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Phys. Rev. D **85**, 064041 — Published 27 March 2012

DOI: [10.1103/PhysRevD.85.064041](https://doi.org/10.1103/PhysRevD.85.064041)

Gravitational radiation from compact binary systems in the massive Brans-Dicke theory of gravity

Justin Alsing,^{1,*} Emanuele Berti,^{2,3,†} Clifford Will,^{4,‡} and Helmut Zaglauer⁵

¹*Department of Physics, University of Oxford, Keble Road, Oxford OX1 3RH, UK*

²*Department of Physics and Astronomy, The University of Mississippi, University, MS 38677, USA*

³*California Institute of Technology, Pasadena, CA 91109, USA*

⁴*McDonnell Center for the Space Sciences, Department of Physics,*

Washington University, St. Louis, MO 63130, USA

⁵*Astrium GmbH, 88039 Friedrichshafen, Germany*

We derive the equations of motion, the periastron shift, and the gravitational radiation damping for quasicircular compact binaries in a massive variant of the Brans-Dicke theory of gravity. We also study the Shapiro time delay and the Nordtvedt effect in this theory. By comparing with recent observational data, we put bounds on the two parameters of the theory: the Brans-Dicke coupling parameter ω_{BD} and the scalar mass m_s . We find that the most stringent bounds come from Cassini measurements of the Shapiro time delay in the Solar System, that yield a lower bound $\omega_{\text{BD}} > 40000$ for scalar masses $m_s < 2.5 \times 10^{-20} \text{ eV}$ (or Compton wavelengths $\lambda_s = h/(m_s c) > 5 \times 10^{10} \text{ km}$), to 95% confidence. In comparison, observations of the Nordtvedt effect using Lunar Laser Ranging (LLR) experiments yield $\omega_{\text{BD}} > 1000$ for $m_s < 2.5 \times 10^{-20} \text{ eV}$. Observations of the orbital period derivative of the quasicircular white dwarf-neutron star binary PSR J1012+5307 yield $\omega_{\text{BD}} > 1250$ for $m_s < 10^{-20} \text{ eV}$ ($\lambda_s > 1.2 \times 10^{11} \text{ km}$). A first estimate suggests that bounds comparable to the Shapiro time delay may come from observations of radiation damping in the eccentric white dwarf-neutron star binary PSR J1141-6545, but a quantitative prediction requires the extension of our work to eccentric orbits.

General relativity (GR) occupies a well earned place next to the standard model as one of the two pillars of modern physics. All observational evidence to date supports GR as the correct classical theory of gravitation, but there are countless attempts at developing alternative theories of gravity. Two of the main motivations for these efforts are the desire to formulate a fully quantizable theory of gravity, and the quest to uncover the mechanisms underlying the dark energy problem in cosmology. In addition, the vast majority of tests of GR that have been carried out to date are in the weak-field, low energy regime, but it is widely believed that GR may indeed break down at higher energies. The direct observation of gravitational waves with Earth- and space-based detectors will mark the dawn of a new era, allowing us to probe gravity in the dynamical, strong-field regime. For these reasons, the study of gravitational radiation in modified theories of gravity has become a central issue.

One of the most popular and simple alternative theories of gravity is scalar-tensor theory, in which gravity is mediated by both a scalar and a tensor field, coupled together in a nontrivial manner through the presence of a nonminimal coupling term in the action [1–3]. The existence of scalar partners to the graviton is predicted in all extra-dimensional theories, and scalar fields play a crucial role in modern cosmology. Scalar-tensor theories are consistent, have a well-posed Cauchy problem, and respect many of the symmetries of GR. They are also

conformally equivalent to GR (if the coupling with matter is nonstandard), allowing us to employ the same techniques used to solve the Einstein field equations as long as we work in the Einstein frame [1, 3]. Finally, generic scalar-tensor theories can be shown to be equivalent to $f(R)$ theories [4, 5]. A good account of the motivations behind scalar-tensor theories, including their historical development, can be found in [1, 3].

String theory suggests the existence of massive but light scalar fields (“axions”) with masses possibly as small as the Hubble scale ($\sim 10^{-33} \text{ eV}$). If we do indeed live in a “string axiverse”, CMB observations, galaxy surveys and measurements of black hole spins may offer exciting experimental opportunities to set constraints on the mass of these scalars [6, 7].

Here we are interested in the possibility of constraining the mass and coupling of massive scalars via present (electromagnetic) and future (gravitational-wave) observations of compact binaries. Until recently, calculations of gravitational radiation damping in scalar-tensor theories (see e.g. [8–11]) have focused mostly on the *massless* case. Due to the interest of light scalars in cosmology and high-energy physics, this restriction has been dropped in more recent work. For example it has been shown that resonant, superradiant effects induced by light, massive scalars may produce “floating orbits” when small compact objects inspiral into rotating black holes, leaving a distinct signature in gravitational waves [12, 13].

A commonly held belief is that only *mixed binaries* (i.e., binaries whose members have different gravitational binding energy) can produce significant amounts of scalar gravitational radiation. There are two reasons for this. The first is that, under standard assumptions, dipole ra-

* justin.alsing@seh.ox.ac.uk

† berti@phy.olemiss.edu

‡ cmw@wuphys.wustl.edu

diation is produced due to violations of the strong equivalence principle when the binary members have unequal “sensitivities”: $s_1 \neq s_2$. These sensitivities are defined in Eq. (11) below, and they are related to the gravitational binding energy of each binary member. In other words, dipole radiation is produced when the system’s center of mass is offset with respect to the center of inertia (see e.g. [3]), so that mixed binaries and eccentric binaries would be the best target to constrain scalar-tensor theories. The second reason is the black hole no-hair theorem, i.e. the fact that black hole solutions in scalar-tensor theories are the same as in GR (see [14] and references therein). Building on earlier work by Jacobson [15], Horbatsch and Burgess recently pointed out that slowly varying scalar fields may violate the no-hair theorem, so that even black hole-black hole binaries may produce dipole radiation [16]. They also developed a formalism to test generic scalar-tensor theories using binary pulsars [17].

For all these reasons, a study of gravitational radiation in massive scalar-tensor theories is quite timely. In this paper we derive the period derivative due to scalar and tensor radiation in theories with a massive scalar field. For simplicity we focus on circular binaries, but (as we will see below) the generalization of our results to eccentric binaries would be of great observational interest¹.

For the reader’s convenience, here we give an executive summary of our main results. Consider a compact binary in circular orbit with component masses m_i and sensitivities s_i ($i = 1, 2$). Then the period derivative due to the emission of scalar and tensor gravitational waves in the massive Brans-Dicke theory is

$$\frac{\dot{P}}{P} = -\frac{8}{5} \frac{\mu m^2}{r^4} \kappa_1 - \frac{\mu m}{r^3} \kappa_D \mathcal{S}^2, \quad (1)$$

where

$$\begin{aligned} \kappa_1 &= \mathcal{G}^2 \left[12 - 6\xi + \xi \Gamma^2 \left(\frac{4\omega^2 - m_s^2}{4\omega^2} \right)^2 \Theta(2\omega - m_s) \right], \\ \kappa_D &= 2\mathcal{G}\xi \frac{\omega^2 - m_s^2}{\omega^2} \Theta(\omega - m_s), \end{aligned} \quad (2)$$

Θ is the Heaviside function, r is the separation of the binary members, m_s is the mass of the scalar field, $m = m_1 + m_2$ and $\mu = m_1 m_2 / m$ are the total and reduced masses of the system, $\mathcal{S} \equiv s_2 - s_1$ and furthermore

$$\begin{aligned} \xi &= \frac{1}{2 + \omega_{\text{BD}}}, \\ \mathcal{G} &= 1 - \xi (s_1 + s_2 - 2s_1 s_2), \\ \Gamma &= 1 - 2 \frac{s_1 m_2 + m_1 s_2}{m}. \end{aligned}$$

Note that scalar dipole radiation is emitted only when the binary’s orbital frequency $\omega > m_s$ and the difference in sensitivities $\mathcal{S} \neq 0$, while scalar quadrupole/monopole radiation is emitted only when $2\omega > m_s$ and it also vanishes for two black holes (since in that case $s_1 = s_2 = 1/2$ and $\Gamma = 0$). This result is only strictly valid in the limit of a very massive ($m_s r \gg 1$) or very light ($m_s r \ll 1$) scalar. However corrections due to an intermediate mass scalar always enter with at least a factor of the small parameter ξ , so this should be a relatively good approximation for the full range of scalar masses.

In addition to deriving the orbital period derivative due to gravitational radiation, we also revisit the calculations of the Shapiro time delay and of the Nordtvedt effect in the massive Brans-Dicke theory. As we will see, the presence of the massive scalar does not allow a straightforward implementation of the parametrized post-Newtonian formalism. By comparing our results for the orbital period derivative, Shapiro time delay and Nordtvedt parameter against recent observational data, we put constraints on the parameters of the theory: the scalar mass m_s and the Brans-Dicke coupling parameter ω_{BD} . Our bounds are summarized in Figure 1.

We find that the most stringent bounds come from the observations of the Shapiro time delay in the Solar System provided by the Cassini mission (these bounds were already studied by Perivolaropoulos, although he used a slightly different notation [18]). From the Cassini observations we obtain $\omega_{\text{BD}} > 40000$ for $m_s < 2.5 \times 10^{-20} \text{eV}$, to 95% confidence. Observations of the Nordtvedt effect using the Lunar Laser Ranging (LLR) experiment yield a slightly weaker bound of $\omega_{\text{BD}} > 1000$ for $m_s < 2.5 \times 10^{-20} \text{eV}$. Observations of the orbital period derivative of the circular white-dwarf neutron-star (WD-NS) binary system PSR J1012+5307 yields $\omega_{\text{BD}} > 1250$ for $m_s < 10^{-20} \text{eV}$. The limiting factor here is our ability to obtain precise measurements of the masses of the component stars as well as of the orbital period derivative, once kinematic corrections have been accounted for. However, there is considerably more promise in the eccentric binary system PSR J1141-6545. This system has allowed for remarkably precise measurements of the orbital period derivative, of the component star masses and of the periastron shift, making it a promising candidate for constraining alternative theories of gravity. Unfortunately the system has nonnegligible eccentricity. Generalizing our result for the orbital period derivative to eccentric binaries is a significant (but worthy) algebraic undertaking.

The plan of the paper is as follows. In section I we describe and motivate the Brans-Dicke theory with a massive scalar field. In section II we perform a post-Newtonian expansion of the field equations. In section III we deal with the Shapiro time delay. In section IV we proceed to obtain the equations of motion of a binary system as well as the periastron shift. In section V we discuss the Nordtvedt effect. In section VI we give details of the derivation of the gravitational radiation damping

¹ We will be working in units $\hbar = c = G = 1$ throughout the paper. Greek indices will span both spatial and time components 0, 1, 2, 3. Roman indices run over the spatial components 1, 2, 3 only. We will adopt the metric signature $(-, +, +, +)$.

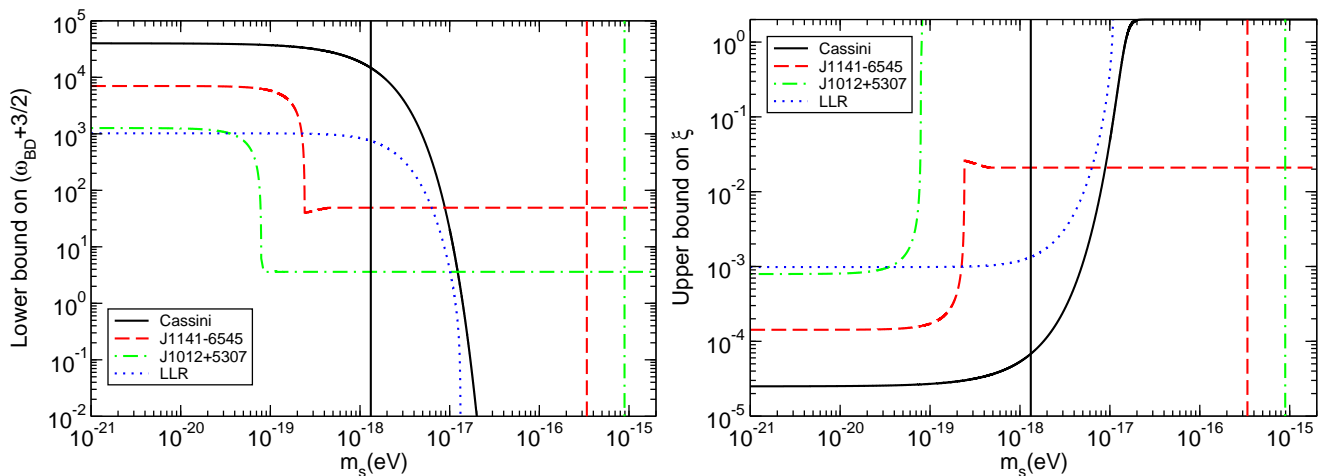


FIG. 1. Left: Lower bound on $(\omega_{\text{BD}} + 3/2)$ as a function of the mass of the scalar m_s from the Cassini mission data (black solid line; cf. [18]), period derivative observations of PSR J1141-6545 (dashed red line) and PSR J1012+5307 (dot-dashed green line), and Lunar Laser Ranging experiments (dotted blue line). Vertical lines indicate the masses corresponding to the typical radii of the systems: 1AU (black solid line) and the orbital radii of the two binaries (dashed red and dot-dashed green lines). Right panel: upper bound on ξ as a function of m_s . Linestyles are the same as in the left panel. Note that the theoretical bounds on the coupling parameters are $\omega > -3/2$ and $\xi < 2$.

of a compact binary system due to scalar and tensor gravitational radiation. In section VII we use these results to put bounds on the parameters of the theory. In the conclusions we point out possible future extensions of our work. Appendix A outlines a step-by-step derivation of the post-Newtonian expansion of the scalar field and of the metric. Appendix B provides details on certain integrals that appear in the calculation of the energy flux. Finally, Appendix C contains a short summary of compact binary observations relevant to this work.

I. THE BRANS-DICKE THEORY WITH A MASSIVE SCALAR FIELD

A. The generic scalar-tensor theory with a single scalar field

A general class of scalar-tensor theories containing a single scalar field in addition to the tensor field was studied by Bergmann and Wagoner [8, 19]. We can characterize the Bergmann-Wagoner theory via the following postulates:

- 1) The principle of general covariance is imposed, leading to tensorial equations.
- 2) The field equations are derived from the action

$$S = \int (\mathcal{L}_G + \mathcal{L}_M) d^4x, \quad (3)$$

where \mathcal{L}_G and \mathcal{L}_M are the Lagrangian densities for the gravitational and matter fields, respectively.

- 3) We postulate that the long-range forces of nature are mediated by the three lowest spin bosons, and assume that the electromagnetic field is the only vector field.

This leaves a scalar degree of freedom ϕ and a tensor degree of freedom (the metric $g_{\mu\nu}$) to describe the dynamics of the gravitational field.

- 4) The field equations are of at most of second differential order, and the tensor and scalar fields are nonminimally coupled; this leads us to the general form

$$\mathcal{L}_G = (-g)^{\frac{1}{2}} [h(\phi)R + l(\phi)g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \lambda(\phi)] \quad (4)$$

for the gravitational Lagrangian density, where $h(\phi)$, $l(\phi)$ and $\lambda(\phi)$ are arbitrary functions of the scalar field ϕ .

- 5) We postulate a principle of *mutual coupling*, in which the matter Lagrangian density depends on the gravitational fields according to

$$\mathcal{L}_M = \mathcal{L}_M(\psi^2(\phi)g^{\mu\nu}, \Psi), \quad (5)$$

where $\psi(\phi)$ is a fourth arbitrary function of ϕ , and Ψ represents the collective matter fields. This guarantees consistency with the strong equivalence principle [1].

Now let us make the conformal transformation $g_{\mu\nu} \rightarrow \psi^2(\phi)g_{\mu\nu}$, and in doing so move into a conformal frame in which the matter fields do not couple directly (but only indirectly, via the metric) to the scalar field; this is commonly referred to as the Jordan frame [1, 3]. Furthermore, without loss of generality we can redefine the scalar field such that $h(\phi) \rightarrow \phi$. These two redefinitions recast the action into the form

$$S = \frac{1}{16\pi} \int \left[\phi R - \frac{\omega(\phi)}{\phi} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + M(\phi) \right] (-g)^{\frac{1}{2}} d^4x + \int \mathcal{L}_M(g^{\mu\nu}, \Psi) d^4x, \quad (6)$$

which has the additional advantage that the resulting weak-field equations for $g_{\mu\nu}$ and ϕ decouple from one

another. The generic theory now contains two undetermined functions: the *cosmological function* $M(\phi)$ and the *coupling function* $\omega(\phi)$ (in the language of [20]). The effect of the coupling function on compact binary dynamics has been studied extensively, and it can lead to interesting consequences if “spontaneous scalarization” occurs [11, 21–24]. Here we focus on the cosmological function, which has three major effects in the generic theory. Firstly, in the resulting field equations for $g_{\mu\nu}$ it plays the role of a cosmological constant. Secondly, it endows the scalar with mass: this manifests itself most clearly in the fact that solutions for ϕ for isolated systems contain Yukawa-like terms $e^{-m_s r}/r$, where m_s is the mass of the scalar field, which in turn gives the field a characteristic range $\ell \sim 1/m_s$ [20]. Finally, the cosmological function may introduce nonlinearities in the dynamics of the scalar field.

B. The matter action and the field equations

Let us now turn to the matter action. Throughout this paper we will make the assumption that all bodies in our system can be treated as point masses. Einstein, Infeld and Hoffmann (EIH) [25] developed a method for obtaining the equations of motion for a system of gravitating point-like masses. In their approach, one begins by obtaining the local gravitational field of a single body (in a comoving frame), under the assumption that the body is small and nearly spherical. One then proceeds to match the interbody gravitational fields onto the obtained local field of the single body under inspection; imposing self consistency yields the EIH equations of motion. The same equations of motion can be obtained with significantly less effort, albeit at the sacrifice of some rigor, by taking the stress-energy tensor to be a distribution of delta functions and neglecting any infinite self-energy terms as they arise [20]. In scalar-tensor theory, however, we must deal with the additional complication that the inertial mass and internal structure of a gravitating body will depend on the local value of the scalar field (i.e. the local value of the effective gravitational “constant”). Variations in internal structure may act back on the motion of the body, leading to violations of the (weak) equivalence principle. Eardley [26] showed that these effects could be accounted for by simply supposing that the masses of the bodies are in general functions of

the scalar field, such that the matter action for a system of point-like masses can be written as

$$S_M = - \sum_a \int m_a(\phi) d\tau_a, \quad (7)$$

where the particles (labeled by a) have inertial masses $m_a(\phi)$, and τ_a is the proper time of particle a measured along its worldline x_a^λ . The distributional stress-energy tensor $T^{\mu\nu}$ and its trace T hence take the form

$$T^{\mu\nu}(x^\lambda) = (-g)^{-\frac{1}{2}} \sum_a m_a(\phi) \frac{u_a^\mu u_a^\nu}{u_a^0} \delta^4(x^\lambda - x_a^\lambda), \quad (8)$$

$$T = g_{\mu\nu} T^{\mu\nu} = -(-g)^{-\frac{1}{2}} \sum_a \frac{m_a(\phi)}{u_a^0} \delta^4(x^\lambda - x_a^\lambda). \quad (9)$$

Far from the system, the scalar will take on its cosmologically imposed value, denoted by ϕ_0 . The relationship between the effective gravitational constant, G , and the scalar field ϕ is therefore (in our chosen system of units) $G = \phi_0/\phi$. In the post-Newtonian limit, we expand ϕ about its asymptotic value and define the small perturbation φ such that $\phi = \phi_0 + \varphi$. In this case, we can write the variation of the inertial masses m_a with ϕ as

$$m_a(\phi) = m_a(\ln G) = m_a(\phi_0) \left[1 + s_a \left(\frac{\varphi}{\phi_0} \right) - \frac{1}{2} (s'_a - s_a^2 + s_a) \left(\frac{\varphi}{\phi_0} \right)^2 + O\left(\left(\frac{\varphi}{\phi_0} \right)^3 \right) \right], \quad (10)$$

where we have defined the “first and second sensitivities” s_a and s'_a to be²

$$s_a = - \frac{\partial(\ln m_a)}{\partial(\ln G)} \Big|_{\phi_0}, \quad s'_a = - \frac{\partial^2(\ln m_a)}{\partial(\ln G)^2} \Big|_{\phi_0}. \quad (11)$$

The full action is now given by

$$S = \frac{1}{16\pi} \int [\phi R - \frac{\omega(\phi)}{\phi} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + M(\phi)] (-g)^{\frac{1}{2}} d^4x - \sum_a \int m_a(\phi) d\tau_a. \quad (12)$$

By varying the action (12) with respect to the tensor and scalar fields, respectively, we obtain the full field equations of the generic theory described above:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{2} \phi^{-1} M(\phi) g_{\mu\nu} = 8\pi \phi^{-1} T_{\mu\nu} + \omega(\phi) \phi^{-2} (\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi_{,\alpha} \phi^{,\alpha}) + \phi^{-1} (\phi_{,\mu\nu} - g_{\mu\nu} \square_g \phi), \quad (13)$$

$$\square_g \phi + \frac{\phi \frac{dM(\phi)}{d\phi} - 2M(\phi)}{(3 + 2\omega(\phi))} = \frac{1}{3 + 2\omega(\phi)} \left(8\pi T^* - \frac{d\omega(\phi)}{d\phi} \phi_{,\alpha} \phi^{,\alpha} \right),$$

² White-dwarfs typically have sensitivities $s_a \sim 10^{-4}$, neutron stars have sensitivities $s_a \sim 0.2$, and black holes have $s_a = 1/2$:

see [27] for detailed calculations.

where we have defined $T^* = T - 2\phi\frac{\partial T}{\partial\phi}$ and \square_g is the curved space d'Alembertian, defined by

$$\square_g = (-g)^{-\frac{1}{2}}\partial_\nu((-g)^{\frac{1}{2}}g^{\mu\nu}\partial_\mu). \quad (14)$$

A detailed derivation of this result can be found in [1].

C. Massive Brans-Dicke theory: The field equations and their weak-field limit

As we recalled earlier, the effects of a generic coupling function on the dynamics of compact binaries have been studied fairly extensively by Damour and Esposito-Farèse [11, 21–24]. Here we are primarily interested in the effects of a nonzero mass of the scalar field. In the limit where $m_s \rightarrow 0$, our final result for the dipolar and quadrupolar flux can be shown to match Eq. (6.40) in [21] (the monopole contribution vanishes for circular orbits).

It would be interesting to study a theory with generic functional forms for both $\omega(\phi)$ and $M(\phi)$, but for simplicity here we will consider a constant coupling function: $\omega(\phi) = \omega_{\text{BD}} = \text{constant}$, as in the usual Brans-Dicke theory [28]. The scalar field equation then reduces to

$$\square_g\phi + \frac{\phi\frac{dM(\phi)}{d\phi} - 2M(\phi)}{3 + 2\omega_{\text{BD}}} = \frac{8\pi T^*}{3 + 2\omega_{\text{BD}}}. \quad (15)$$

In order to get a handle on the effects of the cosmological function $M(\phi)$, let us expand the metric about a Minkowski background $\eta_{\mu\nu}$ and the scalar field around its (cosmologically determined) constant background value ϕ_0 . Following closely the method of [20], we define small

perturbations φ , $h_{\mu\nu}$ and $\theta_{\mu\nu}$ such that

$$\begin{aligned} \phi &= \phi_0 + \varphi, \\ g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}, \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \\ \theta^{\mu\nu} &= h^{\mu\nu} - \frac{1}{2}h\eta^{\mu\nu} - \left(\frac{\varphi}{\phi_0}\right)\eta^{\mu\nu}, \end{aligned} \quad (16)$$

Let us also expand $M(\phi)$ in a Taylor series about ϕ_0 :

$$M(\phi) = M(\phi_0) + M'(\phi_0)\varphi + \frac{1}{2}M''(\phi_0)\varphi^2 + \dots \quad (17)$$

We require that the expanded field equations are consistent at all orders in $(v/c)^n$. Substituting the weak field perturbations (16) into the field equations (13) and (14) and examining the leading-order terms under the assumption of asymptotic flatness, we find that $M(\phi_0) = M'(\phi_0) = 0$. We are therefore left with the quadratic term, that endows the scalar field with mass. To see this, let us substitute $M(\phi) = \frac{1}{2}M''(\phi_0)\varphi^2$ into the scalar field equation, yielding

$$\square_g\phi - m_s^2(\phi - \phi_0) = \frac{8\pi T^*}{3 + 2\omega_{\text{BD}}}, \quad (18)$$

where we have defined the mass of the scalar field

$$m_s^2 \equiv -\frac{\phi_0}{3 + 2\omega_{\text{BD}}}M''(\phi_0). \quad (19)$$

We will see shortly that m_s is precisely the parameter appearing in Yukawa-type corrections $\sim e^{-m_s r}$ to the Newtonian gravitational potential, as well as the ordinary mass parameter in the Klein-Gordon equation. Since the scalar field is expected to be small, we will neglect cubic and higher-order terms in $M(\phi)$, that would introduce additional nonlinearities into the scalar field equation.

In summary, with our choice of coupling and cosmological functions, the field equations of the massive Brans-Dicke theory read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{3 + 2\omega_{\text{BD}}}{4\phi_0\phi}m_s^2(\phi - \phi_0)^2g_{\mu\nu} + \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega_{\text{BD}}}{\phi^2}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}\right) + \frac{1}{\phi}(\phi_{,\mu\nu} - g_{\mu\nu}\square_g\phi), \quad (20)$$

$$\square_g\phi - m_s^2(\phi - \phi_0) = \frac{8\pi T^*}{(3 + 2\omega_{\text{BD}})}. \quad (21)$$

D. The weak-field limit

Let us use the weak-field perturbations (16) to obtain the field equations in the weak-field limit. Expanding the left hand side of (20) and imposing the harmonic gauge condition $\theta^{\mu\nu}_{;\nu} = 0$ we find

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{1}{2}\square_\eta\theta_{\mu\nu} + \frac{\varphi_{,\mu\nu}}{\phi_0} - \eta_{\mu\nu}\square_\eta\left(\frac{\varphi}{\phi_0}\right), \quad (22)$$

where \square_η is the flat-space d'Alembertian, and we neglected quadratic and higher-order terms. The tensor field equation can hence be written as

$$\square_\eta\theta^{\mu\nu} = -16\pi\tau^{\mu\nu}, \quad (23)$$

where $\tau_{\mu\nu} = \phi_0^{-1}T_{\mu\nu} + t_{\mu\nu}$. We have collected the quadratic and higher-order terms in the perturbations φ and $\theta_{\mu\nu}$ into the gravitational stress-energy pseudotensor $t_{\mu\nu}$. By virtue of the gauge condition on $\theta^{\mu\nu}$, we have the

useful result that

$$\tau^{\mu\nu}_{,\nu} = 0. \quad (24)$$

Following a similar procedure for the scalar field equation, we expand $\square_g \phi$ in the weak-field perturbations:

$$\begin{aligned} \square_g \phi = & \left(1 + \frac{1}{2}\theta + \frac{\varphi}{\phi_0}\right) \square_\eta \frac{\varphi}{\phi_0} - \theta^{\mu\nu} \frac{\varphi_{,\mu\nu}}{\phi_0} - \frac{\varphi_{,\alpha} \varphi^{,\alpha}}{\phi_0^2} \\ & + O(\theta^3, \theta^2 \varphi, \varphi^2 \theta \dots). \end{aligned} \quad (25)$$

Substituting this back into the scalar field equation we find, as anticipated, the standard Klein-Gordon equation

$$(\square_\eta - m_s^2)\varphi = -16\pi S, \quad (26)$$

where we have defined the source S as

$$\begin{aligned} S \equiv & -(6 + 4\omega_{\text{BD}})^{-1} \left(T - 2\phi \frac{\partial T}{\partial \phi}\right) \left(1 - \frac{1}{2}\theta - \frac{\varphi}{\phi_0}\right) \\ & - \frac{1}{16\pi} (\theta^{\mu\nu} \varphi_{,\mu\nu} + \phi_0^{-1} \phi_{,\alpha} \phi^{,\alpha} - m_s^2 \phi_0^{-1} \varphi^2 - \frac{1}{2} m_s^2 \theta \varphi) \\ & + O(\theta^3, \theta^2 \varphi, \theta \varphi^2, \varphi^3). \end{aligned} \quad (27)$$

II. POST-NEWTONIAN EXPANSION OF THE MASSIVE BRANS-DICKE THEORY

We will now perform a post-Newtonian expansion of the scalar and tensor fields. This will allow us to derive the Shapiro time delay (section III), the equations of motion and periastron shift of compact binaries (section IV) the Nordtvedt effect (section V), and will be required for the derivation the period derivative due to gravitational radiation (section VI). Before we proceed, it will be convenient to define some auxiliary combinations containing ω_{BD} that show up repeatedly throughout the calculation:

$$\xi \equiv \frac{1}{2 + \omega_{\text{BD}}}, \quad (28)$$

$$\gamma \equiv \frac{1 + \omega_{\text{BD}}}{2 + \omega_{\text{BD}}}, \quad (29)$$

$$\alpha \equiv \frac{1}{3 + 2\omega_{\text{BD}}}. \quad (30)$$

Furthermore, for our choice of units the cosmologically imposed ϕ_0 is given by

$$\phi_0 = \frac{4 + 2\omega_{\text{BD}}}{3 + 2\omega_{\text{BD}}}. \quad (31)$$

Following very closely the method described in [20] (see Appendix A for details) we obtain

$$\begin{aligned} \frac{\phi}{\phi_0} = & \xi \sum_a \frac{m_a}{r_a} (1 - 2s_a) e^{-m_s r_a} + \xi^2 \sum_{a \neq b} \frac{m_a m_b}{r_a r_{ab}} (s_a + 2s'_a - 2s_a^2) \times e^{-m_s r_a} (1 - 2s_b) e^{-m_s r_{ab}} \\ & + \frac{1}{2} \xi^2 \sum_{a,b} \frac{m_a m_b}{r_a r_b} \times (1 - 2s_a) e^{-m_s r_a} (1 - 2s_b) e^{-m_s r_b} - \xi \sum_{a \neq b} \frac{m_a m_b}{r_a r_{ab}} e^{-m_s r_a} \phi_0^{-1} \times \left(1 + \alpha(1 - 2s_b) e^{-m_s r_{ab}}\right) \\ & - \frac{1}{2} \xi \sum_a \frac{m_a v_a^2}{r_a} (1 - 2s_a) e^{-m_s r_a} - \frac{1}{2} \xi \sum_a \frac{\partial^2}{\partial t^2} \left(\frac{e^{-m_s r_a}}{m_s}\right) (1 - 2s_a) + O(6), \end{aligned} \quad (32)$$

$$\begin{aligned} g_{00} = & -1 + 2\phi_0^{-1} \sum_a \frac{m_a}{r_a} \left(1 + \alpha(1 - 2s_a) e^{-m_s r_a}\right) - 2\phi_0^{-2} \sum_{a,b} \frac{m_a m_b}{r_a r_b} \left(1 + \alpha(1 - 2s_a) e^{-m_s r_a}\right) \times \left(1 + \alpha(1 - 2s_b) e^{-m_s r_b}\right) \\ & - 2 \sum_{a \neq b} \frac{m_a m_b}{r_a r_{ab}} \left[\phi_0^{-2} \left(1 + \alpha e^{-m_s r_a}\right) \times \left(1 + \alpha(1 - 2s_b) e^{-m_s r_{ab}}\right) - \xi \phi_0^{-1} s_a (e^{-m_s r_a} + (1 - 2s_b) e^{-m_s r_{ab}})\right] \\ & - \xi^2 (s_a + 2s'_a - 2s_a^2) (1 - 2s_b) e^{-m_s r_a} e^{-m_s r_{ab}} - \sum_a \frac{m_a v_a^2}{r_a} \left[1 + 2\gamma + \xi s_a e^{-m_s r_a} + \frac{1}{2} \xi (1 - e^{-m_s r_a})\right] \\ & + \sum_a m_a \xi (1 - 2s_a) \times \frac{\partial^2}{\partial t^2} \left(\frac{(2 + m_s r_a)(1 - e^{-m_s r_a}) - 2m_s r_a}{2m_s^2 r_a}\right) + O(6), \end{aligned} \quad (33)$$

$$g_{0i} = -2(1 + \gamma) \sum_a \frac{m_a v_a^i}{r_a} - \frac{1}{2} \sum_a m_a \phi_0^{-1} \times \frac{\partial^2}{\partial t \partial x^i} \left(r_a + 2\alpha(1 - 2s_a) \frac{e^{-m_s r_a} + m_s r_a - 1}{m_s^2 r_a}\right) + O(5), \quad (34)$$

$$g_{ij} = \delta_{ij} + 2\phi_0^{-1} \delta_{ij} \sum_a \frac{m_a}{r_a} \left(1 - \alpha(1 - 2s_a) e^{-m_s r_a}\right) + O(4). \quad (35)$$

In the limit $m_s \rightarrow 0$, the above results reduce to those

obtained in the massless Brans-Dicke case [9].

Substituting these results into (27) we find an expres-

sion for the source S in the near zone, to $O(2)$:

$$S(x^\lambda) = \frac{\alpha}{2} \sum_a m_a \left[(1 - 2s_a) \left(1 - \frac{1}{2} v_a^2 - \frac{1}{\phi_0} \sum_b \frac{m_b}{r_b} \right) + \xi (2s'_a - 2s_a^2 + 2s_a - \frac{1}{2}) \sum_b \frac{m_b (1 - 2s_b)}{r_b} e^{-m_s r_b} \right] \delta^4(x^\lambda - x_a^\lambda)$$

III. SHAPIRO TIME DELAY

Using the post-Newtonian expansion of the metric, we can derive an expression for the Shapiro time delay of a light ray passing near a massive body. We note first that the parametrized post-Newtonian (PPN) formalism is not viable when dealing with theories that contain massive fields. In fact, Newtonian order terms are modified by the presence of massive fields, in the sense that the Newtonian potential acquires a Yukawa-like correction of the form

$$\tilde{U}(\mathbf{x}, t) = \frac{1}{\phi_0} \int \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} (1 + \alpha e^{-m_s |\mathbf{x} - \mathbf{x}'|}) d^3 \mathbf{x}'. \quad (36)$$

The impact of this fact for our current purpose is significant: the above potential cannot be expanded in powers of $1/r$, and the coefficients of modified post-Newtonian potentials in the post-Newtonian metric are not constants, but they have a spatial dependence. Nonetheless, we can use the derived metric to obtain an expression for the equations of motion of a photon, and use this to obtain an expression for the Shapiro delay. We will follow closely the method described in [20]. A similar calculation was carried out by Perivolaropoulos [18]; he used a different definition of the mass of the scalar field, but his results are consistent with those derived here.

For a photon traveling along a null geodesic,

$$g_{\mu\nu} u^\mu u^\nu = 0. \quad (37)$$

To requisite order, $O(2)$, the equation of motion can be written as

$$-1 + h_{00}^{(2)} + (\delta_{ij} + h_{ij}^{(2)}) u^i u^j = 0, \quad (38)$$

where $h_{\mu\nu}^{(n)}$ is the $O(n)$ order correction to the metric. Specializing to a single spherically symmetric source of mass M (and negligible sensitivity) at the origin, the post-Newtonian corrections to the metric are (from equations (33) and (35))

$$\begin{aligned} h_{00}^{(2)} &= 2\phi_0^{-1} \frac{M}{r} (1 + \alpha e^{-m_s r}) = 2\tilde{U}, \\ h_{ij}^{(2)} &= 2\phi_0^{-1} \frac{M}{r} (1 - \alpha e^{-m_s r}) \delta_{ij} = 2U (1 - \alpha e^{-m_s r}) \delta_{ij}. \end{aligned} \quad (39)$$

Substituting these into (38), the equation of motion for the photon now reads

$$-1 + 2\tilde{U} + (1 + 2(1 - e^{-m_s r})U) |\mathbf{u}|^2 = 0. \quad (40)$$

The unperturbed Newtonian trajectory of the photon will simply be $x^i(t) = x_e^i + n^i(t - t_e)$, where the photon is emitted from \mathbf{x}_e in direction \mathbf{n} at time t_e . Let us now parametrize the post-Newtonian correction to the trajectory by $x_{\text{PN}}^i(t)$, where the corrected trajectory is then given by $x^i(t) = x_e^i + n^i(t - t_e) + x_{\text{PN}}^i(t)$. Substituting this into the above, we find that the post-Newtonian correction to the trajectory satisfies

$$\mathbf{n} \cdot \frac{d\mathbf{x}_{\text{PN}}}{dt} = \frac{dx_{\text{PN}}^\parallel}{dt} = -2U. \quad (41)$$

Integrating with respect to time, we obtain

$$x_{\text{PN}}^\parallel(t) = -2 \int_{t_e}^t U dt'. \quad (42)$$

The time taken for the photon to travel from \mathbf{x}_e to some other point \mathbf{x} and back again is hence given by

$$\Delta t = 2|\mathbf{x} - \mathbf{x}_e| + 4 \int_{t_e}^t U dt'. \quad (43)$$

The travel time correction δt due to the Shapiro delay corresponds to the second term on the right-hand side. Performing the integration, we find for the Shapiro delay term

$$\delta t = 4M \ln \left[\frac{(r_e + \mathbf{r}_e \cdot \mathbf{n})(r_p - \mathbf{r}_p \cdot \mathbf{n})}{r_b^2} \right], \quad (44)$$

where the photon is emitted from \mathbf{r}_e in direction \mathbf{n} , travels to \mathbf{r}_p and back again, M is the mass of the body causing the time-delay and r_b is the impact parameter of the photon with respect to the source. The mass appearing in (44) is not a measurable quantity; what is actually measured is the Keplerian mass $M_K = M(1 + \alpha e^{-m_s r})$, where r should be thought of as a fixed quantity which depends on how the Keplerian mass of the body was determined. In terms of M_K we have

$$\begin{aligned} \delta t &= \frac{4M_K}{1 + \alpha e^{-m_s r}} \ln \left[\frac{(r_e + \mathbf{r}_e \cdot \mathbf{n})(r_p - \mathbf{r}_p \cdot \mathbf{n})}{r_b^2} \right] \\ &= 2(1 + \tilde{\gamma}) M_K \ln \left[\frac{(r_e + \mathbf{r}_e \cdot \mathbf{n})(r_p - \mathbf{r}_p \cdot \mathbf{n})}{r_b^2} \right], \end{aligned} \quad (45)$$

where in the second line we have defined

$$\tilde{\gamma} = \frac{1 - \alpha e^{-m_s r}}{1 + \alpha e^{-m_s r}}. \quad (46)$$

In the case of the solar system, the r appearing in the definition of $\tilde{\gamma}$ should be set to 1AU, since this is the scale associated with the determination of the Keplerian mass of the Sun. In any metric theory of gravity where the PPN formalism can be applied in a straightforward manner, the obtained expression for the Shapiro delay is identical to (45), only with $\tilde{\gamma}$ replaced by the PPN parameter γ (see for example [20]). We can therefore compare $\tilde{\gamma}$ directly with the observational constraints on γ from Shapiro time delay measurements to obtain an exclusion region in the $(\omega_{\text{BD}}, m_s)$ -plane. In section VII B we will do precisely this, comparing the derived expression for $\tilde{\gamma}$ to the constraints on γ from time-delay measurements obtained by the Cassini mission.

Note that in the limit where $m_s \rightarrow \infty$, $\tilde{\gamma} \rightarrow 1$, i.e. the GR value of the PPN parameter γ . In the limit where $m_s \rightarrow 0$ we have instead $\tilde{\gamma} \rightarrow \gamma = (1 + \omega_{\text{BD}})/(2 + \omega_{\text{BD}})$, i.e. the value of γ in the massless Brans-Dicke theory.

IV. EQUATIONS OF MOTION AND PERIASTRON ADVANCE

Armed with the post-Newtonian expansion of the fields, we are now in a position to obtain the EIH equations of motion. From (7), the matter Lagrangian for the a th body in the system is given by

$$L_a = m_a(\phi) \left(-g_{00} - 2g_{0i}v_a^i - g_{ij}v_a^i v_a^j \right)^{\frac{1}{2}}. \quad (47)$$

To obtain an n -body action we follow the procedure detailed after Eq. (11.90) of [20]. We substitute the post-Newtonian expressions for the metric and scalar fields obtained in the previous section and use the expansion of $m_a(\phi)$ in (10). We first make the gravitational terms in L_a manifestly symmetric under interchange of all pairs of particles, then we take one of each such term generated in L_a , and sum over a . To $O(4)$ we find

$$\begin{aligned} L_{\text{EIH}} = & - \sum_a m_a \left(1 - \frac{1}{2}v_a^2 - \frac{1}{8}v_a^4 \right) \\ & + \frac{1}{2} \sum_{a \neq b} \frac{m_a m_b}{r_{ab}} \left[\mathcal{G}_{ab} + 3\mathcal{B}_{ab}v_a^2 - \sum_{c \neq a} \mathcal{D}_{abc} \frac{m_c}{r_{ac}} \right. \\ & \left. - \frac{1}{2}(\mathcal{G}_{ab} + 6\mathcal{B}_{ab})\mathbf{v}_a \cdot \mathbf{v}_b - \frac{1}{2}\mathcal{G}_{ab}(\mathbf{v}_a \cdot \mathbf{n}_{ab})(\mathbf{v}_b \cdot \mathbf{n}_{ab}) \right], \end{aligned} \quad (48)$$

where we have defined

$$\begin{aligned} \mathbf{n}_{ab} &= \frac{\mathbf{r}_a - \mathbf{r}_b}{r_{ab}}, \\ \mathcal{G}_{ab} &= 1 - \frac{1}{2}\xi \left[1 - (1 - 2s_a)(1 - 2s_b)e^{-m_s r_{ab}} \right], \\ \mathcal{B}_{ab} &= \frac{1}{3}(2\gamma + 1) + \frac{1}{6}\xi \left[1 - (1 - 2s_a)(1 - 2s_b)e^{-m_s r_{ab}} \right], \end{aligned} \quad (49)$$

and

$$\begin{aligned} \mathcal{D}_{abc} = & 1 - \frac{1}{2}\xi \left[2 - (1 - 2s_a)(1 - 2s_b)e^{-m_s r_{ab}} \right. \\ & \left. - (1 - 2s_a)(1 - 2s_c)e^{-m_s r_{ac}} \right] \\ & + \frac{1}{4}\xi^2 \left[1 - (1 - 2s_a)(1 - 2s_b)e^{-m_s r_{ab}} \right. \\ & \left. - (1 - 2s_a)(1 - 2s_c)e^{-m_s r_{ac}} \right. \\ & \left. + (1 - 4(s_a + s'_a - s_a^2)) \right. \\ & \left. \times (1 - 2s_b)(1 - 2s_c)e^{-m_s r_{ab}} e^{-m_s r_{ac}} \right]. \end{aligned} \quad (50)$$

Now let us now specialize to a two-body system with the center of mass at the origin; to this end let us define

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1, \quad m = m_1 + m_2, \\ \delta m &= m_2 - m_1, \quad \mu = \frac{m_1 m_2}{m}. \end{aligned} \quad (51)$$

We also write $\mathcal{G}_{12} = \mathcal{G}$ and $\mathcal{B}_{12} = \mathcal{B}$. With this specialization made, the equations of motion are found to be

$$\begin{aligned} \mathbf{a} = & - \frac{m\mathbf{r}}{r^3} \left[\tilde{\mathcal{G}} - 3\tilde{\mathcal{G}}\mathcal{B}\frac{m}{r} - \frac{1}{2}(\tilde{\mathcal{G}} - 3\tilde{\mathcal{B}})v^2 \right. \\ & \left. - \frac{1}{2}(\mathcal{D}_{211} + \tilde{\mathcal{D}}_{211})\frac{m_1}{r} - \frac{1}{2}(\mathcal{D}_{122} + \tilde{\mathcal{D}}_{122})\frac{m_2}{r} \right. \\ & \left. - 2\mathcal{G}\tilde{\mathcal{G}}\frac{\mu}{r} + (\mathcal{G} + 2\tilde{\mathcal{G}})\frac{\mu}{m}v^2 - \frac{1}{2}(4\mathcal{G} - \tilde{\mathcal{G}})\frac{\mu}{m}(\mathbf{v} \cdot \mathbf{n})^2 \right] \\ & + \frac{m(\mathbf{r} \cdot \mathbf{v})\mathbf{v}}{r^3} \left[\tilde{\mathcal{G}} + 3\mathcal{B} + (\mathcal{G} - 3\tilde{\mathcal{G}})\frac{\mu}{m} \right], \end{aligned} \quad (52)$$

where

$$\begin{aligned} \tilde{\mathcal{G}} &= 1 - \frac{1}{2}\xi \left[1 - (1 - 2s_1)(1 - 2s_2)(1 + m_s r)e^{-m_s r} \right], \\ \tilde{\mathcal{B}} &= \frac{1}{3}(2\gamma + 1) + \frac{1}{6}\xi \left[1 - (1 - 2s_1)(1 - 2s_2)(1 + m_s r)e^{-m_s r} \right] \end{aligned} \quad (53)$$

and

$$\begin{aligned} \tilde{\mathcal{D}}_{122} = & 1 - \xi \left[1 - (1 - 2s_1)(1 - 2s_2)(1 + m_s r)e^{-m_s r} \right] \\ & + \frac{1}{4}\xi^2 \left[1 - 2(1 - 2s_1)(1 - 2s_2)(1 + m_s r)e^{-m_s r} \right. \\ & \left. + (1 - 4(s_1 + s'_1 - s_1^2))(1 - 2s_2)^2(1 + 2m_s r)e^{-2m_s r} \right]. \end{aligned} \quad (54)$$

A. Periastron advance

With the equation of motion in hand, we can view the post-Newtonian corrections together with the scalar Yukawa-like terms as perturbations of the Keplerian orbit and employ the method of osculating elements [20] to obtain an expression for the periastron advance of the binary system. In contrast to the massless Brans-Dicke case (treated in [20]), the integrals that appear in this

perturbation expansion cannot be written in closed form, so an expansion in powers of the eccentricity e is required to obtain closed-form expressions. Fortunately for our current purposes we will only require the result in the two limiting cases of very light and very massive scalars. In the former limit $m_s r \ll 1$, the periastron advance reduces to the massless Brans-Dicke result [20]

$$\dot{\omega} = \frac{6\pi m}{a(1-e^2)P} \mathcal{P} \mathcal{G}^{-1}, \quad (55)$$

where a and e are the semi-major axis and eccentricity, P is the period and \mathcal{P} is given by

$$P = \mathcal{G} \mathcal{B} + \frac{1}{6} \mathcal{G}^2 - \frac{1}{6} \frac{m_1 \mathcal{D}_{211} + m_2 \mathcal{D}_{122}}{m}. \quad (56)$$

In the limit of a very massive scalar $m_s r \gg 1$ the expression for the periastron advance reduces instead to the familiar GR result:

$$\dot{\omega} = \frac{6\pi m}{a(1-e^2)P} \mathcal{G}. \quad (57)$$

V. NORDTVEDT EFFECT

Scalar-tensor theories of gravity predict that massive bodies with a significant amount of gravitational self-energy do not follow geodesics of the background metric; in fact, massive bodies with different gravitational self-energies will follow different trajectories, leading to direct violation of the strong equivalence principle. This is known as the Nordtvedt effect, and leads to detectable effects in the Solar System. Most notably, it leads to a polarization of the Moon's orbit around the Earth [20, 29], which can be constrained using lunar ranging experiments. Let us look at how this effect arises in the massive Brans-Dicke theory.

The effect is usually parametrized by the Nordtvedt parameter η_N , which can be determined directly from the PPN metric of a given theory, and it turns out to be some simple combination of PPN parameters. However, as we have seen previously, in the case of the massive Brans-Dicke theory the PPN formalism is not directly applicable. However we can extract an "effective" Nordtvedt parameter from the equations of motion. To do this, let us consider the relative acceleration of a pair of bodies A and B , $\mathbf{a}_{AB} = \mathbf{a}_A - \mathbf{a}_B$, in the field of a third body C , with $r_{AB} \ll r_{AC}$ and $r_{AC} \simeq r_{BC}$. The Nordtvedt effect will result in an anomalous difference in the accelerations of A and B towards C , proportional to the difference in the specific gravitational self-energies of the two bodies A and B [20, 29, 30]. Since the sensitivity s_a of a body is related to its gravitational self-energy Ω_a by $s_a = \Omega_a/m_a$ (in the weak field limit), the extra term arising in \mathbf{a}_{AB} due to the Nordtvedt effect will be proportional to the the difference in sensitivities $\mathcal{S} = s_B - s_A$.

To Newtonian order, the n -body Lagrangian (48) is given by

$$L_{\text{EIH}} = - \sum_a m_a \left(1 - \frac{1}{2} v_a^2\right) + \frac{1}{2} \sum_{a \neq b} \frac{m_a m_b}{r_{ab}} \mathcal{G}_{ab}, \quad (58)$$

and the n -body equations of motion are hence

$$\begin{aligned} \mathbf{a}_a &= - \sum_{b \neq a} \frac{m_b}{r_{ab}^2} \mathcal{G}_{ab} \hat{\mathbf{r}}_{ab} \\ &\quad - \frac{1}{2} \sum_{b \neq a} \frac{m_b}{r_{ab}^2} \xi (1 - 2s_a) (1 - 2s_b) m_s r_{ab} e^{-m_s r_{ab}} \hat{\mathbf{r}}_{ab}. \end{aligned} \quad (59)$$

The relative acceleration of two bodies A and B in the field of a third body C is then

$$\begin{aligned} \mathbf{a}_{AB} &= \mathbf{a}_B - \mathbf{a}_A \\ &= \frac{\mathcal{G}_{AB}(m_A + m_B)}{r_{AB}^2} \hat{\mathbf{r}}_{AB} - \frac{\mathcal{G}_{BC} m_C}{r_{BC}^2} \hat{\mathbf{r}}_{BC} + \frac{\mathcal{G}_{AC} m_C}{r_{AC}^2} \hat{\mathbf{r}}_{AC} \\ &\quad + \frac{1}{2} \left(\frac{m_A + m_B}{r_{AB}} \xi (1 - 2s_A) (1 - 2s_B) m_s e^{-m_s r_{AB}} \right) \hat{\mathbf{r}}_{AB} \\ &\quad - \frac{1}{2} \frac{m_C}{r_{BC}^2} \xi (1 - 2s_B) (1 - 2s_C) m_s r_{BC} e^{-m_s r_{BC}} \hat{\mathbf{r}}_{BC} \\ &\quad + \frac{1}{2} \frac{m_C}{r_{AC}^2} \xi (1 - 2s_A) (1 - 2s_C) m_s r_{AC} e^{-m_s r_{AC}} \hat{\mathbf{r}}_{AC}. \end{aligned} \quad (60)$$

Regrouping terms together appropriately and assuming that $r_{AB} \ll r_{AC}$, $r_{AC} \simeq r_{BC}$, we can rewrite this as

$$\begin{aligned} \mathbf{a}_{AB} &= - \frac{m^* \hat{\mathbf{r}}_{AB}}{r_{AB}^2} + \frac{1}{\phi_0} \left(\frac{m_C \hat{\mathbf{r}}_{AC}}{r_{AC}^2} - \frac{m_C \hat{\mathbf{r}}_{BC}}{r_{BC}^2} \right) \\ &\quad + [\xi (1 - 2s_C) (1 + m_s r_{AC}) e^{-m_s r_{AC}}] (s_B - s_A) \frac{m_C \hat{\mathbf{r}}_{AC}}{r_{AC}^2} \\ &= - \frac{m^* \hat{\mathbf{r}}_{AB}}{r_{AB}^2} + \frac{1}{\phi_0} \left(\frac{m_C \hat{\mathbf{r}}_{AC}}{r_{AC}^2} - \frac{m_C \hat{\mathbf{r}}_{BC}}{r_{BC}^2} \right) + \eta_N \mathcal{S} \frac{m_C \hat{\mathbf{r}}_{AC}}{r_{AC}^2}, \end{aligned} \quad (61)$$

where the first term is the Newtonian acceleration between the two bodies, the second term is the tidal correction to the orbit of the system (A, B) and the final term (proportional to $\mathcal{S} = s_B - s_A$) is the difference in the accelerations of A and B towards the third body C due to the Nordtvedt effect (cf. [20]). In the second line we have rewritten the third term in the conventional form from which the Nordtvedt parameter is usually defined; we can then simply read off the effective Nordtvedt parameter

$$\eta_N = \xi (1 + m_s r) (1 - 2s_C) e^{-m_s r}, \quad (62)$$

where r is now taken to be the distance from C to the system (A, B). Note that if the Sun were replaced by a black hole ($s_C = 1/2$), there would be no Earth-Moon Nordtvedt effect. In section VII C we will compare the effective η_N to the measured value of the Nordtvedt parameter provided by Lunar Laser Ranging experiments to obtain bounds on $(\omega_{\text{BD}}, m_s)$.

VI. GRAVITATIONAL RADIATION FROM COMPACT BINARIES

A. Tensor radiation

In this section we will follow very closely the general method described in [20]. The power radiated in gravitational waves due to tensor radiation in the Brans-Dicke theory is given by

$$\dot{E} = -\frac{R^2}{32\pi}\phi_0 \left\langle \oint \theta_{\text{TT},0}^{ij} \theta_{\text{TT},0}^{ij} d\Omega \right\rangle, \quad (63)$$

where the angular brackets represent an average over one orbital period and θ_{TT}^{ij} is the transverse-traceless (TT) part of θ^{ij} .

In order to obtain a formal solution to the linearized tensor wave equation (23), we simply fold the source $\tau_{\mu\nu}$ with the retarded Green's function of the flat-space d'Alembertian operator

$$G(t-t', \mathbf{R}-\mathbf{r}') = \frac{\delta(t-t'-|\mathbf{R}-\mathbf{r}'|)}{|\mathbf{R}-\mathbf{r}'|}, \quad (64)$$

with the result

$$\theta^{\mu\nu}(t, \mathbf{R}) = 4 \int_{\mathcal{N}} \frac{\tau^{\mu\nu}(t-|\mathbf{R}-\mathbf{r}'|, \mathbf{r}')}{|\mathbf{R}-\mathbf{r}'|} d^3\mathbf{r}'. \quad (65)$$

Here the integral over t' has been carried out immediately, and the spatial integration region \mathcal{N} is over the near zone. If we make the assumption that the field point in is the radiation zone, such that $|\mathbf{r}'| \ll |\mathbf{R}|$, and make the slow-motion approximation, we can expand the \mathbf{r}' dependence of the integrand and write

$$\theta^{\mu\nu} = \frac{4}{R} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m}{\partial t^m} \int_{\mathcal{M}} \tau^{\mu\nu}(t-R, \mathbf{r}') (\mathbf{n} \cdot \mathbf{r}')^m d^3\mathbf{r}', \quad (66)$$

where $\mathbf{n} = \mathbf{R}/R$, and the integration is now over \mathcal{M} , which is the intersection of the world tube of the near zone with the constant time hypersurface $t_{\mathcal{M}} = t - R$ [31]. For the purpose of obtaining the power loss due to gravitational radiation, we are ultimately interested in $\theta^{\mu\nu}_{,0}$. Due to our choice of gauge $\theta^00_{,0} = \theta^{\mu0}_{,0} = 0$, and hence we only require the spatial components θ^{ij} , which are given (to leading order) by

$$\begin{aligned} \theta^{ij} &= \frac{4}{R} \int \tau^{ij}(t-R, \mathbf{r}') d^3\mathbf{r}' \\ &= \frac{2}{R} \frac{\partial^2}{\partial t^2} \int \tau^{00}(t-R, \mathbf{r}') r'^i r'^j d^3\mathbf{r}'. \end{aligned} \quad (67)$$

Here we have written the monopole moment of τ^{ij} as the time derivative of the quadrupole moment of τ^{00} , by exploiting the conservation law $\tau^{\mu\nu}_{,\nu}$ together with the slow-motion approximation. There can be no contribution from the dipole moment of τ^{00} in (67) to order $O(\frac{m\omega}{R})$,

since the time derivative $\frac{\partial x}{\partial t} \sim v$. The quadrupole moment of τ^{ij} only comes in at higher order, and hence we only require the leading-order contribution from τ^{00} :

$$\tau^{00} = \frac{1}{2}(1+\gamma) \sum_a m_a \delta^3(\mathbf{r}' - \mathbf{r}_a). \quad (68)$$

Substituting this into (67) we obtain

$$\theta^{ij} = (1+\gamma) R^{-1} \frac{d^2}{dt^2} \sum_a m_a r_a^i r_a^j. \quad (69)$$

Specializing to a two-body system with the center of mass at the origin using (51), we obtain to the requisite order

$$\theta^{ij}(t, \mathbf{R}) = 2(1+\gamma) R^{-1} \mu \left(v^i v^j - \tilde{\mathcal{G}} m \frac{r^i r^j}{r^3} \right), \quad (70)$$

where we have used (52) to replace \ddot{r}^i (to leading order) where necessary.

We now need to project (70) onto the TT gauge by applying the projector

$$\begin{aligned} \Lambda(\hat{\mathbf{n}})_{ij,kl} &= \delta_{ik} \delta_{jl} - \frac{1}{2} \delta_{ij} \delta_{kl} - n_i n_k \delta_{jl} \\ &\quad + \frac{1}{2} n_k n_l \delta_{ij} + \frac{1}{2} n_i n_j \delta_{kl} + \frac{1}{2} n_i n_j n_k n_l, \end{aligned} \quad (71)$$

which satisfies $\Lambda_{ij,kl} \Lambda_{kl,nm} = \Lambda_{ij,nm}$ [32] to θ^{kl} :

$$\theta_{\text{TT}}^{ij} = \Lambda(\hat{\mathbf{n}})_{ij,kl} \theta^{kl}. \quad (72)$$

The result is

$$\dot{E} = -\frac{R^2}{32\pi} \phi_0 \left\langle \oint \Lambda_{ij,kl} \theta_{,0}^{ij} \theta_{,0}^{kl} d\Omega \right\rangle. \quad (73)$$

We now note that the only $\hat{\mathbf{n}}$ dependence in the integrand of (73) is contained in the $\Lambda_{ij,kl}$. Performing the integral over the solid angle we find

$$\oint \Lambda_{ij,kl} d\Omega = \frac{2\pi}{15} (11\delta_{ik} \delta_{jl} - 4\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk}), \quad (74)$$

where we have used the identity

$$\oint n^{i_1} n^{i_2} \dots n^{i_{2l}} d\Omega = \frac{4\pi \delta^{(i_1 i_2} \delta^{i_3 i_4} \dots \delta^{i_{2l-1} i_{2l})}}{(2l+1)!}. \quad (75)$$

Substituting this result back into (73) we obtain

$$\dot{E} = -\frac{R^2}{32\pi} \phi_0 \frac{2\pi}{15} \left\langle 12 \theta_{,0}^{ij} \theta_{,0}^{ij} - 4 \theta_{,0}^i \theta_{,0}^i \right\rangle. \quad (76)$$

At this point we will specialize to a circular orbit, which we will parametrize by

$$r_1 = r \cos(\omega(t-R)), \quad (77)$$

$$r_2 = r \sin(\omega(t-R)),$$

$$r_3 = 0,$$

$$v_1 = -v \sin(\omega(t-R)), \quad (78)$$

$$v_2 = v \cos(\omega(t-R)),$$

$$v_3 = 0,$$

where ω is the orbital frequency. In addition, let us suppose that the mass of the scalar is either sufficiently large or sufficiently small that variations of $\tilde{\mathcal{G}}$ over an orbital period can be neglected. Then $\tilde{\mathcal{G}}$ will reduce to the massless Brans-Dicke value in the limit of a low mass scalar [9], or to the GR value in the limit of a very massive scalar. With these two approximations made, we perform the average over one period and obtain the final result for the power emitted in tensor gravitational waves in the Brans-Dicke theory:

$$\dot{E} = -\frac{8}{15} \frac{\mathcal{G}^2 \mu^2 m^2 v^2}{r^4} (12 - 6\xi). \quad (79)$$

Using the relation $(\dot{P}/P) = -\frac{3}{2}(\dot{E}/E)$ as well as the Newtonian result (following from the virial theorem) that $E = T + V = -\frac{1}{2}\mu v^2$ to eliminate v , we finally obtain the fractional period decay due to the emission of tensor gravitational radiation

$$\frac{\dot{P}}{P} = -\frac{8}{5} \frac{\mathcal{G}^2 \mu m^2}{r^4} (12 - 6\xi). \quad (80)$$

We stress again that this result is only valid in the limit where m_s is such that either $e^{-m_s r} \approx 1$, in which case \mathcal{G} reduces to the massless Brans-Dicke value [9], or $e^{-m_s r} \rightarrow 0$, in which case \mathcal{G} reduces to the GR value.

B. Scalar radiation

The general expression for the radiated power due to scalar radiation in Brans-Dicke theory is [20]

$$\dot{E} = -\frac{R^2}{32\pi} \phi_0^{-1} (4\omega_{\text{BD}} + 6) \langle \oint \varphi_{,0} \varphi_{,0} d\Omega \rangle, \quad (81)$$

where the angular brackets represent the average over one orbital period.

We can solve Eq. (26) by using the retarded Green's function for the massive wave operator $\square - m_s^2$:

$$G(t-t', \mathbf{R}-\mathbf{r}') = \frac{\delta(t-t'-|\mathbf{R}-\mathbf{r}'|)}{|\mathbf{R}-\mathbf{r}'|} - \Theta(t-t'-|\mathbf{R}-\mathbf{r}'|) \frac{m_s J_1(m_s \sqrt{(t-t')^2 - |\mathbf{R}-\mathbf{r}'|^2})}{\sqrt{(t-t')^2 - |\mathbf{R}-\mathbf{r}'|^2}}, \quad (82)$$

where J_1 is the Bessel function of the first kind, and Θ is the Heaviside function (see [33] for a detailed derivation

of this result). Now we can write the general solution to (26) as $\varphi = \varphi_B + \varphi_m$, where

$$\begin{aligned} \varphi_B(t, \mathbf{R}) &= 4 \int \int_{\mathcal{N}} \frac{S(t', \mathbf{r}') \delta(t-t'-|\mathbf{R}-\mathbf{r}'|)}{|\mathbf{R}-\mathbf{r}'|} d^3 \mathbf{r}' dt', \\ \varphi_m(t, \mathbf{R}) &= -4 \int \int_{\mathcal{N}} \frac{m_s S(t', \mathbf{r}') J_1(m_s \sqrt{(t-t')^2 - |\mathbf{R}-\mathbf{r}'|^2})}{\sqrt{(t-t')^2 - |\mathbf{R}-\mathbf{r}'|^2}} \Theta(t-t'-|\mathbf{R}-\mathbf{r}'|) d^3 \mathbf{r}' dt' \\ &= -4 \int_{\mathcal{N}} d^3 \mathbf{r}' \times \int_0^\infty \frac{J_1(z) S(t - \sqrt{|\mathbf{R}-\mathbf{r}'|^2 + (\frac{z}{m_s})^2}, \mathbf{r}')}{\sqrt{|\mathbf{R}-\mathbf{r}'|^2 + (\frac{z}{m_s})^2}} dz, \end{aligned} \quad (83)$$

the spatial integration is over the near zone \mathcal{N} , and in the last line we have made the substitution $z = m_s \sqrt{(t-t')^2 - |\mathbf{R}-\mathbf{r}'|^2}$.

Taking the field point to be in the radiation zone

($|\mathbf{R}| \gg |\mathbf{r}'|$) and making the slow-motion approximation, we can expand the \mathbf{r}' dependence of the integrand and write the general solutions (83) as

$$\varphi_B = \frac{4}{R} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m}{\partial t^m} \int_{\mathcal{M}} d^3 \mathbf{r}' S(t-R, \mathbf{r}') (\mathbf{n} \cdot \mathbf{r}')^m, \quad (84)$$

$$\varphi_m = -\frac{4}{R} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m}{\partial t^m} \int_{\mathcal{M}} d^3 \mathbf{r}' (\mathbf{n} \cdot \mathbf{r}')^m \times \int_0^\infty dz \frac{S(t - \sqrt{R^2 + (\frac{z}{m_s})^2}, \mathbf{r}') J_1(z)}{(1 + (\frac{z}{m_s R})^2)^{\frac{m+1}{2}}}. \quad (85)$$

We are now in a position to substitute the post-Newtonian expression for the source S into (84) and (85) and obtain an expression for the gravitational waveform $\varphi(t, \mathbf{R})$ in the far-field, slow-motion limit. We must first specialize to a two-body system with the center of mass at the origin, using (51). Performing the integration and

retaining terms up to order $O(\frac{mv^2}{R})$ and $O(\frac{m^2}{Rr'})$ in the monopole ($m = 0$) and quadrupole ($m = 2$) terms, and $O(\frac{mv}{R})$ in the dipole terms ($m = 1$), we obtain (modulo time-independent terms that are uninteresting, as we ultimately require $\varphi_{,0}$ in order to calculate the radiated power)

$$\varphi_B = 2\alpha R^{-1}\mu \left[\Gamma(\mathbf{n} \cdot \mathbf{v})^2 - \frac{1}{2}\Gamma v^2 - \tilde{\mathcal{G}}\Gamma m \frac{(\mathbf{n} \cdot \mathbf{r})^2}{r^3} - (2 - \xi)\Gamma' \frac{m}{r} - (2\Lambda - (2 - \xi)\Gamma') \frac{m}{r} e^{-m_s r} - 2\mathcal{S}(\mathbf{n} \cdot \mathbf{v}) \right], \quad (86)$$

$$\begin{aligned} \varphi_m = & -2\alpha R^{-1}\mu \left(\Gamma I_3[(\mathbf{n} \cdot \mathbf{v})^2] - \frac{1}{2}\Gamma I_1[v^2] - \Gamma I_3[\tilde{\mathcal{G}}m \frac{(\mathbf{n} \cdot \mathbf{r})^2}{r^3}] - (2 - \xi)\Gamma' I_1\left[\frac{m}{r}\right] \right. \\ & \left. - (2\Lambda - (2 - \xi)\Gamma') I_1\left[\frac{m}{r} e^{-m_s r}\right] - 2\mathcal{S} I_2[(\mathbf{n} \cdot \mathbf{v})] \right). \end{aligned} \quad (87)$$

Here we have defined

$$\begin{aligned} \Gamma &\equiv 1 - 2\frac{s_1 m_2 + m_1 s_2}{m}, \\ \Gamma' &\equiv 1 - s_1 - s_2, \\ \Lambda &\equiv \mathcal{G}\Gamma' - \xi((1 - 2s_1)s'_2 + (1 - 2s_2)s'_1), \end{aligned} \quad (88)$$

and the terms $I_n[f(t)]$ represent the integrals

$$I_n[f(t)] = \int_0^\infty \frac{f(t - \sqrt{R^2 + (\frac{z}{m_s})^2}) J_1(z)}{(1 + (\frac{z}{m_s R})^2)^{\frac{n}{2}}} dz, \quad (89)$$

where the integration over z has yet to be performed, and it is understood that the time-dependent terms in (86) and (87) (replacing $f(t)$ in (89)) are the components of \mathbf{r} and \mathbf{v} . As in the calculation of the tensor component, we assume that $\tilde{\mathcal{G}}$ is approximately constant over an orbital

period ($\tilde{\mathcal{G}} \rightarrow \mathcal{G}$), and we specialize to a circular orbit parametrized by (77). Taking the partial time derivative of φ we find

$$\begin{aligned} \varphi_{,0} = & 2\alpha R^{-1}\mathcal{G}m\mu \left[2\mathcal{S} \left(\frac{n^i r^i}{r^3} - I_2 \left[\frac{n^i r^i}{r^3} \right] \right) \right. \\ & \left. - 4\Gamma \left(\frac{n^i n^j v^i r^j}{r^3} - I_3 \left[\frac{n^i n^j v^i r^j}{r^3} \right] \right) \right], \end{aligned} \quad (90)$$

where we have used (52) to replace i^i (to leading order) where necessary. The first and second terms represent the dipole and quadrupole contributions respectively; note that there is no monopole contribution to leading order in the circular orbit case. Substituting this into (81) and performing the integration over the solid angle via the identity (75), we obtain

$$\begin{aligned} \dot{E} = & -\mathcal{G}^2 m^2 \mu^2 \xi \left\langle \frac{2}{3} \mathcal{S}^2 \left(\frac{r^i}{r^3} - I_2 \left[\frac{r^i}{r^3} \right] \right) \left(\frac{r^i}{r^3} - I_2 \left[\frac{r^i}{r^3} \right] \right) + \frac{8}{15} \Gamma^2 \left(\frac{r^i v^j}{r^3} - I_3 \left[\frac{r^i v^j}{r^3} \right] \right) \left(\frac{r^i v^j}{r^3} - I_3 \left[\frac{r^i v^j}{r^3} \right] \right) \right. \\ & \left. + \frac{8}{15} \Gamma^2 \left(\frac{r^i v^j}{r^3} - I_3 \left[\frac{r^i v^j}{r^3} \right] \right) \left(\frac{r^j v^i}{r^3} - I_3 \left[\frac{r^j v^i}{r^3} \right] \right) \right\rangle \\ = & -\frac{\mathcal{G}^2 \mu^2 m^2 \xi}{r^4} \left[\frac{2}{3} \mathcal{S}^2 (1 - 2Z_2(R; m_s, \omega) + W_2(R; m_s, \omega)) + \frac{8}{15} \Gamma^2 v^2 (1 - 2Z_3(R; m_s, 2\omega) + W_3(R; m_s, 2\omega)) \right], \end{aligned} \quad (91)$$

where in the second line we have performed the average

over one orbital period and we have defined

$$\begin{aligned} Z_n(R; m_s, \omega) &\equiv \cos(\omega R) C_n(R; m_s, \omega) + \sin(\omega R) S_n(R; m_s, \omega), \\ W_n(R; m_s, \omega) &\equiv |C_n(R; m_s, \omega)|^2 + |S_n(R; m_s, \omega)|^2, \\ C_n(R; m_s, \omega) &\equiv \int_0^\infty \cos\left(\omega R \sqrt{1 + \left(\frac{z}{m_s R}\right)^2}\right) \frac{J_1(z)}{\left(1 + \left(\frac{z}{m_s R}\right)^2\right)^{\frac{n}{2}}} dz, \\ S_n(R; m_s, \omega) &\equiv \int_0^\infty \sin\left(\omega R \sqrt{1 + \left(\frac{z}{m_s R}\right)^2}\right) \frac{J_1(z)}{\left(1 + \left(\frac{z}{m_s R}\right)^2\right)^{\frac{n}{2}}} dz. \end{aligned} \quad (92)$$

To get the total power radiated we must perform the integrals in the limit $R \rightarrow \infty$ in which they have closed form solutions. The evaluation of these integrals is discussed in Appendix B. Performing the integrals, we obtain

$$\begin{aligned} \dot{E} = & -\frac{\mathcal{G}^2 m^2 \mu^2 \xi}{r^4} \left[\frac{8}{15} \Gamma^2 v^2 \left(\frac{4\omega^2 - m_s^2}{4\omega^2} \right)^2 \Theta(2\omega - m_s) \right. \\ & \left. + \frac{2}{3} \mathcal{S}^2 \frac{\omega^2 - m_s^2}{\omega^2} \Theta(\omega - m_s) \right]. \end{aligned} \quad (93)$$

Using again $(\dot{P}/P) = -\frac{3}{2}(\dot{E}/E)$, and $E = -\frac{1}{2}\frac{\mathcal{G}m\mu}{r} = -\frac{1}{2}\mu v^2$, we can eliminate v and find for the fractional period derivative due to scalar radiation:

$$\begin{aligned} \frac{\dot{P}}{P} = & -\frac{96}{5} \frac{\mathcal{G}^2 \mu m^2 \Gamma^2}{r^4} \xi \left(\frac{4\omega^2 - m_s^2}{4\omega^2} \right)^2 \Theta(2\omega - m_s) \\ & - \frac{2\mathcal{G}\mu m}{r^3} \mathcal{S}^2 \xi \frac{\omega^2 - m_s^2}{\omega^2} \Theta(\omega - m_s). \end{aligned} \quad (94)$$

Combining this with the result for the tensor gravitational radiation contribution (80), we finally obtain the result quoted in Eq. (1) of the introduction.

VII. OBTAINING BOUNDS ON $(\omega_{\text{BD}}, m_s)$

In this section we compare our results for the period derivative of compact binaries, the Shapiro delay and the Nordtvedt effect against recent observational data to draw exclusion plots in the two-dimensional parameter space of the theory, $(m_s, \omega_{\text{BD}})$. Figure 1 in the introduction summarizes our main results.

A. Bounds from \dot{P} in compact binaries

Due to the presence of the difference in sensitivities ($\mathcal{S} = s_1 - s_2$) in the dipole contribution to the period decay (1), the best candidate systems for drawing exclusion plots in the $(\omega_{\text{BD}}, m_s)$ plane are mixed binaries. White dwarf-neutron star (WD-NS) binaries are particularly suitable due to the large difference in sensitivities ($\sim 10^{-4}$ and ~ 0.2 for WDs and NSs, respectively [27]). To our knowledge, there are three such systems for which accurate measurements of \dot{P} and the other necessary parameters have been made (to date): PSRs J0751+1807, J1012+5307, J1141-6545. A summary of the observations and the relevant references are provided in Appendix C.

In principle, there are two more systems that are of interest to our current purposes: PSR J1738+0333 [34] and PSR J1802-2124 [35]. Both of these systems have very small eccentricities, which means that the result derived in this paper can be used “out of the box”, with no need to generalize our calculation to eccentric binaries. In the case of PSR J1738+0333, a relatively accurate measurement of \dot{P} has been achieved (with error $\sim 30\%$), but the masses of the component stars have yet to be determined

[34]. PSR J1802-2124 is in precisely the opposite situation: the masses of the components have been measured to reasonable precision, but a precise measurement of \dot{P} has yet to be achieved. This is anticipated in the near future [35].

The general approach to obtaining bounds on $(\omega_{\text{BD}}, m_s)$ using observations of the period derivative of mixed binaries is as follows. Firstly, we need to write (1) in terms of the observables relevant to the system under inspection. For circular binaries, the relevant observables are the stellar masses (including the mass ratio q) and the period. Recasting (1) into these observables we obtain

$$\dot{P} = \dot{P}_{\text{GR}} \left[\mathcal{G}^{-\frac{4}{3}} \frac{\kappa_1}{12} + \frac{5}{96} m^{-\frac{2}{3}} \left(\frac{2\pi}{P} \right)^{-\frac{2}{3}} 2\mathcal{S}^2 \kappa_D \right], \quad (95)$$

where \dot{P}_{GR} is the prediction from GR, given by

$$\dot{P}_{\text{GR}} = -\frac{192\pi}{5} \frac{q m^{\frac{5}{3}}}{(1+q)^2} \left(\frac{2\pi}{P} \right)^{\frac{5}{3}}. \quad (96)$$

For mildly eccentric binaries, provided the eccentricity is small enough, we can get approximate bounds using the results obtained here for the circular case. In these instances, we can use the measured periastron shift $\dot{\omega}$, period, and the mass ratio q of the binary. Recasting (1) in terms of these observables (using the results for the periastron advance quoted in section IV A and Kepler’s third law to eliminate m and r) we obtain

$$\dot{P} = \dot{P}_{\text{GR}} \left[\frac{\mathcal{G}^2}{\mathcal{P}^{-\frac{5}{2}}} \frac{\kappa_1}{12} + \frac{5}{16} \frac{2\pi}{P\dot{\omega}} \mathcal{S}^2 \frac{\kappa_D}{2} \right], \quad (97)$$

where

$$\dot{P}_{\text{GR}} = -\frac{4qP}{(1+q)^2} \frac{8}{15\sqrt{3}} \left(\frac{P}{2\pi} \right)^{\frac{3}{2}} \dot{\omega}^{\frac{5}{2}}. \quad (98)$$

With the predicted \dot{P} written in terms of the relevant set of parameters, we are in a position to compare it to observations; the predicted \dot{P} and observed \dot{P}_{obs} are consistent to $n\sigma$ confidence provided that

$$|\dot{P}_{\text{obs}} - \dot{P}(\xi, m_s)| \leq n\sigma, \quad (99)$$

where σ is the combined uncertainty of \dot{P}_{obs} and the predicted \dot{P} , and where we should remember that the latter is uncertain due to uncertainties in the observables (such as stellar masses and period). In order to obtain an upper bound on ξ (and hence a lower bound on ω_{BD}) to 95% confidence for a range of scalar masses, in Figure 1 we simply plot the contour in the $(\omega_{\text{BD}}, m_s)$ -plane associated with $|\dot{P}_{\text{obs}} - \dot{P}(\xi, m_s)| = 2\sigma$.

1. Bounds from neutron star-neutron star binaries

The presence of dipole radiation in mixed binaries suggests that these should be the best candidates for obtaining the most stringent bounds on $(\omega_{\text{BD}}, m_s)$. However, it

is worth looking into the bounds that could be obtained from observations of neutron star-neutron star (NS-NS) binaries. Since the sensitivities of the two component stars are nearly identical in this case ($\mathcal{S} \simeq 0$: cf. [27]), the expression for the period derivative reduces to

$$\dot{P} = \dot{P}_{\text{GR}} \frac{\kappa_1}{12}. \quad (100)$$

Expanding to linear order in ξ we can write this as

$$\dot{P} = \dot{P}_{\text{GR}} (1 + \xi\chi), \quad (101)$$

where

$$\begin{aligned} \chi = & \frac{\Gamma^2}{12} \left(\frac{4\omega^2 - m_s^2}{4\omega^2} \right)^2 \Theta(2\omega - m_s) - \frac{5}{6} \\ & + \frac{1}{3}(1 - 2s_1)(1 - 2s_2). \end{aligned} \quad (102)$$

We can also write the observed period derivative as

$$\dot{P}_{\text{obs}} = \dot{P}_{\text{GR}}(1 + \delta), \quad (103)$$

where δ is the fractional deviation of the observed value from the GR prediction. Applying the condition (99) to the above two equations we find that the predicted \dot{P} is consistent with the observed \dot{P}_{obs} to 2σ confidence provided that

$$|\xi\chi - \delta| \leq 2\sigma, \quad (104)$$

where σ is now the combined uncertainty of χ and the observed deviation from GR, δ . Since the correction χ in the above is of order unity, we conclude that bounds competitive with the most stringent bounds obtained here (from the Cassini Shapiro delay measurements, and those that are expected from a rigorous analysis of J1141-6545) could only be obtained from a \dot{P} measurement to a precision of $\sim 0.01\%$. Since the current best measurements of \dot{P} for NS-NS binaries are not yet close to this precision, we conclude that the bounds that would be obtained by analyzing such systems would be significantly weaker than the most stringent bounds obtained here. For NS-NS systems for which \dot{P} has been measured to a precision of $\sim 1\%$ (such as PSR J0737-3039 [36]), we would expect to obtain relatively weak bounds, comparable to those obtained here from the quasi-circular WD-NS binary J1012+5307.

B. Bounds from Cassini time-delay data

The Shapiro delay has been measured in the Solar System to remarkable precision by radio tracking of the Cassini spacecraft in 2002 [37]. In theories containing only massless fields, these observations are tantamount to a measurement of the PPN parameter γ . This has been measured to be

$$\gamma^{\text{Cassini}} = 1 + (2.1 \pm 2.3) \times 10^{-5} = 1 + \delta \pm \epsilon. \quad (105)$$

As discussed in section III, the mass of the scalar in the massive Brans-Dicke theory prohibits us from using the PPN formalism in the conventional manner, and the concept of constant PPN parameters breaks down (see also [18]). In section III we derived an expression for the Shapiro delay in the massive Brans-Dicke theory, and we defined a quantity $\tilde{\gamma}$ which is analogous to the PPN parameter γ (at least in the context of Shapiro delay) and can be directly compared to the measured value of γ to obtain an exclusion region in the $(m_s, \omega_{\text{BD}})$ plane. Comparing the derived expression for $\tilde{\gamma}$ (46) with the Cassini measurement of γ (105), we require that

$$\alpha < e^{m_s r} \frac{2\epsilon - \delta}{(2 + \delta - 2\epsilon)} \quad (106)$$

to 95% confidence. The resulting bounds on ξ and ω_{BD} are plotted in Figure 1 by solid black lines (cf. also [18]).

We find that $\omega_{\text{BD}} > 40000$ for a range of scalar masses $m_s < 2.5 \times 10^{-20} \text{eV}$, to 95% confidence. This is around one order of magnitude more stringent than the bounds provided by the observations of gravitational radiation damping in binary systems. In the limit $1/m_s \ll 1\text{AU}$, ω_{BD} can take on any value (as long as $\omega_{\text{BD}} > -3/2$).

C. Bounds from Lunar Laser Ranging observations

The most precise measurement of the Nordtvedt effect to date comes from the Lunar Laser Ranging experiment [38]

$$\eta_{\text{N}}^{\text{LLR}} = (0.6 \pm 5.2) \times 10^{-4} = \delta \pm \epsilon. \quad (107)$$

Comparing this observed value to Eq. (62) and neglecting the small sensitivity of the Sun, we require that

$$|\xi(1 + m_s r)e^{-m_s r} - \delta| \leq 2\epsilon \quad (108)$$

to 95% confidence, from which we obtain exclusion regions in the $(m_s, \omega_{\text{BD}})$ plane. These are displayed in Figure 1 by dotted blue lines.

VIII. CONCLUSIONS

In this paper we set constraints on massive Brans-Dicke (or Bergmann-Wagoner) theories with an action of the form (6) with $\omega(\phi) = \omega_{\text{BD}}$, assuming that only a mass term is present in the expansion of $M(\phi)$ around some cosmologically imposed value ϕ_0 . In particular we computed the orbital period derivative for quasicircular binaries. From an observational standpoint it will be important to generalize our work to eccentric (and possibly spinning) binaries, that could yield more stringent constraints on scalar-tensor theories. It will also be interesting to explore possible bounds on massive scalar-tensor theories that could result from Earth- and space-based gravitational-wave observations of compact binaries, along the line of Refs. [39–41].

A second obvious generalization of our work will consist in relaxing our assumptions on the form of $\omega(\phi)$ and $M(\phi)$. More generic assumptions on these functions are necessary for a deeper understanding of binary dynamics in the context of modified gravity models that try to explain cosmological observations. It will be interesting to verify whether time-varying boundary conditions on the scalar field may lead to interesting binary dynamics [16].

Last but not least, full numerical relativity simulations of compact binaries in scalar-tensor theories are under investigation by several groups (see e.g. [42, 43]). Numerical progress in evolving binary dynamics in alternative theories is important, as it could reveal strong-field effects that may be inaccessible to post-Newtonian or perturbative calculations.

ACKNOWLEDGMENTS

We are grateful to Vitor Cardoso, Yanbei Chen, Samaya Nissanke, Ulrich Sperhake, Michele Vallisneri and Helvi Witek for discussions. We are particularly grateful to Leonardo Gualtieri, Michael Horbatsch and Paolo Pani for suggestions and detailed comments on the manuscript, and to Michael Horbatsch for checking that our dipolar and quadrupolar fluxes match Eq. (6.40) in [21] when $m_s \rightarrow 0$. J.A. was supported by a LIGO SURF Fellowship at Caltech. E.B. was supported by NSF Grant PHY-0900735 and by NSF CAREER Grant PHY-1055103. C.M.W. was supported by NSF Grant PHY-0965133. C.M.W. is grateful for the hospitality of the Institut d'Astrophysique de Paris, where part of this work was carried out.

Appendix A: Post-Newtonian expansion of the scalar field and of the metric

Here we provide details of the derivation of the post-Newtonian expansions (32), (33), (34) and (35). We follow very closely the method outlined in [20]. For our current purposes, we must solve the field equations (20) and (21) to the following orders:

$$\begin{aligned} \varphi &\sim O(2) + O(4), \\ h_{00} &\sim O(2) + O(4), \\ h_{0i} &\sim O(3), \\ h_{ij} &\sim O(2). \end{aligned} \quad (\text{A1})$$

We will do this in a number of steps, as described in the following.

1. Step 1: φ to order $O(2)$

To the lowest order, the scalar field equation (21) reduces to

$$(\nabla^2 - m_s^2) \frac{\varphi}{\phi_0} = 8\pi\alpha\phi_0^{-1}T^*. \quad (\text{A2})$$

Expanding the modified stress-energy tensor T^* to lowest order we obtain

$$\begin{aligned} T^* &= \sum_a m_a(2s_a - 1)\delta^3(\mathbf{x} - \mathbf{x}_a) \\ &= -\frac{1}{4\pi} \sum_a m_a(2s_a - 1) (\nabla^2 - m_s^2) \frac{e^{-m_s r_a}}{r_a}. \end{aligned} \quad (\text{A3})$$

Substituting this into (A2), we find the solution for φ to $O(2)$

$$\frac{\varphi}{\phi_0} = \xi \sum_a \frac{m_a}{r_a} e^{-m_s r_a} (1 - 2s_a). \quad (\text{A4})$$

2. Step 2: h_{00} to $O(2)$

The 00-component of the tensor field equation (20) to $O(2)$ is given by

$$R_{00} = -\frac{1}{2}\nabla^2 h_{00} = 8\pi \left(T_{00} + \frac{T}{2} \right) - \frac{1}{2}\nabla^2 \left(\frac{\varphi}{\phi_0} \right). \quad (\text{A5})$$

In a similar fashion to (A3), we write for the stress-energy tensor (to lowest order)

$$\begin{aligned} T &= -T_{00} = -\sum_a m_a \delta^3(\mathbf{x} - \mathbf{x}_a) \\ &= \frac{1}{4\pi} \sum_a m_a \nabla^2 \left(\frac{1}{r_a} \right). \end{aligned} \quad (\text{A6})$$

Substituting this along with the derived $O(2)$ expression for φ into (A5), we obtain the $O(2)$ solution for h_{00} :

$$h_{00} = 2\phi_0^{-1} \sum_a \frac{m_a}{r_a} [1 + \alpha(1 - 2s_a)e^{-m_s r_a}]. \quad (\text{A7})$$

3. Step 3: h_{ij} to $O(2)$

The ij -component of the Ricci tensor to $O(2)$ is given by

$$R_{ij} = -\frac{1}{2}(\nabla^2 h_{ij} - h_{00,ij} + h_{k,ij}^k - h_{i,kj}^k - h_{j,ki}^k). \quad (\text{A8})$$

Imposing the gauge condition

$$h_{i,\mu}^\mu - \frac{1}{2}h_{\mu,i}^\mu = \left(\frac{\varphi}{\phi_0} \right)_{,i}, \quad (\text{A9})$$

we can write the ij -component of tensor field equation (20) to $O(2)$ as

$$\nabla^2 h_{ij} = 8\pi\phi_0^{-1} T\delta_{ij} + \delta_{ij}\nabla^2 \left(\frac{\varphi}{\phi_0} \right). \quad (\text{A10})$$

Using Eq. (A6) for the stress-energy tensor and substituting the derived $O(2)$ expression for φ into the expression above, we obtain the solution for h_{ij} to $O(2)$:

$$h_{ij} = \delta_{ij} 2\phi_0^{-1} \sum_a \frac{m_a}{r_a} [1 - \alpha(1 - 2s_a)e^{-m_s r_a}]. \quad (\text{A11})$$

4. Step 4: h_{0i} to $O(3)$

The $0i$ -component of the Ricci tensor to $O(3)$ is given by

$$R_{0i} = -\frac{1}{2} \left(\nabla^2 h_{0i} - h_{0k}{}^{,k}{}_{,i} + h_{k,0i}^k - h_{ki}{}^{,k}{}_{,0} \right). \quad (\text{A12})$$

Imposing the further gauge condition

$$h_{0,\mu}^\mu - \frac{1}{2} h_{\mu,0}^\mu = -\frac{1}{2} h_{00,0} + \left(\frac{\varphi}{\phi_0} \right)_{,0} \quad (\text{A13})$$

this reduces to

$$R_{0i} = -\frac{1}{2} \nabla^2 h_{0i} + \frac{1}{2} \left(\frac{\varphi}{\phi_0} \right)_{,0i} - \frac{1}{12} h_{k,0i}^k. \quad (\text{A14})$$

We can hence write the $0i$ -component of the tensor field equation (20) to $O(3)$ as

$$-\frac{1}{2} \nabla^2 h_{0i} = 8\pi\phi_0^{-1} T_{0i} + \frac{1}{2} \left(\frac{\varphi}{\phi_0} \right)_{,0i} + \frac{1}{12} h_{k,0i}^k. \quad (\text{A15})$$

The $0i$ -component of the stress-energy tensor to lowest order is given by

$$\begin{aligned} T_0^i &= - \sum_a m_a v_a^i \delta^3(\mathbf{x} - \mathbf{x}_a) \\ &= \frac{1}{4\pi} \sum_a m_a v_a^i \nabla^2 \left(\frac{1}{r_a} \right). \end{aligned} \quad (\text{A16})$$

In order to write the $\varphi_{,00}$ term in the form $\nabla^2 \chi$, we must find a particular solution to $\nabla^2 \chi = e^{-m_s r}/r$. Taking care to ensure that the chosen solution χ is such that the correct limit is obtained as $m_s \rightarrow 0$, we write

$$\nabla^2 \left(\frac{e^{-m_s r_a} + m_s r_a - 1}{m_s^2 r_a} \right) = \frac{e^{-m_s r_a}}{r_a}, \quad (\text{A17})$$

Noting also that $\nabla^2(r_a/2) = r_a$, we can re-write the second and third terms in (A15) as

$$\begin{aligned} h_{k,0i}^k &= \frac{6}{\phi_0} \sum_a m_a \partial_i \partial_0 \left[\frac{r_a}{2} - \alpha(1 - 2s_a) \frac{e^{-m_s r_a} + m_s r_a - 1}{m_s^2 r_a} \right], \\ \left(\frac{\varphi}{\phi_0} \right)_{,0i} &= \xi \sum_a m_a (1 - 2s_a) \partial_i \partial_0 \left(\frac{e^{-m_s r_a} + m_s r_a - 1}{m_s^2 r_a} \right). \end{aligned} \quad (\text{A18})$$

Substituting these into (A15), we obtain the solution for h_{0i} to $O(3)$ given in Eq. (34).

5. Step 5: φ to $O(4)$

Expanding $\square_g \varphi$ to $O(4)$ and recalling the definition of $\theta^{\mu\nu}$ in Eq. (16), we obtain

$$\square_g \varphi = \left(1 + \frac{1}{2} \theta + \frac{\varphi}{\phi_0} \right) \square_\eta \frac{\varphi}{\phi_0} - \theta^{\mu\nu} \frac{\varphi_{,\mu\nu}}{\phi_0} - \frac{\varphi_{,\alpha} \varphi^{,\alpha}}{\phi_0^2}. \quad (\text{A19})$$

The scalar field equation (21) to $O(4)$ can hence be written as

$$\begin{aligned} (\nabla^2 - m_s^2) \frac{\varphi}{\phi_0} &= 8\pi\alpha\phi_0^{-1} T^* \left(1 - \frac{1}{2} \theta - \frac{\varphi}{\phi_0} \right) \\ &\quad + \left(\frac{\varphi}{\phi_0} \right)_{,00} + \left(\nabla \frac{\varphi}{\phi_0} \right)^2. \end{aligned} \quad (\text{A20})$$

Expanding the modified stress-energy tensor to the required order we find

$$\begin{aligned} T^* &= \sum_a m_a \left[(2s_a - 1) + \frac{1}{2} (1 - 2s_a) v_a^2 \right. \\ &\quad \left. - \frac{3}{4} (1 - 2s_a) \theta - \left(2s'_a - 2s_a^2 + \frac{3}{2} \right) \frac{\varphi}{\phi_0} \right]. \end{aligned} \quad (\text{A21})$$

In a similar fashion to step 4, we wish to write the $\varphi_{,00}$ term in the form $(\nabla^2 - m_s^2)\chi$, and we require a particular solution to $(\nabla^2 - m_s^2)\chi = e^{-m_s r}/r$ such that the correct limit is obtained as $m_s \rightarrow 0$. To this end we write

$$(\nabla^2 - m_s^2) \left(\frac{1 - e^{-m_s r}}{2m_s} \right) = \frac{e^{-m_s r}}{r}, \quad (\text{A22})$$

and hence write the second term on the right hand side of Eq. (A20) as

$$\left(\frac{\varphi}{\phi_0} \right)_{,00} = (\nabla^2 - m_s^2) \xi \sum_a m_a (1 - 2s_a) \partial_0 \partial_0 \left(\frac{1 - e^{-m_s r}}{2m_s} \right). \quad (\text{A23})$$

The third term on the right hand side of Eq. (A20) can be re-written (to the required order) as

$$\begin{aligned} \left(\nabla \frac{\varphi}{\phi_0} \right)^2 &= \frac{1}{2} (\nabla^2 - m_s^2) \left(\frac{\varphi}{\phi_0} \right)^2 \\ &\quad - \left(\frac{\varphi}{\phi_0} \right) (\nabla^2 - m_s^2) \left(\frac{\varphi}{\phi_0} \right). \end{aligned} \quad (\text{A24})$$

Substituting the above along with the derived $O(2)$ expressions for φ , h_{00} and h_{ij} into (A20), we obtain the result given in Eq. (32).

6. Step 6: h_{00} to $O(4)$

The 00-component of the tensor field equation (20) to $O(4)$ is given by

$$\begin{aligned}
R_{00} &= -\frac{1}{2}\nabla^2 h_{00} + \left(\frac{\varphi}{\phi_0}\right)_{,00} + \frac{1}{2}\nabla h_{00}\nabla\frac{\varphi}{\phi_0} \\
&\quad - \frac{1}{2}(\nabla h_{00})^2 + \frac{1}{2}h_{00}\nabla^2 h_{00} - \frac{\varphi}{\phi_0}\nabla^2 h_{00} \\
&= 8\pi\phi_0^{-1}\left(1 - \frac{\varphi}{\phi_0}\right)(T_{00} - \frac{1}{2}g_{00}T) \\
&\quad + \left(1 - \frac{\varphi}{\phi_0}\right)\left(\left(\frac{\varphi}{\phi_0}\right)_{,00} + \frac{1}{2}g_{00}\square_g\left(\frac{\varphi}{\phi_0}\right)\right), \tag{A25}
\end{aligned}$$

where we have used the gauge conditions (A9) and (A13) to reduce the expression for R_{00} into a convenient form. The term involving the stress-energy tensor on the right hand side is given (to the required order) by

$$\begin{aligned}
T_{00} - \frac{1}{2}g_{00}T &= \frac{1}{2}\sum_a m_a \delta^3(\mathbf{x} - \mathbf{x}_a) \left[1 + \frac{3}{4}v_a^2 + \frac{5}{4}\theta\right. \\
&\quad \left. + \frac{1}{2}(2s_a + 1)\left(\frac{\varphi}{\phi_0}\right)\right]. \tag{A26}
\end{aligned}$$

Using this along with

$$\begin{aligned}
(\nabla^2 h_{00})^2 &= \frac{1}{2}\nabla^2 h_{00}^2 - h_{00}\nabla^2 h_{00}, \\
(\nabla^2 \frac{\varphi}{\phi_0})^2 &= \frac{1}{2}\nabla^2 \left(\frac{\varphi}{\phi_0}\right)^2 - \frac{\varphi}{\phi_0}\nabla^2 \frac{\varphi}{\phi_0}, \tag{A27}
\end{aligned}$$

Eq. (A25) can be re-written as

$$\begin{aligned}
&2\phi_0^{-1}\sum_a m_a \nabla^2 \left(\frac{1}{r_a}\right) \left[1 + \frac{3}{2}v_a^2 + \frac{5}{4}\theta + (s_a - \frac{1}{2})\frac{\varphi}{\phi_0}\right] \\
&+ \nabla^2 \frac{\varphi}{\phi_0} + \left(\frac{1}{2}\theta - h_{00}\right)\nabla^2 \frac{\varphi}{\phi_0} - \nabla^2 \left(\frac{\varphi}{\phi_0}\right)^2 \\
&- \frac{1}{2}\nabla^2 h_{00}^2 + 2h_{00}\nabla^2 h_{00} - 2\frac{\varphi}{\phi_0}\nabla^2 h_{00} - \frac{1}{2}\left(\frac{\varphi}{\phi_0}\right)_{,00} \\
&+ 2\frac{\varphi}{\phi_0}\nabla^2 \frac{\varphi}{\phi_0} = \nabla^2 h_{00} \tag{A28}
\end{aligned}$$

Using Eq. (A17) to re-write the term involving $\varphi_{,00}$ in the form $\nabla^2\chi$, and substituting in the derived $O(2)$ expressions for h_{00} , h_{ij} and φ , and the $O(4)$ expression for φ , we obtain the result presented in Eq. (33). Note that there are two contributions to the term involving the second time derivative in Eq. (33): one contribution from φ (to $O(4)$) and one from $\varphi_{,00}$.

Appendix B: Evaluation of integrals $C_n(R; m_s, \omega)$ and $S_n(R; m_s, \omega)$ arising in the derivation of the period derivative due to scalar radiation

In reaching the final expression for the power emitted in scalar gravitational radiation (94), we were required

to evaluate the integrals $C_n(R; m_s, \omega)$ and $S_n(R; m_s, \omega)$ defined in (92). In this appendix we give details of the evaluation of these integrals.

Since we are interested in the gravitational radiation in the far zone, we only need to determine the asymptotic behavior of these integrals for $R \rightarrow \infty$. Substituting $u = \sqrt{1 + (z/m_s R)^2}$ into (92) we obtain

$$\begin{aligned}
C_n(R; m_s, \omega) &= m_s R \int_1^\infty du \frac{\cos(\omega R u)}{u^{n-1}} \frac{J_1(m_s R \sqrt{u^2 - 1})}{\sqrt{u^2 - 1}}, \\
S_n(R; m_s, \omega) &= m_s R \int_1^\infty du \frac{\sin(\omega R u)}{u^{n-1}} \frac{J_1(m_s R \sqrt{u^2 - 1})}{\sqrt{u^2 - 1}}. \tag{B1}
\end{aligned}$$

We will discuss the evaluation of C_n only, as the evaluation of S_n proceeds in exactly the same way. To begin with, let us choose some ϵ such that $m_s R \epsilon \gg 1$ while $\omega R \epsilon^2 \ll 1$ and split up the integral into an integration from 1 to $1 + \epsilon^2/2$ and from $1 + \epsilon^2/2$ to ∞ . In the first integral, as the argument of the cosine is nearly constant we can approximate

$$\begin{aligned}
m_s R \int_1^{1+\epsilon^2/2} du \frac{\cos(\omega R u)}{u^{n-1}} \frac{J_1(m_s R \sqrt{u^2 - 1})}{\sqrt{u^2 - 1}} \\
\approx \cos(\omega R) (1 - J_0(m_s R \epsilon)), \tag{B2}
\end{aligned}$$

with the zeroth order Bessel function J_0 given by its asymptotic value

$$J_0(m_s R \epsilon) \sim \sqrt{\frac{2}{\pi}} \frac{\cos(m_s R \epsilon - \pi/4)}{\sqrt{m_s R \epsilon}}. \tag{B3}$$

For the second integral, we can approximate the Bessel function J_1 by its asymptotic value

$$J_1(x) \sim \sqrt{\frac{2}{\pi}} \frac{\cos(x - 3\pi/4)}{\sqrt{x}}, \tag{B4}$$

and hence the integral can be approximated by

$$\sqrt{\frac{2}{\pi}} \sqrt{m_s R} \int_{1+\epsilon^2/2}^\infty du \frac{\cos(\omega R u)}{u^{n-1}} \frac{\cos(m_s R \sqrt{u^2 - 1} - 3\pi/4)}{\sqrt{u^2 - 1}^3}. \tag{B5}$$

Performing an integration by parts exactly cancels the corresponding boundary term in (B2); this is not surprising, since we expect that the result should not depend on the value of ϵ . In analyzing the above integral, then, we can neglect all terms arising from the lower endpoint (since a full analysis will show that they will exactly cancel the terms arising from the upper endpoint in (B2)). We are interested in the leading asymptotic behavior of the above integral; we hence require the asymptotic behavior of integrals of the type

$$I = \frac{1}{4} \sqrt{\frac{2}{\pi}} \sqrt{m_s R} \int_{1+\epsilon^2/2}^\infty \frac{du}{u^{n-1} \sqrt{u^2 - 1}^3} e^{\rho(u)}, \tag{B6}$$

where

$$\rho(u) = iR(n_1\omega u + n_2m_s\sqrt{u^2-1}) - in_23\pi/4, \quad (\text{B7})$$

with $n_{1,2} = \pm 1$. The part of the integration contour which gives the dominant contribution is determined by $\rho(u)$ and the relative sizes of ω and m_s . Let us deal with the two cases $\omega > m_s$ and $\omega < m_s$ in turn.

a. $\omega > m_s$

For $n_1 = -n_2$, $\rho(u)$ has a stationary point at $a = \omega/\sqrt{\omega^2 - m_s^2}$ and we can apply the method of stationary phase (see e.g. [44]). Since only a small region around the stationary point contributes to the integral, expanding the exponent around a gives the leading-order behavior

$$\begin{aligned} I &\sim \frac{1}{4} \sqrt{\frac{2}{\pi}} \sqrt{m_s R} \frac{e^{\rho(a)}}{a^{n-1} \sqrt[4]{a^2-1}^3} \int_{-\delta}^{+\delta} ds e^{\frac{1}{2}\rho''(a)s^2} \\ &\sim \frac{1}{2} \left(\frac{\sqrt{\omega^2 - m_s^2}}{\omega} \right)^{n-1} e^{iRn_1\sqrt{\omega^2 - m_s^2}} e^{in_1\pi}. \end{aligned} \quad (\text{B8})$$

For $n_1 = n_2$, $\rho(u)$ no longer has any stationary points in the integration domain, so the leading order behavior is obtained by integration by parts. Since the integrand goes to zero at the upper endpoint $+\infty$, the only contribution will come from the lower endpoint, which as we have discussed must exactly cancel with the corresponding terms from (B2). The complete leading order behavior of $C_n(R; m_s, \omega)$ (and similarly $S_n(R; m_s, \omega)$) for $\omega > m_s$ is hence given by

$$\begin{aligned} C_n(R; m_s, \omega) &\sim \cos(\omega R) - \left(\frac{\sqrt{\omega^2 - m_s^2}}{\omega} \right)^{n-1} \cos(R\sqrt{\omega^2 - m_s^2}), \\ S_n(R; m_s, \omega) &\sim \sin(\omega R) - \left(\frac{\sqrt{\omega^2 - m_s^2}}{\omega} \right)^{n-1} \sin(R\sqrt{\omega^2 - m_s^2}). \end{aligned} \quad (\text{B9})$$

b. $\omega < m_s$

The case $\omega < m_s$ is somewhat more subtle. Now the first derivative of $\rho(u)$ can only vanish on the imaginary axis. We must therefore consider the analytic properties of $\rho(u)$ and use the method of steepest descent [44]. The central idea behind this method is to deform the integration contour in such a way that it follows lines of constant phase (lines of steepest descent), in the hope that along the new contour the integral may be evaluated asymptotically. Firstly, we must take care of the fact that $\rho(u)$ is not analytic in the complex plane, owing to the fact that the square root term makes it double-valued. Working on a two-sheeted Riemann surface we can still apply the method of steepest descent, provided the deformed contour does not include either of the branch points that appear on the real axis at $u = \pm 1$. Since we are extending our exponent into the complex plane, we must first fix

the branch of $\sqrt{u^2-1}$ that we are using. This will then determine the constant phase contours and the location of the saddle points. The asymptotic behavior obtained in the end will of course be independent of the choice of branch. Writing $u = w + iv$ and $\sqrt{u^2-1} = U + iV$, we define the principal branch of the square root to be

$$\sqrt{u^2-1} = \begin{cases} U + iV & w, v > 0 \\ U - iV & w > 0, v < 0 \\ -U - iV & w, v < 0 \\ -U + iV & w < 0, v > 0 \end{cases} \quad (\text{B10})$$

The second branch of the square root is then simply the negative of (B10).

For $n_1 = n_2$, there are no saddle points in the principal Riemann sheet, so we proceed with a straightforward integration by parts. As before, since the integrand vanishes at the upper endpoint $+\infty$ the only contribution comes from the lower endpoint, and this will exactly cancel the corresponding contribution from (B2).

The case $n_1 = -n_2$ is somewhat more challenging. In the principal Riemann sheet, we find two saddle points on the imaginary axis at $u = b_{\pm} = \pm i\omega/\sqrt{m_s^2 - \omega^2}$. At these saddle points the imaginary axis intersects another constant phase contour that closes on the real axis at $u = \pm m_s/\sqrt{m_s^2 - \omega^2}$.

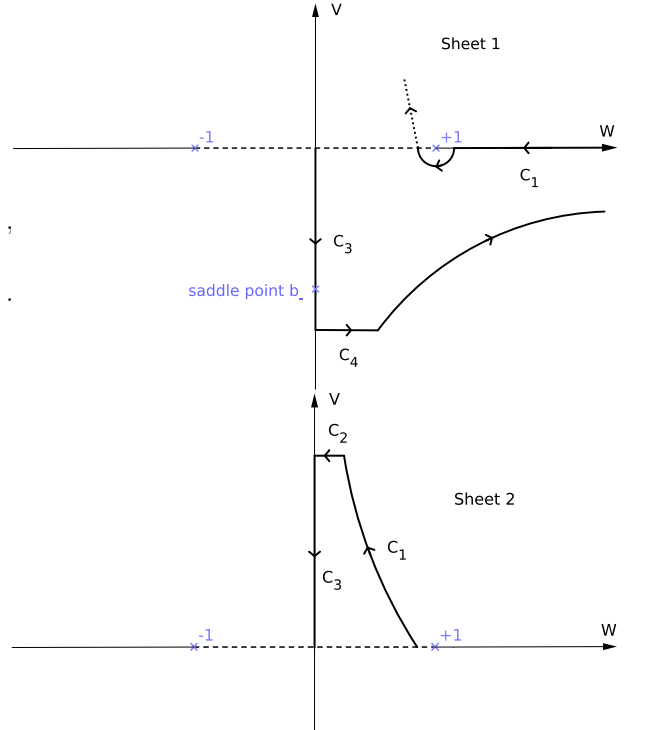


FIG. 2. Deformed integration contour $C_1+C_2+C_3+C_4$ along lines of steepest descent for $\rho(u) = iR(\omega u - m_s\sqrt{u^2-1}) + i3\pi/4$ (corresponding to $n_1 = +1$, $n_2 = -1$) on the two sheets of the Riemann surface associated with $\sqrt{u^2-1}$. Here only the saddle point on the negative imaginary axis contributes.

Let us consider the case $n_1 = +1$. The original inte-

gration contour runs along the real axis from $1 + \epsilon^2/2$ to $+\infty$. We now deform the contour by going along constant phase lines in the direction in which the real part of the exponent $\rho(u)$ decreases (see figure 2). Starting from the lower endpoint $1 + \epsilon^2/2$, we follow the contour C_1 with phase $\omega - m_s \epsilon$ into the lower half of the complex plane, and then through the branch cut onto the second sheet of the Riemann surface. On this sheet there are no saddle points, and our constant phase contour approaches $(\omega - m_s \epsilon)/(m_s + \omega) \pm i\infty$. We can then connect it to the imaginary axis by a path C_2 parametrized by $t + iT$, with t running from $(\omega - m_s \epsilon)/(m_s + \omega)$ to 0 and constant $T \gg 1$. The contribution from this path vanishes as $T \rightarrow \infty$. The integration contour then follows C_3 along the imaginary axis through the branch cut onto the first sheet, through the saddle point at $b_- = -i\omega/\sqrt{m_s^2 - \omega^2}$ and towards $-\infty$. From there it can be closed onto the positive real axis by the path C_4 (analogous to C_2) that ultimately makes no contribution. Since the integrand vanishes as $u \rightarrow +\infty$ the upper endpoint of the integration is once again unimportant. For the case $n_1 = -1$, we can proceed in a similar fashion, only this time the deformed contour will pass through the saddle point b_+ .

We now have all of the ingredients we need to evaluate the integral. Since our deformed contour together

with the original contour does not include any of the branch points, we can still apply the Cauchy theorem and approximate the integration along the contour $C_1 + C_2 + C_3 + C_4$ to obtain the asymptotic behavior of I as $R \rightarrow \infty$. The two crucial regions of the deformed contour are the lower endpoint on the contour C_1 and the saddle point on the imaginary axis (b_- for $n_1 = +1$ or b_+ for $n_1 = -1$). Even though the contribution from the saddle point is sub-dominant, it *does* give the asymptotic behavior of the original integrals (92) that we require; recall again that the contribution from the lower endpoint will be exactly canceled by the contribution from (B2). Integrating through the saddle point b_{\pm} along the imaginary axis and parameterizing $u = i(b_{\pm} + t)$, we obtain

$$\begin{aligned} I_{\pm} &\sim \frac{1}{4} \sqrt{\frac{2}{\pi}} \sqrt{m_s R} \frac{e^{\rho(b_{\mp})}}{b_{\mp}^{n-1} \sqrt[4]{b_{\mp}^2 - 1}} \int_{-\delta}^{\delta} dt i e^{-\frac{1}{2}\rho''(b_{\mp})t^2} \\ &\sim -\frac{1}{2} \left(\mp \frac{\sqrt{m_s^2 - \omega^2}}{i\omega} \right)^{n-1} e^{-R\sqrt{m_s^2 - \omega^2}}. \end{aligned} \quad (\text{B11})$$

where I_{\pm} corresponds to $n_1 = \pm 1$. The complete leading-order behavior of $C_n(R; m_s, \omega)$ (and similarly $S_n(R; m_s, \omega)$) for $\omega < m_s$ is then given by

$$\begin{aligned} C_n(R; m_s, \omega) &\sim \cos(\omega R) - \left(\frac{\sqrt{m_s^2 - \omega^2}}{\omega} \right)^{n-1} e^{-R\sqrt{m_s^2 - \omega^2}} \frac{i^{n-1} + (-i)^{n-1}}{2} \\ S_n(R; m_s, \omega) &\sim \sin(\omega R) - \left(\frac{\sqrt{m_s^2 - \omega^2}}{\omega} \right)^{n-1} e^{-R\sqrt{m_s^2 - \omega^2}} \frac{i^{n-1} - (-i)^{n-1}}{2}. \end{aligned} \quad (\text{B12})$$

Appendix C: Observational data on compact binaries used in this paper

TABLE I. Parameters relevant to the binary system PSR J1012+5307 [45].

Period, P (days)	0.60467271355(3)
Period derivative (observed), \dot{P}^{obs}	5.0(1.4) 10^{-14}
Period derivative (intrinsic), \dot{P}^{intr}	-1.5(1.5) 10^{-14}
Mass ratio, q	10.5(5)
NS Mass, m_1 (M_{\odot})	1.64(22)
WD Mass, m_2 (M_{\odot})	0.16(2)
Eccentricity, e (10^{-6})	1.2(3)

c. PSR J1012+5307

PSR J1012+5307 is a 5.3ms pulsar in a 14.5hr quasicircular binary system with a low-mass WD companion [46].

The relevant parameters for this system are listed in Table I. The parameter values are taken directly from [45]. The mass ratio and individual masses were determined in [47], and the intrinsic period derivative, corrected for Doppler effects, was determined in [45].

Using the parameters listed in Table I and Eq. (96), the value of the period derivative predicted by GR is given by

$$\dot{P}_{GR} = -\frac{192\pi}{5} \frac{q m^{\frac{5}{3}}}{(1+q)^2} \left(\frac{2\pi}{P} \right)^{\frac{5}{3}} = -1.1(2) \times 10^{-14}. \quad (\text{C1})$$

Using the method described in section VII A we obtain the bound on ξ (and hence ω_{BD}) as a function of the scalar mass which is displayed in Figure 1 by a solid green line. In particular, we find a lower bound $\omega_{\text{BD}} > 1250$ for $m_s < 10^{-20}$ eV. The limiting factor here is our ability to obtain a precise value for the intrinsic period derivative, once Doppler effects have been accounted for.

d. PSR J0751+1807

PSR J0751+1807 is a millisecond pulsar in a 6hr circular binary system with a helium WD companion [48]. The period derivative has been measured to $\sim 15\%$ precision, after kinematic corrections have been made. However, the determination of the masses of the stars in this system has proved to be more of an issue. Assuming GR to be true, Nice et al. [49] used combined observations of the Shapiro delay and orbital period derivative to constrain the masses of the component stars to a precision of $\sim 10\%$. Unfortunately, in the context of using the measured period derivative to constrain modified theories of gravity, we cannot assume GR in the calculation of the masses. The solution to this issue is to use only the observations of the Shapiro delay to constrain the masses, and use these masses in conjunction with the observed \dot{P} to compare theory with predictions. The problem with this is that using the Shapiro delay alone provides a very weak constraint on the masses, with $\sim 100\%$ uncertainty for each of the two components. Nonetheless, we could perform an analysis similar to that done for PSR J1012+5307. Given the large uncertainties associated with this system, however, we expect that the bounds obtained from such an analysis would be very weak and would not provide us with any further insight, and for this reason we have neglected this system.

TABLE II. Parameters relevant to the binary system PSR J1141-6545 [50].

Period, P (days)	0.1976509593(1)
Period derivative (observed), \dot{P}^{obs}	$-4.03(25) 10^{-13}$
Period derivative (intrinsic), \dot{P}^{intr}	$-4.01(25) 10^{-13}$
Mass ratio, q	1.245(14)
NS Mass, m_1 (M_\odot)	1.27(1)
WD Mass, m_2 (M_\odot)	1.02(1)
Eccentricity, e	0.171884(2)
Periastron advance, $\dot{\omega}$ ($^\circ \text{yr}^{-1}$)	5.3096(4)

e. PSR J1141-6545

PSR J1141-6545 is a 394ms pulsar in a moderately eccentric binary system with a WD companion [51]. The relevant parameters for this system are displayed in Table II, and are taken directly from [50]. The masses of the WD and NS were determined by [50].

This system is comfortably the most useful in the context of putting bounds on $(\omega_{\text{BD}}, m_s)$, and in constraining alternative theories of gravity using observations of the orbital period derivative in general. \dot{P} has been measured to remarkable precision, currently $\sim 6\%$, and this is expected to improve further to $\sim 2\%$ by 2012 [50]. The other necessary parameters for our purposes, the masses and the periastron shift, have also been measured to excellent precision, so the total uncertainty in the system is (relatively) very small. Unfortunately this system does not have negligible eccentricity, so the result for \dot{P} derived here does not strictly hold. In order to do a full and accurate analysis of this system, the result (1) must be generalized to cover eccentric binaries. For the moment we present a rather crude analysis of this system where we neglect the eccentricity, to find at least a ball park estimate of the bounds that we may expect to obtain once a full analysis is performed. Using the above parameters and equation (98), the value of the period derivative predicted by GR is given by

$$\dot{P}_{GR} = -\frac{4q}{(1+q)^2} \frac{8}{15\sqrt{3}} \left(\frac{P}{2\pi}\right)^{\frac{3}{2}} \dot{\omega}^{\frac{5}{2}} = -3.440(3) 10^{-13}. \quad (\text{C2})$$

Using the method described in section VII A, we obtain the bound on ξ (and hence ω_{BD}) displayed in Figure 1 by a solid blue line. Once we account for eccentricity in a proper way, this system is very likely to provide the most stringent bounds among all of the binaries observed so far.

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