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Relativistic (Lattice) Boltzmann Equation with Non-Ideal Equation of State

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Abstract

The relativistic Boltzmann equation for a single particle species generally implies a fixed, unchangeable equation of state that corresponds to that of an ideal gas. Real-world systems typically have more complicated equation of state which cannot be described by the Boltzmann equation. The present work derives a 'Boltzmann-like' equation that gives rise to a conserved energy-momentum tensor with an arbitrary (but thermodynamically consistent) equation of state. Using this, a Lattice Boltzmann scheme for diagonal metric tensors and arbitrary equations of state is constructed. The scheme is verified for QCD in the Milne metric by comparing to viscous fluid dynamics.

I. MOTIVATION

The Boltzmann equation is a tool that has proven to be very useful in many different areas of physics. Despite its usefulness, there are some properties of the Boltzmann equation that are not optimal for modelling physics systems. In particular, for a single particle species the equation of state is fixed by one parameter alone, namely the particle's mass. Since in the limit of a small particle mean free path the Boltzmann equation describes fluid dynamics, this implies that the equation of state for the fluid hence described is unchangeable and (typically) not realistic. This is a problem in particular for the so-called Lattice Boltzmann Approach to fluid dynamics [1–3], where the Boltzmann equation serves as a convenient algorithm for computing the behavior of fluids. In the non-relativistic context, ways to circumvent this problem are known, e.g. a modification of the equilibrium distribution function, modifying only the pressure components or introducing a new force term [4–8], but it is not obvious how to generalize those to the relativistic case [9, 10].

On the other hand, it is known that particle masses change when considering a heat bath: for instance, photons acquire temperature-dependent masses in a plasma, which leads to a corresponding change of the plasma equation of state [11]. In high-temperature Quantum-Chromodynamics (QCD), these medium-dependent quasiparticles have been successfully used to model the QCD equation of state [12]. Is it thus possible to write down a 'Boltzmann-like' equation for a single particle species with a medium-dependent mass that can reproduce any thermodynamically consistent equation of state? The objective of the present work is to give an affirmative answer to this question by means of an explicit construction.

Note that in the context of quasiparticle and Nambu-Jona-Lasinio models, essentially all the relevant parts of the present derivation can be found [13–17]. In this sense, the results presented here are not new. However, as far as I can tell, all published results employ multiple species of particles, while the results below are for a single species of a 'virtual' particle, and therefore probably computationally cheaper. Also, to my knowledge the present work is the first to provide a concrete example for an algorithm outside equilibrium with an arbitrary equation of state. The article is structured as follows: in Sec. II, I give a textbook-style review of the Boltzmann equation in curved spaces. In Sec. III, a framework for arbitrary non-ideal equations of state is set up and subsequently tested for the case of QCD at high temperature. In Sec. IV, a relativistic lattice Boltzmann scheme for matter

with a non-ideal equation of state in curved spacetime is given, with the particular example of QCD in a Milne spacetime that may be of relevance for high energy nuclear collisions. Finally, I conclude in section V.

II. BOLTZMANN EQUATION IN CURVED SPACE: A REVIEW

This section gives a text-book style review of the Boltzmann equation in curved space, introducing the usual particle current and energy-momentum tensor. Expert readers may want to skip this section and read on in Sec. III.

The Boltzmann equation specifies the evolution of the single particle distribution function $f(X^\mu, P^\mu)$, which is dependent on space-time $X^\mu \equiv (t, \mathbf{x})$ and four-momentum $P^\mu \equiv (E, \mathbf{p})$. If collisions are absent, but forces such as gravity are present, particles are assumed to propagate along geodesics which can be parameterized by an affine parameter \mathcal{T} . Accordingly, the single particle distribution f does not change along geodesics,

$$\frac{df}{d\mathcal{T}} = \frac{dt}{d\mathcal{T}} \frac{\partial f}{\partial t} + \frac{d\mathbf{x}}{d\mathcal{T}} \frac{\partial f}{\partial \mathbf{x}} + \frac{dP^\alpha}{d\mathcal{T}} \frac{\partial f}{\partial P^\alpha} = 0.$$

Multiplying with the mass m one can recognize $m dt/d\mathcal{T} = E$, $m d\mathbf{x}/d\mathcal{T} = \mathbf{p}$, the energy and momentum of a relativistic particle. When re-instating collisions, particles will no longer follow geodesics, so $df/d\mathcal{T}$ will no longer be vanishing. Hence in the general case one has

$$P^\mu \partial_\mu f + F^\alpha \partial_\alpha^{(p)} f = -\mathcal{C}[f], \quad (2.1)$$

where $\mathcal{C}[f]$ is the collision term and $F^\alpha \equiv m \frac{dP^\alpha}{d\mathcal{T}}$ the force felt by individual particles. For gravity, the force is given by $F^\alpha = -\Gamma_{\mu\nu}^\alpha P^\mu P^\nu$ where $\Gamma_{\mu\nu}^\alpha$ are the Christoffel symbols that are calculated as derivatives of the underlying metric tensor $g_{\mu\nu}$. For electromagnetism, the force is given by the Lorentz force $F^\alpha = q F^{\alpha\beta} P_\beta$ where $F^{\alpha\beta}$ is the electromagnetic field strength tensor that can be specified in terms of electric and magnetic fields, and q is the particle's charge.

Including both the gravitational and electromagnetic force terms, let us now take an integral moment of Eq. (2.1) with weight

$$\int d\chi \equiv \int \frac{d^4 P}{(2\pi)^4} \sqrt{-g} 2\Theta(p^0) (2\pi) \delta(g_{\mu\nu} P^\mu P^\nu - m^2), \quad (2.2)$$

where for clarity $d^4 P = \prod_{\mu=0}^3 dP^\mu$ and Θ denotes the Heaviside step-function, g denotes the determinant of the metric tensor $g_{\mu\nu}$ and I have adopted the 'mostly-minus' sign convention

for the metric. The delta-function in $d\chi$ places particles on the mass shell and the step-function picks out positive energy states. Apart from the appearance of p^0 , which could be replaced by a scalar product with a future pointing four-vector, this form of $d\chi$ is Lorentz covariant (cf. [18]). Using $\partial_\mu \sqrt{-g} = \sqrt{-g} \Gamma_{\alpha\mu}^\alpha$ and $\partial_\lambda g_{\mu\nu} = \Gamma_{\lambda\mu}^\rho g_{\rho\nu} + \Gamma_{\lambda\nu}^\rho g_{\rho\mu}$ one has

$$\begin{aligned} \sqrt{-g} P^\mu \partial_\mu f &= \nabla_\mu (\sqrt{-g} P^\mu f) - 2\sqrt{-g} P^\mu \Gamma_{\alpha\mu}^\alpha f, \\ \nabla_\mu [2\Theta(p^0) \delta(P^2 - m^2)] &= 2\Theta(p^0) \delta'(P^2 - m^2) 2P^\alpha P_\beta \Gamma_{\alpha\mu}^\beta, \end{aligned}$$

where ∇_μ denotes the (geometric) covariant derivative. Rewriting $2P_\beta \delta'(P^2 - m^2) = \partial_\beta^{(p)} \delta(P^2 - m^2)$ and using partial integration one finds

$$\int d\chi P^\mu \partial_\mu f = \nabla_\mu \int d\chi P^\mu f + \int d\chi \Gamma_{\alpha\mu}^\beta P^\alpha P^\mu \partial_\beta^{(p)} f.$$

Also, it is straightforward to show that $\int d\chi F^{\alpha\beta} P_\beta \partial_\alpha^{(p)} f = 0$ via partial integration and the fact that $F^{\alpha\beta} = -F^{\beta\alpha}$. Hence the Boltzmann equation implies

$$\nabla_\mu \int d\chi P^\mu f = - \int d\chi \mathcal{C}[f]. \quad (2.3)$$

Defining the particle number current as $N^\mu \equiv \int d\chi P^\mu f$ one finds that the Boltzmann equation implies the covariant conservation of particle number, $\nabla_\mu N^\mu = 0$, if $\int d\chi \mathcal{C}[f] = 0$.

Taking the integral moment $\int d\chi P^\nu$ of Eq. (2.1), one finds

$$\nabla_\mu \int d\chi P^\mu P^\nu f - q F^{\nu\beta} \int d\chi P_\beta = - \int d\chi P^\nu \mathcal{C}[f]. \quad (2.4)$$

For uncharged particles ($q = 0$), and defining the energy-momentum tensor as $T^{\mu\nu} \equiv \int d\chi P^\mu P^\nu f$, the Boltzmann equation implies covariant conservation of energy and momentum if

$$\int d\chi P^\nu \mathcal{C}[f] = 0. \quad (2.5)$$

I will assume the collision term to fulfill Eq. (2.5) for the remainder of this work. For charged particles, the Boltzmann equation implies

$$\nabla_\mu T^{\mu\nu} = q F^{\nu\beta} N_\beta,$$

or the change of energy and momentum being caused by the Lorentz force for a current $J_\beta \equiv q N_\beta$. For the remainder of this work, I will deal with uncharged particles ($q = 0$). However, the generalization to charged particles should be straightforward.

A. Equation of State for Uncharged Boltzmann Gas

In equilibrium, the energy-momentum tensor is given by ideal hydrodynamics,

$$T_{\text{eq}}^{\mu\nu} = \epsilon U^\mu U^\nu - p \Delta^{\mu\nu}, \quad (2.6)$$

where U^μ is the fluid velocity obeying $U^2 = 1$ and $\Delta^{\mu\nu} \equiv g^{\mu\nu} - U^\mu U^\nu$. The equilibrium energy density ϵ and pressure p of the system are related by the equation of state. Since Eq. (2.6) must correspond to the particle's energy-momentum tensor in equilibrium, one has

$$\epsilon = U_\mu U_\nu T_{\text{eq}}^{\mu\nu} = \int d\chi (P^\mu U_\mu)^2 f_{\text{eq}}, \quad p = -\frac{\Delta_{\mu\nu}}{3} T_{\text{eq}}^{\mu\nu} = -\frac{1}{3} \int d\chi [P^2 - (P^\mu U_\mu)^2] f_{\text{eq}},$$

which may be conveniently evaluated by performing a Lorentz boost to the frame where $P^\mu U_\mu = p^0$ (recall that $d\chi$ is Lorentz covariant).

Let us now consider a specific equilibrium distribution function for a system of uncharged particles (cf. [18]),

$$f_{\text{eq}}(X^\alpha, P^\alpha) = Z \times \exp \left[- \left(\frac{P^\alpha U_\alpha - \mu}{T} \right) \right], \quad (2.7)$$

where Z denotes the number of degrees of freedom and μ and T are the chemical potential and temperature, respectively. In Eq. (2.7), U_α is a macroscopic velocity that can be identified with the fluid velocity in Eq. (2.6). In this case, ϵ, p and the number density $n \equiv U_\mu N^\mu$ may be evaluated as¹

$$\epsilon = \frac{Z e^{\mu/T} m^2 T}{2\pi^2} \left(3TK_2 \left(\frac{m}{T} \right) + mK_1 \left(\frac{m}{T} \right) \right), \quad p = \frac{Z e^{\mu/T} m^2 T^2}{2\pi^2} K_2 \left(\frac{m}{T} \right), \quad n = p/T,$$

by using the identity $\int_m^\infty (x^2 - m^2)^{n+1/2} e^{-x/T} = (2n+1)!! K_{1+n}(m/T) (mT)^{n+1}$ for modified Bessel functions K_α . It is straightforward to show that these results obey the basic thermodynamic relations

$$\epsilon + p = sT + \mu n, \quad d\epsilon = Tds + \mu dn, \quad (2.8)$$

where s denotes the entropy density. From the equation of state, an interesting quantity to calculate is the speed of sound squared $c_s^2 \equiv dp/d\epsilon$. For illustration, at $\mu = 0$ it can be calculated from the above expressions as

$$c_s^2(T, \mu = 0) = \left(3 + \frac{m}{T} \frac{K_2(m/T)}{K_3(m/T)} \right)^{-1},$$

¹ Note that this definition of n corresponds to $\partial p / \partial \mu$. To see this, first go to the local rest frame where $P^2 - (P^\alpha U_\alpha)^2 = \mathbf{p}^2$ and then rewrite $\partial f_{\text{eq}} / \partial \mu = -\partial_0^{(p)} f_{\text{eq}}$. Integrate by parts and rewrite $2\mathbf{p}^2 \delta'(P^2 - m^2) = -p^i \partial_i^{(p)} \delta(P^2 - m^2)$. Another integration by parts then gives $\partial p / \partial \mu = U_\mu N^\mu$.

which increases monotonically with temperature from zero to $1/3$. Also, the relation $p = nT$ is the equation of state of an ideal gas. Clearly, non-ideal equations of state with a non-monotonic behavior of c_s or $p \neq nT$ are not describable in this framework.

In particular, note that changing the behavior of the equilibrium distribution function f_{eq} will not change the relation $p = nT$, and hence does not provide the freedom needed to describe a particular non-ideal equation of state that is dictated by nature.

III. NON-IDEAL EQUATIONS OF STATE

As shown in the preceding section, the Boltzmann equation (2.1) for a single uncharged particle species leads to equations of state that depend only on one parameter, namely the particle's mass. In order to describe arbitrary equations of state with a single uncharged particle species I therefore want to consider temperature (and density) dependent masses $m \rightarrow M(T, \mu)$, motivated by the fact that in a plasma at high temperature or density this approach is physically sound [11]. The particles described by the Boltzmann equation should then be regarded as virtual or 'quasi'-particles, but for sufficiently non-ideal equations of state, they will no longer correspond to any real excitations found in nature. However, the virtue of introducing this virtual particles will be that no long-range forces or particle mixtures will be necessary to describe the macroscopic system dynamics.

One immediate problem that arises when considering medium-dependent masses is that thermodynamic consistency is no longer guaranteed. Specifically, basic thermodynamic relations imply that

$$\epsilon + p = T \left. \frac{\partial p}{\partial T} \right|_{\mu} + \mu \left. \frac{\partial p}{\partial \mu} \right|_T, \quad (3.1)$$

which would be violated when inserting $m \rightarrow M(T, \mu)$ in the formulas from Sec. II A. To fix thermodynamic consistency, I propose the following alternate definition for the energy-momentum tensor:

$$T^{\mu\nu} \equiv \int d\chi P^{\mu} P^{\nu} f + B(T, \mu) g^{\mu\nu}, \quad (3.2)$$

where $B(T, \mu)$ is a function that will be determined by requiring thermodynamic consistency in equilibrium, cf. (3.1). Calculating energy density and pressure from (3.2), one finds that $B(T, \mu)$ drops out in $\epsilon + p$ and that thermodynamic consistency requires

$$0 = dB + \frac{1}{2} \int d\chi f_{\text{eq}} dM^2, \quad (3.3)$$

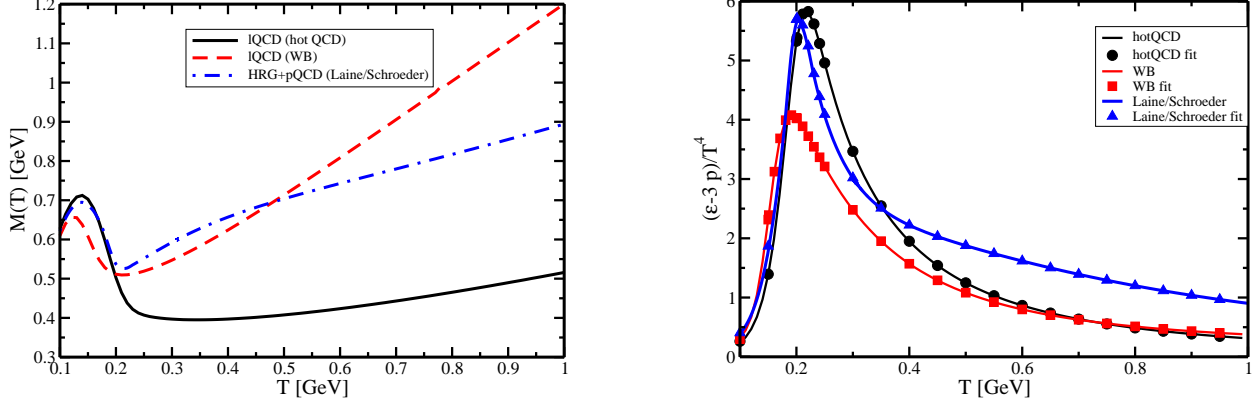


FIG. 1: Left: Results for $M(T)$ when fitting the entropy density from lattice QCD collaborations (hotQCD [19] and Wuppertal-Budapest (WB) [20], respectively) or an interpolation from hadron-resonance gas to perturbative QCD (Laine/Schröder, Ref. [21]). Right: quality of the fit (symbols) when comparing the trace anomaly $\epsilon - 3p$ to the original lattice QCD results (full lines).

where I used $2\mathbf{p}^2\delta'(P^2 - M^2) = -p^i\partial_i^{(p)}\delta(P^2 - M^2)$ and integration by parts.

Considering the concrete example (2.7), one has explicitly

$$\begin{aligned}\epsilon &= \frac{Ze^{\mu/T}M^2T^2}{2\pi^2} \left[3K_2\left(\frac{M}{T}\right) + \frac{M}{T}K_1\left(\frac{M}{T}\right) \right] + B(T, \mu), \\ p &= \frac{Ze^{\mu/T}M^2T^2}{2\pi^2} K_2\left(\frac{M}{T}\right) - B(T, \mu), \quad n = \frac{Ze^{\mu/T}M^2T}{2\pi^2} K_2\left(\frac{M}{T}\right).\end{aligned}\quad (3.4)$$

A. Example: QCD at Small Densities

Let us consider the above construction for the QCD equation of state at zero baryon chemical potential. In order for the Boltzmann energy-momentum tensor to correctly reproduce the high temperature limit of QCD with $N_c = 3, N_f = 3$, one has to set

$$Z = \frac{\pi^4}{180} (4(N_c^2 - 1) + 7N_cN_f).$$

Then, one can determine $M(T)$ by inverting $\frac{\epsilon+p}{T} = \frac{Z}{2\pi^2} M^3 K_3\left(\frac{M}{T}\right) = s_{lQCD}$, where s_{lQCD} may be obtained from the lattice QCD results (cf. [19, 20], both $N_f = 3$) or an interpolation from hadron resonance gas results to perturbative QCD (cf. [21], $N_f = 4$). Thermodynamic consistency requires $-\frac{ZM^2T}{2\pi^2} K_1\left(\frac{M}{T}\right) \frac{dM}{dT} = \frac{dB(T)}{dT}$, which can be solved for $B(T)$ numerically by integrating up from small temperatures where $B \simeq 0$. The resulting fits for the masses and the quality of the fit for the quantity $\epsilon - 3p$ for three 'physical' QCD equations of state are shown in Fig. 1.

B. The 'Boltzmann-like' Equation

The modified energy-momentum tensor (3.2) is no longer expected to correspond to a moment of the Boltzmann equation (2.1), because of the extra term in (3.2). However, one can ask if there is a modified 'Boltzmann-like' equation that will give $\nabla_\mu T^{\mu\nu} = 0$ (for uncharged particles). Inverting the steps leading to this equation in Sec. II, and rewriting $P^\mu \delta'(P^2 - M^2) = \frac{1}{2} \partial_{(p)}^\mu \delta(P^2 - M^2)$ and integrating by parts I find that

$$\nabla_\mu T^{\mu\nu} = \int d\chi P^\nu \left[P^\mu \partial_\mu - \Gamma_{\alpha\beta}^\lambda P^\alpha P^\beta \partial_\lambda^{(p)} + \frac{1}{2} \partial_\mu M^2 \partial_{(p)}^\mu \right] f = 0,$$

where the term involving $B(T, \mu)$ cancels if

$$0 = dB + \frac{1}{2} \int d\chi f dM^2. \quad (3.5)$$

Note that this is the same as the thermodynamic consistency condition (3.3), except that it is promoted to hold also out of equilibrium. As a consequence, the 'Boltzmann-like' equation

$$P^\mu \partial_\mu f - \Gamma_{\alpha\beta}^\lambda P^\alpha P^\beta \partial_\lambda^{(p)} f + \frac{1}{2} \partial_\mu M^2 \partial_{(p)}^\mu f = -\mathcal{C}[f] \quad (3.6)$$

is guaranteed to reproduce a conserved energy-momentum tensor that allows arbitrary equations of state parameterized by medium-dependent masses $M(T, \mu)$. Note that the 'new' term $\frac{1}{2} \partial_\mu M^2 \partial_{(p)}^\mu f$ precisely corresponds to the result found when deriving the Boltzmann equation from quantum field theory (cf. [22]). Finally, repeating the steps in Sec. II, one finds that for $\int d\chi \mathcal{C}[f] = 0$, Eq. (3.6) leads to

$$\nabla_\mu N^\mu = 0, \quad N^\mu \equiv \int d\chi P^\mu f, \quad (3.7)$$

so that the current is formally unchanged when considering medium-dependent masses.

C. On-shell-ization

For some applications, it is useful to explicitly perform the dp^0 integral of the Boltzmann-like equation. The reason is that if one is interested in moments of the Boltzmann equation with respect to the integral measure $d\chi$, this will allow one to work with a distribution function \hat{f} that then only depends on \mathbf{p} rather than the four momentum P^μ . When discretizing momenta (see below), one thus only has to deal with three dimensions rather than

four. Note that any factors of p^0 that one may have wanted to include *before* integration will simply turn into multiplicative factors of $E \equiv \sqrt{(-g_{ij}p^ip^j + M^2)/g_{00}}$ because of the delta-function that is part of $d\chi$. Defining

$$\hat{f}(X^\mu, \mathbf{p}) \equiv \int dp^0 2p^0 \Theta(p^0) \delta(P^2 - M^2) f(X^\mu, p^0, \mathbf{p}) \quad (3.8)$$

which is in accordance with [23] up to a factor of g_{00} , one can integrate Eq. (3.6) with $\int dp^0 2\Theta(p^0) \delta(P^2 - M^2)$, finding

$$\partial_\mu \left(P^\mu \frac{\hat{f}}{E} \right) - \Gamma_{\alpha\mu}^i \partial_i^{(p)} \left(P^\alpha P^\mu \frac{\hat{f}}{E} \right) + \frac{1}{2} \partial_i M^2 \partial_{(p)}^i \frac{\hat{f}}{E} + 2\Gamma_{\alpha\mu}^\alpha P^\mu \frac{\hat{f}}{E} = -\frac{1}{E} \mathcal{C}[\hat{f}], \quad (3.9)$$

where P^μ here is to be understood as on-shell momentum, $P^\mu = (E, \mathbf{p})$. In terms of \hat{f} , the particle current and energy-momentum tensor are given as

$$N^\mu = \int \frac{d^3p}{(2\pi)^3} \sqrt{-g} P^\mu \frac{\hat{f}}{E}, \quad T^{\mu\nu} = \int \frac{d^3p}{(2\pi)^3} \sqrt{-g} P^\mu P^\nu \frac{\hat{f}}{E} + B(T, \mu) g^{\mu\nu}. \quad (3.10)$$

As a specific example (cf. [23]), consider a metric with a line element of the form $ds^2 = dt^2 - R(t)^2 (d\mathbf{x}^2)$. Then $\Gamma_{\alpha\mu}^i P^\alpha P^\mu = 2E p^i R'/R$, $\Gamma_{\alpha\mu}^\alpha P^\mu = 3E R'/R$ and Eq. (3.9) becomes

$$P^\mu \partial_\mu \hat{f} - \frac{2p^i E R'}{R} \partial_i^{(p)} \hat{f} + \frac{1}{2} \partial_i M^2 \partial_{(p)}^i \hat{f} = -\mathcal{C}[\hat{f}].$$

As another example, consider the line element $ds^2 = d\tau^2 - dx^2 - dy^2 - \tau^2 dY^2$ (Milne metric). Then one has $\Gamma_{\alpha\mu}^i P^\alpha P^\mu = -2E p^Y g^{iY} \tau$, $\Gamma_{\alpha\mu}^\alpha P^\mu = E/\tau$, so that one finds

$$P^\mu \partial_\mu \hat{f} - \frac{2p^Y E}{\tau} \partial_Y^{(p)} \hat{f} + \frac{1}{2} \partial_i M^2 \partial_{(p)}^i \hat{f} = -\mathcal{C}[\hat{f}]. \quad (3.11)$$

IV. LATTICE BOLTZMANN-EQUATIONS

The main idea behind Lattice Boltzmann equations is to have a minimum sampling of momentum space given by a discrete set of N vectors P_n^μ with $n = 0, \dots, N-1$ such that the conservation equations for the current and energy-momentum tensor are reproduced exactly. For maximum efficiency, one uses a linear ansatz for the collision term

$$\mathcal{C}[\hat{f}] = \frac{P^\mu U_\mu}{\tau_R} (\hat{f} - \hat{f}_{\text{eq}}) \quad (4.1)$$

with τ_R the relaxation time. Taking the first two moments of the Boltzmann equation this leads to the conservation of the current and energy momentum tensor provided that

$$U_\mu T^{\mu\nu} = U_\mu T_{\text{eq}}^{\mu\nu} = \epsilon(T, \mu) U^\nu \quad U_\mu N^\mu = U_\mu N_{\text{eq}}^\mu = n(T, \mu), \quad (4.2)$$

where the equilibrium energy and particle densities are given in Eqns. (3.4). The function B , which is required to always match its equilibrium value, is determined from Eq. (3.3).

Before discretizing momentum space on a lattice, it is instructive to first consider the shear and bulk viscosity coefficients that the collision term (4.1) corresponds to.

A. Chapman-Enskog Expansion

In the hydrodynamic (close to equilibrium) limit, the particle distribution function can be expanded around equilibrium in powers of space-time gradients,

$$f = f_{\text{eq}} + f_1 + f_2 + \dots,$$

where f_1 is of first order in gradients, f_2 of second order, and so on. In the absence of external forces ($\Gamma_{\alpha\beta}^\lambda = 0$), the Boltzmann Equation (3.6) with the collision term (4.1) can then be solved iteratively in powers of gradients. Specifically, to first order in gradients one finds

$$f_1 = -\frac{\tau_R}{P \cdot U} \left[P^\mu \partial_\mu f_{\text{eq}} + \frac{1}{2} \partial_\mu M^2 \partial_{(p)}^\mu f_{\text{eq}} \right],$$

which can be evaluated easily using $f_{\text{eq}} = Z e^{-P \cdot U/T}$. Since a small-gradient expansion corresponds to an expansion around ideal hydrodynamics, we may use the equations of ideal hydrodynamics to simplify the above equations. Specifically, for a metric signature of the form $+- --$ one has (c.f. [25])

$$D \ln s = -\nabla \cdot U, \quad D U^\alpha = c_s^2 \nabla^\alpha \ln s,$$

where c_s is the speed of sound and $D \equiv U^\mu \partial_\mu$ and $\nabla^\alpha \equiv \Delta^{\alpha\beta} \partial_\beta$, $\Delta^{\alpha\beta} = g^{\alpha\beta} - U^\alpha U^\beta$. Using the thermodynamic relation $\frac{dP}{ds} = c_s^2 T = \frac{dT}{ds} s$ and consistently ignoring higher order gradient term corrections, one finds

$$f_1 = f_{\text{eq}} \frac{\tau_R}{P \cdot U} \left[\frac{P^\mu P^\nu}{T} \sigma_{\mu\nu} + \frac{1}{3T} \left(P^2 - (1 - 3c_s^2)(P \cdot U)^2 - 3c_s^2 M T \frac{dM}{dT} \right) \nabla \cdot U \right]$$

where $\sigma_{\mu\nu} = \nabla_{(\mu} U_{\nu)} - \frac{1}{3} \Delta_{\mu\nu} \nabla \cdot U$. Decomposing the full energy momentum tensor into

$$T^{\mu\nu} \equiv \int d\chi P^\mu P^\nu f + g^{\mu\nu} B = \epsilon U^\mu U^\nu - P \Delta^{\mu\nu} + \pi^{\mu\nu} + \Delta^{\mu\nu} \Pi,$$

where $\pi^{\mu\nu} = 2\eta \sigma^{\mu\nu}$ and $\Pi = \zeta \nabla \cdot U$, one identifies the shear and bulk parts of the dissipative tensor with

$$\pi^{\mu\nu} \equiv T^{<\mu\nu>} = \int d\chi P^{<\mu} P^{\nu>} f, \quad \Pi \equiv \frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu} + P = \frac{1}{3} \int d\chi \Delta_{\mu\nu} P^\mu P^\nu f_1.$$

Note that $\int d\chi (P \cdot U)^2 f_1 = 0$ because of Eq. (2.5). Using usual the decomposition of the integrals in a tensor basis spanned by $U^\mu U^\nu$ and $\Delta^{\mu\nu}$ one finds the shear and bulk viscosity coefficients from $\pi^{\mu\nu}$ and Π as

$$\eta = \frac{\tau_R}{15T} Z \int \frac{d^3p}{(2\pi)^3} \frac{(M^2 - E^2)^2}{E^2} e^{-E/T}, \quad (4.3)$$

$$\zeta = \frac{\tau_R}{9T} Z \int \frac{d^3p}{(2\pi)^3} p^2 \frac{-M^2 + (1 - 3c_s^2)E^2 + 3c_s^2 M T dM/dT}{E^2} e^{-E/T}, \quad (4.4)$$

where $E = \sqrt{M^2 + \mathbf{p}^2}$. After a little bit of algebra it is possible to show that in the massless limit $\eta = \tau_R \frac{(\epsilon + P)}{5}$ (c.f. [26]), while for constant masses $\eta = \frac{\tau_R}{T} \int_0^T dT' (\epsilon + P)$, $\zeta = \frac{\tau_R}{3T} (-3c_s^2 T (\epsilon + P) + 5 \int_0^T dT' (\epsilon + P))$. No simple formulae seem to exist for medium-dependent masses. Note also that these results differ from Ref. [24, 27] (and many others using the Israel-Stewart ansatz) because non-linearities were not taken account there properly.

Pushing the Chapman-Enskog expansion to second order or following Ref. [28] would be desirable to extract all the second-order hydrodynamic transport coefficients [25]. While this is left for future work, it is possible to extract the value of the hydrodynamic relaxation times for the shear sector, τ_π . Identifying the relaxation time with the coefficient that is multiplying $-U^\alpha \partial_\alpha (\eta \sigma_{\mu\nu})$ in $\pi_{\mu\nu}$, one finds after a little algebra

$$\tau_\pi = \tau_R$$

from the derivative of f_1 . Note that it is therefore possible to use the same value of τ_π in numerical simulations using the Lattice Boltzmann with medium-dependent masses and second-order hydrodynamics.

B. Lattice Boltzmann with Medium-Dependent Masses

In the following, I will present a minimal set of vectors P_n^μ that is usable for a general relativistic Boltzmann equation with medium dependent masses, albeit only for metric tensors that are diagonal. The scheme is constructed by noting that the (on-shell) equilibrium distribution function for a Boltzmann gas can be expanded as

$$\hat{f}_{\text{eq}}(X^\mu, \mathbf{p}) = e^{\mu/T - E u_0/T} \sum_{n=0}^{\infty} \left(p^i u_i / T \right)^n / n!,$$

and I recall the definition of E given in Sec. III C. If the metric is diagonal, one may rescale the space-like momentum components such that $-g_{ij}p^i p^j \rightarrow \delta_{ij}\tilde{p}^i \tilde{p}^j \equiv |\tilde{\mathbf{p}}|^2$. (Note that this rescaling also changes the form of the Boltzmann equation.) Now $\tilde{\mathbf{p}}/|\tilde{\mathbf{p}}|$ is a unit vector that may be parameterized by spherical coordinates (angles ϕ, θ). Setting furthermore $|\tilde{\mathbf{p}}| = M \sinh \xi$ (implying $E = M \cosh \xi / \sqrt{g_{00}}$) one has the parametrization

$$P^\mu \equiv M \sinh \xi \left(\coth \xi / \sqrt{g_{00}}, \sin \theta \cos \phi / \sqrt{-g_{11}}, \sin \theta \sin \phi / \sqrt{-g_{22}}, \cos \theta / \sqrt{-g_{33}} \right)$$

for the momentum in terms of the variables ξ, θ, ϕ . Therefore one has

$$\hat{f}_{\text{eq}}(X^\mu, P^\mu) = e^{-p^0 u_0 / T} \sum_{n=0}^{\infty} (\mathbf{v}^i)^n a^{(n)}(X^\mu),$$

where $\mathbf{v}^i \equiv \tilde{\mathbf{p}}^i / M \equiv (\sinh \xi \sin \theta \cos \phi, \sinh \xi \sin \theta \sin \phi, \sinh \xi \cos \theta)$ and $a^{(n)}(X^\mu)$ are some coefficients that are only space-time (but not momentum-) dependent. Powers of the velocities² \mathbf{v} may be represented using the polynomials $P_{i_1 \dots i_n}^{(n)}(\mathbf{v}/|\mathbf{v}|)$ that are orthogonal with respect to the angular integral $d\Omega$ (see Ref. [10] for details).

This then motivates the ansatz for the general distribution function:

$$\hat{f}(X^\mu, \xi, \theta, \phi) = e^{-\frac{M_0}{T_0} \cosh \xi} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P_{i_1 \dots i_n}^{(n)} \left(\frac{\mathbf{v}}{|\mathbf{v}|} \right) \sinh^n \xi R_{(k)}(\cosh \xi) a_{i_1, \dots, i_n}^{(nk)}(X^\mu), \quad (4.5)$$

where T_0, M_0 are some reference temperature and mass, respectively, and R^k are polynomials of degree k that will be defined below. In practice, the infinite sums above are truncated at some finite order. Furthermore, it turns out that for any even n , the $\sinh \xi$ terms may be represented by the sum over polynomials $R_{(k)}$, so Eq. (4.5) may be modified such that there is a single inverse power of $\sinh \xi$ for every n odd. Replacing continuum momenta P^μ by a discrete set requires the condition that the integrals in Eqns. (3.10) are represented exactly. For the angular integrals, this requirement is identical to that of massless particles discussed in Ref. [10]. Note that Eqns. (3.10) are then evaluated for fixed values of ξ rather than fixed E, \mathbf{p} , implying another change in the Boltzmann equation coming from the space-time dependent masses. For convenience, a concrete example will be given below.

² Note that \mathbf{v} really is equal to velocity times the Lorentz factor.

C. Deriving the Momentum Lattice

Let us quickly review the derivation of the discrete set of momenta: the angles ϕ can be found from the requirement that

$$\int_0^{2\pi} d\phi (\sin \phi)^a (\cos \phi)^{N_\phi - a - 1} = \frac{\pi}{N_\phi} \sum_l (\sin \phi_l)^a (\cos \phi_l)^{N_\phi - a - 1},$$

where a is assumed to be a non-negative integer smaller than $2N_\phi - 1$. Namely, the above integrand can be recast as a Fourier series involving as highest harmonics $\cos[(N_\phi - 1)\phi]$ and $\sin[(N_\phi - 1)\phi]$. Exact representation of the integral as a sum is possible if the angles are chosen as the nodes of functions orthogonal to the integrand. For the Fourier series above, there are actually two sets of orthogonal functions: $\cos[N_\phi \phi]$ and $\sin[N_\phi \phi]$. Choosing $\sin[N_\phi \phi]$, the nodes are given by $\phi = \phi_l = \frac{l\pi}{N_\phi}$, $l = 0, 1, \dots, 2N_\phi - 1$, which fixes the set for ϕ . For the discrete set of angles θ , note that the integrands Eq. (3.10) only depend on θ through P^μ , so that $\sin \theta$ always comes with either $\cos \phi$ or $\sin \phi$. Since any odd power of $\cos \phi$, $\sin \phi$ integrates to zero, any non-vanishing contribution must involve $\sin^2 \theta = 1 - \cos^2 \theta$. Hence, it is sufficient to consider only integrands with powers of $\cos \theta$, which may be recast as a sum:

$$\int_{-1}^1 d(\cos \theta) \cos^{2N_\theta - 1} \theta = \sum_j w_j^\theta \cos^{2N_\theta - 1} \theta_j,$$

where the discrete angles θ_j are given as the roots of the Legendre polynomial $L_{N_\theta}(\cos \theta_j) = 0$, $j = 0, 1, \dots, N_\theta - 1$ and the weight factors w_j^θ are given as $w_j^\theta = 2 / \left[(1 - \cos^2 \theta_j) \left(L'_{N_\theta}(\cos \theta_j) \right)^2 \right]$.

Similarly, the integrands in Eq. (3.10) then only depend on $\cosh \xi$ and $\sinh^2 \xi$, since any odd power of $\sinh \xi$ would have integrated to zero already. Therefore, it is sufficient to consider integrals of the form

$$\int_0^\infty d\xi e^{-z_0 \cosh \xi} \sinh^2 \xi \cosh^{2N_\xi - 1} \xi = \sum_k w_k^\xi(z_0) \cosh^{2N_\xi - 1}(\xi_k(z_0)), \quad z_0 \equiv M_0/T_0,$$

and the nodes ξ_k and weights w_k^ξ are calculated from the set of polynomials R_k which are orthogonal on $\int d\xi e^{-z_0 \cosh \xi} \sinh^2 \xi$. Specifically, one finds

$$R_0(\xi) = 1, \quad R_1(\xi) = \cosh \xi - \frac{K_2(z_0)}{K_1(z_0)},$$

$$R_2(\xi) = \cosh^2 \xi + \frac{6K_1(-z_0^2 K_0^2 + (4+z_0^2)K_1^2)}{z_0(z_0^3 K_0^3 + 8z_0^2 K_0^2 K_1 + 14z_0 K_0 K_1^2 + 2(2-z_0^2)K_1^3 - z_0^3 K_2^3)} R_1(\xi) - \frac{3K_2 + z_0 K_1}{z_0 K_1}, \quad \text{etc.}$$

Hence the discrete values ξ_k are calculated from $R_{N_\xi}(\xi_k) = 0$ and the weights w_k^ξ fulfill

$$\sum_{k=0}^{N_\xi-1} w_k^\xi R_0(\xi_k) = \frac{K_1(z_0)}{z_0}, \quad \sum_{k=0}^{N_\xi-1} w_k^\xi R_m(\xi_k) = 0, \quad m = 1, \dots, N_\xi - 1.$$

Thus, one finds the following representation of the momentum integrals:

$$\int \frac{d\Omega}{4\pi} \int_0^\infty d\xi \sinh^2 \xi \hat{f}(\xi, \theta, \phi) = \sum_{k=0}^{N_\xi-1} \sum_{j=0}^{N_\theta-1} \sum_{l=0}^{2N_\phi-1} w_{kj} \hat{f}(\xi_k, \theta_j, \phi_l) = \sum_n w_n \hat{f}(P_n^\mu),$$

with the weights $w_{kj} = e^{z_0 \cosh \xi_k} w_k^\xi w_j^\theta / (4N_\phi) \equiv w_n$, and where the collective index n runs over all discrete momenta P_n^μ constructed from the ensemble ϕ_l, θ_j, ξ_k .

D. Lattice Boltzmann Algorithm for Milne Spacetime

In this subsection I give a detailed construction of a lattice Boltzmann algorithm with non-ideal QCD equation of state in an expanding spacetime with $ds^2 = d\tau^2 - dx^2 - dy^2 - \tau^2 dY^2$ (Milne). For simplicity, I will limit myself to neglecting space dependencies, which are algorithmically easy to program (cf. [10] for a practical example).

Starting with the Boltzmann-equation (3.11) for the on-shell distribution function, let us first rescale momenta $p^Y = \tilde{p}^Y / \tau$ so that $E = \delta_{ij} \tilde{p}^i \tilde{p}^j$. Next, replacing $\tilde{\mathbf{p}}^i = M(\tau) \mathbf{v}^i$ and using Eq. (4.1), Eq. (3.11) becomes

$$\partial_\tau \hat{f} \Big|_{\mathbf{v}} - \frac{\tilde{p}^Y}{\tau} \partial_Y^{(\tilde{p})} \hat{f} - \partial_\tau \ln M \tilde{\mathbf{p}} \cdot \partial_{(\tilde{p})} \hat{f} = - \frac{\hat{f} - \hat{f}_{\text{eq}}}{\tau_R}. \quad (4.6)$$

Neglecting space dependencies, one has cylindrical symmetry and hence $\hat{f} = \hat{f}(\tau, \xi, \theta)$. Therefore the ansatz for the general distribution function can be simpler than (4.5), namely

$$\hat{f}(X^\mu, \xi, \theta, \phi) = e^{-M^0/T_0 \cosh \xi} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} L_n(\cos \theta) R_{(k)}(\cosh \xi) a^{(nk)}(\tau). \quad (4.7)$$

Using the nodes and weights from Sec. IV C, one immediately finds

$$a^{(ml)} = \frac{(2m+1)}{I(l)} \int \frac{d\Omega}{4\pi} \int d\xi \sinh^2 \xi L_m(\cos \theta) R_l(\cosh \xi) \hat{f} = \frac{(2m+1)}{I(l)} \sum_n w_n L_m R_l \hat{f} \Big|_{P_n^\mu},$$

where $I(l) = \int d\xi \sinh^2 \xi R_l^2(\cosh \xi)$.

I will not consider conserved particle number, so the only quantity of interest is

$$T^{\mu\nu} = \frac{M^2}{2\pi^2} \sum_n w_n \hat{f}(P_n^\mu) P_n^\mu P_n^\nu + B(T) g^{\mu\nu}. \quad (4.8)$$

Since all spatial dependencies have been neglected, the fluid velocity is trivial, $U^\mu = (1, \mathbf{0})$, and the equilibrium energy density is given by the 00 component of Eq. (4.8).

A change in the energy density can be calculated via the Boltzmann equation (4.6):

$$\partial_\tau \epsilon = \frac{M^4}{2\pi^2} \sum_n w_n \cosh \xi_n^2 \left[\frac{\tilde{p}^Y}{\tau} \partial_Y^{(\tilde{p})} \hat{f} - \frac{\hat{f} - \hat{f}_{\text{eq}}}{\tau_R} \right], \quad (4.9)$$

where I used partial integration and the identity (3.5), which becomes

$$\frac{\partial_\tau M^4}{8\pi^2} \sum_n w_n \hat{f}(P_n^\mu) + \partial_\tau B(T) = 0. \quad (4.10)$$

More specifically, a lattice Boltzmann algorithm may hence be constructed as follows: a valid initial condition at time τ consists of specifying $\hat{f} = \hat{f}_{\text{cur}}$ and an initial temperature and particle mass, $T_{\text{cur}}, M_{\text{cur}}$. Then, make a prediction of the change in distribution function and (logarithm of) mass:

$$\begin{aligned} \delta f_{\text{pred}} &= \left(\frac{\hat{S}_1}{\tau} + \frac{\delta \ln M_{\text{pred}}}{\delta \tau} \hat{S}_2 - \frac{\hat{f}_{\text{cur}} - \hat{f}_{\text{eq}}(T_{\text{cur}})}{\tau_R(T_{\text{cur}})} \right) \delta \tau, \\ \delta \ln M_{\text{pred}} &= - \left. \frac{dB}{d\epsilon} \right|_{T_{\text{cur}}} \frac{\sum w_n \left(\hat{S}_1/\tau - \frac{\hat{f}_{\text{cur}} - \hat{f}_{\text{eq}}(T_{\text{cur}})}{\tau_R(T_{\text{cur}})} \right)}{\sum w_n \hat{f}_{\text{cur}}} \delta \tau \end{aligned} \quad (4.11)$$

where \hat{S}_1 and \hat{S}_2 are representations of the momentum derivatives $\tilde{p}^Y \partial_Y^{(\tilde{p})}$ and $\tilde{\mathbf{p}} \cdot \partial_{(\tilde{p})} \hat{f}$ in the form of (4.7) with coefficients

$$\begin{aligned} s_1^{ml} &= -\frac{2m+1}{I(l)} \sum_n w_n \hat{f}_{\text{cur}} \left[(P_m R'_l \cosh \xi - R_l) \cos^2 \theta \tanh \xi^2 + R_l (P_m + P'_m (\cos \theta - \cos^3 \theta)) \right] \\ s_2^{ml} &= -\frac{2m+1}{I(l)} \sum_n w_n \hat{f}_1 \left[P_m R_l \left(2 + \frac{1}{\cosh^2 \xi} \right) + P_m R'_l \cosh \xi \tanh^2 \xi \right], \end{aligned}$$

respectively. Via Eq. (4.9), this leads to a prediction for the new temperature T_{pred} . These predictions are then corrected using the trapezoid integration formula

$$\delta f_{\text{corr}} = \frac{\delta \tau}{2} \left(\partial_\tau \hat{f} \Big|_\tau + \partial_\tau \hat{f} \Big|_{\tau+\delta \tau} \right) + \mathcal{O}(\delta \tau)^3, \quad \delta \ln M_{\text{corr}} = \frac{\delta \tau}{2} \left(\partial_\tau \ln M \Big|_\tau + \partial_\tau \ln M \Big|_{\tau+\delta \tau} \right) + \mathcal{O}(\delta \tau)^3,$$

where the values at time $\tau + \delta \tau$ are calculated using δf_{pred} and $\delta \ln M_{\text{pred}}$. Note that the resulting mass M_{corr} does not necessarily correspond to the equilibrium particle mass $M(T_{\text{new}})$. As a consequence, I use $f_{\text{eq}} = Z e^{-\sqrt{M_{\text{eq}}^2 + M_{\text{corr}}^2 \sinh^2 \xi}/T}$ for the equilibrium distribution function in the algorithm. The above steps may be repeated to solve the Boltzmann equation (4.6) for arbitrary times. The resulting algorithm leads to time integrated quantities that are accurate to $\mathcal{O}(\delta \tau)^2$ (cf. [29]).

E. Results for Milne Spacetime

In this section I provide tests of the above Lattice Boltzmann algorithm by comparing results to viscous fluid dynamics for the QCD equation of state of Ref. [21] and a Milne metric. The fluid dynamics equations for the energy density and quantity $\Phi \equiv T_Y^Y - p$ fulfill the coupled equations [24]

$$\partial_\tau \epsilon = -\frac{\epsilon + P}{\tau} + \frac{\Phi}{\tau}, \quad \partial_\tau \Phi = -\frac{\Phi}{\tau_\pi} + \frac{4\eta}{3\tau_\pi \tau} - \frac{4\Phi}{3\tau} - \frac{\lambda_1}{2\tau_\pi \eta^2} \Phi^2, \quad (4.12)$$

where τ_π is the relaxation time and λ_1 is a self-coupling parameter. While I found $\tau_\pi = \tau_R$ in Sec. IV A, λ_1 is currently not known. For simplicity, for the hydrodynamic calculation I will use the values $\tau_\pi = 5\frac{\eta}{\epsilon+P}$ and $\lambda_1 = \frac{5}{7}\eta\tau_\pi$ that are reported for the massless gas case [26].

One should keep in mind that — since the correct values for τ_π, λ_1 will differ from this choice in view of the findings in section IV A — this implies that the hydrodynamic and Lattice Boltzmann results will not agree in practice. Note, however, that there is another issue that prevents perfect agreement between (second-order) hydrodynamics and Lattice Boltzmann theory even in principle: the reason is that, even if one were to use the same second-order transport coefficients in a numerical second-order hydrodynamics and a Lattice Boltzmann solver, the two would still disagree because of the different *third* order gradient terms. However, for all practical purposes when hydrodynamics itself can be considered applicable, the difference between the two numerical schemes could be considered small.

For the Lattice-Boltzmann framework, the QCD equation of state is parameterized as in Sec. III A with a reference value $M_0/T_0 = 1$ for (4.5). Note that this reference value corresponds to a reference temperature of $T \sim 0.82$ GeV, This means that results will be most accurate for this temperature (cf. [10]), and in particular will break down if applied to problems involving fluid cells with temperatures exceeding two times this reference temperature. The results at temperatures different than this reference value can be improved by increasing the value of N_ξ , but in practice I find that $N_\xi \geq 3$ gives adequate results. In Fig. 2, I show the temperature evolution in viscous fluid dynamics and the above lattice Boltzmann algorithm for $N_\xi = 5, N_\theta = 5$ and $\delta\tau = 0.2$ fm/c. As can be seen from this figure, the non-ideal equation of state time evolution in fluid dynamics for $\eta/s = 0.5$ is described reasonably well throughout the whole simulation time, even though it differs markedly from the ideal equation of state time evolution (shown in Fig. 2 for comparison). Overall, the algorithm

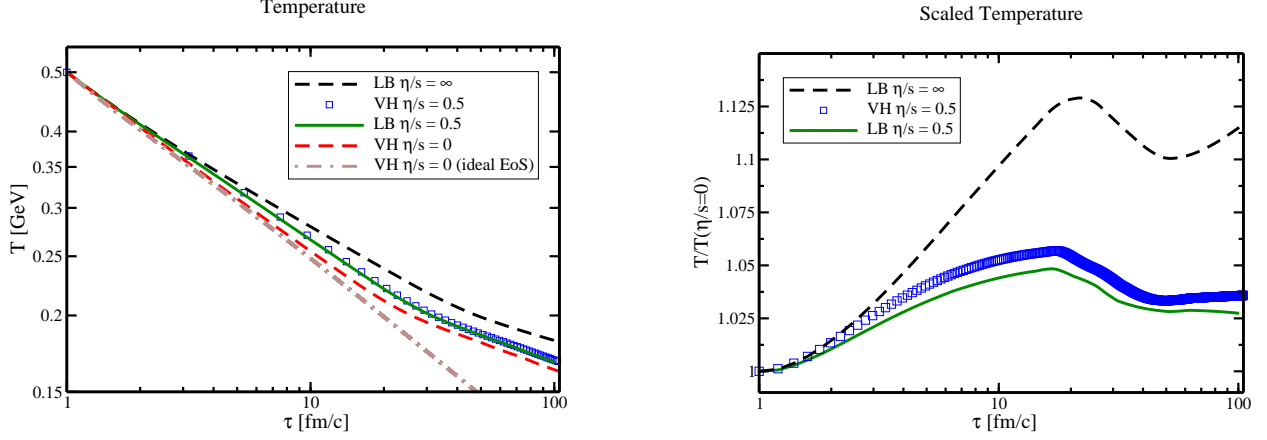


FIG. 2: Temperature evolution for various viscosities, from fluid dynamics ('VH') and Lattice Boltzmann ('LB') respectively. Results are for a QCD equation of state except for the ideal equation of state evolution (left plot). Transport coefficients for VH are only approximate, so perfect agreement is not expected. Right: results normalized with respect to the ideal fluid dynamics to highlight differences. The $\eta/s = \infty$ results are obtained by setting $\tau_R = \infty$ (free-streaming) and updating the medium-dependent mass according to the change in energy density. These results do not correspond to an actual physics situation and are presented for illustrative purposes only.

seems to perform rather well and provides a concrete example for simulating the Boltzmann equation for a system with non-ideal equation of state and non-Minkowski geometry.

V. CONCLUSIONS

In this note I have set up a general relativistic transport equation for a single species of uncharged particles with a medium-dependent mass. This 'Boltzmann-like' equation allows for a conserved particle current and energy-momentum tensor. The latter explicitly allows arbitrary (thermodynamically consistent) equations of state when using the medium-dependent mass as fitting parameter. I expect this formulation to be useful for relativistic fluid dynamics simulation in the Lattice Boltzmann framework with arbitrary equations of state. Possibly, it can also have relevance for non-relativistic computational non-ideal fluid dynamics (cf. [30]). Furthermore, it can be applicable in the context of simulating parton dynamics with non-ideal equations of state, cf. [31, 32] or quasiparticle models of QCD [16].

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