependent. This issue is not new, and not specific to Higgs models; it was discussed in the supersymmetric case in [35] in the context of the effective potential for σ . However, the effective Lagrangian L_{eff} represents only the long-distance behavior of the model; in fact, the cosine interaction is not even renormalizable in three dimensions. The underlying gauge theory is of course renormalizable, and the ultraviolet renormalization of instanton effects is well-understood. In the case of pure gauge theory, the renormalizability of monopole gas effects has been confirmed by detailed analysis of the relevant functional determinants [36, 37]. On the other hand, the effective Lagrangian represents only the long-ranged interaction mediated by σ . This interaction falls off very slowly with distance, because it is induced by non-perturbative effects. Interactions mediated by particles with masses obtained from perturbation theory must be integrated out to obtain L_{eff} [9]. This induces a dependence of the parameters of L_{eff} on some intermediate momentum scale on the order of the lightest perturbative mass. In the case at hand, this will be either the mass associated with A_4^3 or ϕ^3 , which are obtained by minimizing the effective potential with respect to θ and v . Thus L_{eff} is only valid up to the lightest perturbative scale, and its finite parameters depend implicitly on that scale, which in turn depend on θ and v . Thus the monopole activities are not simple functional determinants, but include the effects of integrating the instanton gas down to a scale where only the σ interaction remains. For notational simplicity, we will continue to denote the monopole activities by $\xi_a(\theta) \exp(-S_a(\theta))$. We now turn to consideration of the deconfined, mixed confined and Higgs phase in turn.

In the deconfined phase, we have $v = 0$, but $Z(2)_C$ is broken so $\theta \neq \pi/2$ and $Tr_F P \neq 0$. As we cross from the confined to the deconfined phase, the second-order character of the deconfinement transition means that θ will move continuously from its $Z(2)_C$ -symmetric value of $\pi/2$ towards 0 as a is decreased below 0. Throughout this phase, $v = 0$ and thus we have for the BPS action

$$S_{BPS} = \frac{4\pi}{g^2} \cdot 2\theta \quad (90)$$

and for the KK solution, we have instead

$$S_{KK} = \frac{4\pi}{g^2} (2\pi - 2\theta). \quad (91)$$

There is a natural region in the deconfined phase where the monopole dynamics is essentially identical to that in the confined phase. We begin by expanding the monopole actions around $\theta = \pi/2$. The BPS action in this limit becomes

$$S_{BPS} = \frac{8\pi}{g^2} \theta = \frac{4\pi^2}{g^2} + \frac{8\pi}{g^2} \delta \quad (92)$$

where we have made the substitution $\theta = \pi/2 + \delta$. The KK action becomes in the same limit

$$S_{KK} = \frac{4\pi}{g^2} (2\pi - 2\theta) = \frac{4\pi^2}{g^2} - \frac{8\pi}{g^2} \delta. \quad (93)$$

In order to obtain monopole physics similar to that of the confined phase we must require

$$S_{BPS} = S_{KK} = \frac{4\pi^2}{g^2} + \mathcal{O}(1) \quad (94)$$

which in turn implies that δ is no larger than $\mathcal{O}(g^2)$. From the effective action we constructed in section III, we have in the deconfined phase

$$U_{eff} = \frac{2}{\pi^2 L^4} \delta^4 + \frac{a}{L^4} \delta^2 \quad (95)$$

we see that δ will be non-zero only if a is negative. In that case, we must have

$$|a| \propto \delta^2 \lesssim g^4. \quad (96)$$

Thus the approximation that, $S_{BPS} = S_{KK} = \frac{4\pi^2}{g^2} + \mathcal{O}(1)$, is valid only in a very narrow region in the deconfined phase where $\theta = \pi/2 - \mathcal{O}(g^2)$ and $|a| \lesssim g^4$. We also expect that the functional determinants of the BPS and KK monopoles are approximately equal in this region. Thus, in this region all of the monopole physics which we worked out for the confined phase is valid: the monopole plasma is equivalent to a sine-Gordon field theory, and the string tension is obtained from the sine-Gordon kink solution.

In the region where $a < -1$, θ is zero, and we know that the interpretation of a finite-temperature instanton in terms of monopole constituents is probably lost. In pure gauge theories, the monopole constituent picture of the instanton breaks down at the classical level when $\theta \rightarrow 0$. As shown in [11, 12], in the pure $SU(2)$ gauge theory the instanton action density is well-localized into two separate lumps when $\theta = \pi/2$, but only one lump persists when $\theta \rightarrow 0$. This is reflected in the behavior of the formula for S_{BPS} as θ approaches zero. Nevertheless, the total action of a BPS-KK pair stays exactly at $S_{\text{instanton}} = S_{BPS} + S_{KK} = 8\pi^2/g^2$ for all values of θ . This suggests that the bulk of the confined phase, where $a < -1$, might be best interpreted in terms of an instanton gas rather than as a gas of monopoles. This region would then naturally extend to the right of the line segment $a = -1$ by a factor of $\mathcal{O}(g^2)$. However, it should be noted that the work of Rossi [38] showed that for pure gauge theories with $\theta = 0$, an infinite line of four-dimensional instantons with spacing L and scale parameter $2\pi/L$ is exactly equivalent to a monopole solution of the field equations. This solution was later realized to be equivalent to the $\theta = 0$ limit of the KK monopole. We will return to the relation of the Euclidean and Minkowski solutions below when we discuss a certain duality present in the system. The region where $0 < \theta < \pi/2$, corresponding to $-1 < a < 0$, appears to be a crossover region where the interpretation of the topological content is not yet clear, as the system moves smoothly from a dilute monopole gas near $a = 0$ to a phase where $S_{BPS} = 0$ for $a \leq -1$.

In the mixed phase, $\theta = \pi/2$ and v is non-zero. We have

$$S_{BPS} = S_{KK} = \frac{4\pi}{g^2} \sqrt{\pi^2 + g^2 L^2 v^2} \quad (97)$$

as in the confined phase. The functional determinants $\xi_{BPS}(\pi/2)$ and $\xi_{KK}(\pi/2)$ are equal as well. Because $v \neq 0$, the handling of ultraviolet divergences is not as simple as in the confined phase, but can be carried out in principle [39, 40]. The analysis of the string tension performed for the confined phase carries over, and σ_{3d} is given by

$$\sigma_{3d} = \frac{4g}{\pi} \sqrt{\frac{\xi_M}{L}} e^{-S_M} \quad (98)$$

where as before S_M and ξ_M are the common monopoles in this phase. There is a natural region next to the confined phase where $gLv < \pi$. In that region, we again have $S_M = 4\pi^2/g^2 + \mathcal{O}(1)$ and the renormalization group arguments used in the confined phase work here as well. Although not natural in the case $g(L) \ll 1$, there is a region far from the confined region where $gLv \gg 1$, where $S_M \approx 4\pi Lv/g$. This is precisely the action of a Minkowski-space monopole of mass $4\pi v/g$ with a worldline of length L ; we return to this point in the discussion of duality below.

In the Higgs phase, we have $\theta \neq \pi/2$ and $v \neq 0$ so both $Z(2)$ symmetries are broken. The action of a BPS monopole solution is

$$S_{BPS} = \frac{4\pi}{g^2} \sqrt{4\theta^2 + g^2 L^2 v^2} \quad (99)$$

but for the KK solution, we have instead

$$S_{KK} = \frac{4\pi}{g^2} \sqrt{(2\pi - 2\theta)^2 + g^2 L^2 v^2}. \quad (100)$$

There are several regions of interest with the Higgs phase. Near the critical line where θ is close to $\pi/2$, the behavior is similar to that of the mixed confined phase; the argument is exactly the same as for the deconfined phase when $\theta \approx \pi/2$ in relation to the confined phase. We also expect behavior similar to that of the mixed confined phase when $gLv \gg 1$. The region $gLv \gg 1$ may be treated in a manner very similar to Polyakov's original treatment of the three-dimensional Georgi-Glashow model, except that there is an additional factor of 2 in the monopole fugacity, and the three-dimensional instanton action S_{3d} is replaced by $4\pi Lv/g$. In both these regions, we have the approximate equality $S_{BPS} \approx S_{KK}$, and we expect the dilute monopole gas picture is valid. There is also a region where $gLv \leq \mathcal{O}(g^2)$ and $\theta = 0$ (for $a < -1$) which has the behavior of the $\theta = 0$ region of the deconfined phase.

In Figure 5, we show a final version of the phase diagram. The figure shows the large region where the dilute monopole gas description should be valid, and either $S_{BPS} = S_{KK}$ or $S_{BPS} \simeq S_{KK}$. Note that this region includes all of the confined and mixed confined regions, a large part of the Higgs phase, and a small part of the deconfined phase. The region where the dilute gas approximation is valid is somewhat larger. However, we have also indicated the region where the dilute gas approximation breaks down, because $S_{BPS} \approx 0$ and $S_{KK} \approx 8\pi^2/g^2$. For obvious reasons, we have labeled this region as an instanton region, although the correct treatment of topological excitations in this region is no clearer in the Higgs system than in the pure gauge case.

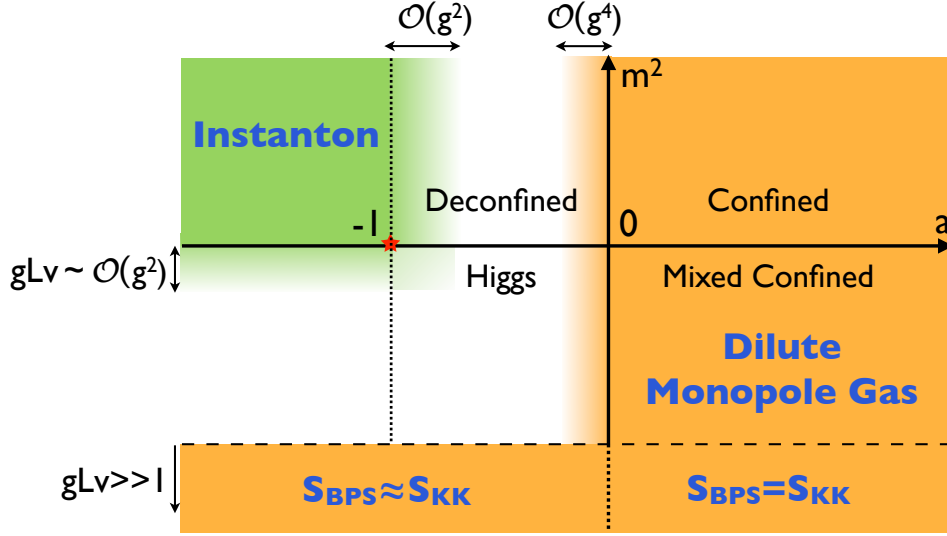


Figure 5: The phase diagram showing the region where the dilute monopole gas approximation is valid and $S_{BPS} \approx S_{KK}$. The dilute gas region itself is somewhat larger than the shaded region. The region labeled instanton is where the dilute monopole picture does not hold.

C. Duality

As we have seen, the regions where various approximations hold are not necessarily coincident with the phase boundaries. Essentially, the mixed confined phase mediates between the confined and Higgs phases, producing a broad band where the confining behavior of Wilson loops can be ascribed to a dilute monopole gas. Across each phase boundary (except possibly for the Higgs-deconfined boundary), the semiclassical expression for the string tension measured by Wilson loops varies smoothly. This would not be expected if the phase transitions were first-order, and singular corrections to the semiclassical picture are possible for second-order transitions due to coupling between the order parameters and the dual field σ . This sort of coupling of different order parameters is familiar in the PNJL model [26]. More important than the smooth behavior of the string tension, however, is the continuity of the monopole confinement mechanism across the confined, mixed and Higgs phases.

We can understand the role of topological excitations from a different point of view by invoking duality in a form similar to that used by Poppitz and Unsal in their analysis of the Seiberg-Witten model [1]; their work also serves as an introduction to duality in this context. The general issue in their work and here is the relation between topologically-stable solutions of the classical field equations in Euclidean space and Minkowski space. These are respectively solutions with finite action (instantons) and finite energy (monopoles). Higgs models with adjoint scalars have both, and two different approaches for computing the partition function on $R^3 \times S^1$ suggest themselves. We have extensively discussed the use of instantons, but another approach would be to consider the statistical mechanics of Minkowski-space solutions with finite energy, which are monopoles or more generally Julia-Zee dyons [41]. Such dyons will make contributions to the overall partition function proportional to $\exp(-LM)P$, where M is the monopole mass and P is a Polyakov loop factor. As we will see below, there is evidence that summing over finite-action instanton contributions to the partition function is equivalent to summing over finite-energy dyon contributions, extending the ideas in [1] to the case of non-supersymmetric Higgs models on $R^3 \times S^1$.

Our approach is somewhat different from that of Poppitz and Unsal, in that we relate a finite rather than infinite sum over Euclidean monopoles to an infinite sum of Minkowski-space dyons. We begin with an easy variant of the Poisson summation formula associated with $Z(N)_C$. Let $f(\theta)$ be a function defined on the interval $-\pi < \theta < \pi$. We define the Fourier series in the usual way:

$$f(\theta) = \sum_{n \in \mathbb{Z}} \tilde{f}(n) e^{in\theta} \quad (101)$$

$$\tilde{f}(n) = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} f(\theta) e^{-in\theta}. \quad (102)$$

Then we have that

$$\begin{aligned}
\sum_{k=0}^{N-1} f\left(\theta - \frac{2\pi k}{N}\right) &= \sum_{n \in \mathbb{Z}} \tilde{f}(n) \sum_{k=0}^{N-1} e^{in(\theta - \frac{2\pi k}{N})} \\
&= \sum_{n \in \mathbb{Z}} \tilde{f}(n) e^{in\theta} N \delta(n \equiv 0 \pmod{N}) \\
&= \sum_{n \in \mathbb{Z}} N \tilde{f}(nN) e^{inN\theta}
\end{aligned} \tag{103}$$

so that for $N = 2$ only the even coefficients $\tilde{f}(2n)$ contribute. Let us apply this identity to the combination

$$\xi_{BPS}(\theta) e^{-S_{BPS}(\theta)} + \xi_{KK}(\theta) e^{-S_{KK}(\theta)} = \xi_{BPS}(\theta) e^{-S_{BPS}(\theta)} + \xi_{BPS}(\pi - \theta) e^{-S_{BPS}(\pi - \theta)} \tag{104}$$

which occurs in the dual Lagrangian and in the formula for σ_{3d} so we have

$$f(\theta) = \xi_{BPS}(\theta) e^{-S_{BPS}(\theta)}. \tag{105}$$

For small g^2 , $S_{BPS}(\theta)$ is strongly peaked at $\theta = 0$, so we can make the approximation

$$\tilde{f}(n) \simeq \int_0^\infty \frac{d\theta}{\pi} \xi_{BPS}(0) e^{-S_{BPS}(\theta)} e^{in\theta}. \tag{106}$$

Although this integral, with the limits taken to infinity, can be evaluated in a saddle point approximation, it can also be evaluated exactly [1], giving

$$\tilde{f}(2n) \simeq \xi_{BPS}(0) \frac{gLv}{2\pi} \cdot \frac{\frac{4\pi}{g^2}}{\sqrt{\left(\frac{4\pi}{g^2}\right)^2 + n^2}} K_1 \left[gLv \sqrt{\left(\frac{4\pi}{g^2}\right)^2 + n^2} \right]. \tag{107}$$

The Higgs phase represents the most general domain of applicability of the duality transformation, because in the Higgs phase $v \neq 0$ and $0 \leq \theta < \pi/2$. It is natural to introduce $M(n)$ the mass of a Minkowski-space Julia-Zee dyon [41] of magnetic charge $4\pi/g$ and electric charge ng

$$M(n) = gv \sqrt{\left(\frac{4\pi}{g^2}\right)^2 + n^2} \tag{108}$$

so that we can write

$$\tilde{f}(2n) \simeq \xi_{BPS}(0) \frac{LM(0)}{2\pi} \cdot \frac{1}{\sqrt{\left(\frac{4\pi}{g^2}\right)^2 + n^2}} K_1[LM(n)]. \tag{109}$$

The asymptotic expansion of the Bessel function for large argument gives a factor of $\exp[-LM(n)]$:

$$\tilde{f}(2n) \simeq \xi_{BPS}(0) \frac{LM(0)}{2\pi} \cdot \frac{1}{\sqrt{\left(\frac{4\pi}{g^2}\right)^2 + n^2}} \sqrt{\frac{\pi}{2LM(n)}} \exp[-LM(n)]. \tag{110}$$

Thus each term in the sum carries a factor of $\exp[-LM(n) + i2n\theta]$. This suggests an obvious interpretation of the finite sum over BPS and KK monopoles, which are constituents of instantons, as being equivalent to a gas of Julia-Zee dyons, each carrying a Polyakov loop factor appropriate to its charge. This interpretation is valid throughout most of the Higgs and mixed confined phases, except in the region near $m^2 = 0$ where the mass of the lightest dyon $M(0) = 4\pi v/g$, which is a Minkowski-space monopole, becomes light. Within this framework, the only significant difference between the mixed confined and Higgs phases is that in the mixed confined phase, θ is restricted to $\pi/2$.

When we cross the phase boundary $m^2 = 0$, we move into a region where $v = 0$. As long as we stay away from the region where θ is zero or $\mathcal{O}(g^2)$, the approximate form of the Fourier coefficients is valid, and we have

$$\tilde{f}(2n) \simeq \frac{2g^2}{16\pi^2 + g^4 n^2} \tag{111}$$

which tells us that

$$\exp\left(-\frac{8\pi}{g^2}\theta\right) + \exp\left(-\frac{8\pi}{g^2}(\pi - \theta)\right) \approx \sum_{n \in \mathbb{Z}} \frac{4g^2}{16\pi^2 + g^4 n^2} e^{i2n\theta}. \quad (112)$$

Although the right-hand side is a good approximation to the left-hand side as θ varies, it is striking how different the two forms are. However, an exact evaluation of $\tilde{f}(2n)$ in the limit $v = 0$ gives

$$\tilde{f}(2n) = \frac{1}{\pi} \int_0^\pi d\theta \exp\left[-\left(\frac{8\pi}{g^2} + i2n\right)\theta\right] = \frac{1}{\frac{8\pi}{g^2} + i2n} \left(1 - e^{-8\pi^2/g^2}\right) \quad (113)$$

showing that the non-perturbative behavior has not totally disappeared.

VII. CONCLUSIONS

We have shown that the phase structure of the deformed $SU(2)$ adjoint Higgs model on $R^3 \times S^1$ is rich, with four different phases distinguished by the behavior of the three gauge-invariant order parameters associated with the global symmetries of the model. We have used a particular deformation which makes the phase diagram simple, but the appearance of four distinct phases is general. Despite the $Z(2)_C \times Z(2)_H$ global symmetry, the phase transitions separating the different phases may be of second order or of first order. In addition to the known confined, deconfined and Higgs phases, we have found a fourth phase, the mixed confined phase, which takes the place of what would be a confining phase with a Higgs mechanism. In the mixed confined phase, the behavior of A_4 and ϕ become entwined in such a way that the global symmetry group $Z(2)_C \times Z(2)_H$ breaks spontaneously to a $Z(2)$ symmetry which acts non-trivially on both A_4 and ϕ . This behavior, found using perturbation theory, extends to the topological properties of the model, where the BPS and KK monopole solutions are constructed using a linear combination of A_4 and ϕ . The area-law behavior of Wilson loops orthogonal to the compact S^1 direction can be attributed to a dilute magnetic monopole gas in at least part of all four phases. There are several unresolved issues. The correct treatment of topology in the deconfined phase when $\theta = 0$, corresponding to the high-temperature limit $T \gg \Lambda$ in the case of finite temperature, remains elusive. A detailed calculation of the monopole activities in the effective Lagrangian which determines σ_{3d} would be useful in comparing with lattice results. The correct interpretation of the duality between Euclidean space monopoles, which are constituents of monopoles, and Minkowski-space dyons is compelling, but incomplete. There is also the question of generalizing our $SU(2)$ results to $SU(N)$ adjoint Higgs models on $R^3 \times S^1$. For $SU(N)$ gauge theories on $R^3 \times S^1$, the natural set of order parameters is $Tr_F P^k$, and the $Z(N)$ center symmetry can break to a subgroup $Z(p)$ [4, 7]. With the addition of an adjoint scalar, there is the additional set of order parameters of the form $Tr_F P^k \phi$ available. This suggests a very rich phase structure is possible. Finally, we note that many of the predictions we have made may be difficult to test, because lattice simulations of the three-dimensional Higgs model are consistent with Polyakov's semiclassical results only over a narrow region [42]. However, the overall phase structure we have predicted in our four-dimensional model should be relatively easy to test with lattice simulations.

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