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M. Shifman

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Degeneracies in Supersymmetric Gluodynamics and its Orientifold Daughters at large N

M. Shifman

*William I. Fine Theoretical Physics Institute, University of Minnesota,
Minneapolis, MN 55455, USA*

Abstract

I use the Nicolai map and ensuing (super)locality of appropriate correlation functions to prove the existence of an infinite number of degeneracies in the mass spectra and decay coupling constants in supersymmetric gluodynamics and its daughter orientifold theory at large N .

In this paper an infinite number of parity degeneracies of the mass spectra and decay coupling constants will be shown to exist in supersymmetric gluodynamics and its daughter orientifold theory at large N . For instance, the masses of the glueballs with $J^P = k^\pm$ ($k = 0, 1, 2, \dots$) are degenerate.

Consider the simplest $\mathcal{N} = 1$ supersymmetric Yang–Mills theory (referred to as supersymmetric gluodynamics). For definiteness we assume the gauge group to be $SU(N)$. The Lagrangian includes the gluon and gluino fields,

$$\begin{aligned}\mathcal{L} &= \left\{ \frac{1}{4g^2} \int d^2\theta \, W^{a\alpha} W_\alpha^a + \text{H.c.} \right\} \\ &= -\frac{1}{4g^2} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{i}{g^2} \lambda^{a\alpha} \mathcal{D}_{\alpha\dot{\beta}} \bar{\lambda}^{a\dot{\beta}},\end{aligned}\tag{1}$$

(this is in Minkowski space, for a review see [1]). This theory is supposed to be confining, i.e. its physical spectrum consists of color-singlet mesons and baryons, e.g. glueballs.

The basic object of our analysis is a local *single-trace* operator

$$\mathcal{O}_{\alpha_1\alpha_2\dots} = \text{Tr}(\mathcal{F}\mathcal{F}\dots)_{\alpha_1\alpha_2\dots} - \text{Lorentz traces}\tag{2}$$

where \mathcal{F} is the self-dual gluon field strength tensor,

$$\mathcal{F}_{\mu\nu} \equiv F_{\mu\nu} + i\tilde{F}_{\mu\nu},\tag{3}$$

and the operator $\mathcal{O}_{\alpha_1\alpha_2\dots}$ polynomially depends on a number of the gluon self-dual operators $\mathcal{F}_{\alpha\beta}$. None of the Lorentz indices are assumed to be contracted. Moreover, we will assume that the operator $\mathcal{O}_{\alpha_1\alpha_2\dots}$ has the maximal possible Lorentz spin compatible with its composition. Since $\mathcal{F}_{\mu\nu}$ belongs to the $(1, 0)$ representation of the Lorentz group, the maximal spin produced by the operator $\mathcal{O}_{\alpha_1\alpha_2\dots}$ is k , where k is the number of \mathcal{F} factors in $\mathcal{O}_{\alpha_1\alpha_2\dots}$.

Our consideration will consist of several steps. The first step is based on the Nicolai map [2] in supersymmetric gluodynamics [3]. After performing the Nicolai mapping the theory becomes free, i.e. the partition function can be written as the following path integral:¹

$$\int \mathcal{D}\mathcal{F}_{\mu\nu} \exp \left[\int d^4x \left(-\frac{1}{2g^2} \text{Tr} \mathcal{F}_{\mu\nu}^2 \right) \right].\tag{4}$$

¹The path integral is written in Euclidean space.

In deriving Eq. (4) one performs functional integration over the gluino fields and then, from the functional integration over $\mathcal{D}A$ (in the light-cone gauge) one proceeds to the functional integration over $\mathcal{D}\mathcal{F}_{\mu\nu}$. Three local variables in $A_\mu(x)$ are traded for three variables residing in $\mathcal{F}_{\mu\nu}(x)$. The functional determinant obtained from the integration over the gluino fields exactly cancels the Jacobian $\delta A(x)/\delta \mathcal{F}(y)$. The Nicolai mapping and its applications were rarely discussed (if at all) in the last two decades. A revival of interest is due to the recent paper [4].

In terms of \mathcal{F} the gluon potential A_μ can be written as nonlocal operator through the inversion of the Nicolai map. For instance, in a symbolic form

$$A(x) = \int d^4y (x-y)^{-3} \left\{ \mathcal{F}(y) - \int d^4z d^4z' (y-z)^{-3} (y-z')^{-3} [\mathcal{F}(z), \mathcal{F}(z')] + \dots \right\}, \quad (5)$$

see Fig. 1

The diagram illustrates the nonlocal expression for $A(x)$ in terms of \mathcal{F} . It shows the equation $A(x) = x \text{---} y + x \text{---} y + \dots$. In the first term, $x \text{---} y$ is a dashed line. In the second term, $x \text{---} y$ is a dashed line, and a thick solid line branches off to the right from the point y . This thick solid line is labeled with z at the top and z' at the bottom, indicating it represents the gluon field \mathcal{F} . The dashed line represents the nonlocal operator ∂^{-1} .

Figure 1: A nonlocal expression for A in terms of \mathcal{F} . The dashed line stands for the nonlocal operator ∂^{-1} . The thick solid line denotes \mathcal{F} .

In the momentum space, say, in the linear approximation

$$A_\alpha = \frac{k_\beta}{k^2} \mathcal{F}_{\mu\nu} C^{\mu\nu\beta\alpha},$$

with purely numerical coefficients C .

The inversion of the Nicolai mapping is problematic near self-dual points where $\tilde{\mathcal{F}}$ vanishes [4]. However, in the 't Hooft limit self-dual field configurations are expected to be suppressed [5] in the functional integrals. In particular, the θ dependence is always suppressed by $1/N$ and is not seen in the limit $N \rightarrow \infty$. Moreover, self-dual fields are certainly unimportant for excited states in the spectrum (for which predictions of the type (14)

below apply equally well). The irrelevance of self-dual fields at $N = \infty$ in super-Yang–Mills is akin to the same observation done long ago by Witten in a two-dimensional example [6]. This motivates us to limit ourselves to the large- N limit. Another reason for sticking to this limit will be indicated below.

Summarizing, irrelevance of the self-dual fields in the mass spectrum at $N = \infty$ must be viewed as a well-motivated physical assumption which is hard to avoid (this is also in agreement with some results in [4]). If so, all predictions for the masses and coupling constants following from the Nicolai map are solid.

Now, let us consider the two-point function

$$\begin{aligned}\Pi_{\alpha_1\alpha_2\dots\alpha'_1\alpha'_2\dots}(q) &= i \int d^4x \exp(iqx) \langle \mathcal{O}_{\alpha_1\alpha_2\dots}(x) \mathcal{O}_{\alpha'_1\alpha'_2\dots}(0) \rangle \\ &= i \int d^4x \exp(iqx) \int \mathcal{DF} \exp \left[\int d^4x \left(-\frac{1}{2g^2} \text{Tr } \mathcal{F}_{\mu\nu}^2 \right) \right] \\ &\quad \times \mathcal{O}_{\alpha_1\alpha_2\dots}(x) \mathcal{O}_{\alpha'_1\alpha'_2\dots}(0). \end{aligned} \tag{6}$$

In analyzing the above equation we will focus on the kinematic structure with the highest spin, namely,

$$\Pi_{\alpha_1\alpha_2\dots\alpha'_1\alpha'_2\dots}(q) = \left(g_{\alpha_1\alpha'_1} g_{\alpha_2\alpha'_2} \dots + \text{permutations} \right) + \dots \tag{7}$$

Since the functional integration in (6) runs over \mathcal{DF} , treated as independent variables, it is obvious that the correlation function (6) will vanish unless $x = 0$. More exactly,²

$$\begin{aligned} &\int \mathcal{DF} \mathcal{F}_{\mu\nu}(x) \mathcal{F}_{\rho\sigma}(y) \exp \left[\int d^4x \left(-\frac{1}{2g^2} \text{Tr } \mathcal{F}_{\mu\nu}^2 \right) \right] \\ &\propto (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}) \delta^4(x - y). \end{aligned} \tag{8}$$

We will refer to this property of the specifically designed correlation functions as to superlocality.

²The pre-exponent in Eq. (8) is not gauge invariant. One should understand this relation somewhat symbolically, as a building block for two-point functions with the pre-exponent composed of expressions (2) which are gauge invariant.

The same is valid for all operators \mathcal{O} which are polynomially expressed in terms of \mathcal{F} . Since the action is quadratic in \mathcal{F} , and the same \mathcal{F} is the functional integration variable, in evaluating (6) one can use the pairwise Wick contraction (after expressing the operators $\mathcal{O}_{\alpha_1\alpha_2\dots}$ in terms of \mathcal{F} 's).

If so, upon the functional integration the correlation function (6) contracts and becomes a (generalized) tadpole, with no imaginary part,

$$\text{Im } \Pi_{\alpha_1\alpha_2\dots\alpha'_1\alpha'_2\dots}(q) = \left(g_{\alpha_1\alpha'_1} g_{\alpha_2\alpha'_2} \dots + \text{permutations} \right) \times \text{zero}. \quad (9)$$

Equation (9) should be trivial in perturbation theory. However, let us not forget that supersymmetric gluodynamics confines, and its physical spectrum at large N consists of massive stable mesons with certain spins and parities³ (there is no massless states in the physical spectrum). Then (9) imposes constrains on the spectra.

Let us examine these constrains first in the example of, say, cubic in \mathcal{F} operators. Then the operator \mathcal{O} takes the form

$$\mathcal{O} \rightarrow \text{Tr} \left(E^{k_1} - iB^{k_1} \right) \left(E^{k_2} - iB^{k_2} \right) \left(E^{k_3} - iB^{k_3} \right) \quad (10)$$

where E and B are chromoelectric and chromomagnetic fields, respectively, and $k_{1,2,3} = 1, 2, 3$. Complete symmetrization over k_1, k_2, k_3 is assumed. It can be split in two parts with definite parities,

$$\begin{aligned} \mathcal{O}_- &= \text{Tr} E^{k_1} E^{k_2} E^{k_3} - \left(\text{Tr} B^{k_1} B^{k_2} E^{k_3} + \text{perm.} \right), \\ \mathcal{O}_+ &= \text{Tr} B^{k_1} B^{k_2} B^{k_3} - \left(\text{Tr} E^{k_1} E^{k_2} B^{k_3} + \text{perm.} \right). \end{aligned} \quad (11)$$

The operator \mathcal{O}_- creates from the vacuum $J^P = 3^-$ meson (in its rest frame $\langle 0 | \text{Tr} E^{k_1} E^{k_2} E^{k_3} + \dots | J^P = 3^- \rangle = \text{const} \times \varepsilon^{k_1 k_2 k_3}$ where $\varepsilon^{k_1 k_2 k_3}$ is polarization vector; full symmetrization is assumed). A similar expression can be written for $\langle 0 | \text{Tr} B^{k_1} B^{k_2} B^{k_3} + \dots | J^P = 3^+ \rangle$. The two-point function (6) reduces to

$$\langle \mathcal{O}_-, \mathcal{O}_- \rangle - \langle \mathcal{O}_+, \mathcal{O}_+ \rangle \quad (12)$$

where we take into account the fact that the P -parity is unbroken in supersymmetric gluodynamics, and, hence, the cross correlator $\langle \mathcal{O}_-, \mathcal{O}_+ \rangle$ must

³At this point the large- N limit is useful but not crucial. At finite N the spectrum is no longer represented by an infinite sum of poles. Instead, the resonances acquire finite widths, and multiparticle states contribute to the two-point correlation function (6).

vanish. The imaginary part of $\langle \mathcal{O}_-, \mathcal{O}_- \rangle$ is positive-definite, and so is the imaginary part of $\langle \mathcal{O}_+, \mathcal{O}_+ \rangle$. Since the overall imaginary part vanishes, the first one should cancel the second.

Thus, we conclude that the masses and residues of positive and negative parity states of spin 3 must be degenerate. This is obviously valid for arbitrary spins.

We observe parity degeneracy of the physical spectra produced by two distinct operators obtained from each other by the substitution, $E \leftrightarrow B$. In addition to parity degeneracy discussed above, we certainly have supersymmetric degeneracies, following from the fact that massive supersymmetry representations contain three subsequent spin states, for instance $(0, 1/2, 1/2, 1)$ in the vector superfield. This general degeneracy must be superimposed with the parity degeneracy specific to supersymmetric gluodynamics.

Degeneracies following from superlocality of the correlation functions of the type (6) are valid not only for the mass spectrum, but for the decay constants too. For instance, consider a three-point correlator

$$\langle \mathcal{O}(x), \mathcal{O}(y), \mathcal{O}(0) \rangle \quad (13)$$

where for simplicity I limit myself to the spin-zero operator $\mathcal{O} = \text{Tr } \mathcal{F}^2$. Acting on the vacuum, the operator \mathcal{O} produces either scalar (S) or pseudoscalar (P) glueballs, whose masses are degenerate at every level. Following the same line of reasoning as above, it is easy to see that the absence of physical cuts in (13) implies that at every level

$$g_{SSS} = -3 g_{SPP}, \quad (14)$$

where the coupling constants g in the expression above are the S -wave decay constants $S \rightarrow SS$ and $S \rightarrow PP$, respectively. One can consider off-diagonal transitions too (i.e. from one level to another).

Because of the planar equivalence between supersymmetric gluodynamics and the orientifold theory [8] the same conclusion applies to Yang–Mills theory with one Dirac quark in the two-index antisymmetric or symmetric representation. In the former case at $N = 3$ we get just one-flavor QCD. It would be interesting to study $1/N$ corrections to these predictions, perhaps, on lattices.

Finally, I would like to mention that two particular results of the same nature had been established in the literature previously on different grounds. First, the fact of degeneracy of spectra (imaginary parts) in the scalar and

pseudoscalar channels, $\langle F^2(x), F^2(0) \rangle$ and $\langle F\tilde{F}(x), F\tilde{F} \rangle$, respectively, dates back to [9]. Second, it was shown [10] that the spectral functions associated with the (nonchiral!) operator $\text{Tr} \left(\mathcal{F}_{\mu\nu} \bar{\lambda}^2 \right)$ are fully degenerate in the $J^{PC} = 1^\pm$ channels. This statement follows from $\mathcal{N} = 1/2$ supersymmetry discovered in [11], to which the above operator is related. It is probable that extensions of Seiberg's construction [11] can be worked out providing an alternative derivation of the results presented in this note.

Verification of the predicted degeneracies seems to be an excellent testing ground for lattice studies of supersymmetry at strong coupling, both in the Lagrangian and Hamiltonian formulations.

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