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We present a fully nonlinear study of long wavelength cosmological perturbations within the framework of the projectable Hořava-Lifshitz gravity, coupled to a single scalar field. Adopting the gradient expansion technique, we explicitly integrate the dynamical equations up to any order of the expansion, then restrict the integration constants by imposing the momentum constraint. While the gradient expansion relies on the long wavelength approximation, amplitudes of perturbations do not have to be small. When the $\lambda \to 1$ limit is taken, the obtained nonlinear solutions exhibit a continuous behavior at any order of the gradient expansion, recovering general relativity in the presence of a scalar field and the "dark matter as an integration constant". This is in sharp contrast to the results in the literature based on the "standard" (and naive) perturbative approach where in the same limit, the perturbative expansion of the action breaks down and the scalar graviton mode appears to be strongly coupled. We carry out a detailed analysis on the source of these apparent pathologies and determine that they originate from an improper application of the perturbative approximation in the momentum constraint. We also show that there is a new branch of solutions, valid in the regime where $|\lambda - 1|$ is smaller than the order of perturbations. In the limit $\lambda \to 1$, this new branch allows the theory to be continuously connected to general relativity, with an effective component which acts like pressureless fluid.

Recently, Hořava [1] proposed a new theory of quantum gravity in the framework of quantum field theory. One of the essential ingredients of the theory is inclusion of higher-dimensional operators, so that they dominate the ultraviolet (UV) behavior and render the theory power counting renormalizable. Improvement of the UV behavior by higher-dimensional operators has been known for some time [2] but in those previous attempts, higher time derivative terms led to ghost degrees of freedom. The major modification put forward by Hořava's theory is that the power-counting renormalizability is achieved without inclusion of higher time derivative terms. This is realized by invoking the anisotropic scaling between time and space,

$$t \to b^{-z}t, \quad \vec{x} \to b^{-1}\vec{x}, \tag{1}$$

so that higher-dimensional operators include spatial derivatives only. This is reminiscent of Lifshitz scalars [3] in condensed matter physics, hence the theory is often referred to as the Hořava-Lifshitz (HL) gravity. For the $3 + 1$ dimensional theory to be power-counting renormalizable, the dynamical critical exponent z has to be larger than or equal to 3 [1] (see also [4]). Because of the anisotropic scaling, the theory cannot be invariant under the spacetime diffeomorphism, $x^\mu \to x'^\mu(x^\nu)$, $(\mu, \nu = 0, 1, 2, 3)$. Instead, the fundamental symmetry of the theory is the invariance under the so called foliation-preserving diffeomorphism,

$$t \to t'(t), \quad \vec{x} \to \vec{x}'(t, \vec{x}), \tag{2}$$

denoted usually by Diff$(M, \mathcal{F})$. The basic variables of the theory are the lapse function N, the shift vector N^i, and the 3-dimensional spatial metric g_{ij} [5]. Since the lapse function N corresponds to a gauge degree of freedom associated with the space-independent time reparametrization, it is natural to restrict the lapse function to be independent of the spatial coordinates:

$$N = N(t). \tag{3}$$

This condition, imposed in the original formulation of the theory, is called the *projectability condition*.

Since its introduction, there has been many cosmological applications of the HL gravity and various remarkable features have been found (see [6, 7] for reviews). In particular, the higher-order spatial curvature terms can give rise

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to a bouncing universe [8], may ameliorate the flatness problem [9] and lead to caustic avoidance [10]; the anisotropic scaling provides a solution to the horizon problem and generation of scale-invariant perturbations without inflation [11], a new mechanism for generation of primordial magnetic seed field [12], and also a modification of the spectrum of gravitational wave background via a peculiar scaling of radiation energy density [13]; with the projectability condition, the lack of a local Hamiltonian constraint leads to "dark matter as an integration constant" [14]; in the parity-violating version of the theory, circularly polarized gravitational waves can also be generated in the early universe [15].

Despite all of its remarkable features, the theory has been challenged by significant questions. In particular, the Diff(M, $\mathcal{F}$) symmetry allows the existence of an additional spin-0 degree of freedom, often called *scalar graviton*, and its fate is one of important open issues. Actually, the scalar graviton is known to be unstable either in the UV due to ghost instability or in the infrared (IR) due to gradient instability [16–18], depending on the value of a coupling constant λ. In order to avoid the ghost instability, λ must satisfy either $\lambda < 1/3$ or $\lambda > 1$. Precisely in these two ranges, the scalar graviton exhibits gradient instability at long distances. We then have to tame this IR instability by expansion of the universe [19, 20] or have to hide it by the standard Jeans instability. One can formulate a condition under which one of these happens [6]. Essentially, the condition says that λ must be sufficiently close to 1 in the IR.

However, in the limit $\lambda \to 1$, the scalar graviton appears to be strongly coupled [20–22]. That is, the "standard" (and naive) perturbative expansion breaks down in the sense that nonlinear terms dominate linear terms in the $\lambda \to 1$ limit. Note that this does not necessarily imply loss of predictability: if the theory is renormalizable, all coefficients of infinite number of nonlinear terms can be written in terms of finite parameters in the action, as several well-known theories with strong coupling (e.g. [23]) indicate. However, because of the breakdown of the (naive) perturbative expansion, we need to employ nonperturbative methods to analyze the fate of the scalar graviton in the limit $\lambda \to 1$. Such an analysis was performed in [6] for spherically symmetric, static, vacuum configurations and it was shown that the limit is continuously connected to general relativity (GR). [1] This may be considered as an analogue of the Vainshtein effect [24, 25]. A similar consideration for cosmology was given in [26], where a fully nonlinear analysis of superhorizon cosmological perturbations was carried out.

One of limitations of the analysis in [26] is that it is for a purely gravitational system in the absence of ordinary matter (but with "dark matter as an integration constant"). Since the naive perturbative expansion is known to break down not only in the gravity sector but also in the matter sector [20, 21], it is rather important to extend the analysis of [26] to the system with ordinary matter. Technically speaking, however, this kind of extention is indeed a nontrivial challenge since the system now has multi components (ordinary matter and "dark matter as an integration constant") and the gradient expansion technique has not been developed for multi-component systems even in the standard cosmology in GR.

Thus, one of the main objectives of the present paper is to extend the analysis of [26] to the case where HL gravity is coupled to a single scalar field, and provide yet another example indicating that general relativity (plus "dark matter as an integration constant") is restored in the $\lambda \to 1$ limit by nonlinear dynamics. Another goal is to point out the source of the discrepancy between perturbative and nonperturbative results. As we will see, the solution of the momentum constraint in the naive application of the "standard" perturbative expansion is not valid in the regime where $|\lambda - 1|$ is smaller than the order of perturbations.

The paper is organized as follows. In Sec. I, we briefly review the basic equations in the HL gravity with the projectability condition (3), while in Sec. II, we analyze the inhomogeneous cosmology in HL gravity using the gradient expansion method [27], and present the solutions to the equations of motion. In Sec.III, we present a discussion on the source of the divergences in the naive perturbative expansion and show that the momentum constraint is dominated by nonlinear terms in the $\lambda \to 1$ limit. The results are summarized and discussed in Sec.IV. The paper is supplemented by two Appendices, in which we present some of the technical steps of our calculations.

I. BASIC EQUATIONS

In this section, we review the basic equations of the HL gravity coupled with a scalar field [9, 28], following the notation in [6], and reformulate them in a way suitable for gradient expansion [26]. In order to make the present paper self-contained, some repetition of the material in [26] is inevitable in Secs.I and II, although we shall try our best to limit them to a minimum.

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[1] Specifically, the solutions are continuously connected to the $\lambda = 1$ theory, whose action has the exact same form as the Einstein-Hilbert term (up to high curvature terms negligible at low energies). However, due to the different symmetries, the resulting theory is not exactly GR, but GR with an effective component which acts like dark-matter [14]. This is what we mean by "continuity with GR" throughout the present paper. In the case considered in [6], however, the "dark matter" component is automatically set to zero by the assumed staticity.

With Diff(M, $\mathcal{F}$) and the projectability, the building blocks of the theory are g_{ij}, K_{ij}, D_i and R_{ij}, where K_{ij} denotes the extrinsic curvature of constant time hypersurfaces, D_i is the covariant derivative compatible with the the 3-dimensional spatial metric g_{ij}, and R_{ij} is the three-dimensional Ricci tensor built out of g_{ij}. (This is in contrast to GR or any other theory with general covariance whose building blocks are the 4-dimensional metric and its Riemann tensor.) For the critical exponent $z = 3$, their momentum dimensions are, respectively, $[K_{ij}] = [k]^3$ and $[R_{ij}] = [k]^2$. Throughout the present paper, we shall impose the projectability condition as well as invariance under the spatial parity ($x^i \to -x^i$) and the time reflection ($t \to -t$). The number of independent coupling constants in this setup is 11 for $z = 3$ [9, 16]. In fact, with the foliation-preserving diffeomorphisms (2), the projectability condition (3), and the additional requirements of parity and time reflection symmetry, the most general gravitational action can be specified as

$$I_g = \frac{M_{Pl}^2}{2} \int N dt \sqrt{g} \, d^3 \vec{x} \left(K_{ij} K^{ij} - \lambda K^2 - 2\Lambda + R + L_{z>1} \right), \tag{4}$$

where g is the determinant of g_{ij}, and the extrinsic curvature K_{ij} is defined as

$$K_{ij} = \frac{1}{2N} \left(\partial_t g_{ij} - D_i N_j - D_j N_i \right), \tag{5}$$

K ($= g^{ij} K_{ij}$) is the trace of K_{ij}, and R is the Ricci scalar constructed from g_{ij}. To lower and raise an index, g_{ij} and its inverse g^{ij} are used. For the sake of simplicity and clarity, in the remainder of this paper, we choose our units such that $M_{Pl} = 1$.

In contrast to GR, the less restricting symmetry allows both kinetic terms $K_{ij} K^{ij}$ and K^2 to be invariant independently, giving rise to the extra parameter λ, which assumes the value 1 in GR, as mentioned above. Furthermore, in order to realize the power-counting renormalizability, the higher curvature Lagrangian $L_{z>1}$ should include up to sixth spatial derivatives. For the analysis in the present paper, the concrete form of $L_{z>1}$ is not needed. Adding the scalar field action I_ϕ that is invariant under spatial parity and time reflection, as well as the foliation preserving diffeomorphism, the total action is

$$\begin{aligned} I &= I_g + I_\phi, \\ I_\phi &= \int N \, dt \, \sqrt{g} \, d^3 \vec{x} \left[\frac{1}{2} (\partial_\perp \phi)^2 - V(\phi, D_i, g_{ij}) \right], \end{aligned} \tag{6}$$

where we define the derivative along vector normal to the hypersurface

$$\partial_\perp \equiv \frac{1}{N} (\partial_t - N^k \partial_k), \tag{7}$$

and we decompose the scalar field potential as

$$V(\phi, D_i, g_{ij}) = V_0(\phi) + V_{z \geq 1}(\phi, D_i, g_{ij}). \tag{8}$$

Here, $V_{z \geq 1}$ summarizes terms with two or more spatial derivatives and like $L_{z>1}$ above, its concrete form is not needed for the purposes of the present paper.

Variation of the total action with respect to the 3-dimensional metric g_{ij} leads to the dynamical equation

$$\mathcal{E}_{gij} + \mathcal{E}_{\phi ij} = 0, \tag{9}$$

where

$$\begin{aligned} \mathcal{E}_{gij} &\equiv g_{ik} g_{jl} \frac{2}{N\sqrt{g}} \frac{\delta I_g}{\delta g_{kl}} \\ &= -\frac{1}{N} (\partial_t - N^k D_k) p_{ij} + \frac{1}{N} (p_{ik} D_j N^k + p_{jk} D_i N^k) \\ &\quad - K p_{ij} + 2 K_i^k p_{kj} + \frac{1}{2} g_{ij} K^{kl} p_{kl} - \Lambda g_{ij} - G_{ij} + \mathcal{E}_{g,z>1,ij}, \end{aligned} \tag{10}$$

$$\begin{aligned} \mathcal{E}_{\phi ij} &\equiv g_{ik} g_{jl} \frac{2}{N\sqrt{g}} \frac{\delta I_\phi}{\delta g_{kl}} \\ &= g_{ij} \left[\frac{1}{2} (\partial_\perp \phi)^2 - V_0(\phi) \right] + \mathcal{E}_{\phi, z \geq 1, ij}. \end{aligned} \tag{11}$$

Here, $\mathcal{E}_{g,z>1,ij}$ and $\mathcal{E}_{\phi,z\geq1,ij}$ are contributions from $L_{z>1}$ and $-V_{z\geq1}$, respectively, $p_{ij} \equiv K_{ij} - \lambda K g_{ij}$, and G_{ij} is Einstein tensor of g_{ij}. The trace part and traceless part of Eq.(9) are, respectively,

$$(3\lambda - 1)\left(\partial_\perp K + \frac{1}{2}K^2\right) + \frac{3}{2}A^i{}_j A^j{}_i + \frac{3}{2}(\partial_\perp \phi)^2 + Z = 0\,, \tag{12}$$

and

$$\partial_\perp A^i{}_j + KA^i{}_j + \frac{1}{N}(A^k{}_j \partial_k N^i - A^i{}_k \partial_j N^k) - \left(Z^i{}_j - \frac{1}{3}Z\delta^i_j\right) = 0\,, \tag{13}$$

where

$$A^i{}_j \equiv K^i{}_j - \frac{1}{3}K\delta^i_j\,, \tag{14}$$

is the traceless part of $K^i{}_j$ and we defined

$$\begin{aligned}
Z^i{}_j &\equiv Z^i_{g,j} + Z^i_{\phi,j}\,, \qquad Z \equiv Z^i{}_i\,, \\
Z^i_{g,j} &\equiv -\Lambda\,\delta^i_j - G^i{}_j + g^{ik}\mathcal{E}_{z>1,g,kj}\,, \\
Z^i_{\phi,j} &\equiv -V_0(\phi)\delta^i_j + g^{ik}\mathcal{E}_{z\geq1,\phi,kj}\,.
\end{aligned} \tag{15}$$

Here, $Z^i_{g,j}$ is the variation of the potential part of the gravitational action with respect to the spatial metric; it is a generalization of (minus) the Einstein tensor of g_{ij} to include higher curvature terms, as well as the cosmological constant. The quantity $Z^i_{\phi,j}$ is obtained similarly from the potential part of the scalar field action.

The variation of the total action with respect to ϕ yields the remaining dynamical equation

$$0 = -\frac{1}{N\sqrt{g}}\frac{\delta I_\phi}{\delta\phi} = \frac{1}{N\sqrt{g}}\partial_t(\sqrt{g}\,\partial_\perp\phi) - \frac{1}{N\sqrt{g}}D_i(\sqrt{g}\,N^i\partial_\perp\phi) + E_\phi\,, \tag{16}$$

where

$$E_\phi \equiv \frac{1}{\sqrt{g}}\frac{\delta}{\delta\phi}\int \sqrt{g}\,dt\,d^3\vec{x}\,V(\phi, D_i, g_{ij}) = V_0'(\phi) + E_{\phi,z\geq1}\,, \tag{17}$$

and $E_{\phi,z\geq1}$ is the contribution from $V_{z\geq1}$.

Since the 3-dimensional spatial diffeomorphism is a subgroup of the foliation preserving diffeomorphism, $Z^i_{g,j}$ and $Z^i_{\phi,j}$ satisfy the generalized Bianchi identity and matter conservation,

$$D_j Z^j_{g,i} = 0\,, \qquad D_j Z^j_{\phi,i} + E_\phi \partial_i \phi = 0\,. \tag{18}$$

For convenience, we decompose the spatial metric and the extrinsic curvature as

$$g_{ij} = a^2(t)\,e^{2\zeta(t,\vec{x})}\,\gamma_{ij}(t,\vec{x})\,, \tag{19}$$

$$K^i{}_j = \frac{1}{3}K(t,\vec{x})\,\delta^i{}_j + A^i{}_j(t,\vec{x})\,, \tag{20}$$

where we have defined $\zeta(t,\vec{x})$ so that $\det\gamma = 1$, and $a(t)$ (up to an overall normalization) is defined later in Eq. (37). The trace part and the traceless part of the definition of the extrinsic curvature lead, respectively, to

$$\partial_\perp \zeta + \frac{\partial_t a}{N\,a} = \frac{1}{3}\left(K + \frac{1}{N}\partial_i N^i\right)\,, \tag{21}$$

and

$$\partial_\perp \gamma_{ij} = 2\,\gamma_{ik}A^k{}_j + \frac{1}{N}\left(\gamma_{jk}\partial_i N^k + \gamma_{ik}\partial_j N^k - \frac{2}{3}\gamma_{ij}\partial_k N^k\right)\,. \tag{22}$$

The momentum constraint is obtained by varying the action with respect to N^i:

$$D_j K^j{}_i - \lambda\,\partial_i K = \partial_\perp \phi\,\partial_i\phi\,. \tag{23}$$

According to the decomposition (20), the momentum constraint is rewritten as

$$\partial_j A^j{}_i + 3\,A^j{}_i\,\partial_j\zeta - \frac{1}{2}A^j{}_l\,(\gamma^{-1})^{lk}\,\partial_i\gamma_{jk} - \frac{1}{3}\,(3\lambda - 1)\,\partial_i K = \partial_\perp\phi\,\partial_i\phi\,. \tag{24}$$

It can be shown that the evolution equations we have derived are consistent with vanishing $A^i{}_i$, $\ln\det\gamma$, $\gamma_{ij} - \gamma_{ji}$ and $\gamma_{ik}A^k{}_j - \gamma_{jk}A^k{}_i$ [26].

II. GRADIENT EXPANSION

In this section, we analyze the dynamics of nonlinear superhorizon perturbations in the spatial gradient expansion approach. This approach is valid as long as the characteristic length scale L of the perturbations is much larger than the Hubble length H^{-1}. By the introduction of small parameter $\epsilon \sim 1/(H L)$, we perform a series expansion on all relevant quantities and equations. For instance, a spatial derivative acting on a quantity at order ϵ^p raises the order to ϵ^{p+1} and thus is counted as $\mathcal{O}(\epsilon)$. We then solve the equations order by order in gradient expansion, extending the calculations of [26] in a spatially flat Friedmann-Robertson-Walker background, to include a single scalar field as the source.

A. Gauge fixing

The foliation preserving diffeomorphism invariance, like all other gauge symmetries, reflects a redundancy in the descriptions of the theory. By an appropriate choice of gauge conditions, these degrees can be eliminated and physical quantities can be extracted. In the present paper we adopt the synchronous gauge, or the Gaussian normal coordinate system, by setting the lapse function to unity and the shift vector to zero:

$$N = 1, \qquad N^i = 0. \tag{25}$$

This choice fixes the time coordinate but in the spatial coordinates, there remains a gauge freedom of time-independent spatial diffeomorphism, corresponding to the change of coordinates on the initial constant-time hypersurface. This residual gauge degree of freedom will be discussed later in Subsection II F.

After the gauge fixing, our basic equations (12), (13), (16), (21) and (22) are simplified to

$$(3\lambda - 1)\partial_t K = -\frac{1}{2}(3\lambda - 1)K^2 - \frac{3}{2}A^i{}_j A^j{}_i - \frac{3}{2}(\partial_t \phi)^2 - Z, \tag{26}$$

$$\partial_t A^i{}_j = -K A^i{}_j + Z^i{}_j - \frac{1}{3} Z \, \delta^i{}_j, \tag{27}$$

$$0 = \partial_t^2 \phi + K \partial_t \phi + E_\phi, \tag{28}$$

$$\partial_t \zeta = -\frac{\partial_t a}{a} + \frac{1}{3} K, \tag{29}$$

$$\partial_t \gamma_{ij} = 2 \gamma_{ik} A^k{}_j, \tag{30}$$

while the momentum constraint (24) has the form

$$\partial_j A^j{}_i + 3 A^j{}_i \partial_j \zeta - \frac{1}{2} A^j{}_l (\gamma^{-1})^{lk} \partial_i \gamma_{jk} - \frac{1}{3} (3\lambda - 1) \partial_i K = \partial_t \phi \, \partial_i \phi. \tag{31}$$

Hereafter, we assume that $\lambda \neq 1/3$; this is consistent with the regime of physical interest $\lambda \geq 1$, discussed in the Introduction section.

B. Basic assumptions and order analysis

We begin by determining the order of all relevant variables. In the limit $\epsilon \to 0$, we expect a universe that looks locally like a Friedmann universe, leading to our starting assumption

$$\partial_t \gamma_{ij} = \mathcal{O}(\epsilon). \tag{32}$$

For the scalar field, a similar assumption leads to $\partial_i \phi = \mathcal{O}(\epsilon)$. However, in order to simplify the analysis, we impose the stronger condition

$$\partial_i \phi = O(\epsilon^2). \tag{33}$$

That is, we assume that $\phi^{(0)}$, which is the leading order term of ϕ, is only time dependent:

$$\phi^{(0)} = \phi^{(0)}(t). \tag{34}$$

The first assumption (32) then implies, from Eq. (30),

$$A^i{}_j = O(\epsilon), \tag{35}$$

leading, using the constraint equation (31), to

$$\partial_i K = O(\epsilon^2). \tag{36}$$

In other words, the zero-th order part $K^{(0)}$ of K depends on t only. This fact enables us to define $a(t)$ by

$$3 \frac{\partial_t a(t)}{a(t)} = K^{(0)} (\equiv 3H(t)). \tag{37}$$

With this definition of $a(t)$, Eq. (29) leads to

$$\partial_t \zeta = O(\epsilon). \tag{38}$$

To summarize, the relevant quantities in the analysis are expanded as follows:

$$\begin{aligned}
\zeta &= \zeta^{(0)}(\vec{x}) + \epsilon\,\zeta^{(1)}(t,\vec{x}) + \epsilon^2 \zeta^{(2)}(t,\vec{x}) + \mathcal{O}(\epsilon^3)\,, & (39)\\
\gamma_{ij} &= f_{ij}(\vec{x}) + \epsilon\,\gamma^{(1)}_{ij}(t,\vec{x}) + \epsilon^2 \gamma^{(2)}_{ij}(t,\vec{x}) + \mathcal{O}(\epsilon^3)\,, & (40)\\
K &= 3\,H(t) + \epsilon\,K^{(1)}(t,\vec{x}) + \epsilon^2 K^{(2)}(t,\vec{x}) + \mathcal{O}(\epsilon^3)\,, & (41)\\
A^i{}_j &= \epsilon\,A^{(1)\,i}{}_j(t,\vec{x}) + \epsilon^2 A^{(2)\,i}{}_j(t,\vec{x}) + \mathcal{O}(\epsilon^3)\,, & (42)\\
\phi &= \phi^{(0)}(t) + \epsilon\,\phi^{(1)}(t,\vec{x}) + \epsilon^2 \phi^{(2)}(t,\vec{x}) + \mathcal{O}(\epsilon^3)\,, & (43)
\end{aligned}$$

where a quantity with the upper index (n) corresponds to the n-th order term in the gradient expansion.

C. Equations in each order

After determining the orders of all physical quantities, we now use this information in the evolution equations (26)–(30) to obtain the evolution equations at each order. In the zero-th order of gradient expansion we have

$$\begin{aligned}
(3\,\lambda - 1)\left(\partial_t H + \frac{3}{2}\,H^2\right) &= -\frac{1}{2}\,(\partial_t \phi^{(0)})^2 + V_0(\phi^{(0)}) + \Lambda\,, \\
\partial_t^2 \phi^{(0)} + 3\,H\,\partial_t \phi^{(0)} + V'(\phi^{(0)}) &= 0\,,
\end{aligned} \tag{44}$$

where a prime denotes the ordinary derivative with respect to the indicated argument. By using the second of the above, the first equation can be integrated to give

$$3\,H^2 = \frac{2}{3\,\lambda - 1}\left[\frac{1}{2}\,(\partial_t \phi^{(0)})^2 + V(\phi^{(0)}) + \Lambda\right] + \frac{\widetilde{C}}{a^3}\,, \tag{45}$$

where $\widetilde{C}$ is an integration constant. The last term in the right hand side of this equation is the "dark matter as an integration constant" [14], a direct consequence of the projectability condition.

The dynamical equations at order $\mathcal{O}(\epsilon^n)$ with $n \geq 1$, are written as

$$a^{-3}\partial_t \left[a^3 \left(K^{(n)} + \frac{3\,\phi^{(n)}\partial_t\phi^{(0)}}{3\,\lambda - 1}\right)\right] = -\frac{1}{2}\sum_{p=1}^{n-1} K^{(p)} K^{(n-p)} - \frac{3}{2\,(3\,\lambda - 1)} \sum_{p=1}^{n-1} \Big[A^{(p)\,i}{}_j A^{(n-p)\,j}{}_i$$

$$+ \partial_t\phi^{(p)}\partial_t\phi^{(n-p)}\Big] - \frac{\bar{Z}^{(n)}}{3\,\lambda - 1}\,, \tag{46}$$

$$a^{-3}\partial_t\left(a^3 A^{(n)\,i}{}_j\right) = -\sum_{p=1}^{n-1} K^{(p)} A^{(n-p)\,i}{}_j + \bar{Z}^{(n)\,i}{}_j - \frac{1}{3}\,\bar{Z}^{(n)}\delta^i{}_j\,, \tag{47}$$

$$a^{-3}\partial_t\left(a^3\partial_t\phi^{(n)}\right) + \left[V_0''(\phi^{(0)}) - \frac{3\,(\partial_t\phi^{(0)})^2}{3\,\lambda - 1}\right]\phi^{(n)} = -\left(K^{(n)} + \frac{3\,\phi^{(n)}\partial_t\phi^{(0)}}{3\,\lambda - 1}\right)\partial_t\phi^{(0)}$$

$$- \sum_{p=1}^{n-1} K^{(p)}\partial_t\phi^{(n-p)} - \bar{E}_\phi^{(n)}\,, \tag{48}$$

$$\partial_t\zeta^{(n)} = \frac{1}{3}\left(K^{(n)} + \frac{3\,\phi^{(n)}\partial_t\phi^{(0)}}{3\,\lambda - 1}\right) - \frac{\phi^{(n)}\partial_t\phi^{(0)}}{3\,\lambda - 1}\,, \tag{49}$$

$$\partial_t\gamma_{ij}^{(n)} = 2\sum_{p=0}^{n-1}\gamma_{ik}^{(p)} A^{(n-p)\,k}{}_j\,, \tag{50}$$

where for later convenience, we introduced new (barred) quantities

$$\bar{Z}^{(n)\,i}{}_j \equiv Z^{(n)\,i}{}_j + V_0'(\phi^{(0)})\,\phi^{(n)}\delta_j^i\,, \quad \bar{Z}^{(n)} \equiv Z^{(n)} + 3\,V_0'(\phi^{(0)})\,\phi^{(n)}\,, \quad \bar{E}_\phi^{(n)} \equiv E_\phi^{(n)} - V_0''(\phi^{(0)})\,\phi^{(n)}\,, \tag{51}$$

to subtract the terms depending on $\phi^{(n)}$ from (unbarred) $Z_j^{(n)\,i}$ and $E_j^{(n)\,i}$, defined in Eqs.(15) and (17). Here, $Z^{(n)\,i}{}_j$, $Z^{(n)}$ and $E_\phi^{(n)}$ are the n-th order terms of $Z^i{}_j$, Z, E_ϕ, respectively. With this definition, $\bar{Z}^{(n)\,i}{}_j$, $\bar{Z}^{(n)}$ and $\bar{E}_\phi^{(n)}$ do not depend on $\zeta^{(n)}$, $\gamma_{ij}^{(n)}$, $K^{(n)}$, $A^{(n)\,i}{}_j$, nor on $\phi^{(n)}$.

Similarly, from Eq. (31) we obtain the order $\mathcal{O}(\epsilon^{n+1})$ $(n \geq 1)$ momentum constraint as

$$\partial_j A^{(n)\,j}{}_i + 3\sum_{p=1}^n A^{(p)\,j}{}_i\partial_j\zeta^{(n-p)} - \frac{1}{2}\sum_{p=1}^n\sum_{q=0}^{n-p} A^{(p)\,j}{}_l(\gamma^{-1})^{(q)\,lk}\partial_i\gamma_{jk}^{(n-p-q)}$$

$$- \frac{1}{3}\,(3\,\lambda - 1)\,\partial_i\left(K^{(n)} + \frac{3\,\phi^{(n)}\partial_t\phi^{(0)}}{3\,\lambda - 1}\right) - \sum_{p=1}^{n-1}\partial_t\phi^{(p)}\partial_i\phi^{(n-p)} = 0\,, \tag{52}$$

where $(\gamma^{-1})^{(n)\,ij}$ is the n-th order term of the inverse of γ_{ij}, i.e. the inverse $(\gamma^{-1})^{ij}$ is expanded as

$$(\gamma^{-1})^{ij} = f^{ij} + \epsilon\,(\gamma^{-1})^{(1)\,ij} + \epsilon^2(\gamma^{-1})^{(2)\,ij} + \dots\,, \tag{53}$$

where $f^{ij} = (\gamma^{-1})^{(0)\,ij}$ is the inverse of f_{ij}. It is straightforward to show that $(\gamma^{-1})^{(n)\,ij}$ $(n \geq 1)$ satisfies the following differential equation:

$$\partial_t(\gamma^{-1})^{(n)\,ij} = -2\sum_{p=1}^n A^{(p)\,i}{}_k(\gamma^{-1})^{(n-p)\,kj}\,. \tag{54}$$

In addition to the dynamical equations and momentum constraint, there are also some useful identities. First, we expand the generalized Bianchi identity (18) to obtain

$$\partial_j\bar{Z}^{(n)\,j}{}_i + 3\sum_{p=1}^n\left(\bar{Z}^{(p)\,j}{}_i - \frac{1}{3}\,\bar{Z}^{(p)}\delta^j{}_i\right)\partial_j\zeta^{(n-p)} - \frac{1}{2}\sum_{p=1}^n\sum_{q=0}^{n-p}\bar{Z}^{(p)\,j}{}_l(\gamma^{-1})^{(q)\,lk}\partial_i\gamma_{jk}^{(n-p-q)}$$

$$+ \sum_{p=1}^{n-1}\left[\bar{E}_\phi^{(n-p)} + V_0''(\phi^{(0)})\,\phi^{(n-p)}\right]\partial_i\phi^{(p)} = 0\,, \tag{55}$$

for $n \geq 1$. Next, expanding the conditions $A^i{}_i = 0$, $\partial_i \ln \det \gamma = 0$, $\gamma_{ij} - \gamma_{ji} = 0$, $\gamma_{ik} A^k{}_j - \gamma_{jk} A^k{}_i = 0$ and $A^i{}_j - \gamma_{jk} A^k{}_l (\gamma^{-1})^{li} = 0$ leads to the following identities:

$$A^{(n) i}{}_i = 0, \qquad \sum_{p=0}^{n} (\gamma^{-1})^{(p) jk} \partial_i \gamma_{jk}^{(n-p)} = 0, \qquad \gamma_{ij}^{(n)} - \gamma_{ji}^{(n)} = 0,$$

$$\sum_{p=0}^{n-1} \left(\gamma_{ik}^{(p)} A^{(n-p) k}{}_j - \gamma_{jk}^{(p)} A^{(n-p) k}{}_i \right) = 0, \qquad A^{(n) i}{}_j - \sum_{p=0}^{n-1} \sum_{q=0}^{n-p-1} \gamma_{jk}^{(p)} A^{(n-p-q) k}{}_l (\gamma^{-1})^{(q) li} = 0. \qquad (56)$$

D. $\mathcal{O}(\epsilon)$ solution

For $\mathcal{O}(\epsilon)$, Eqs.(46)–(50) reduce to

$$\partial_t \left[a^3 \left(K^{(1)} + \frac{3 \, \phi^{(1)} \partial_t \phi^{(0)}}{3 \, \lambda - 1} \right) \right] = 0, \qquad (57)$$

$$\partial_t \left(a^3 A^{(1) i}{}_j \right) = 0, \qquad (58)$$

$$a^{-3} \partial_t \left(a^3 \partial_t \phi^{(1)} \right) + \left[V_0''(\phi^{(0)}) - \frac{3 \, (\partial_t \phi^{(0)})^2}{3 \, \lambda - 1} \right] \phi^{(1)} = - \left(K^{(1)} + \frac{3 \, \phi^{(1)} \partial_t \phi^{(0)}}{3 \, \lambda - 1} \right) \partial_t \phi^{(0)}, \qquad (59)$$

$$\partial_t \zeta^{(1)} = \frac{1}{3} \left(K^{(1)} + \frac{3 \, \phi^{(1)} \partial_t \phi^{(0)}}{3 \, \lambda - 1} \right) - \frac{\phi^{(1)} \partial_t \phi^{(0)}}{3 \, \lambda - 1}, \qquad (60)$$

$$\partial_t \gamma_{ij}^{(1)} = 2 \, f_{ik} A^{(1) k}{}_j, \qquad (61)$$

where from equations (15), (17) and (51), we have at first order, $\bar{Z}^{(1) i}{}_j = \bar{Z}^{(1)} = \bar{E}_\phi^{(1)} = 0$. Integrating the above equations, we obtain

$$K^{(1)} + \frac{3 \, \phi^{(1)} \partial_t \phi^{(0)}}{3 \, \lambda - 1} = \frac{C^{(1)}(\vec{x})}{a(t)^3}, \qquad (62)$$

$$A^{(1) i}{}_j = \frac{C^{(1) i}{}_j(\vec{x})}{a(t)^3}, \qquad (63)$$

$$\phi^{(1)} = \left[C^{(1)}(\vec{x}) \int_{t_{\rm in}}^t dt' \, \frac{f_2(t') \, \partial_{t'} \phi^{(0)}(t')}{a(t')^3 W(t')} + \phi_{\rm in}^{(1)}(\vec{x}) \right] f_1(t)$$
$$+ \left[-C^{(1)}(\vec{x}) \int_{t_{\rm in}}^t dt' \, \frac{f_1(t') \, \partial_{t'} \phi^{(0)}(t')}{a(t')^3 W(t')} + \dot{\phi}_{\rm in}^{(1)}(\vec{x}) \right] f_2(t), \qquad (64)$$

$$\zeta^{(1)} = \frac{C^{(1)}(\vec{x})}{3} \int_{t_{\rm in}}^t \frac{dt'}{a^3(t')} - \frac{1}{3 \, \lambda - 1} \int_{t_{\rm in}}^t dt' \, \phi^{(1)}(t') \, \partial_{t'} \phi^{(0)}(t') + \zeta_{\rm in}^{(1)}(\vec{x}), \qquad (65)$$

$$\gamma_{ij}^{(1)} = 2 \, f_{ik}(\vec{x}) \, C^{(1) k}{}_j(\vec{x}) \int_{t_{\rm in}}^t \frac{dt'}{a^3(t')} + \gamma_{{\rm in} \, ij}^{(1)}(\vec{x}), \qquad (66)$$

where the integration "constants" $C^{(1)}$, $C^{(1) i}{}_j$, $\phi_{\rm in}^{(1)}$, $\dot{\phi}_{\rm in}^{(1)}$, $\zeta_{\rm in}^{(1)}$ and $\gamma_{{\rm in} \, ij}^{(1)}$ depend only on the spatial coordinates $\vec{x}^i$ and satisfy

$$C^{(1) i}{}_i = 0, \qquad f_{ik} C^{(1) k}{}_j = f_{jk} C^{(1) k}{}_i. \qquad (67)$$

The functions $f_i(t)$ $(i = 1, 2)$ are two independent solutions of the homogeneous equation

$$a^{-3} \partial_t (a^3 \partial_t f_i) + \left[V_0''(\phi^{(0)}) - \frac{3 \, (\partial_t \phi^{(0)})^2}{3 \, \lambda - 1} \right] f_i = 0; \qquad f_1(t_{\rm in}) = 1, \quad f_1'(t_{\rm in}) = 0; \qquad f_2(t_{\rm in}) = 0, \quad f_2'(t_{\rm in}) = 1, \quad (68)$$

and

$$W(t) \equiv f_1(t) \, \partial_t f_2(t) - f_2(t) \, \partial_t f_1(t). \qquad (69)$$

The two first order integration "constants", $\zeta_{\mathrm{in}}^{(1)}$ and $\gamma_{\mathrm{in}\,ij}^{(1)}$, can be absorbed into their zero-th order counterparts, $\zeta_{\mathrm{in}}^{(0)}$ and $\gamma_{\mathrm{in}\,ij}^{(0)}$. Thus, without loss of generality, we can set

$$\zeta_{\mathrm{in}}^{(1)} = 0\,, \qquad \gamma_{\mathrm{in}\,ij}^{(1)} = 0\,. \tag{70}$$

Finally, the momentum constraint equation (52) with $n = 1$ leads to the following relation among the remaining integration constants, $C^{(1)}$, $C^{(1)\,i}{}_j$, $\zeta^{(0)}$ and f_{ij},

$$\partial_j C^{(1)\,j}{}_i + 3\,C^{(1)\,j}{}_i\partial_j\zeta^{(0)} - \frac{1}{2}\,C^{(1)\,j}{}_l f^{lk}\partial_i f_{jk} - \frac{1}{3}\,(3\,\lambda - 1)\,\partial_i C^{(1)} = 0. \tag{71}$$

Note that $\phi_{\mathrm{in}}^{(1)}(\vec{x})$ and $\dot{\phi}_{\mathrm{in}}^{(1)}(\vec{x})$ do not appear in this equation. The physical meaning of $\phi_{\mathrm{in}}^{(1)}(\vec{x})$ and $\dot{\phi}_{\mathrm{in}}^{(1)}(\vec{x})$ are obvious:

$$\phi^{(1)}\Big|_{t=t_{\mathrm{in}}} = \phi_{\mathrm{in}}^{(1)}(\vec{x})\,, \quad \partial_t\phi^{(1)}\Big|_{t=t_{\mathrm{in}}} = \dot{\phi}_{\mathrm{in}}^{(1)}(\vec{x})\,. \tag{72}$$

E. $\mathcal{O}(\epsilon^n)$ solution $(n \geq 1)$

Equipped with the zero-th and first order solution, we can now determine the general solutions at arbitrary order in gradient expansion. For any $n \geq 1$, the solution to Eqs.(46)–(50) is

$$K^{(n)} + \frac{3\,\phi^{(n)}\partial_t\phi^{(0)}}{3\lambda - 1} = \frac{1}{a^3(t)}\int_{t_{\mathrm{in}}}^{t} dt'a^3(t')\left\{ -\frac{\bar{Z}^{(n)}(t',\vec{x})}{3\lambda - 1} - \frac{1}{2}\sum_{p=1}^{n-1} K^{(p)}(t',\vec{x})K^{(n-p)}(t',\vec{x}) \right.$$
$$\left. -\frac{3}{2\,(3\lambda - 1)}\sum_{p=1}^{n-1}\left[A^{(p)\,i}{}_j(t',\vec{x})\,A^{(n-p)\,j}{}_i(t',\vec{x}) + \partial_{t'}\phi^{(p)}(t',\vec{x})\,\partial_{t'}\phi^{(n-p)}(t',\vec{x}) \right] \right\}\,, \tag{73}$$

$$A^{(n)\,i}{}_j = \frac{1}{a^3(t)}\int_{t_{\mathrm{in}}}^{t} dt'a^3(t')\left[-\sum_{p=1}^{n-1} K^{(p)}(t',\vec{x})A^{(n-p)\,i}{}_j(t',\vec{x}) + \bar{Z}^{(n)\,i}{}_j(t',\vec{x}) - \frac{1}{3}\bar{Z}^{(n)}(t',\vec{x})\delta^i{}_j \right]\,, \tag{74}$$

$$\phi^{(n)} = f_1(t)\int_{t_{\mathrm{in}}}^{t} dt'\frac{f_2(t')r^{(n)}(t',\vec{x})}{W(t')} - f_2(t)\int_{t_{\mathrm{in}}}^{t} dt'\frac{f_1(t')r^{(n)}(t',\vec{x})}{W(t')}\,, \tag{75}$$

$$\zeta^{(n)} = \int_{t_{\mathrm{in}}}^{t} dt'\left[\frac{1}{3}\left(K^{(n)}(t',\vec{x}) + \frac{3\phi^{(n)}(t',\vec{x})\partial_{t'}\phi^{(0)}(t')}{3\lambda - 1} \right) - \frac{\phi^{(n)}(t',\vec{x})\partial_{t'}\phi^{(0)}(t')}{3\lambda - 1} \right]\,, \tag{76}$$

$$\gamma_{ij}^{(n)} = 2\int_{t_{\mathrm{in}}}^{t} dt'\sum_{p=0}^{n-1} \gamma_{ik}^{(p)}(t',\vec{x})A^{(n-p)\,k}{}_j(t',\vec{x})\,, \tag{77}$$

where

$$r^{(n)}(t,\vec{x}) \equiv \left(K^{(n)}(t,\vec{x}) + \frac{3\,\phi^{(n)}(t,\vec{x})\partial_t\phi^{(0)}(t)}{3\lambda - 1} \right)\partial_t\phi^{(0)}(t) + \sum_{p=1}^{n-1} K^{(p)}(t,\vec{x})\,\partial_t\phi^{(n-p)}(t,\vec{x}) + \bar{E}_\phi^{(n)}(t,\vec{x})\,, \tag{78}$$

and by redefining $C^{(1)}$, $C^{(1)\,i}{}_j$, $\phi_{\mathrm{in}}^{(1)}$, $\dot{\phi}_{\mathrm{in}}^{(1)}$, $\zeta^{(0)}$ and f_{ij}, we have set, respectively,

$$K^{(n)}\Big|_{t=t_{\mathrm{in}}} = A^{(n)\,i}{}_j\Big|_{t=t_{\mathrm{in}}} = \phi^{(n)}\Big|_{t=t_{\mathrm{in}}} = \partial_t\phi^{(n)}\Big|_{t=t_{\mathrm{in}}} = \zeta^{(n)}\Big|_{t=t_{\mathrm{in}}} = \gamma_{ij}^{(n)}\Big|_{t=t_{\mathrm{in}}} = 0\,. \tag{79}$$

We remind that the first order constants have already been fixed in Eq.(70) by redefinition of $\zeta^{(0)}$ and f_{ij}, respectively.

The initial condition for $\gamma_{ij}^{(n)}$ $(n \geq 1)$ implies that $\gamma_{ij}|_{t=t_{\mathrm{in}}} = f_{ij}$, $(\gamma^{-1})^{ij}\big|_{t=t_{\mathrm{in}}} = f^{ij}$ and $(\gamma^{-1})^{(n)\,ij}\big|_{t=t_{\mathrm{in}}} = 0$ $(n \geq 1)$. Therefore, for $n \geq 1$, the solution to Eq.(54) is

$$(\gamma^{-1})^{(n)\,ij} = -2\int_{t_{\mathrm{in}}}^{t} dt'\sum_{p=1}^{n} A^{(p)\,i}{}_k(\gamma^{-1})^{(n-p)\,kj}\,. \tag{80}$$

As shown in Appendix A, the solution (73)-(77) automatically satisfies the $(n+1)$-th order momentum constraint equation (52), provided that the redefined integration constants $(C^{(1)}, C^{(1)\,i}{}_j, \zeta^{(0)}, f_{ij})$ satisfy (71) up to $\mathcal{O}(\epsilon^{n+1})$.

F. Number of physical degrees of freedom

The solution we obtained in the previous subsection involves a number of functions depending only on spatial coordinates, $\zeta^{(0)}(\vec{x})$, $f_{ij}(\vec{x})$, $C^{(1)}(\vec{x})$, $C^{(1)\,i}{}_j(\vec{x})$, $\phi_{\rm in}^{(1)}(\vec{x})$ and $\dot{\phi}_{\rm in}^{(1)}(\vec{x})$ which emerged as integration "constants". However, not all of the components are independent nor physical. Firstly, they are subject to the constraint (71). Secondly, as stated just after Eq. (25), our gauge condition (25) leaves time-independent spatial diffeomorphism as a residual gauge freedom. Therefore, the number of physical degrees of freedom included in each integration "constant" is

$$
\begin{aligned}
\zeta^{(0)}(\vec{x}) \ &\dots \ \text{1 scalar growing mode} \ = 1 \text{ component} , \\
f_{ij}(\vec{x}) \ &\dots \ \text{2 tensor growing modes} \ = 5 \text{ components} \ - 3 \text{ gauge} , \\
C^{(1)}(\vec{x}) \ &\dots \ \text{1 scalar decaying mode} \ = 1 \text{ component} , \\
C^{(1)\,i}{}_j(\vec{x}) \ &\dots \ \text{2 tensor decaying modes} \ = 5 \text{ components} \ - 3 \text{ constraints} , \\
\phi_{\rm in}^{(1)}(\vec{x}), \dot{\phi}_{\rm in}^{(1)}(\vec{x}) \ &\dots \ \text{2 scalar modes} .
\end{aligned}
\tag{81}
$$

This is consistent with the fact that the HL gravity includes not only a tensor graviton (2 propagating degrees of freedom) but also a scalar graviton (1 propagating degree of freedom) and that our system includes a scalar field (1 propagating degree of freedom) as well.

III. PERTURBATIVE VS NONPERTURBATIVE APPROACHES

In the previous section, we have derived solutions for nonlinear perturbations in any order of gradient expansion. While gradient expansion relies on the long wavelength approximation, amplitudes of perturbations do not have to be small. Thus, our analysis in the previous section is totally nonperturbative with respect to amplitudes of perturbations. The dynamical equations and their solutions do not suffer from any divergences in the $\lambda \to 1$ limit, and GR coupled with a scalar field and dark matter is safely recovered in this limit.

This is in sharp contrast with results known in the literature based on the "standard" (and naive) perturbative approach, in which pathologies such as divergences and strong coupling are found in the $\lambda \to 1$ limit. In this section, we shall see how this problem arises in the "standard" perturbative approach and why it becomes under control in our nonperturbative approach. In the "standard" perturbative approach, all relevant equations are expanded with respect to amplitudes of perturbations, irrespective of sizes of coefficients in the expansion. We shall see that, in the momentum constraint, coefficients of terms linear in perturbations actually vanish in the $\lambda \to 1$ limit and thus, for sufficiently small but nonvanishing $|\lambda - 1|$, the linear terms become less important than nonlinear terms. Hence, neglecting nonlinear terms, blindly solving the linearized momentum constraint and then taking the $\lambda \to 1$ limit would be totally nonsense and lead to inconsistencies. This is precisely the situation in the "standard" perturbative approach. Clearly, this is a breakdown of the treatment based on the "standard" perturbative expansion but not of the HL theory itself. Indeed, as already stated above, our nonperturbative analysis in the previous section does not show any pathologies in the $\lambda \to 1$ limit. In the rest of this section, we shall investigate these issues explicitly. For simplicity we shall consider the cases without the scalar field (but with the built-in "dark matter as integration constant").

A. Breakdown of standard perturbative expansion in the $\lambda \to 1$ limit

In this subsection let us briefly review the standard perturbative approach and see that, contrary to the nonperturbative approach based on the gradient expansion in the previous section, it breaks down in the $\lambda \to 1$ limit.

Let us adopt the following metric ansatz in the transverse gauge,

$$
N = 1, \quad N_i = \partial_i B + n_i, \quad g_{ij} = a^2 e^{2\zeta_T} \left(e^h \right)_{ij} ,
\tag{82}
$$

where n_i is transverse and h_{ij} is transverse and traceless: $\partial^i n_i = 0$, $\partial^i h_{ij} = 0$ and $h^i{}_i = 0$. Throughout this subsection, indices are raised and lowered by δ^{ij} and δ_{ij}. We introduce a small parameter $\bar{\epsilon}$, consider ζ_T, B, n_i and h_{ij} as quantities of $O(\bar{\epsilon})$, and perform perturbative expansion with respect to $\bar{\epsilon}$.

In the regime of validity of the standard perturbative expansion, in order to calculate the action up to cubic order, it suffices to solve the momentum constraint up to the first order, which can be written in the form,

$$
\partial_i \left[a^2 (3\lambda - 1) \partial_t \zeta_T - (\lambda - 1) \triangle B \right] + \frac{1}{2} \triangle n_i = 0,
\tag{83}
$$

leading to

$$a^{-2}\triangle B = \frac{3\lambda - 1}{\lambda - 1}\partial_t\zeta_T, \quad n_i = 0, \tag{84}$$

where $\triangle \equiv \partial^i\partial_i$.

It is straightforward to calculate the kinetic action up to the third order. The quadratic part $I_{kin}^{(2)}$ and the cubic part $I_{kin}^{(3)}$ are [29]

$$
\begin{aligned}
I_{kin}^{(2)} &= \int dt d^3\vec{x}\, a^3 \left(a^{-2}\partial_t\zeta_T\triangle B + \frac{1}{8}\partial_t h^{ij}\partial_t h_{ij} \right), \\
I_{kin}^{(3)} &= \int dt d^3\vec{x}\, a^3 \left[3\zeta_T \left(a^{-2}\partial_t\zeta_T\triangle B + \frac{1}{8}\partial_t h^{ij}\partial_t h_{ij} \right) + \frac{1}{2}a^{-4}\zeta_T\partial^i(\partial_i B\triangle B + 3\partial^j B\partial_i\partial_j B) \right. \\
&\qquad \left. + \frac{1}{2}(a^{-2}\partial^k h^{ij}\partial_k B - 3\partial_t h^{ij}\zeta_T)a^{-2}\partial_i\partial_j B - \frac{1}{4}a^{-2}\partial_t h^{ij}\partial_k h_{ij}\partial^k B \right].
\end{aligned}
\tag{85}
$$

When B is eliminated by using (84), one can easily see that the quadratic part $I_{kin}^{(2)}$ written in terms of $\tilde{\zeta}_T = \sqrt{\frac{2(3\lambda-1)}{\lambda-1}}\zeta_T$ is regular. On the other hand, the cubic part $I_{kin}^{(3)}$ written in terms of $\tilde{\zeta}_T$ is divergent in the limit $\lambda \to 1$. Thus, the perturbative expansion breaks down in this limit. More precisely, the regime of validity of the standard perturbative expansion is

$$|\zeta_T| \ll \min(|\lambda - 1|, 1), \tag{86}$$

and disappears in the $\lambda \to 1$ limit.

Evidently, the breakdown of the standard perturbative expansion in the $\lambda \to 1$ limit originates from the denominator $\lambda - 1$ in the solution (84) to the linearized momentum constraint.

B. Transformation from transverse to synchronous gauge

In the standard perturbative approach summarized in the previous subsection, we have adopted the transverse gauge (82). Instead, in the nonperturbative approach based on the gradient expansion presented in Sec.II, we have adopted the synchronous gauge (25). In this subsection, we shall investigate the spatial coordinate transformation between the two gauges. (Note that in both gauges the space-independent time reparametrization is already fixed by the condition $N = 1$.) The transformation is nonlinear but we treat it perturbatively. As we shall see below, this provides an alternative way to see the breakdown of the standard perturbative expansion.

As described in Appendix B, we start with the transverse gauge, carry out the spatial gauge transformation to the synchronous gauge, and use the momentum constraint (in the transverse gauge) to eliminate the nondynamical degree of freedom. In this way, we can express the perturbation in the synchronous gauge in terms of that in the transverse gauge. Up to the second order, the result is

$$
\begin{aligned}
\zeta = -\frac{2}{3}\frac{1}{\lambda - 1}\Bigg\{ & \zeta_T - \frac{(3\lambda - 1)}{(\lambda - 1)}(\partial_i\zeta_T)(\partial^i\triangle^{-1}\zeta_T) + \frac{(3\lambda - 1)}{(\lambda - 1)}\int^t dt'\,(\partial_i\zeta_T)(\partial^i\triangle^{-1}\partial_{t'}\zeta_T) \\
& -\frac{(3\lambda - 1)}{2(\lambda - 1)}\int^t dt'\triangle^{-1}\left[2(\partial^i\triangle\zeta_T)(\partial_i\triangle^{-1}\partial_{t'}\zeta_T) + \left(\partial^i\partial^j\zeta_T + \frac{1}{2}\triangle h^{ij}\right)(\partial_i\partial_j\triangle^{-1}\partial_{t'}\zeta_T) + (\triangle\zeta_T)(\partial_{t'}\zeta_T)\right] \\
& +\frac{1}{4}\int^t dt'\,\triangle^{-1}\left[\frac{1}{2}(\partial_i\partial_{t'}h_{jk})(\partial^i h_{jk}) + \frac{1}{2}(\partial_{t'}h^{ij})(\triangle h_{ij}) - 3(\partial_i\partial_j\zeta_T)(\partial_{t'}h^{ij})\right] + \mathcal{O}(\bar{\epsilon}^3)\Bigg\},
\end{aligned}
\tag{87}
$$

where ζ is the perturbation in the synchronous gauge defined in (19), ζ_T and h_{ij} are the metric perturbations in the transverse gauge defined in (82). One can easily see that the terms quadratic in ζ_T are suppressed with respect to the linear term under the condition (86).

Conversely, for a fixed amplitude of ζ_T, the expansion with respect to $\bar{\epsilon}$ in (87) breaks down in the $\lambda \to 1$ limit. This is very similar to the way how the standard perturbative expansion of the action (85) breaks down in the $\lambda \to 1$ limit. It is apparent from the intermediate steps of the calculation (shown explicitly in Appendix B) that the terms with negative powers of $(\lambda - 1)$ are introduced by the solution of the momentum constraint.

C. Linear vs nonlinear terms in the momentum constraint

Having understood that the origin of the breakdown of the standard perturbative expansion is the treatment of the momentum constraint, we now discuss the regime of validity of the standard perturbative expansion in the momentum constraint. Importantly, we shall see that a new branch of solution emerges at the edge of the regime of validity of the standard perturbative expansion.

For this purpose, we adopt the transverse gauge (82) and expand the momentum constraint with respect to ζ_T and h_{ij}, considering them as small quantities but keeping B and n_i as nonlinear quantities [2]:

$$\zeta_T = O(q), \quad h_{ij} = O(q), \quad B = O(q^0), \quad n_i = O(q^0), \tag{88}$$

where we have introduced a small parameter q to count the order of perturbations ζ_T and h_{ij}. In the absence of the scalar field, the momentum constraint is [26]

$$0 = \mathcal{H}_j \equiv \partial_j A^j{}_i + 3\, A^j{}_i\, \partial_j \zeta_T - \frac{1}{2} A^j{}_l\, (\gamma^{-1})^{lk}\, \partial_i \gamma_{jk} - \frac{1}{3}\, (3\lambda - 1)\, \partial_i K, \tag{89}$$

where $\gamma_{ij} = (e^h)_{ij}$, $(\gamma^{-1})^{ij} = (e^{-h})^{ij}$, while the trace of the extrinsic curvature (21) becomes

$$
\begin{aligned}
K &= 3\left(\partial_\perp \zeta_T + \frac{\partial_t a}{Na}\right) - \frac{1}{N}\partial_i \left[(g^{-1})^{ij} N_j\right] \\
&= 3H - a^{-2}\triangle B + 3\partial_t \zeta_T + a^{-2}\left[-(\partial^k \zeta_T)(\partial_k B) + 2\zeta_T \triangle B + h^{kl}\partial_k \partial_l B\right] \\
&\quad + O(q) \times n_k + O(q^2),
\end{aligned}
\tag{90}
$$

and the traceless part (22) is

$$
\begin{aligned}
A^j{}_i &= \frac{1}{2}(\gamma^{-1})^{jk}\partial_\perp \gamma_{ki} - \frac{1}{2N}\left\{(\gamma^{-1})^{jk}\gamma_{il}\partial_k\left[(g^{-1})^{lm} N_m\right] + \partial_i\left[(g^{-1})^{jk} N_k\right] - \frac{2}{3}\delta_i^j \partial_k\left[(g^{-1})^{kl} N_l\right]\right\} \\
&= \frac{1}{2}\partial_t h^j{}_i - \frac{1}{a^2}\left\{\frac{1}{2}(\partial^j n_i + \partial_i n^j) + \left(\partial^j \partial_i B - \frac{1}{3}\delta_i^j \triangle B\right)\right\} \\
&\quad + \frac{1}{a^2}\left\{(\partial^j \zeta_T)(\partial_i B) + (\partial_i \zeta_T)(\partial^j B) + 2\zeta_T(\partial^j \partial_i B) - \frac{2}{3}\delta_i^j\left[(\partial^k \zeta_T)(\partial_k B) + \zeta_T \triangle B\right]\right\} \\
&\quad + \frac{1}{a^2}\left\{\frac{1}{2}\left(\partial_i h^{jk} + \partial^j h_i{}^k - \partial^k h^j{}_i\right)(\partial_k B) + h^{jk}(\partial_i \partial_k B) - \frac{1}{3}\delta_i^j h^{kl}(\partial_k \partial_l B)\right\} \\
&\quad + O(q) \times n_k + O(q^2).
\end{aligned}
\tag{91}
$$

Here, it is understood that $(g^{-1})^{ij}$ is the inverse of g_{ij}, that derivatives do not act beyond parentheses and that indices are raised and lowered by δ^{ij} and δ_{ij}. A straightforward calculation results in the following expansion of $\mathcal{H}_i$,

$$
\begin{aligned}
\mathcal{H}_i &= -(3\lambda - 1)\,\partial_i \partial_t \zeta_T + \mathcal{O}(q^2) - \frac{1}{2\,a^2}\left[\triangle + \mathcal{O}(q)\right] n_i \\
&\quad + \frac{1}{a^2}\left\{(\lambda - 1)\left[\delta_i^j \triangle + \mathcal{O}(q)\right] + \left(\frac{1}{2}\triangle h^j{}_i + \partial^j \partial_i \zeta_T + \delta_i^j \triangle \zeta_T\right) + \mathcal{O}(q^2)\right\}\partial_j B.
\end{aligned}
\tag{92}
$$

Notice that in the above, no assumption is made for B and n_i, which are still considered to be nonlinear quantities. It is now clear that the leading term in the coefficient of B relies not only on the order of perturbations, but also on the value of $\lambda - 1$.

In the regime $q \ll \min(1, \lambda - 1)$, the momentum constraint becomes

$$a^{-2}\triangle B = \frac{3\lambda - 1}{\lambda - 1}\, \partial_t \zeta_T + \mathcal{O}(q^2), \qquad n_i = \mathcal{O}(q^2), \qquad \text{for } q \ll \min(1, \lambda - 1), \tag{93}$$

---

[2] Although the discussion in this subsection employs the transverse gauge, the general argument holds in any gauge. In particular, in the synchronous gauge, where $B = n_i = 0$, the issue arising in B gets transferred to the longitudinal part of h_{ij}. However, for the sake of clarity, we chose to keep find solutions for the nondynamical fields, gauging away the longitudinal mode instead.

and agrees with the result of the standard perturbative expansion (84). Naively using this expression in the action, then taking the $\lambda \to 1$ limit would lead to breakdown of the standard perturbative expansion as already seen in (85) and (87).

On the other hand, if λ is sufficiently close to 1 and the condition

$$\lambda - 1 \ll q \ll 1 \tag{94}$$

is met, then the coefficient of B in Eq.(92) is dominated by the $\mathcal{O}(q)$ terms instead of the $\mathcal{O}(\lambda - 1)$ term. Note that this is a nonlinear regime but is still consistent with the assumed smallness of the metric perturbations ζ_T and h_{ij}. In this regime, the constraint can be written as

$$\mathcal{H}_j = -2\partial_j \partial_t \zeta_T + \mathcal{O}(q^2) - \frac{1}{2\,a^2} \left(\triangle + \mathcal{O}(q)\right) n_j + \frac{1}{a^2} \left[M_j{}^i + \mathcal{O}(\lambda - 1) + \mathcal{O}(q^2) \right] \partial_i B \,, \tag{95}$$

where we have defined

$$M_j{}^i \equiv \frac{1}{2} \triangle h^i{}_j + \partial^i \partial_j \zeta_T + \delta^i_j \triangle \zeta_T = O(q)\,. \tag{96}$$

The transverse part of the Eq.(95) can be found by evaluating $\partial_{[k} \mathcal{H}_{j]}$,

$$\begin{aligned}
\partial_{[k} \triangle n_{j]} &= (\triangle h^i{}_{[j})(\partial_{k]} \partial_i B) + (\partial_{[k} \triangle h^i{}_{j]})(\partial_i B) \\
&\quad + 2(\partial^i \partial_{[j} \zeta_T)(\partial_{k]} \partial_i B) + 2(\partial_{[k} \triangle \zeta_T)(\partial_{j]} B) + \mathcal{O}(q^2).
\end{aligned} \tag{97}$$

On the other hand, the longitudinal part can be computed from $\partial^j \mathcal{H}_j$ as

$$-2 \triangle \partial_t \zeta_T + \frac{1}{a^2} \bar{M} B = 0\,, \tag{98}$$

where we define the operator $\bar{M}$ as,

$$\bar{M} \equiv M^{ij} \partial_i \partial_j + 2 \left(\partial^i \triangle \zeta_T \right) \partial_i\,. \tag{99}$$

If this operator is invertible, then we obtain

$$B = 2\,a^2\,\bar{M}^{-1} \triangle \partial_t \zeta_T + \mathcal{O}\left(\frac{\lambda - 1}{q}\right) + \mathcal{O}(q)\,. \tag{100}$$

We note that either the $(\lambda - 1)/q$ or q term can provide the largest correction to B, depending on the value of $\lambda - 1$.

In summary, we have seen that there are two branches of solutions to the momentum constraint, depending on the value of $\lambda - 1$. One is (93) in the linear regime, and the other is (100) in the nonlinear regime. The standard perturbative expansion in the previous subsections corresponds to the solution (93). On the other hand, what is relevant for the nonperturbative recovery of GR (plus "dark matter") in the $\lambda \to 1$ limit is the solution (100). The two regimes are mutually exclusive.

D. Yet another consideration

In the previous subsection we have seen that there are two mutually exclusive branches of solution to the momentum constraint. This explains the reason why the standard perturbative approach breaks down in the $\lambda \to 1$ limit and why the theory itself can be still regular and continuous in the limit.

For λ away from 1, the standard perturbative expansion is valid in the transverse gauge and we have the expansion of the kinetic action as given in subsection III A. On the other hand, for λ sufficiently close to 1, i.e. in the regime (94), it is the nonlinear solution (100) that should be substituted to the kinetic action.

Unlike the kinetic action, the potential part of the action does not depend on λ, when written in terms of ζ_T and h_{ij}. This is because B does not appear in the potential part of the action. Therefore, if we could somehow define a field ζ_c in such a way that the series of terms in the kinetic action for ζ_T sums up to form a standard canonical kinetic term for ζ_c, then the whole action written in terms of ζ_c should remain finite in the $\lambda \to 1$ limit. Since each term in the kinetic action (after eliminating B) includes exactly two time derivatives, such a field redefinition should be possible in principle. In practice, however, the field redefinition is not easy to perform since it would be nonlinear and highly nonlocal in space. Nonetheless, this consideration already suggests that the $\lambda \to 1$ limit should be regular and continuous nonperturbatively.

IV. SUMMARY AND DISCUSSION

In this paper, we have performed a fully nonlinear analysis of superhorizon perturbations in the HL gravity coupled to a scalar field, by using the gradient expansion technique [27]. After applying the long wavelength expansion to the set of field equations, we integrated these explicitly to the second order. We then showed that the solutions can be extended to any order in gradient expansion, while satisfying the momentum constraint at each order. These solutions are continuous in the GR limit $\lambda \to 1$ for any order in the expansion, both in the gravity sector, which consists of the "dark matter as an integration constant", and in the matter sector, which contains a scalar field in the present work. The form of the equations suggests that our qualitative result should remain the same when additional matter fields are introduced.

This is in sharp contrast with the results obtained in the framework of the "standard" (and naive) perturbation theory, in which pathologies such as divergences and strong coupling are found in the $\lambda \to 1$ limit [20, 21]. We determined that the results of the standard perturbative expansion are valid only in the region where $|\lambda - 1|$ is larger than the order of perturbations. In other words, the range of validity (86) of these solutions has zero measure in the limit $\lambda \to 1$. We found that the divergences are originating from the momentum constraint, where the coefficients of the terms linear in perturbations vanish in the $\lambda \to 1$ limit. Thus, for sufficiently small but nonvanishing $|\lambda - 1|$, the linear terms become less important in comparison to the nonlinear ones. Neglecting nonlinear terms and naively solving the linearized momentum constraint, then taking the $\lambda \to 1$ limit turns out to be the main source of the said pathologies. Once their origins were understood, we carried out a detailed analysis of the nonlinear momentum constraint in the perturbative approach. In addition to the known result which is valid when $|\zeta| \ll \min(|\lambda - 1|, 1)$, we found a second branch of solution valid in the regime where $|\lambda - 1| \ll |\zeta| \ll 1)$. The presence of the latter solution justifies the recovery of GR obtained in our nonperturbative approach, in the $\lambda \to 1$ limit.

Our results, together with the similar examples studied in [6, 26], discernibly support that the apparent strong coupling found previously in the HL gravity may only indicate the breakdown of the treatment based on the naive perturbative expansion but not of the theory itself. General relativity should be recovered by nonlinear effects, analogously to the Vainshtein mechanism [24] that were first encountered in the massive gravity theories.

We note that the present analysis was limited to the discussion of the classical (and superhorizon) evolution of perturbations; their quantum mechanical origin were not considered. In contrast, in Ref.[29], the HL gravity in the $\lambda \to \infty$ limit was found to be weakly coupled under a certain condition, and the spectrum of perturbations that were generated from quantum fluctuations was calculated in this limit. However, the presence of regular behavior both in $\lambda \to \infty$ and in $\lambda \to 1$ limits is not sufficient to draw conclusions on the transition between the two regimes. This is because of our lack of an understanding on the details of the renormalization group (RG) flow. Specifically, to be able to match these two results, one needs to define a conserved quantity (like the comoving curvature perturbations in relativistic cosmology). However, since the matching involves a wide range of varying λ, one needs to know how the flow of λ is realized and how such a flow affects the evolution of cosmological perturbations. With these considerations, we refrain from exploring the cosmological implications of our results for now.

On the other hand, a quantum mechanical extension of our analysis may have a chance to address such issues. One of the major concerns with a proper renormalization analysis in HL gravity is the strong coupling problem in the $\lambda \to 1$ limit, or more specifically, the breakdown of the perturbative expansion. However, we have shown in Sec. III C that the full nonlinear analysis in the limit $\lambda \to 1$ is still consistent with small perturbations, except for the nondynamical mode B which becomes nonlinear. The solution to the momentum constraint in the two regimes, (93) and (100), gives the nondynamical mode as

$$B \simeq \begin{cases} \frac{3\lambda - 1}{\lambda - 1} a^2 \triangle^{-1} \partial_t \zeta = O(\zeta), & \text{for } |\zeta| \ll \min(\lambda - 1, 1) \\ 2 a^2 \bar{M}^{-1} \triangle \partial_t \zeta = O(1), & \text{for } \lambda - 1 \ll |\zeta| \ll 1 \end{cases}. \tag{101}$$

Note that both cases are compatible with small ζ. Thus, substituting this nonlinear solution for B in the action, then applying the perturbative expansion for ζ may provide a healthy perturbative action. (However, the reduced action is nonlocal in space while it is local in time).

We also note that the breakdown of the naive perturbative expansion does not necessarily result in loss of renormalizability. We know that in the regime $|\zeta| \ll \min(\lambda - 1, 1)$, the leading UV contributions in the action are invariant under the scaling (1) with $z = 3$, provided that the scalar graviton is assigned a vanishing scaling dimension, i.e. $\zeta \to \zeta$. This fact is nothing but the power-counting renormalizability of the theory, and is seen after replacing B in the action with the linear solution (93), or the first line of (101). Note also that coefficients of all possible terms in the perturbative expansion are expressed in terms of 11 coupling constants in the action (4). On the other hand, for the regime $\lambda - 1 \ll |\zeta| \ll 1$, we need to replace B in the action with the nonlinear solution (100), or the second line of (101). What is important here is that the scaling dimension of B from the nonlinear solution and that from

the linear solution are exactly the same: $B \to b\,B$ under the scaling (1) with $z = 3$ in both cases. Therefore, after substituting the nonlinear solution to B in the action, we still conclude that the leading UV contributions in the action are invariant under the scaling (1) with $z = 3$, provided that the scalar graviton is assigned a vanishing scaling dimension. In other words, the conditions for power-counting renormalizability of the theory continue to hold in the nonlinear regime.

In the present paper, we have considered the projectable version of the HL theory and showed that the general relativity (plus dark matter) is safely recovered in the $\lambda \to 1$ limit. If we relax the projectability condition and thus allow the lapse function to depend on spatial coordinates then we should include as the building blocks of the theory not only g_{ij}, K_{ij}, D_i and R_{ij} but also $a_i \equiv \partial_i \ln(N)$ (with $[a_i] = [k]^1$ for $z = 3$) [17]. This gives rise to a proliferation of independent coupling constants. For example, for the minimal value of the dynamical critical exponent $z = 3$, the number of independent terms in the gravitational action of the non-projectable extension turns out to be more than 70 [22]. In some regime of parameters, the non-projectable extension is claimed to be free from the breakdown of the standard perturbative expansion method in the $\lambda \to 1$ limit, while in other regime the expansion breaks down. It is certainly worthwhile performing a nonperturbative analysis of the non-projectable theory in the regime of parameters where the standard perturbative expansion breaks down and then identifying the observationally viable regime of parameters.

There is yet another extension of the HL theory, with an additional local $U(1)$ symmetry, $U(1) \ltimes \mathrm{Diff}(M,\,\mathcal{F})$ [30]. It has been shown that the standard perturbative expansion does not break down in the gravitational sector, but does break down in the matter sector, at least apparently [31] (see also [32–36] for more on this extension). It is intriguing to see if a nonperturbative analysis similar to those presented in the present paper can resolve this problem.

Finally, the biggest obstacle in front of the HL gravity, and in general, any Lorentz symmetry breaking theory, is the restoration of the Lorentz invariance in the matter sector at low energies [37, 38]. Even if the Lorentz violation is restricted only to the gravity sector, the radiative corrections from graviton loops will generate Lorentz violation in the matter sector. Such terms can be under control provided that the Lorentz breaking scale is much lower than the Planck scale [39]. Another approach is to introduce a mechanism, or symmetry to suppress the Lorentz violating operators at low energies, such as supersymmetry [40]. Such an approach was adopted in [41] where a SUSY theory with anisotropic scaling was constructed. On the other hand, this seems to be a highly nontrivial task for the case of interacting models [42, 43].

Appendix A: Order $\mathcal{O}(\epsilon^{n+1})$ momentum constraint for $n \geq 2$

In this Appendix, we prove by induction that the order $\mathcal{O}(\epsilon^n)$ solution (73)–(77) satisfies the order $\mathcal{O}(\epsilon^{n+1})$ momentum constraint equation (52) for $n \geq 2$.

The proof extends the method presented in Ref.[26] to include a scalar field source: we rewrite the left hand side of the $(n+1)$-th order constraint (52) as a linear combination of lower order constraints by using the explicit solution (73)–(77). To achieve this, we make use of the generalized Bianchi identity (55) as well as the identities in Eq.(56). We also use the following identity for functions $f(t)$ and $g(t)$ satisfying $a^3(t_{\mathrm{in}})\,f(t_{\mathrm{in}})\,g(t_{\mathrm{in}}) = 0$,

$$f(t)\,g(t) = \frac{1}{a^3(t)} \int_{t_{\mathrm{in}}}^{t} dt'\, a^3(t') \left[a(t')^{-3} \partial_{t'}(a^3(t')\,f(t')) \cdot g(t') + f(t') \cdot \partial_{t'} g(t') \right]. \tag{A1}$$

By applying the identity (A1) to $(f(t), g(t)) = (A^{(p)\,j}{}_i, \partial_j \zeta^{(n-p)})$, $(f(t), g(t)) = (A^{(p)\,j}{}_l, (\gamma^{-1})^{(q)\,lk} \partial_i \gamma_{jk}^{(n-p-q)})$ and $(f(t), g(t)) = (\partial_t \phi^{(p)}, \partial_i \phi^{(n-p)})$, the left hand side of the $(n+1)$-th order momentum constraint equation (52) is

rewritten as

$$
\begin{aligned}
\mathcal{C}_i^{(n+1)} &\equiv \partial_j A^{(n)\,j}{}_i + 3\sum_{p=1}^{n} A^{(p)\,j}{}_i \partial_j \zeta^{(n-p)} - \frac{1}{2}\sum_{p=1}^{n}\sum_{q=0}^{n-p} A^{(p)\,j}{}_l (\gamma^{-1})^{(q)\,lk}\partial_i \gamma_{jk}^{(n-p-q)} \\
&\quad - \frac{1}{3}(3\lambda-1)\partial_i\left(K^{(n)} + \frac{3\,\phi^{(n)}\partial_t\phi^{(0)}}{3\lambda-1}\right) - \sum_{p=1}^{n-1}\partial_t\phi^{(p)}\partial_i\phi^{(n-p)} \\
&= \partial_j A^{(n)\,j}{}_i + \frac{1}{a^3(t)}\int_{t_{\rm in}}^{t} dt'\, a^3(t')\left\{ 3\sum_{p=1}^{n}\left[a^{-3}\partial_{t'}\left(a^3 A^{(p)\,j}{}_i\right)\partial_j\zeta^{(n-p)} + A^{(p)\,j}{}_i\partial_j\left(\partial_{t'}\zeta^{(n-p)}\right)\right] \right. \\
&\quad - \frac{1}{2}\sum_{p=1}^{n}\sum_{q=0}^{n-p}\left[a^{-3}\partial_{t'}\left(a^3 A^{(p)\,j}{}_l\right)(\gamma^{-1})^{(q)\,lk}\partial_i\gamma_{jk}^{(n-p-q)} + A^{(p)\,j}{}_l\partial_{t'}\left((\gamma^{-1})^{(q)\,lk}\right)\partial_i\gamma_{jk}^{(n-p-q)} \right. \\
&\quad \left. + A^{(p)\,j}{}_l(\gamma^{-1})^{(q)\,lk}\partial_i\left(\partial_{t'}\gamma_{jk}^{(n-p-q)}\right)\right] - \sum_{p=1}^{n-1}\left[a^{-3}\partial_{t'}(a^3\partial_{t'}\phi^{(p)})\partial_i\phi^{(n-p)} + \partial_{t'}\phi^{(p)}\partial_{t'}\partial_i\phi^{(n-p)}\right]\right\} \\
&\quad - \frac{1}{3}(3\lambda-1)\partial_i\left(K^{(n)} + \frac{3\,\phi^{(n)}\partial_t\phi^{(0)}}{3\lambda-1}\right).
\end{aligned}
\tag{A2}
$$

Using Eqs.(73)–(74), (48)–(50) and (54), this is further rewritten as

$$
\begin{aligned}
\mathcal{C}_i^{(n+1)} &= \frac{1}{a^3(t)}\int_{t_{\rm in}}^{t} dt'\, a^3(t')\left\{ \partial_j\left(-\sum_{p=1}^{n-1} K^{(p)} A^{(n-p)\,j}{}_i\right) \right. \\
&\quad + 3\left[\sum_{p=2}^{n}\left(-\sum_{q=1}^{p-1} K^{(q)} A^{(p-q)\,j}{}_i\right)\partial_j\zeta^{(n-p)} + \sum_{p=1}^{n-1} A^{(p)\,j}{}_i\partial_j\left(\frac{1}{3}K^{(n-p)}\right)\right] \\
&\quad - \frac{1}{2}\sum_{p=1}^{n}\sum_{q=0}^{n-p}\left[\left(-\sum_{r=1}^{p-1} K^{(r)} A^{(p-r)\,j}{}_l\right)(\gamma^{-1})^{(q)\,lk}\partial_i\gamma_{jk}^{(n-p-q)} \right. \\
&\quad + A^{(p)\,j}{}_l\left(-2\sum_{r=1}^{q} A^{(r)\,l}{}_m(\gamma^{-1})^{(q-r)\,mk}\right)\partial_i\gamma_{jk}^{(n-p-q)} \\
&\quad \left. + A^{(p)\,j}{}_l(\gamma^{-1})^{(q)\,lk}\partial_i\left(2\sum_{r=0}^{n-p-q-1}\gamma_{jm}^{(r)} A^{(n-p-q-r)\,m}{}_k\right)\right] \\
&\quad - \frac{1}{6}(3\lambda-1)\partial_i\left(-\sum_{p=1}^{n-1} K^{(p)} K^{(n-p)}\right) - \frac{1}{2}\partial_i\left(-\sum_{p=1}^{n-1} A^{(p)\,j}{}_k A^{(n-p)\,k}{}_j\right) \\
&\quad \left. + \sum_{p=1}^{n-1}\sum_{q=1}^{p} K^{(q)}\partial_{t'}\phi^{(p-q)}\partial_i\phi^{(n-p)} + \frac{1}{6}\sum_{p=1}^{n}\bar{Z}^{(p)}\sum_{q=0}^{n-p}(\gamma^{-1})^{(q)\,jk}\partial_i\gamma_{jk}^{(n-p-q)}\right\},
\end{aligned}
\tag{A3}
$$

where we have used the generalized Bianchi identity (55). By using the identities (56) we finally obtain

$$
\mathcal{C}_i^{(n+1)} = -\frac{1}{a^3(t)}\int_{t_{\rm in}}^{t} dt'\, a^3(t')\sum_{p=1}^{n-1} K^{(n-p)}\mathcal{C}_i^{(p+1)}.
\tag{A4}
$$

Since the $\mathcal{O}(\epsilon^2)$ constraint in Eq. (71) is already satisfied, i.e. $\mathcal{C}_i^{(2)} = 0$, the above relation implies that $\mathcal{C}^{(n+1)} = 0$ for $n \geq 2$.

Appendix B: Expansion of the nonlinear perturbations

Here, we present the detail of the calculations to obtain the expression of ζ in terms of ζ_T and h_{ij}, given in Eq.(87). While the former ζ is defined in the synchronous gauge $N_i = 0$, the latter ζ_T and h_{ij} are defined in in the transverse

gauge $\delta^{ik}\partial_k h_{ij} = 0$. In both gauges, the freedom in the time coordinate is fixed by the choice $N = 1$. For the perturbative expansion of the spatial metric, we use

$$g_{ij} = a^2\, e^{2\zeta}\, \left(e^h\right)_{ij} = a^2\delta_{ij} + \left(a^2\left(2\,\zeta\delta_{ij} + h_{ij}\right)\right) + \left[\frac{a^2}{2}\left(4\,\zeta^2\delta_{ij} + 4\,\zeta h_{ij} + h_{il}h^l{}_j\right)\right] + \mathcal{O}(\bar{\epsilon}^3)\,, \tag{B1}$$

where $\bar{\epsilon}$ denotes the order of perturbations and the indices of h_{ij} are raised and lowered with Kronecker delta. Throughout this Appendix, when the expansion of a quantity is shown, the terms outside parentheses, in parentheses and in square brackets are of order $\bar{\epsilon}^0$, $\bar{\epsilon}^1$ and $\bar{\epsilon}^2$, respectively.

1. Expansion of the momentum constraint

We first concentrate on the linear perturbation in the transverse gauge. To remove the nondynamical degrees in the shift vector, we solve the constraint equation order by order. We expand the shift vector while separating the contributions from each order, as

$$N_i = \left(\partial_i\, B^{(1)} + N_i^{T\,(1)}\right) + \left[\frac{1}{2}\left(\partial_i\, B^{(2)} + N_i^{T\,(2)}\right)\right] + \mathcal{O}(\bar{\epsilon}^3) \tag{B2}$$

With these decompositions, we expand the momentum constraint in vacuum

$$\mathcal{H}_j \equiv D_i K^i{}_j - \lambda\,\partial_j K = 0\,, \tag{B3}$$

as a series in perturbations. At first order, we get

$$\mathcal{H}_j^{(1)} = -\,(3\,\lambda - 1)\,\partial_j\partial_t\zeta_T + \frac{\lambda - 1}{a^2}\,\partial_j\triangle B^{(1)} - \frac{1}{2\,a^2}\triangle N_j^{T\,(1)}\,, \tag{B4}$$

which can be solved by

$$\triangle B^{(1)} = \frac{3\,\lambda - 1}{\lambda - 1}a^2\,\partial_t\zeta_T\,, \qquad N_i^{T\,(1)} = 0\,. \tag{B5}$$

Using the second of these results, the next order constraint yields,

$$\begin{aligned}
\mathcal{H}_j^{(2)} &= (\lambda - 1)\,\partial_j\left[\frac{1}{2}\triangle B^{(2)} + (\partial^k\zeta_T)\,(\partial_k B^{(1)}) - 2\,\zeta_T\triangle B^{(1)} - h^{kl}\partial_k\partial_l B^{(1)}\right] \\
&\quad + \frac{a^2}{4}\left[(\partial_t h^{kl})(\partial_k h_{lj}) - h^{kl}\partial_k\partial_t h_{lj} - (\partial_t h^{kl})(\partial_j h_{kl})\right] + \frac{3\,a^2}{2}\,(\partial^k\zeta_T)\,(\partial_t h_{kj}) \\
&\quad + (\partial_k\partial_j\zeta_T)\,(\partial^k B^{(1)}) + (\triangle\zeta_T)\,(\partial_j B^{(1)}) + \frac{1}{2}\,(\triangle h_{kj})(\partial^k B^{(1)}) - \frac{1}{4}\triangle N_j^{T\,(2)} = 0\,.
\end{aligned} \tag{B6}$$

For the following, only the longitudinal part of this relation is needed; by taking its divergence, then using the first equation of (B5), we end up with

$$\begin{aligned}
\triangle B^{(2)} &= 2\,a^2\left(\frac{3\,\lambda - 1}{\lambda - 1}\right)\left[2\,\zeta_T\,\partial_t\zeta_T + h^{ij}\partial_i\partial_j\triangle^{-1}\partial_t\zeta_T - (\partial^i\zeta_T)\,(\partial_i\triangle^{-1}\partial_t\zeta_T)\right] \\
&\quad - \frac{2\,a^2\,(3\,\lambda - 1)}{(\lambda - 1)^2}\triangle^{-1}\left[2\,(\partial^i\triangle\zeta_T)(\partial_i\triangle^{-1}\partial_t\zeta_T) + \left(\partial^i\partial^j\zeta_T + \frac{1}{2}\triangle h^{ij}\right)(\partial_i\partial_j\triangle^{-1}\partial_t\zeta_T) + (\triangle\zeta_T)\,(\partial_t\zeta_T)\right] \\
&\quad + \frac{a^2}{\lambda - 1}\triangle^{-1}\left[\frac{1}{2}\,(\partial_i\partial_t h_{jk})\,(\partial^i h^{jk}) + \frac{1}{2}\,(\partial_t h_{ij})\,(\triangle h^{ij}) - 3\,(\partial_i\partial_j\zeta_T)\,(\partial_t h^{ij})\right]\,.
\end{aligned} \tag{B7}$$

2. Coordinate transformations

Next, we determine the transformation between the transverse and synchronous gauges. We parametrize the coordinate transformation as

$$\tilde{x}^\mu = x^\mu + \left(\xi^{(1)\,\mu}\right) + \left[\frac{1}{2}(\xi^{(1)\,\nu}\partial_\nu\xi^{(1)\,\mu} + \xi^{(2)\,\mu})\right] + \mathcal{O}(\bar{\epsilon}^3)\,, \tag{B8}$$

where over-tilde denotes quantities in the synchronous gauge. The parameters $\xi^{(n)\,\mu}$ are decomposed as

$$\xi^{(n)\,\mu} = \left(0,\ \xi^{(n)\,i} + \partial^i \xi^{(n)}\right), \tag{B9}$$

with $\partial_i \xi^{(n)\,i} = 0$, while the indices of $\xi^{(n)\,i}$ are raised and lowered by δ^{ij} and δ_{ij}. For any tensor field expanded as

$$T = T^{(0)} + \left(\delta T\right) + \left[\frac{1}{2}\delta^2 T\right] + \mathcal{O}(\bar{\epsilon}^3), \tag{B10}$$

the transformation at linear and quadratic order proceeds through [44]

$$\begin{aligned}
\widetilde{\delta T} &= \delta T + \pounds_{\xi^{(1)}} T^{(0)}, \\
\widetilde{\delta^2 T} &= \delta^2 T + 2\,\pounds_{\xi^{(1)}} \delta T + \pounds_{\xi^{(1)}}^2 T^{(0)} + \pounds_{\xi^{(2)}} T^{(0)}.
\end{aligned} \tag{B11}$$

For the metric tensor, the transformations become

$$\begin{aligned}
\widetilde{\delta g_{\mu\nu}} &= \delta g_{\mu\nu} + g_{\mu\sigma}^{(0)} \partial_\nu \xi^{(1)\,\sigma} + g_{\nu\sigma}^{(0)} \partial_\mu \xi^{(1)\,\sigma}, \\
\widetilde{\delta^2 g_{\mu\nu}} &= \delta^2 g_{\mu\nu} + 2\left(\xi^{(1)\,\sigma}\,\partial_\sigma \delta g_{\mu\nu} + \delta g_{\mu\sigma} \partial_\nu \xi^{(1)\,\sigma} + \delta g_{\nu\sigma} \partial_\mu \xi^{(1)\,\sigma}\right) \\
&\quad + g_{\mu\sigma}^{(0)}\,\partial_\nu \left(\xi^{(1)\,\rho} \partial_\rho \xi^{(1)\,\sigma}\right) + g_{\nu\sigma}^{(0)}\,\partial_\mu \left(\xi^{(1)\,\rho} \partial_\rho \xi^{(1)\,\sigma}\right) + 2\,g_{\rho\sigma}^{(0)}(\partial_\mu \xi^{(1)\,\sigma})(\partial_\nu \xi^{(1)\,\rho}) \\
&\quad + g_{\mu\sigma}^{(0)} \partial_\nu \xi^{(2)\,\sigma} + g_{\nu\sigma}^{(0)} \partial_\mu \xi^{(2)\,\sigma}.
\end{aligned} \tag{B12}$$

We now determine the transformation ξ^μ needed to go from the transverse gauge to the synchronous gauge. For this, we evaluate the $0i$ components of (B12) and set $\widetilde{\delta g_{0i}} = 0$. At first order, we obtain,

$$\partial_i B^{(1)} + N_i^{T\,(1)} + a^2\left(\partial_i \partial_t \xi^{(1)} + \partial_t \xi_i^{(1)}\right) = 0, \tag{B13}$$

where the transverse and longitudinal parts can be easily separated to give,

$$\xi^{(1)} = -\int^t dt'\,\frac{B^{(1)}(t')}{a^2}, \qquad \xi_i^{(1)} = -\int^t dt'\,\frac{N_i^{T\,(1)}(t')}{a^2}. \tag{B14}$$

Using the solutions (B5) to the linear momentum constraint, the transformation parameters become

$$\xi^{(1)} = -\frac{3\lambda - 1}{\lambda - 1}\,\triangle^{-1}\zeta_T, \qquad \xi_i^{(1)} = 0. \tag{B15}$$

Similarly, the $0i$ component of the second order transformation (B12) gives,

$$\partial_i B^{(2)} + N_i^{T\,(2)} + a^2\left(\partial_i \partial_t \xi^{(2)} + \partial_t \xi_i^{(2)}\right) + (\partial^j \xi^{(1)})(\partial_i \partial_j B^{(1)}) - (\partial_i \partial^j \xi^{(1)})(\partial_j B^{(1)}) - 4\,\zeta_T\,\partial_i B^{(1)} - 2\,h_{ij}\partial^j B^{(1)} = 0, \tag{B16}$$

where we used the second equation of (B15). Using also the first equation of (B15) as well as the first order constraint (B5), the longitudinal part of the second order transformation can be obtained as

$$\begin{aligned}
\triangle \xi^{(2)} &= -\int^t dt'\,\frac{\triangle B^{(2)}}{a^2} + \left(\frac{3\lambda - 1}{\lambda - 1}\right)^2 (\partial^i \zeta_T)(\partial_i \triangle^{-1}\zeta_T) + 2\left(\frac{3\lambda - 1}{\lambda - 1}\right)\zeta_T^2 \\
&\quad + 2\left(\frac{3\lambda - 1}{\lambda - 1}\right)\int^t dt'\left[h^{ij}\partial_i\partial_j \triangle^{-1}\partial_{t'}\zeta_T - \left(\frac{\lambda + 1}{\lambda - 1}\right)(\partial^i \zeta_T)(\partial_i \triangle^{-1}\partial_{t'}\zeta_T)\right].
\end{aligned} \tag{B17}$$

Inserting the expression of $B^{(2)}$ from (B7) into the above expression, we finally obtain

$$\begin{aligned}
\triangle \xi^{(2)} &= \left(\frac{3\lambda - 1}{\lambda - 1}\right)^2 (\partial^i \zeta_T)(\partial_i \triangle^{-1}\zeta_T) - \frac{4\,(3\lambda - 1)}{(\lambda - 1)^2}\int^t dt'\,(\partial_i \zeta_T)(\partial^i \triangle^{-1}\partial_{t'}\zeta_T) \\
&\quad + \frac{2\,(3\lambda - 1)}{(\lambda - 1)^2}\int^t dt'\,\triangle^{-1}\left[2\,(\partial^i \triangle\zeta_T)(\partial_i \triangle^{-1}\partial_{t'}\zeta_T) + \left(\partial^i \partial^j \zeta_T + \frac{1}{2}\triangle h^{ij}\right)(\partial_i \partial_j \triangle^{-1}\partial_{t'}\zeta_T) + (\triangle\zeta_T)(\partial_{t'}\zeta_T)\right] \\
&\quad - \frac{1}{\lambda - 1}\int^t dt'\,\triangle^{-1}\left[\frac{1}{2}(\partial_i \partial_{t'}h_{jk})(\partial^i h_{jk}) + \frac{1}{2}(\partial_{t'}h^{ij})(\triangle h_{ij}) - 3\,(\partial_i \partial_j \zeta_T)(\partial_{t'}h^{ij})\right].
\end{aligned} \tag{B18}$$

We note that the second order gauge transformation is more divergent than the first order one (B15), in the limit $\lambda \to 1$.

Finally, we calculate the field ζ in the synchronous gauge. Since we adopted a nonperturbative decomposition for the spatial metric, it is useful to express this quantity as,

$$\zeta = \frac{1}{6} \, \log \left(\frac{\det \, \tilde{g}}{a^6} \right). \tag{B19}$$

Applying the perturbative expansion to the right hand side, we obtain

$$\zeta = \left(\frac{1}{6 \, a^2} \, \widetilde{\delta g_{ii}} \right) + \left[\frac{1}{12 \, a^2} \, \left(\widetilde{\delta^2 g_{ii}} - \frac{1}{a^2} \widetilde{\delta g_{ij}} \, \widetilde{\delta g_{ij}} \right) \right] + \mathcal{O}(\bar{\epsilon}^3). \tag{B20}$$

Using the transformed metric from (B12), the above expression becomes

$$\zeta = \left(\zeta_T + \frac{1}{3} \, \triangle \xi^{(1)} \right) + \left[(\partial^i \zeta_T) \, (\partial_i \xi^{(1)}) + \frac{1}{6} (\partial^i \xi^{(1)}) \, (\partial_i \triangle \xi^{(1)}) + \frac{1}{6} \, \triangle \xi^{(2)} \right] + \mathcal{O}(\bar{\epsilon}^3). \tag{B21}$$

Finally, using the transformations (B15) and (B18), we obtain Eq.(87), which relates the nonlinear perturbation ζ in the synchronous gauge to an expansion series of perturbations ζ_T and h_{ij} in the transverse gauge.

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