Casimir densities for a spherical boundary in de Sitter spacetime

K. A. Milton and A. A. Saharian

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K. A. Milton[*]

Homer L. Dodge Department of Physics and Astronomy,
University of Oklahoma, Norman, Oklahoma 73019-2061, USA

A. A. Saharian[†]

Department of Physics, Yerevan State University,
1 Alex Manoogian Street, 0025 Yerevan, Armenia

Two-point functions, mean-squared fluctuations, and the vacuum expectation value of the energy-momentum tensor operator are investigated for a massive scalar field with an arbitrary curvature coupling parameter, subject to a spherical boundary in the background of de Sitter spacetime. The field is prepared in the Bunch-Davies vacuum state and is constrained to satisfy Robin boundary conditions on the sphere. Both the interior and exterior regions are considered. For the calculation in the interior region, a mode-summation method is employed, supplemented with a variant of the generalized Abel-Plana formula. This allows us to explicitly extract the contributions to the expectation values which come from de Sitter spacetime without boundaries. We show that the vacuum energy-momentum tensor is non-diagonal with the off-diagonal component corresponding to the energy flux along the radial direction. With dependence on the boundary condition and the mass of the field, this flux can be either positive or negative. Several limiting cases of interest are then studied. In terms of the curvature coupling parameter and the mass of the field, two very different regimes are realized, which exhibit monotonic and oscillatory behavior of the vacuum expectation values, respectively, far from the sphere. The decay of the boundary-induced expectation values at large distances from the sphere is shown to be power-law (monotonic or oscillating), independent of the value of the field mass. The expressions for the Casimir densities in the exterior region are generalized for a more general class of spherically-symmetric spacetimes inside the sphere.

I. INTRODUCTION

De Sitter (dS) spacetime is one of the simplest and most interesting spacetimes allowed by general relativity. Quantum field theory in this background has been extensively studied during the past two decades. Much of the early interest was motivated by the questions related to the quantization of fields on curved backgrounds. dS spacetime has a high degree of symmetry and numerous physical problems are exactly solvable on this background. The importance of this theoretical work increased by the appearance of the inflationary cosmology scenario [1]. In most inflationary models, an approximately dS spacetime is employed to solve a number of problems in standard cosmology. During an inflationary epoch, quantum fluctuations in the inflaton field introduce inhomogeneities which play a central role in the generation of cosmic structures from inflation. More recently, astronomical observations of high redshift supernovae, galaxy clusters, and cosmic microwave background [2] indicate that at the present epoch the universe is accelerating and can be well approximated by a world with a positive cosmological constant. If the universe would ac-celerate indefinitely, the standard cosmology would lead to an asymptotic dS universe. Hence, the investigation of physical effects in dS spacetime is important for understanding both the early universe and its future. Another motivation for investigations of dS-based quantum theories is related to the holographic duality between quantum gravity on dS spacetime and a quantum field theory living on a boundary identified with the timelike infinity of dS spacetime [3].

In dS spacetime, the interaction of fluctuating quantum fields with the background gravitational field gives rise to vacuum polarization. The Casimir effect presents another type of vacuum polarization induced by the presence of boundaries. This effect is among the most striking macroscopic manifestations of non-trivial properties of the quantum vacuum. It has important implications on all scales, from subnuclear to cosmological. The reflecting boundaries alter the zero-point modes of a quantized field and shift the vacuum expectation values of quantities, such as the energy density and stresses. As a result, forces arise acting on constraining boundaries. The particular features of these forces depend on the nature of the quantum field, the type of spacetime manifold, the boundary geometry, and the specific boundary conditions imposed on the field. Since the original work by Casimir many theoretical and experimental works have been done on this problem (see, e.g., Ref. [4] and references therein).

In the present paper we consider a problem with both

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[*]Electronic address: milton@nhn.ou.edu
[†]Electronic address: saharian@ysu.am

types of vacuum polarization. Namely, we evaluate the vacuum expectation values for the field squared and the energy-momentum tensor of a scalar field with general curvature coupling parameter induced by a spherical boundary on the background of $(D + 1)$-dimensional dS spacetime. Historically, the investigation of the Casimir effect for a spherical shell was motivated by the Casimir semiclassical model of an electron. In this model Casimir suggested [5] that Poincaré stress, to stabilize the charged particle, could arise from vacuum quantum fluctuations and the fine structure constant could be determined by a balance between the Casimir force (assumed attractive) and the Coulomb repulsion. However, as has been shown by Boyer [6], the Casimir energy for a perfectly conducting sphere is positive, implying a repulsive force. This result was later reconsidered by a number of authors [7]. More recently new methods have been developed for the investigation of the Casimir effect including direct mode summation techniques and the zeta function regularization scheme [8], semiclassical methods [9] in the framework of the Gutzwiller trace formula, the optical approach [10], worldline numerics [11], the path integral approach [12], methods based on scattering theory [13], numerical methods based on evaluation of the stress tensor via the fluctuation-dissipation theorem [14] (for a review see Refs. [4, 15]). The Casimir effect for a spherical shell in an arbitrary number of dimensions is analyzed in Refs. [16, 17] for a massless scalar field satisfying Dirichlet and a special type of Robin (corresponding to the electromagnetic TM modes) boundary conditions using the Green's function method and in Refs. [18, 19] for the electromagnetic field and massless scalar and spinor fields with various boundary conditions on the basis of the zeta regularization technique. The case for a massive vector field has been discussed in Ref. [20].

The investigation of the energy distribution inside a perfectly reflecting spherical shell was made in Ref. [21]. The distribution of the other components for the energy-momentum tensor of the electromagnetic field inside and outside a shell and in the region between two concentric spherical shells is studied in Refs. [22, 23] (see also Ref. [24]). The investigation of the electromagnetic energy density near a conducting boundary was first carried out by DeWitt [25]. The vacuum expectation values for the energy-momentum tensor of a massive scalar field with general curvature coupling parameter and obeying the Robin boundary condition on spherically symmetric boundaries in $(D + 1)$-dimensional spacetime are investigated in Ref. [26]. The Casimir densities for spherical boundaries in the background of global monopole and Rindler-like spacetimes have been discussed in Refs. [27] and [28] respectively.

Previously, the Casimir stresses for spherical boundaries on the background of dS spacetime have been investigated in Ref. [29] for a conformally coupled massless scalar field. In this last case the problem is conformally related to the corresponding problem in Minkowski spacetime and the vacuum characteristics are generated from those for the Minkowski counterpart, just by multiplying with the conformal factor. As it has been shown in Refs. [30] for the geometry of flat boundaries, qualitatively new features arise in the case of non-conformally coupled fields (for the case of a minimally coupled massless field see Ref. [31]). (For flat backgrounds, see also Ref. [32].) The curvature of the background spacetime decisively influences the behavior of boundary-induced vacuum expectation values at distances larger than the curvature scale. Recently, the topological Casimir effect in dS spacetime with toroidally compactified spatial dimensions has been investigated in Ref. [33].

We have organized the paper as follows. In the next section the Wightman function is evaluated inside a spherical boundary in dS spacetime for a scalar field with general curvature coupling parameter and with Robin boundary condition on the sphere. Among the most important quantities describing the local properties of a quantum field and the corresponding quantum back-reaction effects are the expectation values of the field squared and of the energy-momentum tensor. These quantities in the interior of a sphere will be investigated in Secs. III and IV. The Wightman function for the region outside a spherical boundary is considered in Sec. V. The vacuum expectation values of the field squared and the energy-momentum tensor in this region are discussed in Sec. VI. Sec. VII generalizes the results for the exterior region to a more general class of spherically-symmetric spacetimes inside a spherical boundary. Section VIII contains a summary of the work. In Appendix A the expression for the Wightman function in boundary-free dS spacetime is derived by making use the corresponding mode sum. In Appendix B, for the Wightman function, we explicitly demonstrate the limit to the geometry of a spherical boundary in Minkowski spacetime. Appendix C sketches the corresponding calculation of the Green's function for this problem.

II. WIGHTMAN FUNCTION INSIDE A SPHERE

We consider a quantum scalar field $\varphi(x)$ on a $(D + 1)$-dimensional dS spacetime background described in inflationary coordinates. The latter are most appropriate for cosmological applications. The spatial part of the line element we will write in terms of the hyperspherical coordinates $(r, \vartheta, \phi) \equiv (r, \theta_1, \theta_2, \ldots \theta_n, \phi)$, $n = D - 2$:

$$ds^2 = dt^2 - e^{2t/\alpha}(dr^2 + r^2 d\Omega_{D-1}^2), \qquad (2.1)$$

where $d\Omega_{D-1}^2$ is the line element on a $(D-1)$-dimensional sphere with unit radius. In what follows, in addition to the synchronous time coordinate, t, we will also use the conformal time, τ, defined as $\tau = -\alpha e^{-t/\alpha}$, $-\infty < \tau < 0$. In terms of this coordinate the line element takes the conformally flat form:

$$ds^2 = \alpha^2 \tau^{-2} \left(d\tau^2 - dr^2 - r^2 d\Omega_{D-1}^2 \right). \qquad (2.2)$$

Note that the parameter α is related to the cosmological constant Λ through the expression $\alpha = D(D-1)/(2\Lambda)$.

The dynamics of a massive scalar field are governed by the equation [34]

$$(\nabla_l \nabla^l + m^2 + \xi R)\varphi = 0, \qquad (2.3)$$

where ∇_l is the covariant derivative operator, $R = D(D+1)/\alpha^2$ is the Ricci scalar for dS spacetime, and ξ is the curvature coupling parameter. The special values of this parameter $\xi = 0$ and $\xi = \xi_D \equiv (D-1)/4D$ correspond to minimally and to conformally coupled fields. The importance of these two special cases comes from the fact that, in the massless limit, the corresponding fields mimic the behavior of gravitons and photons, respectively. Our main interest in this paper is the study of the changes in the vacuum expectation values (VEVs) of the field squared and the energy-momentum tensor induced by a spherical shell with radius a centered at the origin in dS spacetime. We assume that on the sphere the field obeys Robin boundary condition

$$(\tilde{A} + \tilde{B}\partial_r)\varphi(x) = 0, \quad r = a, \qquad (2.4)$$

with constant coefficients $\tilde{A}$ and $\tilde{B}$, in general, different for the inner and outer regions. The results for Dirichlet and Neumann boundary conditions are obtained as special cases. Robin boundary conditions are an extension of the ones imposed on perfectly conducting boundaries and may, in some geometries, be useful for depicting the finite penetration of the field into the boundary with the "skin-depth" parameter related to the Robin coefficient [35]. These types of conditions naturally arise for scalar and fermion bulk fields in braneworld models.

As the first step in the investigation of the VEVs we will evaluate the Wightman function $W(x,x') = \langle 0|\varphi(x)\varphi(x')|0\rangle$, where $|0\rangle$ stands for the vacuum state (for the Wightman function in the geometry of spherical boundaries in the background of a constant negative curvature space see Ref. [36]). In order to do that we employ the mode sum formula

$$W(x,x') = \sum_\sigma \varphi_\sigma(x)\varphi_\sigma^*(x'), \qquad (2.5)$$

with $\{\varphi_\sigma(x), \varphi_\sigma^*(x)\}$ being a complete set of solutions to the classical field equation, specified by a set of quantum numbers σ, satisfying the boundary condition (2.4). In accordance with the spherical symmetry of the problem under consideration, the angular dependence of the mode functions is given by the spherical harmonic of degree l (see Ref. [37]), $Y(m_p; \vartheta, \phi)$, where $m_p = (m_0 \equiv l, m_1, \ldots, m_n)$, $l = 0, 1, 2, \ldots$, and $m_1, m_2, \ldots, m_n$ are integers such that

$$0 \leqslant m_{n-1} \leqslant m_{n-2} \leqslant \cdots \leqslant m_1 \leqslant l,$$
$$-m_{n-1} \leqslant m_n \leqslant m_{n-1}. \qquad (2.6)$$

Presenting the mode functions in the form $\varphi_\sigma(x) = T(\tau)\mathcal{R}(r)Y(m_p; \vartheta, \phi)$, from the field equation (2.3) it follows that the time and the radial coordinate dependences

are given in terms of cylinder functions as:

$$\mathcal{R}(r) = r^{1-D/2}[b_1 J_\mu(\lambda r) + b_2 Y_\mu(\lambda r)],$$
$$T(\tau) = \eta^{D/2} \sum_{j=1,2} c_j H_\nu^{(j)}(\lambda \eta), \ \eta = |\tau|, \qquad (2.7)$$

where $J_\mu(z)$ and $Y_\mu(z)$ are the Bessel and Neumann functions respectively, $H_\nu^{(j)}(z)$, $j = 1, 2$, are the Hankel functions (we use the notations from Ref. [38]). The orders of the cylinder functions in Eq. (2.7) are defined as:

$$\mu = l + D/2 - 1,$$
$$\nu = \left[D^2/4 - D(D+1)\xi - m^2\alpha^2\right]^{1/2}. \qquad (2.8)$$

Note that ν is either real and nonnegative or purely imaginary. For a conformally coupled massless field $\nu = 1/2$ and the Hankel functions in Eq. (2.7) are expressed in terms of elementary functions.

Different choices of the coefficients c_j in the expression for the function $T(\tau)$ correspond to different choices of the vacuum state in dS spacetime. The choice of the vacuum state is among the most important steps in construction of a quantum field theory in a fixed classical gravitational background. dS spacetime is a maximally symmetric space and it is natural to choose a vacuum state having the same symmetry. In fact, there exists a one-parameter family of maximally symmetric quantum states (see, for instance, Ref. [39] and references therein). Here we will assume that the field is prepared in the dS-invariant Bunch-Davies vacuum state [40] for which $c_2 = 0$. Among the set of dS-invariant quantum states the Bunch-Davies vacuum is the only one for which the ultraviolet behavior of the two-point functions is the same as in Minkowski spacetime.

First we consider the region inside the spherical shell. From the regularity condition at the origin it follows that for this region $b_2 = 0$ and the mode functions realizing the Bunch-Davies vacuum state are written in the form

$$\varphi_\sigma(x) = C_\sigma \frac{\eta^{D/2}}{r^{D/2-1}} H_\nu^{(1)}(\lambda \eta) J_\mu(\lambda r) Y(m_p; \vartheta, \phi). \quad (2.9)$$

From the boundary condition (2.4), it follows that the eigenvalues for λ have to be solutions to the equation

$$A J_\mu(\lambda a) + B \lambda a J_\mu'(\lambda a) = 0, \qquad (2.10)$$

where the prime means the derivative with respect to the argument of the function. In Eq. (2.10) and in what follows we use the notations

$$A = \tilde{A} + (1 - D/2)\tilde{B}/a, \quad B = \tilde{B}/a. \qquad (2.11)$$

For real A, B and $\mu > -1$, all roots of Eq. (2.10) are simple and real, except the case $A/B < -\mu$ when there are two purely imaginary zeros (see, e.g., Ref. [41]). We will denote by $z = \lambda_{\mu,k}$, $k = 1, 2, \ldots$, the zeros of the function $A J_\mu(z) + B z J_\mu'(z)$ in the right half-plane, assuming that they are arranged in ascending order. So,

for the eigenvalues of λ one has $\lambda = \lambda_{\mu,k}/a$. Now we see that the set of quantum numbers σ is specified to $\sigma = (k, l, m_1, \ldots, m_n)$. Note that for Dirichlet boundary condition $A = 1$, $B = 0$, and for Neumann boundary condition $A = (1 - D/2)B$.

The coefficient C_σ in Eq. (2.9) is determined from the orthonormalization condition [34]

$$-i \int d^D x \sqrt{|g|} g^{00} \varphi_\sigma(x) \overleftrightarrow{\partial}_\tau \varphi_{\sigma'}^*(x) = \delta_{\sigma\sigma'}, \qquad (2.12)$$

where the integration goes over the region inside the sphere and $\delta_{\sigma\sigma'}$ is the Kronecker delta. Substituting the functions from Eq. (2.9), the integration over the angular variables is performed by making use of the normalization integral for the spherical harmonics:

$$\int |Y(m_k; \vartheta, \phi)|^2 \, d\Omega = N(m_k), \qquad (2.13)$$

where the explicit form for $N(m_k)$ can be found in Ref. [37]. By using also the relation (for ν purely real or imaginary)

$$H_{\nu^*}^{(2)}(x) H_\nu^{(1)\prime}(x) - H_\nu^{(1)}(x) H_{\nu^*}^{(2)\prime}(x) = \frac{4i}{\pi x} e^{i(\nu^* - \nu)\pi/2}, \qquad (2.14)$$

for the normalization coefficient we get

$$C_\sigma^2 = \frac{\pi \lambda T_\mu(\lambda a) e^{i(\nu - \nu^*)\pi/2}}{2N(m_k) a \alpha^{D-1}}, \qquad (2.15)$$

with the notation

$$T_\mu(z) = \frac{z}{(z^2 - \mu^2) J_\mu^2(z) + z^2 J_\mu'^2(z)}. \qquad (2.16)$$

Substituting the eigenfunctions into the mode sum formula (2.5), for the Wightman function one gets ($n = D - 2$)

$$W(x, x') = \frac{\pi e^{i(\nu - \nu^*)\pi/2}}{2\alpha^{D-1} n S_D a^2} \frac{(\eta\eta')^{D/2}}{(rr')^{n/2}} \sum_{l=0}^{\infty} (2l + n)$$
$$\times C_l^{n/2}(\cos\theta) \sum_{k=1}^{\infty} z T_\mu(z) H_\nu^{(1)}(z\eta/a) H_{\nu^*}^{(2)}(z\eta'/a)$$
$$\times J_\mu(zr/a) J_\mu(zr'/a)\big|_{z=\lambda_{\mu,k}}. \qquad (2.17)$$

Here $C_p^q(x)$ is the Gegenbauer or ultraspherical polynomial of degree p and order q, $S_D = 2\pi^{D/2}/\Gamma(D/2)$ is the surface area of a unit sphere in D-dimensional space, and θ is the angle between the directions (ϑ, ϕ) and (ϑ', ϕ'). In deriving Eq. (2.17), we have used the addition theorem [37] for the spherical harmonics:

$$\sum_{m_p} \frac{Y(m_p; \vartheta, \phi)}{N(m_p)} Y^*(m_p; \vartheta', \phi') = \frac{2l + n}{n S_D} C_l^{n/2}(\cos\theta), \qquad (2.18)$$

where the sum is taken over the integer values m_p, $p = 1, 2, \ldots, n$, in accordance with Eq. (2.6).

Before proceeding to the evaluation of the Wightman function, we comment about the realizability of the Bunch-Davies vacuum state. It is well known that in dS spacetime without boundaries the Bunch-Davies vacuum state is not a physically realizable state for $\operatorname{Re}\nu \geqslant D/2$. The corresponding Wightman function contains infrared divergences arising from long-wavelength modes. As it has been shown in Ref. [42], these divergences lead to inconsistencies with Einstein equations and they cannot arise through dynamical evolution from a state which is initially free of such divergences. In the presence of boundaries, the boundary condition imposed on the quantized field may exclude these modes and the Bunch-Davies vacuum becomes a realizable state. An example is provided by a spherical boundary described above. In the region inside the spheres and for boundary conditions with $\bar{A} \neq 0$, there is a minimum value for λ, $\lambda \geqslant \lambda_{n/2,1}/a$, and the two-point function (2.17) contains no infrared divergences.

The eigenvalues $\lambda_{\mu,k}$ are given implicitly and the Wightman function in the form (2.17) is not convenient for the further evaluation of the VEVs. In order to sum over these eigenvalues we use the summation formula [43]

$$2 \sum_{k=1}^{\infty} T_\mu(\lambda_{\mu,k}) f(\lambda_{\mu,k}) = \int_0^\infty f(x) dx$$
$$+ \frac{\pi}{2} \operatorname{Res}_{z=0} f(z) \frac{\bar{Y}_\mu(z)}{\bar{J}_\mu(z)} - \frac{1}{\pi} \int_0^\infty dx \frac{\bar{K}_\mu(x)}{\bar{I}_\mu(x)}$$
$$\times [e^{-\mu\pi i} f(x e^{\pi i/2}) + e^{\mu\pi i} f(x e^{-\pi i/2})], \qquad (2.19)$$

where $f(z)$ is an analytic function on the right half-plane and, for a given function $F(z)$, the barred notation is defined as:

$$\bar{F}(z) \equiv AF(z) + BzF'(z). \qquad (2.20)$$

Formula (2.19) can be generalized for the case of the existence of purely imaginary zeros of the function $\bar{J}_\nu(z)$ by adding the corresponding residue term and taking the principal value of the integral on the right (see Ref. [43]). In what follows we assume values of A/B for which all roots $\lambda_{\mu,k}$ are real.

As a function $f(z)$ in Eq. (2.19) we take

$$f(z) = z H_\nu^{(1)}(z\eta/a) H_{\nu^*}^{(2)}(z\eta'/a) J_\mu(zr/a) J_\mu(zr'/a). \qquad (2.21)$$

By making use of the properties of the cylinder functions, after the application of Eq. (2.19), the Wightman function is presented in the decomposed form:

$$W(x, x') = W_{\mathrm{dS}}(x, x') + W_{\mathrm{b}}(x, x'). \qquad (2.22)$$

Here, the first term in the right-hand side corresponds to

the first integral on the right of Eq. (2.19),

$$W_{\mathrm{dS}}(x,x') = \frac{\pi e^{i(\nu-\nu^*)\pi/2}}{4\alpha^{D-1}nS_D} \frac{(\eta\eta')^{D/2}}{(rr')^{n/2}} \sum_{l=0}^{\infty} (2l+n)$$

$$\times C_l^{n/2}(\cos\theta) \int_0^{\infty} d\lambda\, \lambda H_\nu^{(1)}(\lambda\eta) H_{\nu^*}^{(2)}(\lambda\eta')$$

$$\times J_\mu(\lambda r) J_\mu(\lambda r'), \qquad (2.23)$$

and

$$W_{\mathrm{b}}(x,x') = -\frac{\alpha^{1-D}}{\pi n S_D} \frac{(\eta\eta')^{D/2}}{(rr')^{n/2}} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos\theta)$$

$$\times \int_0^{\infty} dz\, z \frac{\bar{K}_\mu(za)}{\bar{I}_\mu(za)} I_\mu(zr) I_\mu(zr')$$

$$\times [I_\nu(z\eta')K_\nu(z\eta) + I_{-\nu}(z\eta)K_\nu(z\eta')]. \qquad (2.24)$$

The term (2.23) does not depend on the radius of the sphere whereas the second term on the right-hand side vanishes in the limit $a \to \infty$. From here it follows that $W_{\mathrm{dS}}(x,x')$ is the Wightman function for a scalar field in boundary-free dS spacetime. In Appendix A we show this by direct evaluation. The second term in the right-hand side of Eq. (2.22) is induced by the presence of the spherical shell. In the limit $\alpha \to \infty$ with fixed t, the line element (2.1) goes to the Minkowskian line element in spherical coordinates. In Appendix B it is shown that in this limit, from Eq. (2.24), the Wightman function is obtained for a scalar field inside a spherical boundary in Minkowski spacetime. In Appendix C we derive the corresponding causal Green's function.

In a similar way we can evaluate the Wightman function in a general state described by the modes (2.7) without the specification of the coefficients in the linear combination of the Hankel functions. After the normalization of the corresponding mode functions, we obtain a set of quantum states determined by the ratio c_2/c_1. In general, the latter may depend on l and λ. After the application of the generalized Abel-Plana formula, the boundary-induced part in the corresponding Wigthman function is presented in a form similar to Eq. (2.24) where now, instead of the combination of functions in the square brackets, a more general bilinear combination of the modified Bessel functions appears.

III. VEV OF THE FIELD SQUARED INSIDE A SPHERE

Given the Wightman function, we can proceed to the evaluation of the VEV of the field squared. This VEV is among the most important quantities in discussing the phase transitions in the early universe and the generation of the cosmic structures from inflation. The VEV of the field squared is obtained taking the coincidence limit of the arguments. In this limit the Wightman function is divergent and some renormalization procedure is needed. The important point here is that for points away from the

sphere the divergences are the same as those for dS spacetime without boundaries. As we have already extracted the part $W_{\mathrm{dS}}(x,x')$, the renormalization is reduced to the renormalization of the VEV in the boundary-free dS spacetime, which is already done in the literature. In this way, the renormalized VEV of the field squared inside a sphere is presented in the decomposed form

$$\langle\varphi^2\rangle = \langle\varphi^2\rangle_{\mathrm{dS}} + \langle\varphi^2\rangle_{\mathrm{b}}, \qquad (3.1)$$

where $\langle\varphi^2\rangle_{\mathrm{dS}}$ is the VEV in the boundary-free dS spacetime and the term $\langle\varphi^2\rangle_{\mathrm{b}}$ is induced by the sphere. Due to the maximal symmetry of the Bunch-Davies vacuum state the VEV $\langle\varphi^2\rangle_{\mathrm{dS}}$ does not depend on the spacetime point.

The boundary-induced part is directly obtained from the corresponding part in the Wightman function, given by Eq. (2.24). By taking into account that

$$C_l^{n/2}(1) = \frac{\Gamma(l+n)}{\Gamma(n)l!}, \qquad (3.2)$$

we get

$$\langle\varphi^2\rangle_{\mathrm{b}} = -\frac{\eta^D\alpha^{1-D}}{\pi S_D r^{D-2}} \sum_{l=0}^{\infty} D_l \int_0^{\infty} dx\, x \frac{\bar{K}_\mu(ax)}{\bar{I}_\mu(ax)} I_\mu^2(rx)$$

$$\times K_\nu(x\eta)[I_\nu(x\eta) + I_{-\nu}(x\eta)]. \qquad (3.3)$$

In Eq. (3.3),

$$D_l = (2l+D-2)\frac{\Gamma(l+D-2)}{\Gamma(D-1)\,l!} \qquad (3.4)$$

is the degeneracy of the angular mode with given l. The integral representation (3.3) is valid for $\mathrm{Re}\,\nu < 1$. For large values of x, the integrand in Eq. (3.3) behaves as $e^{-(a-r)x}$. The presence of the spherical boundary breaks the dS-invariance and the mean-squared fluctuation of the field depends on time. As is seen from Eq. (3.3), this dependence appears through the ratios a/η and r/η. The latter property is a consequence of the maximal symmetry of the Bunch-Davies vacuum in the absence of the sphere. Note that a/η and r/η are the proper radius of the sphere and the proper distance from the sphere center measured in units of the dS curvature scale α. The influence of the gravitational field on boundary-induced quantum effects appears through the function $K_\nu(z)[I_\nu(z) + I_{-\nu}(z)]$. The latter is a monotonically decreasing function of z for $\nu^2 \geqslant 0$ and exhibits an oscillatory behavior for $\nu^2 < 0$ and $z \lesssim |\nu|$. As it will be seen below, this feature leads to interesting physical consequences.

For a conformally coupled massless scalar field, $\xi = \xi_D$, $m = 0$, one has $\nu = 1/2$ and

$$K_{1/2}(x)\left[I_{1/2}(x) + I_{-1/2}(x)\right] = 1/x. \qquad (3.5)$$

In this case, from Eq. (3.3), we find

$$\langle\varphi^2\rangle_{\mathrm{b}} = (\eta/\alpha)^{D-1}\langle\varphi^2\rangle_{\mathrm{M}}, \qquad (3.6)$$

where

$$\langle\varphi^2\rangle_{\mathrm{M}} = -\frac{r^{2-D}}{\pi S_D}\sum_{l=0}^{\infty}D_l\int_0^{\infty}dx\,\frac{\bar{K}_\mu(ax)}{\bar{I}_\mu(ax)}I_\mu^2(rx),\quad (3.7)$$

is the corresponding VEV inside a spherical boundary in Minkowski spacetime [26]. Of course, this result could be obtained directly by using the conformal relation between the problems in dS and Minkowski spacetimes.

Now let us consider the asymptotics of the boundary-induced VEV. Near the center of the sphere the dominant contribution comes from the mode with the lowest orbital momentum $l = 0$ and in the leading order we find

$$\langle\varphi^2\rangle_{\mathrm{b}} \approx -\frac{(2\alpha)^{1-D}}{\pi^{D/2+1}\Gamma(D/2)}\int_0^{\infty}dx\,x^{D-1}\frac{\bar{K}_{D/2-1}(ax/\eta)}{\bar{I}_{D/2-1}(ax/\eta)}$$
$$\times K_\nu(x)\left[I_\nu(x) + I_{-\nu}(x)\right].\qquad (3.8)$$

The behavior of the field squared for large values of the sphere proper radius, $a/\eta \gg 1$, assuming that r/η is fixed, is obtained from Eq. (3.8) expanding the function $K_\nu(x)\left[I_\nu(x) + I_{-\nu}(x)\right]$ for small values of the argument. For $x \ll 1$ in the leading order one has

$$K_\nu(x)\left[I_\nu(x) + I_{-\nu}(x)\right] \approx \sigma_\nu\mathrm{Re}\left[\frac{2^{2\nu-1}\Gamma(\nu)}{\Gamma(1-\nu)x^{2\nu}}\right],\quad (3.9)$$

where $\sigma_\nu = 1$ for positive ν and $\sigma_\nu = 2$ for imaginary ν. The case $\nu = 0$ should be considered separately. The behavior of the VEV is qualitatively different for these two cases. For positive ν, one gets

$$\langle\varphi^2\rangle_{\mathrm{b}} \approx -\frac{(2\alpha)^{1-D}}{\pi^{D/2+1}}\frac{(\eta/a)^{D-2\nu}}{\Gamma(D/2)}b(\nu),\qquad (3.10)$$

with the notation

$$b(\nu) = \frac{2^{2\nu-1}\Gamma(\nu)}{\Gamma(1-\nu)}\int_0^{\infty}dx\,x^{D-1-2\nu}\frac{\bar{K}_{D/2-1}(x)}{\bar{I}_{D/2-1}(x)}.\quad (3.11)$$

In this case the boundary-induced VEV monotonically decreases with increasing proper radius of the sphere. For imaginary values of ν, the leading term is in the form

$$\langle\varphi^2\rangle_{\mathrm{b}} \approx -\frac{2(2\alpha)^{1-D}}{\pi^{D/2+1}}\frac{(\eta/a)^D}{\Gamma(D/2)}c(\nu)\cos[2|\nu|\ln(a/\eta) + \phi(\nu)],$$
$$(3.12)$$

where $c(\nu)$ and $\phi(\nu)$ are defined by the relation $b(\nu) = c(\nu)e^{i\phi(\nu)}$. In this case the VEV exhibits a damped oscillatory behavior.

The VEV of the field squared diverges on the boundary. Surface divergences are well known in quantum field theory with boundaries and they have been investigated for various geometries. Near the sphere the dominant contribution in Eq. (3.3) comes from large values of x and l. By taking into account that for large x and for fixed ν one has $K_\nu(x)\left[I_\nu(x) + I_{-\nu}(x)\right] \approx 1/x$, we conclude that near the boundary, to the leading order, one

has $\langle\varphi^2\rangle_{\mathrm{b}} \approx (\eta/\alpha)^{D-1}\langle\varphi^2\rangle_{\mathrm{M}}$. By using the asymptotic expression for $\langle\varphi^2\rangle_{\mathrm{M}}$, we get

$$\langle\varphi^2\rangle_{\mathrm{b}} \approx -\frac{(\eta/\alpha)^{D-1}\Gamma((D-1)/2)}{(4\pi)^{(D+1)/2}(a-r)^{D-1}}\kappa_B,\qquad (3.13)$$

where

$$\kappa_B = 2\delta_{0B} - 1.\qquad (3.14)$$

In deriving Eq. (3.13) we have assumed that $(a - r) \ll \eta/(m\alpha)$ which corresponds to small proper distances from the sphere compared with the Compton wavelength of the scalar particle. As the boundary-free part in the VEV is constant everywhere, we conclude that near the sphere the total VEV is dominated by the boundary-induced part. The factor κ_B in Eq. (3.13) indicates that the sign of the surface divergence is reversed for Dirichlet as opposed to any other Robin boundary condition.

IV. VACUUM ENERGY-MOMENTUM TENSOR INSIDE A SPHERE

Another important quantity characterizing the properties of the quantum vacuum is the VEV of the energy-momentum tensor. In addition to describing the physical structure of the quantum field at a given point, the energy-momentum tensor acts as the source of gravity in the Einstein equations. It therefore plays an important role in modelling a self-consistent dynamics involving the gravitational field. (For the renormalization of the Einstein equations in the presence of energy-momentum tensor divergences, see Ref. [44]; the demonstration that the finite and divergent parts of the Casimir energy obey the equivalence principle appears in Refs. [45, 46].) Similar to the field squared, the VEV of the energy-momentum tensor is decomposed as

$$\langle T_{ik}\rangle = \langle T_{ik}\rangle_{\mathrm{dS}} + \langle T_{ik}\rangle_{\mathrm{b}},\qquad (4.1)$$

where the boundary-free part is presented in the form $\langle T_{ik}\rangle_{\mathrm{dS}} = \mathrm{const}\cdot g_{ik}$. The latter property is a direct consequence of the maximal symmetry of the Bunch-Davies vacuum state. In particular, for $D = 3$ dS spacetime the renormalized boundary-free part is given by the expression [40, 47] (see also Ref. [34])

$$\langle T_{ik}\rangle_{\mathrm{dS}} = \frac{g_{ik}}{32\pi^2\alpha^4}\{m^2\alpha^2(m^2\alpha^2/2 + 6\xi - 1)$$
$$\times[\psi(3/2 + \nu) + \psi(3/2 - \nu) - \ln(m^2\alpha^2)]$$
$$-(6\xi - 1)^2 + 1/30 + (2/3 - 6\xi)m^2\alpha^2\},\ (4.2)$$

where $\psi(x)$ is the logarithmic derivative of the gamma-function. For $m\alpha \gg 1$, to the leading order, $\langle T_{ik}\rangle_{\mathrm{dS}} \approx Cg_{ik}/(384\pi^2m^2\alpha^6)$, where the coefficient C depends on the curvature coupling parameter only. For minimally and conformally coupled fields we have $C = 7$ and $C = -1/5$ respectively.

For the evaluation of the boundary-induced part in the VEV of the energy-momentum tensor, we use the formula

$$\langle T_{ik}\rangle_{\mathrm{b}} = \lim_{x'\to x}\partial_i\partial_k' W_{\mathrm{b}}(x,x') + [(\xi - 1/4)g_{ik}\nabla_l\nabla^l$$
$$-\xi\nabla_i\nabla_k - \xi R_{ik}]\langle\varphi^2\rangle_{\mathrm{b}}, \qquad (4.3)$$

where $R_{ik} = Dg_{ik}/\alpha^2$ is the Ricci tensor for dS spacetime. By taking into account Eqs. (2.24) and (3.3), after long but straightforward calculations, the VEVs for the diagonal components are presented in the form (no summation over i)

$$\langle T_i^i\rangle_{\mathrm{b}} = -\frac{r^{2-D}\eta^2}{2\pi S_D\alpha^{D+1}}\sum_{l=0}^{\infty} D_l\int_0^{\infty} dx\, x^{3-D}\frac{\bar{K}_\mu(ax)}{\bar{I}_\mu(ax)}$$
$$\times\left\{G_i^i[I_\mu(rx)]F_\nu(x\eta) + 2I_\mu^2(rx)F_i^i(x\eta)\right\}, \quad (4.4)$$

where we have introduced the notation

$$F_\nu(z) = z^D K_\nu(z)\left[I_\nu(z) + I_{-\nu}(z)\right]. \qquad (4.5)$$

Other functions in Eq. (4.4) are defined in accordance with (no summation over k)

$$F_0^0(y) = \left(\frac{1}{4}\partial_y^2 - D\frac{\xi + \xi_D}{y}\partial_y - 1\right.$$
$$\left. + \frac{D^2\xi + m^2\alpha^2}{y^2}\right)F_\nu(y),$$
$$F_k^k(y) = \left[\left(\xi - \frac{1}{4}\right)\partial_y^2 + (\xi(2 - D) + D\xi_D)\frac{1}{y}\partial_y\right.$$
$$\left. - \frac{\xi D}{y^2}\right]F_\nu(y), \qquad (4.6)$$

for $k = 1, 2, \ldots, D$, and

$$G_0^0[f(y)] = (1 - 4\xi)\left[f'^2(y) - \frac{D-2}{y}f(y)f'(y)\right.$$
$$\left. + \left(1 + \frac{\mu^2}{y^2}\right)f^2(y)\right],$$
$$G_1^1[f(y)] = f'^2(y) + \frac{\xi_1}{y}f(y)f'(y)$$
$$- \left[1 + \frac{(D/2 - 1)\xi_1 + \mu^2}{y^2}\right]f^2(y), \qquad (4.7)$$
$$G_k^k[f(y)] = (4\xi - 1)f'^2(y) - \frac{\xi_1}{y}f(y)f'(y)$$
$$+ \left[4\xi - 1 + \frac{(D/2 - 1)\xi_1 + \mu^2(1 + \xi_1)}{(D - 1)y^2}\right]f^2(y),$$

for $k = 2, \ldots, D$, where

$$\xi_1 = 4(D - 1)\xi - D + 2. \qquad (4.8)$$

In addition, the boundary-induced VEV has also nonzero off-diagonal component

$$\langle T_0^1\rangle_{\mathrm{b}} = \frac{\eta\alpha^{-D-1}}{\pi S_D r^{D-1}}\sum_{l=0}^{\infty} D_l\int_0^{\infty} dx\, x^{1-D}\frac{\bar{K}_\mu(ax)}{\bar{I}_\mu(ax)}I_\mu(rx)$$
$$\times[2rxI_\mu'(rx) - (D - 2)I_\mu(rx)]$$
$$\times[(\xi - 1/4)\eta\partial_\eta + \xi]F_\nu(x\eta), \qquad (4.9)$$

which corresponds to the energy flux along the radial direction (see below). Note that in the formulas above, the components of the energy-momentum tensor are written in the coordinates $(\tau, r, \vartheta, \phi)$. The components in the system (t, r, ϑ, ϕ) with the synchronous time coordinate will be denoted by $\langle T_{(\mathrm{s})i}^k\rangle_{\mathrm{b}}$. We have the relations (no summation over i)

$$\langle T_{(\mathrm{s})i}^i\rangle_{\mathrm{b}} = \langle T_i^i\rangle_{\mathrm{b}}, \ \langle T_{(\mathrm{s})0}^1\rangle_{\mathrm{b}} = (\eta/\alpha)\langle T_0^1\rangle_{\mathrm{b}}. \qquad (4.10)$$

As an additional check of the expressions for the VEV of the energy-momentum tensor, it can be seen that the boundary-induced parts fulfill the trace relation

$$\langle T_l^l\rangle_{\mathrm{b}} = \left[D(\xi - \xi_D)\nabla_l\nabla^l + m^2\right]\langle\varphi^2\rangle_{\mathrm{b}}. \qquad (4.11)$$

In particular, the boundary-induced part in the VEV of the energy-momentum tensor is traceless for a conformally coupled massless scalar field. The trace anomaly is contained in the boundary-free part only. In addition, the VEVs obey the covariant conservation equation $\nabla_k\langle T_i^k\rangle_{\mathrm{b}} = 0$. For the geometry of interest, this equation is reduced to two equations:

$$\left(\partial_\eta - \frac{D}{\eta}\right)\langle T_0^0\rangle_{\mathrm{b}} - \partial_r\langle T_0^1\rangle_{\mathrm{b}}$$
$$+ \frac{1}{\eta}[\langle T_1^1\rangle_{\mathrm{b}} + (D - 1)\langle T_2^2\rangle_{\mathrm{b}}] = 0,$$
$$\left(\partial_\eta - \frac{D+1}{\eta}\right)\langle T_1^0\rangle_{\mathrm{b}} - \left(\partial_r + \frac{D-1}{r}\right)\langle T_1^1\rangle_{\mathrm{b}}$$
$$+ \frac{D-1}{r}\langle T_2^2\rangle_{\mathrm{b}} = 0. \qquad (4.12)$$

For a conformally coupled massless field one has $\nu = 1/2$ and $F_\nu(z) = z^{D-1}$. With this it is easily seen that the off-diagonal component vanishes and $F_0^0(y) = -y^{D-1}$, $F_k^k(y) = 0$ for $k = 1, 2, \ldots, D$. As a result we find (no summation over i)

$$\langle T_i^i\rangle_{\mathrm{b}} = (\eta/\alpha)^{D+1}\langle T_i^i\rangle_{\mathrm{M}}, \qquad (4.13)$$

where $\langle T_i^i\rangle_{\mathrm{M}}$ is the corresponding VEV for a sphere in Minkowski spacetime [26]. Again, the result (4.13) directly follows from the conformal relation between the problems in dS and Minkowski spacetimes. The electromagnetic field is conformally invariant in 4-dimensional spacetimes. Hence, the electromagnetic Casimir densities for a conducting spherical shell on background of $D = 3$ dS spacetime are obtained from those in Minkowski spacetime by using Eq. (4.13).

Let us denote by $E_V^{(\mathrm{b})}$ the boundary-induced part of the vacuum energy in the spatial volume V with a boundary ∂V:

$$E_V^{(\mathrm{b})} = \int_V d^D x\sqrt{\gamma}\langle T_{(\mathrm{s})0}^0\rangle_{\mathrm{b}}, \qquad (4.14)$$

where γ is the determinant of the spatial metric tensor $\gamma_{\beta\delta}$ with $g_{ik} = (1, -\gamma_{\beta\delta})$ and with the Greek indices run-

ning over $1, 2, \ldots, D$. Now, from the covariant conservation equation $\nabla_k \langle T_{(s)i}^k \rangle_{\mathrm{b}} = 0$ with $i = 0$, it follows that

$$
\begin{aligned}
\partial_t E_V^{(\mathrm{b})} &= -\int_{\partial V} d^{D-1}x \sqrt{h}\, n_\beta \langle T_{(s)0}^\beta \rangle_{\mathrm{b}} \\
&\quad + \frac{1}{\alpha} \int_V d^D x \sqrt{\gamma} \langle T_{(s)\beta}^\beta \rangle_{\mathrm{b}}, \quad (4.15)
\end{aligned}
$$

where n_β, $\gamma^{\beta\delta} n_\beta n_\delta = 1$, is the external normal to the boundary ∂V and h is the determinant of the induced metric $h_{\beta\delta} = \gamma_{\beta\delta} - n_\beta n_\gamma$. The first term in the right-hand side of Eq. (4.15) describes the energy flux through the boundary ∂V. In particular, for a spherical boundary with radius $r_0 < a$ and with the center at $r = 0$ one has $n_\beta = (\alpha/\eta)\delta_\beta^1$ and the flux term is given by $-(\alpha r_0/\eta)^{D-1} S_D (\alpha/\eta) \langle T_{(s)0}^1 \rangle_{\mathrm{b}}$. Now, by taking into account that the proper surface area of the sphere with radius r_0 is given by $(\alpha r_0/\eta)^{D-1} S_D$, we conclude that the quantity $\langle T_0^1 \rangle_{\mathrm{b}} = (\alpha/\eta) \langle T_{(s)0}^1 \rangle_{\mathrm{b}}$, given by Eq. (4.9), is the energy flux density per unit proper surface area.

By using the expression (4.4) for the energy density, for the boundary-induced vacuum energy inside the sphere with radius r_0 one gets

$$
\begin{aligned}
E_{r \leqslant r_0}^{(\mathrm{b})} &= -\frac{(r_0/\eta)^{D-2}}{2\pi\alpha} \sum_{l=0}^{\infty} D_l \int_0^\infty dx\, x^{1-D} \\
&\quad \times \frac{\bar{K}_\mu(ax/r_0)}{\bar{I}_\mu(ax/r_0)} \{ (1-4\xi) F_\nu(x\eta/r_0) I_\mu(x) \\
&\quad \times [xI'_\mu(x) - (D/2-1) I_\mu(x)] - F_0^0(x\eta/r_0) \\
&\quad \times [x^2 I_\mu'^2(x) - (x^2 + \mu^2) I_\mu^2(x)] \}. \quad (4.16)
\end{aligned}
$$

In deriving Eq. (4.16) we have used the formula for the integral involving the square of the modified Bessel function [48]. Related to the surface divergences in the vacuum energy density (see below), the integrated energy $E_{r \leqslant r_0}^{(\mathrm{b})}$ diverges in the limit $r_0 \to a$. In order to obtain a finite result for the vacuum energy inside the sphere, an additional renormalization procedure is needed. Our main interest in the present paper are local characteristics of the vacuum away from the boundary. As we have noted before, the renormalization of the latter is reduced to that for the boundary-free dS geometry.

The general expressions (4.4) and (4.9) are simplified at the sphere's center and near the sphere. Near the center, the dominant contribution to the boundary-induced VEVs comes from the modes with $l = 0$ and $l = 1$. To the leading order, for the diagonal components we get (no summation over i)

$$
\begin{aligned}
\langle T_i^i \rangle_{\mathrm{b}} &\approx -\frac{2^{-D} \alpha^{-D-1}}{\pi^{D/2+1} \Gamma(D/2)} \int_0^\infty dx\, x \left\{ \frac{\bar{K}_{D/2}(ax/\eta)}{\bar{I}_{D/2}(ax/\eta)} \right. \\
&\quad \times G_{(1)}^i F_\nu(x) + \frac{\bar{K}_{D/2-1}(ax/\eta)}{\bar{I}_{D/2-1}(ax/\eta)} \\
&\quad \left. \times [G_{(0)}^i F_\nu(x) + 2F_i^i(x)] \right\}, \quad (4.17)
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
G_{(0)}^0 &= 1 - 4\xi, \quad G_{(0)}^k = \frac{4(D-1)\xi}{D} - 1, \\
G_{(1)}^0 &= D \frac{1-4\xi}{2}, \quad G_{(1)}^k = 4(D-1)\xi - D + 2, \quad (4.18)
\end{aligned}
$$

for $k = 1, 2, \ldots, D$. The first and second terms in the figure braces of Eq. (4.17) come from the modes $l = 1$ and $l = 0$, respectively. As we could expect, the stresses are isotropic at the center. For the off-diagonal component of the VEV one has the following leading term:

$$
\begin{aligned}
\langle T_0^1 \rangle_{\mathrm{b}} &\approx \frac{2^{1-D} \alpha^{-D-1} \eta r}{\pi^{D/2+1} \Gamma(D/2+1)} \int_0^\infty dx\, x \\
&\quad \times \left[\frac{\bar{K}_{D/2-1}(ax)}{\bar{I}_{D/2-1}(ax)} + \frac{\bar{K}_{D/2}(ax)}{\bar{I}_{D/2}(ax)} \right] \\
&\quad \times [(\xi - 1/4)\eta \partial_\eta + \xi] F_\nu(x\eta). \quad (4.19)
\end{aligned}
$$

The first and second terms in the square brackets are the contributions of the modes $l = 0$ and $l = 1$. The off-diagonal component vanishes at the center. The behavior of the energy-momentum tensor components for large values of the sphere proper radius, $a/\eta \gg 1$, assuming fixed values of r/η, is obtained from Eqs. (4.17) and (4.19) by expanding the function $F_\nu(z)$ for small values of the argument. By using Eq. (3.9), similar to the case of the field squared, it can be seen that for positive values of ν the VEVs decay monotonically like $(a/\eta)^{2\nu - D - 2}$. For imaginary ν, the behavior of the VEVs is damped oscillatory with the amplitude decaying as $(a/\eta)^{-D-2}$.

For points near the surface of the sphere, the dominant contribution to the boundary-induced VEVs comes from large values of l and x. By using the uniform asymptotic expansions for the modified Bessel functions, it can be seen that the leading terms in the diagonal components for a scalar field with non-conformal coupling ($\xi \neq \xi_D$) are related to the corresponding terms for a spherical boundary in Minkowski spacetime by (no summation over i) $\langle T_i^i \rangle_{\mathrm{b}} \approx (\eta/\alpha)^{D+1} \langle T_i^i \rangle_{\mathrm{M}}$. These leading terms are given by the expressions (no summation over k)

$$
\begin{aligned}
\langle T_k^k \rangle_{\mathrm{b}} &\approx \frac{D \Gamma((D+1)/2)(\xi - \xi_D)\kappa_B}{2^D \pi^{(D+1)/2} [\alpha(a-r)/\eta]^{D+1}}, \\
\langle T_1^1 \rangle_{\mathrm{b}} &\approx \frac{(D-1)\Gamma((D+1)/2)(\xi - \xi_D)\kappa_B}{2^D \pi^{(D+1)/2} a(a-r)^D (\alpha/\eta)^{D+1}}, \quad (4.20)
\end{aligned}
$$

for $k = 0, 2, \ldots, D$, and κ_B is defined in Eq. (3.14). For the off-diagonal component we have

$$
\langle T_0^1 \rangle_{\mathrm{b}} \approx \frac{D \Gamma(D/2)(\xi - \xi_D)\kappa_B}{2^D \pi^{D/2+1} \alpha [\alpha(a-r)/\eta]^D}. \quad (4.21)
$$

The leading terms do not depend on the mass and have opposite signs for Dirichlet and non-Dirichlet boundary conditions. In particular, for a minimally coupled Dirichlet scalar field the energy density near the sphere is negative and the energy flux is directed away from the boundary.

In Fig. 1, for $D = 3$ dS space, we have plotted the boundary-induced part in the VEV of the energy density for a scalar field with the Dirichlet boundary condition as a function of the proper distance form the center of the sphere measured in units of the dS curvature scale α. (In figures below we plot the VEVs for both interior and exterior regions. The calculations for the exterior region are presented in Sect. VI.) For the sphere's proper radius (in units of α) we have taken the value $a/\eta = 1$. The left/right panel corresponds to comformally/minimally coupled scalar fields. The numbers near the curves are the values of the parameter $m\alpha$. These values are chosen in a way to have cases of both real and purely imaginary ν. For a minimally coupled field the dependence of the VEVs on the mass is weak for points near the sphere. This follows from the asymptotic (4.20). For a conformally coupled field the leading term vanishes and the subleading terms in the asymptotic expansions depend on the mass. Note that for conformally and minimally coupled fields the graphs have different scales. For non-Dirichlet boundary conditions, the VEV of the energy density near the boundary has opposite sign compared with the Dirichlet case.

Fig. 2 displays the dependence of the boundary-induced part in the energy density, for a fixed proper distance from the center of the sphere with the Dirichlet boundary condition, as a function of the mass measured in units of the dS energy scale. As in Fig. 1, the left and right panels are for $D = 3$ conformally and minimally coupled fields respectively and the sphere radius is taken $a/\eta = 1$. The numbers near the curves are the values of the ratio r/η. For large values of $m\alpha$ the VEVs exhibit an oscillatory behavior.

Fig. 3 shows the energy flux as a function of the proper distance from the center of the sphere with the radius $a/\eta = 1$ for $D = 3$ conformally and minimally coupled fields with the Dirichlet boundary condition. Again, for a minimally coupled scalar field the dependence on the mass is weak, which directly follows from the asymptotic expression (4.21). For both cases of minimally and conformally coupled fields the energy flows out of the sphere. For non-Dirichlet boundary conditions the energy flux has the opposite sign.

The dependence of the energy flux on the mass of the field is depicted in Fig. 4 for a fixed value of the proper distance from the sphere's center. As before, the case of a $D = 3$ scalar field with the Dirichlet boundary condition is considered. For a conformally coupled massless field the energy flux vanishes

In this section, we have investigated the VEV of the bulk energy-momentum tensor inside a spherical boundary. For a scalar fields with Robin boundary conditions, it was shown in Ref. [49] that in the discussion of the relation between the mode sum energy, evaluated as the sum of the zero-point energies for each normal mode of frequency, and the volume integral of the renormalized energy density it is necessary to include in the energy a surface term concentrated on the boundary (see also Ref.

[50]). An expression for the surface energy-momentum tensor for a scalar field with a general curvature coupling parameter in the general case of bulk and boundary geometries is derived in Ref. [51]. By making use of this expression and the mode functions from Eq. (2.9), we can evaluate the mode sum for the surface energy-momentum tensor. The latter is located on the sphere and, in addition to the divergences of the boundary-free dS space-time, contains surface divergences. The corresponding renormalization procedure is similar to that for the total Casimir energy and will be discussed elsewhere.

V. WIGHTMAN FUNCTION IN THE EXTERIOR REGION

Now we turn to the evaluation of the VEVs in the region outside a spherical shell with radius a. The boundary condition for a scalar field is in the form (2.4). As we did for the interior region, first we consider the Wightman function. This function is evaluated by using the mode sum formula (2.5). In the exterior region, $r > a$, the radial parts of the mode functions are given by Eq. (2.7), where the ratio of the coefficients in the linear combination of the Bessel and Neumann functions is determined from the boundary condition (2.4). Note that the boundary condition can be written in the covariant form $(1 + \beta n^l \nabla_l)\varphi = 0$, with n^l being the outward normal (with respect to the region under consideration) to the boundary normalized as $g_{il} n^i n^l = -1$. For the interior and exterior regions we have $n^l = (\eta/\alpha)\delta_1^l$ and $n^l = -(\eta/\alpha)\delta_1^l$, respectively. Comparing with the boundary condition in the form (2.4), we see that $\tilde{B}/\tilde{A} = (\eta/\alpha)\beta$ for the interior region and $\tilde{B}/\tilde{A} = -(\eta/\alpha)\beta$ for the exterior one. Hence, when we impose Robin boundary condition with the same coefficient β then the ratio $\tilde{B}/\tilde{A}$ has opposite sign for the exterior and interior region. In general, β could be different for these regions.

In the exterior region, for the mode functions realizing the Bunch-Davies vacuum state, one finds

$$\varphi_\sigma(x) = C_\sigma \frac{\eta^{D/2}}{r^{D/2-1}} H_\nu^{(1)}(\lambda\eta) g_\mu(\lambda a, \lambda r) Y(m_k; \vartheta, \phi),$$
(5.1)

where we have defined

$$g_\mu(\lambda a, \lambda r) = \bar{Y}_\mu(\lambda a) J_\mu(\lambda r) - \bar{J}_\mu(\lambda a) Y_\mu(\lambda r), \quad (5.2)$$

with $0 \leqslant \lambda < \infty$ and with the barred notation (2.20). The normalization coefficient is determined from the condition (2.12), where now the integration goes over the exterior region and in the right-hand side the delta symbol for λ is understood as the Dirac delta function $\delta(\lambda - \lambda')$. In evaluating the normalization integral over the radial coordinate we note that this integral is divergent for $\lambda = \lambda'$ and, hence, the dominant contribution comes from large values of r. By making use of the asymptotic formulas for the Bessel and Neumann functions for
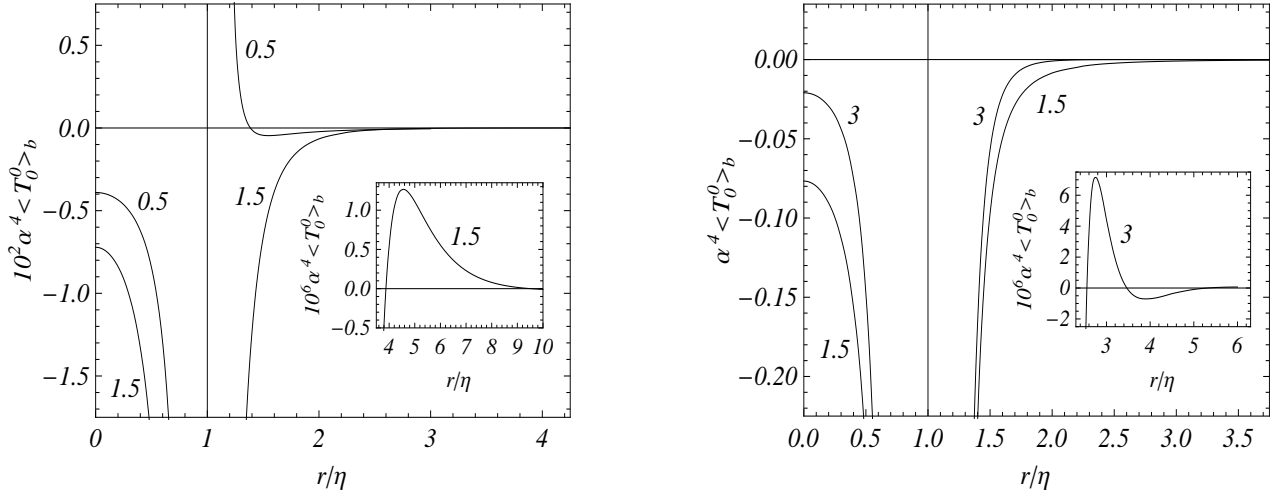
FIG. 1: The boundary-induced part in the VEV of the energy density as a function of the proper distance from the center of the sphere for conformally (left plot) and minimally (right plot) coupled $D = 3$ scalar fields with the Dirichlet boundary condition. The numbers near the curves are the corresponding values of the parameter $m\alpha$ and for the sphere's proper radius we have taken $a/\eta = 1$.
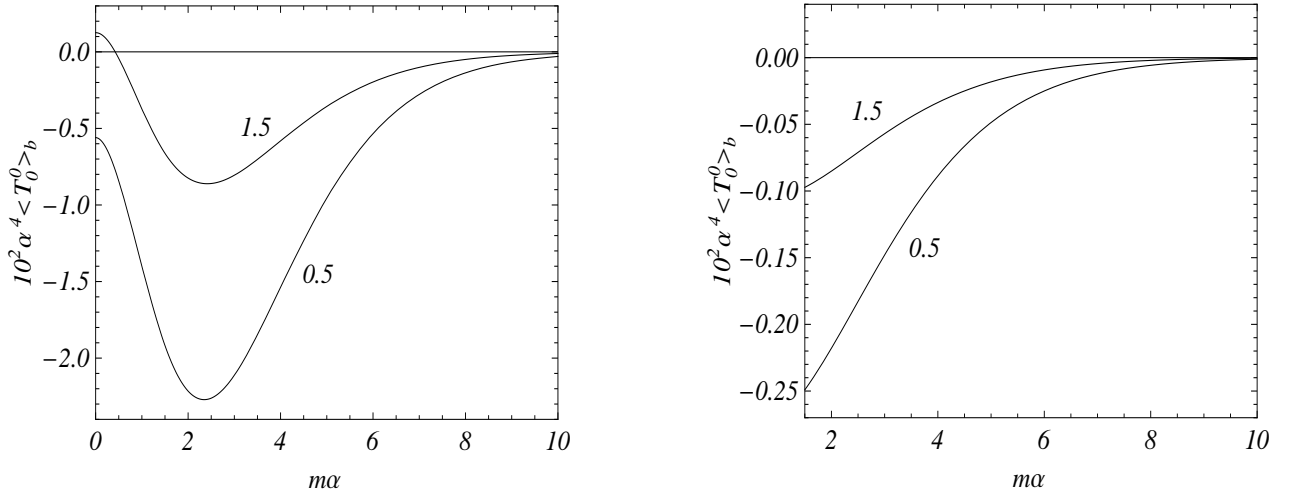

FIG. 2: The boundary-induced part in the VEV of the energy density as a function of the field mass for a fixed proper distance from the center of the sphere in $D = 3$ dS spacetime. The left and right panels correspond to conformally and minimally coupled fields. The numbers near the curves are the values of the proper distance from the sphere's center in units of α.

large arguments, from Eq. (2.12) one gets

$$C_\sigma^2 = \frac{\pi\lambda e^{i(\nu-\nu^*)\pi/2}}{4\alpha^{D-1}N(m_k)}\left[\bar{J}_\mu^2(\lambda a) + \bar{Y}_\mu^2(\lambda a)\right]^{-1}, \qquad (5.3)$$

where we have used Eq. (2.14).

Substituting the functions (5.1) into the mode sum

(2.5), for the Wightman function we find the expression:

$$W(x,x') = \frac{\pi e^{i(\nu-\nu^*)\pi/2}}{4\alpha^{D-1}nS_D}\frac{(\eta\eta')^{D/2}}{(rr')^{n/2}}\sum_{l=0}^{\infty}(2l+n)$$
$$\times C_l^{n/2}(\cos\theta)\int_0^{\infty}d\lambda\frac{g_\mu(\lambda a,\lambda r)g_\mu(\lambda a,\lambda r')}{\bar{J}_\mu^2(\lambda a) + \bar{Y}_\mu^2(\lambda a)}$$
$$\times H_\nu^{(1)}(\lambda\eta)H_{\nu^*}^{(2)}(\lambda\eta'). \qquad (5.4)$$

Similar to the case for the interior region, this function can be written in the decomposed form (2.22). In order to extract the boundary-induced part explicitly, we subtract from Eq. (5.4) the function $W_{\rm dS}(x,x')$, written in the
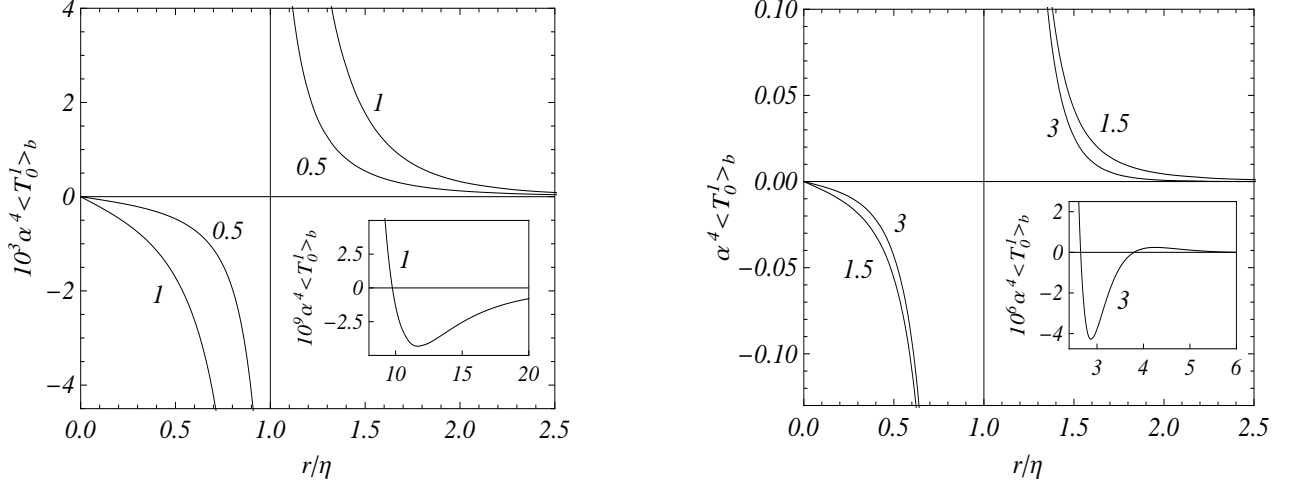
FIG. 3: The energy flux as a function of the proper distance from the center of the sphere for $D = 3$ conformally and minimally coupled scalar fields (left and right plots respectively) with the Dirichlet boundary condition. The numbers near the curves correspond to the values of the parameter $m\alpha$.
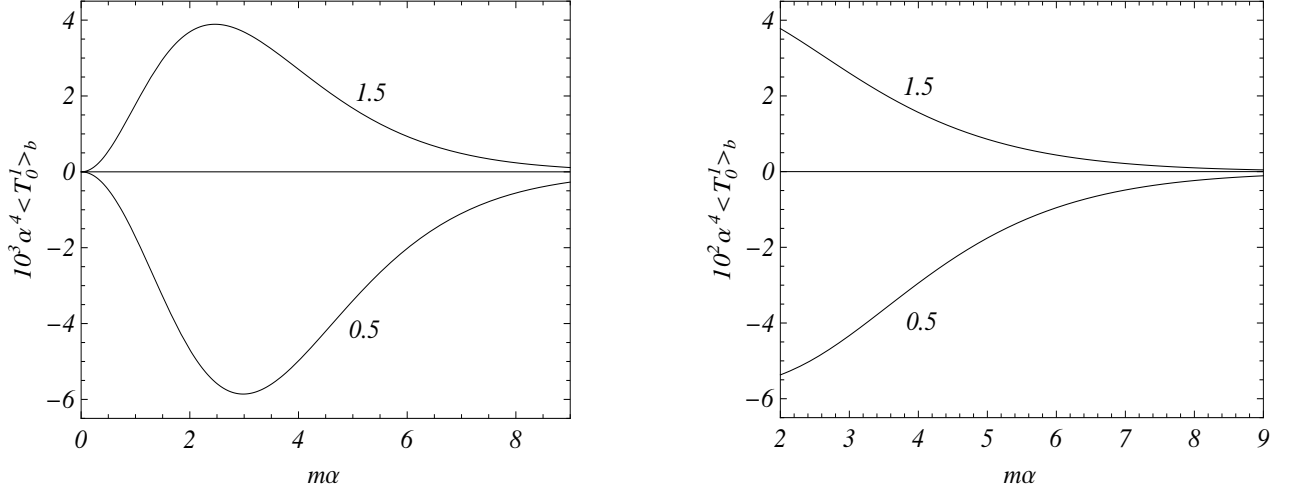

FIG. 4: The energy flux as a function of the mass for a fixed proper distance from the center of the sphere with the proper radius $a/\eta = 1$. The left and right panels are for $D = 3$ conformally and minimally coupled fields with the Dirichlet boundary condition. The numbers near the curves are the values of the ratio r/η (proper distance from the sphere's center in units of α).

form (2.23). For the further evaluation of the difference we use the relation

$$\frac{g_\mu(\lambda a, \lambda r)g_\mu(\lambda a, \lambda r')}{\bar{J}_\mu^2(\lambda a) + \bar{Y}_\mu^2(\lambda a)} - J_\mu(\lambda r)J_\mu(\lambda r')$$
$$= -\frac{1}{2}\sum_{j=1,2}\frac{\bar{J}_\mu(\lambda a)}{\bar{H}_\mu^{(j)}(\lambda a)}H_\mu^{(j)}(\lambda r)H_\mu^{(j)}(\lambda r'). (5.5)$$

As a result, the boundary-induced part is presented in

the form

$$W_{\mathrm{b}}(x, x') = -\frac{\pi e^{i(\nu - \nu^*)\pi/2}(\eta\eta')^{D/2}}{8\alpha^{D-1}nS_D(rr')^{n/2}}\sum_{l=0}^{\infty}(2l + n)$$
$$\times C_l^{n/2}(\cos\theta)\sum_{j=1,2}\int_0^\infty d\lambda\,\lambda\frac{\bar{J}_\mu(\lambda a)}{\bar{H}_\mu^{(j)}(\lambda a)}$$
$$\times H_\mu^{(j)}(\lambda r)H_\mu^{(j)}(\lambda r')H_\nu^{(1)}(\lambda\eta)H_{\nu^*}^{(2)}(\lambda\eta'). \quad (5.6)$$

Assuming that the function $\bar{H}_\mu^{(1)}(z)$, $(\bar{H}_\nu^{(2)}(z))$ has no zeros for $0 < \arg z \leqslant \pi/2$ $(-\pi/2 \leqslant \arg z < 0)$, we rotate the integration contour by the angle $\pi/2$ for the term with $j = 1$ and by the angle $-\pi/2$ for $j = 2$. Introducing

the modified Bessel functions, for the boundary-induced part of the Wightman function in the exterior region we get the expression

$$
\begin{aligned}
W_{\mathrm{b}}(x, x') \; = \; & -\frac{\alpha^{1-D}}{\pi n S_D} \frac{(\eta \eta')^{D/2}}{(rr')^{n/2}} \sum_{l=0}^{\infty} (2l+n) \, C_l^{n/2}(\cos\theta) \\
& \times \int_0^{\infty} dx \, x \frac{\bar{I}_\mu(ax)}{\bar{K}_\mu(ax)} K_\mu(rx) K_\mu(r'x) \\
& \times [I_\nu(x\eta') K_\nu(x\eta) + I_{-\nu}(x\eta) K_\nu(x\eta')] \, (5.7)
\end{aligned}
$$

Comparing with Eq. (2.24), we see that the expressions for the boundary induced parts in the exterior and interior regions are related by the interchange $K_\mu \rightleftarrows I_\mu$.

VI. VACUUM EXPECTATION VALUES IN THE EXTERIOR REGION

A. Field squared

First we consider the VEV for the field squared. It is presented in the decomposed form (3.1), with the boundary-induced part given by

$$
\begin{aligned}
\langle \varphi^2 \rangle_{\mathrm{b}} \; = \; & -\frac{\alpha^{1-D}}{\pi S_D (r/\eta)^{D-2}} \sum_{l=0}^{\infty} D_l \int_0^{\infty} dx \, x^{1-D} \\
& \times \frac{\bar{I}_\mu(ax/\eta)}{\bar{K}_\mu(ax/\eta)} K_\mu^2(rx/\eta) F_\nu(x), \qquad (6.1)
\end{aligned}
$$

where $F_\nu(x)$ is defined by Eq. (4.5). The time dependence in this expression appears through the proper radius of the sphere and the proper distance from the center. For a conformally coupled massless field one has the relation (3.6), where the expression for $\langle \varphi^2 \rangle_{\mathrm{M}}$ is obtained from Eq. (3.7) by the interchange $K_\mu \rightleftarrows I_\mu$. For points near the sphere the leading term in the asymptotic expansion over the distance from the sphere is in the form (3.13) with $(a-r)$ replaced by $(r-a)$. In particular, near the sphere the VEV has the same sign for the interior and exterior regions.

Now we turn to the asymptotic behavior of the boundary-induced VEV at large distances from the sphere, assuming that $r/\eta \gg 1$ for fixed a/η. In this limit, the dominant contribution to the integral in Eq. (6.1) comes from the region near the lower limit of the integration, $x \lesssim \eta/r$, and we can use the relation

$$
\frac{\bar{I}_\mu(z)}{\bar{K}_\mu(z)} \approx 2 \frac{A+\mu B}{A-\mu B} \frac{(z/2)^{2\mu}}{\mu \Gamma^2(\mu)}, \qquad (6.2)
$$

for $z \ll 1$. By using also Eq. (3.9), the integral in Eq. (6.1) involving the square of the Macdonald function is evaluated with the help of a formula from Ref. [48]. The dominant contribution comes from the mode with $l = 0$. The behavior of the boundary-induced VEV at large distances is qualitatively different for positive and

imaginary values of the parameter ν. In the first case, the leading term is given by the expression

$$
\begin{aligned}
\langle \varphi^2 \rangle_{\mathrm{b}} \; \approx \; & -\frac{2\alpha^{1-D}}{\pi S_D} \frac{\Gamma(\nu)\Gamma(n/4+1-\nu)}{n\Gamma^2(n/2)\Gamma(1-\nu)} \frac{A+nB/2}{A-nB/2} \\
& \times \Gamma(3n/4+1-\nu) \frac{(a/\eta)^{D-2}}{(r/\eta)^{2D-2\nu-2}}, \qquad (6.3)
\end{aligned}
$$

and the VEV is a monotonically decreasing function of the radial coordinate.

At large distances from the sphere and for imaginary ν, $\nu = i|\nu|$, the VEV behaves as

$$
\begin{aligned}
\langle \varphi^2 \rangle_{\mathrm{b}} \; \approx \; & -\frac{C_0(\nu)\alpha^{1-D}}{\pi^{D/2+1}\Gamma(D/2-1)} \frac{A+nB/2}{A-nB/2} \\
& \times \frac{(a/\eta)^{D-2}}{(r/\eta)^{2D-2}} \cos[2|\nu|\ln(r/\eta)+\phi_0], \qquad (6.4)
\end{aligned}
$$

where the coefficient C_0 and the phase ϕ_0 are defined by the relation

$$
C_0(\nu)e^{i\phi_0} = \frac{\Gamma(\nu)}{\Gamma(1-\nu)} \Gamma(3n/4-\nu+1) \Gamma(n/4-\nu+1). \tag{6.5}
$$

In particular, for $D = 3$ one has

$$
C_0(\nu) = \pi\sqrt{2} \frac{|3/16 - \nu + \nu^2|}{|\nu| \cosh^{1/2}(2\pi|\nu|)}. \tag{6.6}
$$

As is seen, for imaginary ν the behavior of the boundary-induced VEV is damped oscillatory. For a fixed value of r, the VEV behaves as $\langle \varphi^2 \rangle_{\mathrm{b}} \sim e^{-Dt/\alpha} \cos(2|\nu|t/\alpha + \phi_0')$, where $\phi_0' = 2|\nu| \ln(r/\alpha) + \phi_0$. The oscillation frequency is increasing with increasing mass of the field.

It is of interest to compare the behavior of the VEV at large distances to the corresponding behavior for a spherical boundary in Minkowski spacetime. In the latter case, for a massless field the decay is of power-law: $\langle \varphi^2 \rangle_{\mathrm{M}} \sim a^{D-2}/r^{2D-3}$. For a massive field, assuming that $mr \gg 1$, the VEV behaves as

$$
\langle \varphi^2 \rangle_{\mathrm{M}} \approx -\frac{\sqrt{\pi}}{4 S_D r^{D-1}} \frac{e^{-2mr}}{\sqrt{mr}} \sum_{l=0}^{\infty} D_l \frac{\bar{I}_\mu(am)}{\bar{K}_\mu(am)}, \tag{6.7}
$$

and it is exponentially suppressed. In the case of dS spacetime, for a fixed value of $m\alpha \lesssim 1$, the decay at large distances is of power-law (monotonic or oscillatory) for both massless and massive fields. So, we conclude that the curvature of the background spacetime essentially changes the behavior of the VEVs at distances larger than the curvature radius of the background spacetime.

B. VEV of the energy-momentum tensor

Now we turn to the VEV of the energy-momentum tensor outside the sphere. This VEV is presented in the decomposed form (4.1). By using the formula (4.3) and

Eq. (6.1) for the VEV of the field squared, the diagonal components of the boundary-induced part are presented in the form (no summation over i)

$$\langle T_i^i \rangle_{\rm b} = -\frac{r^{2-D}\eta^2}{2\pi S_D \alpha^{D+1}} \sum_{l=0}^{\infty} D_l \int_0^{\infty} dx\, x^{3-D} \frac{\bar{I}_\mu(ax)}{\bar{K}_\mu(ax)}$$
$$\times \left\{ G_i^i[K_\mu(rx)]F_\nu(x\eta) + 2K_\mu^2(rx)F_i^i(x\eta) \right\}, \quad (6.8)$$

where the functions $F_i^i(y)$ and $G_i^i[f(y)]$ are defined by Eqs. (4.6) and (4.7). For the off-diagonal component we have the expression

$$\langle T_0^1 \rangle_{\rm b} = \frac{\eta \alpha^{-D-1}}{\pi S_D r^{D-1}} \sum_{l=0}^{\infty} D_l \int_0^{\infty} dx\, x^{1-D} \frac{\bar{I}_\mu(ax)}{\bar{K}_\mu(ax)} K_\mu(rx)$$
$$\times \left[2rx K_\mu'(rx) - (D-2)K_\mu(rx) \right]$$
$$\times \left[(\xi - 1/4)\eta \partial_\eta + \xi \right] F_\nu(x\eta). \quad (6.9)$$

These components are written in the coordinate system $(\tau, r, \vartheta, \phi)$. They are related to the components in the system (t, r, ϑ, ϕ) by Eq. (4.10).

The vacuum energy in the region $r \geqslant r_0 > a$ is given by the expression (4.14) with the integration over this region. Making use of the expression for the energy density from Eq. (6.8), one finds the formula

$$E_{r \geqslant r_0}^{({\rm b})} = \frac{(r_0/\eta)^{2-D}}{2\pi\alpha} \sum_{l=0}^{\infty} D_l \int_0^{\infty} dx\, x^{1-D}$$
$$\times \frac{\bar{I}_\mu(ax/r_0)}{\bar{K}_\mu(ax/r_0)} \left\{ (1 - 4\xi) F_\nu(x\eta/r_0) K_\mu(x) \right.$$
$$\times \left[x K_\mu'(x) - (D/2 - 1)K_\mu(x) \right] - F_0^0(x\eta/r_0)$$
$$\times \left. \left[x^2 K_\mu'^2(x) - (x^2 + \mu^2)K_\mu^2(x) \right] \right\}. \quad (6.10)$$

This energy is related to the energy flux and to the vacuum stresses in the exterior region by Eq. (4.15). Similar to the interior region, the integrated energy diverges in the limit $r_0 \to a$.

Let us consider the asymptotics for the VEV of the energy-momentum tensor. For points near the sphere, in a way similar to that for the interior region we find for $r \to a$ (no summation over k)

$$\langle T_k^k \rangle_{\rm b} \approx \frac{D\Gamma((D+1)/2)(\xi - \xi_D)\kappa_B}{2^D \pi^{(D+1)/2}[\alpha(r-a)/\eta]^{D+1}},$$
$$\langle T_1^1 \rangle_{\rm b} \approx -\frac{(D-1)\Gamma((D+1)/2)(\xi - \xi_D)\kappa_B}{2^D \pi^{(D+1)/2} a(r-a)^D (\alpha/\eta)^{D+1}}, (6.11)$$

for $k = 0, 2, \ldots, D$, and

$$\langle T_0^1 \rangle_{\rm b} \approx -\frac{D\Gamma(D/2)(\xi - \xi_D)\kappa_B}{2^D \pi^{D/2+1}\alpha[\alpha(r-a)/\eta]^D}. \quad (6.12)$$

As we see, for a scalar field with non-conformal coupling the components $\langle T_k^k \rangle_{\rm b}$, $k = 0, 2, \ldots, D$, have the same sign near the sphere for exterior and interior regions, whereas the radial stress and the off-diagonal component change signs.

Now we consider the limit of large distances from the sphere, assuming that $r/\eta \gg 1$ for fixed a/η. Similar to the case of the field squared, discussed in the previous subsection, for positive values of ν to the leading order we get:

$$\langle T_i^k \rangle_{\rm b} \approx \frac{(\eta/r)^{D-2\nu+1-\delta_i^k}(a/r)^{D-2}}{2\pi^{D/2+1}\Gamma(D/2-1)\alpha^{D+1}} \frac{\tilde{A}\, B_i^k(\nu)}{\tilde{A} + (D-2)\tilde{B}/a}. \quad (6.13)$$

In this expression,

$$B_0^0(\nu) = \frac{D\Gamma(\nu)}{\Gamma(1-\nu)} \left[(D+1-2\nu)\xi - \frac{D-2\nu}{4} \right]$$
$$\times \Gamma(3n/4 + 1 - \nu)\Gamma(n/4 + 1 - \nu), \quad (6.14)$$

and (no summation over k)

$$B_k^k(\nu) = \frac{2\nu}{D} B_0^0(\nu), \quad k = 1, 2, \ldots, D,$$
$$B_0^1(\nu) = -2\frac{D-1-\nu}{D} B_0^0(\nu). \quad (6.15)$$

In this case the boundary-induced VEV decays monotonically with increasing proper distance from the sphere. At large distances the vacuum stresses are isotropic. If we denote the corresponding effective pressure by $P_{\rm b}$, $P_{\rm b} = -\langle T_1^1 \rangle_{\rm b}$, then the equation of state is of the barotropic type: $P_{\rm b} = -(2\nu/D)\langle T_0^0 \rangle_{\rm b}$.

At large distances from the sphere and for imaginary ν, the leading term in the asymptotic expansion is in the form

$$\langle T_i^k \rangle_{\rm b} \approx \frac{(\eta/r)^{D+1-\delta_i^k}(a/r)^{D-2}C_i^k(\nu)}{\pi^{D/2+1}\Gamma(D/2-1)\alpha^{D+1}}$$
$$\times \frac{\tilde{A}\cos[2|\nu|\ln(r/\eta) + \phi_i^k]}{\tilde{A} + (D-2)\tilde{B}/a}, \quad (6.16)$$

where $C_i^k(\nu)$ and the phases ϕ_i^k are defined by the relation

$$B_i^k(\nu) = C_i^k(\nu)e^{i\phi_i^k}. \quad (6.17)$$

As in the previous case, to the leading order the vacuum stresses are isotropic. The oscillations in the energy density and the stresses are shifted by the phase $\pi/2$. For a fixed comoving radial distance r, the VEVs oscillate like $\langle T_i^k \rangle_{\rm b} \sim e^{-(D+1-\delta_i^k)t/\alpha}\cos(2|\nu|t/\alpha + \phi_i^{k'})$. For the Neumann boundary condition, the leading terms, given by Eqs. (6.13) and (6.16), vanish and it is necessary to take the next terms in the asymptotic expansions. As a result the asymptotic expressions for the Neumann boundary condition contain an additional factor $(a/r)^2$ compared with non-Neumann boundary conditions. In Figs. 1–4, the boundary-induced part of the energy density and the energy flux in the exterior region are depicted as functions of the proper distance from the sphere center and of the mass, for $D = 3$ minimally and conformally coupled fields with the Dirichlet boundary condition and for the sphere's proper radius corresponding to $a/\eta = 1$. Both

cases of real and imaginary ν are considered. The oscillatory behavior at large distances in the latter case is plotted separately in the inset. The first zero decreases with increasing mass, whereas the oscillation frequency increases.

VII. FURTHER GENERALIZATIONS

The approach described above may be used for the evaluation of the Casimir densities for more general background geometries, in particular, in models where the spherical boundary separates spacetime regions with different geometries. Here, as an example, we consider a model in which the spacetime is described by two distinct metric tensors in the regions outside and inside the spherical boundary with radius a. The line element in the exterior region will be taken in the form (2.2). We will assume that inside the sphere the spacetime geometry is regular and is described by the general spherically symmetric line element

$$ds^2 = \alpha^2 \tau^{-2}[e^{2u(r)}dt^2 - e^{2v(r)}dr^2 - e^{2w(r)}d\Omega_D^2], \quad (7.1)$$

where the functions $u(r)$, $v(r)$, $w(r)$ are continuous at the core boundary:

$$u(a) = v(a) = 0, \ w(a) = \ln(a). \quad (7.2)$$

Here we assume that there is no surface energy-momentum tensor located at $r = a$ and, hence, the derivatives of these functions are continuous as well. Note that by introducing the new radial coordinate $\tilde{r} = e^{w(r)}$ with the sphere's center at $\tilde{r} = 0$, the angular part of the line element (7.1) is written in the standard Minkowskian form. With this coordinate, in general, we will obtain a non-standard angular part in the exterior line element (2.2). From the regularity of the interior geometry at the sphere center one has the conditions $u(r), v(r) \to 0$, and $w(r) \sim \ln \tilde{r}$ for $\tilde{r} \to 0$.

The curvature scalar for the metric corresponding to (7.1) is presented in the form

$$R = (\alpha/\eta)^{-2} \check{R} + D(D+1)\alpha^{-2}e^{-2u(r)}, \quad (7.3)$$

where

$$\check{R} = -2e^{-2v}\left[u'' + u'^2 - u'v' + (D-2)(D-1)w'^2/2\right. $$
$$+(D-1)\left(w'' + w'^2 + w'u' - w'v'\right)\bigg] $$
$$+(D-2)(D-1)e^{-2w} \quad (7.4)$$

is the curvature scalar for the static metric corresponding to the line element inside the square brackets in (7.1). Explicitly writing the field equation (2.3) for the metric (7.1), it can be seen that the time and radial variables are separated in two special cases: for $u(r) = 0$ or for a massless field. We will consider the first case. The discussion for the second case is similar.

Taking $u(r) = 0$ and assuming the Bunch-Davies vacuum state, the mode functions in the region inside the sphere, $r < a$, are written in the form

$$\varphi_\sigma(x) = \eta^{D/2} H_\nu^{(1)}(\lambda\eta) f_l(r) Y(m_p; \vartheta, \phi), \quad (7.5)$$

where the radial function is a solution of the equation

$$f_l''(r) + [(D-1)w' - v']f_l'(r) + e^{2v}[\lambda^2 - \xi\check{R} $$
$$-l(l+D-2)e^{-2w}]f_l(r) = 0. \quad (7.6)$$

and

$$\check{R} = -2(D-1)e^{-2v}\left(w'' + Dw'^2/2 - w'v'\right) $$
$$+(D-1)(D-2)e^{-2w}. \quad (7.7)$$

The solution of the radial equation (7.6) regular at the origin we will denote by $R_l(r, \lambda)$. Near the center this solution behaves like $\tilde{r}^l$. Note that the parameter λ enters in the radial equation in the form λ^2. As a result the regular solution can be chosen in such a way that $R_l(r, -\lambda) = \text{const} \cdot R_l(r, \lambda)$. Now for the radial part of the eigenfunctions one has

$$f_l(r) = \begin{cases} R_l(r, \lambda) & \text{for } r < a \\ r^{1-D/2}\left[A_l J_\mu(\lambda r) + B_l Y_\mu(\lambda r)\right] & \text{for } r > a \end{cases}. \quad (7.8)$$

The coefficients A_l and B_l are determined by the conditions of continuity of the radial function and its derivative at $r = a$:

$$A_l = \frac{\pi}{2}a^{D/2-1}R_l(a, \lambda)\check{Y}_\mu(\lambda a),$$
$$B_l = -\frac{\pi}{2}a^{D/2-1}R_l(a, \lambda)\check{J}_\mu(\lambda a), \quad (7.9)$$

with the notation, for a cylinder function $F(z)$,

$$\check{F}(z) \equiv zF'(z) - \left[\frac{D}{2} - 1 + a\frac{R_l'(a, z/a)}{R_l(a, z/a)}\right] F(z), \quad (7.10)$$

where $R_l'(a, \lambda) = \partial_r R_l(r, \lambda)|_{r=a}$. Due to our choice of the function $R_l(r, \lambda)$, the logarithmic derivative in formula (7.10) is an even function of z. Hence, in the region $r > a$ the radial part of the mode functions becomes

$$f_l(r) = \frac{\pi}{2}(a/r)^{D/2-1}R_l(a, \lambda)h_\mu(\lambda a, \lambda r), \quad (7.11)$$

with the notation

$$h_\mu(\lambda a, \lambda r) = \check{Y}_\mu(\lambda a)J_\mu(\lambda r) - \check{J}_\mu(\lambda a)Y_\mu(\lambda r). \quad (7.12)$$

The normalization condition (2.12) is written in terms of the radial eigenfunctions as

$$\int_{r_0}^\infty dr\sqrt{|g_r|}f_l(r, \lambda)f_l(r, \lambda') = \frac{\pi\delta(\lambda - \lambda')}{4N(m_k)\alpha^{D-1}}e^{i(\nu - \nu^*)\pi/2}, \quad (7.13)$$

where r_0 is the value of the radial coordinate r corresponding to the sphere center and g_r is the radial part of the determinant g. As the integral on the left is divergent

for $\lambda' = \lambda$, the main contribution in the coincidence limit comes from large values r. By using the expression (7.11) for the radial part in the region $r > a$ and replacing the Bessel and Neumann functions by the leading terms of their asymptotic expansions for large values of the argument, it can be seen that from (7.13) the following result is obtained:

$$R_l^2(a, \lambda) = \frac{a^{2-D}\alpha^{1-D}\lambda e^{i(\nu-\nu^*)\pi/2}}{\pi N(m_k)\left[\check{J}_\mu^2(\lambda a) + \check{Y}_\mu^2(\lambda a)\right]}. \qquad (7.14)$$

By taking into account relation (7.14), the mode functions in the exterior region are presented in the form

$$\varphi_\sigma(x) = c_\sigma \frac{\eta^{D/2}}{r^{D/2-1}}H_\nu^{(1)}(\lambda\eta)h_\mu(\lambda a, \lambda r)Y(m_k; \vartheta, \phi), \qquad (7.15)$$

where

$$c_\sigma^2 = \frac{\pi\lambda e^{i(\nu-\nu^*)\pi/2}}{4\alpha^{D-1}N(m_k)}\left[\check{J}_\mu^2(\lambda a) + \check{Y}_\mu^2(\lambda a)\right]^{-1}. \qquad (7.16)$$

Now comparing with (5.1), we see that the mode functions (7.15) are obtained from the corresponding functions for the sphere with Robin boundary condition by the replacement

$$\frac{A}{B} \to -\frac{D}{2} + 1 - a\frac{R_l'(a, \lambda)}{R_l(a, \lambda)}. \qquad (7.17)$$

As a result, the Wightman function in the exterior region is given by expression (5.4) with this replacement.

Further evaluation of the Wightman function and the Casimir densities is the same as in the case of the Robin sphere, described in Sections V and VI. The Wightman function in the region $r > a$ is presented in the decomposed form (2.22), where now the part induced by the geometry (7.1) with $u(r) = 0$ in the region $r < a$ is given by (5.7) with

$$\frac{A}{B} = -\frac{D}{2} + 1 - a\frac{R_l'(a, xe^{\pi i/2})}{R_l(a, xe^{\pi i/2})}. \qquad (7.18)$$

The factor $e^{\pi i/2}$ in the argument of the function R_l is related to the complex rotation we have used after formula (5.6). Similarly, the expressions for corresponding parts in the VEVs of the field squared and the energy-momentum tensor are presented as (6.1) and (6.8) with the ratio of the coefficients from (7.18).

VIII. CONCLUSION

In this paper, we have investigated the Casimir densities induced by a spherical boundary in dS spacetime for a massive scalar field with general curvature coupling parameter. On the sphere the field obeys Robin boundary condition with coefficients, in general, different for the interior and exterior regions. We have assumed that the field is prepared in the Bunch-Davies vacuum state which is dS-invariant in the boundary-free spacetime. In free field theories all properties of a quantum field are encoded in two-point functions. We have computed the Wightman function and the Green's function both in the region interior to the sphere and in its exterior. The VEVs for the field squared and the energy-momentum tensor are obtained from these two-point functions in the coincidence limit. In addition, the Wightman function determines the response of particle detectors of the Unruh-DeWitt type. In a similar way other two-point functions can be investigated.

For the evaluation of the Wightman function we have employed the mode summation method. In the region inside the spherical boundary, the mode functions for a scalar field, realizing the Bunch-Davies vacuum state, are given by Eq. (2.9). The eigenvalues for λ are quantized by the boundary condition on the sphere and they are the solutions of Eq. (2.10). The corresponding mode sum for the Wightman function is given by Eq. (2.17) and contains the summation over these eigenvalues. The latter are given implicitly and Eq. (2.17) is not convenient for the further evaluation of the VEVs in the coincidence limit. In order to obtain a more workable form, we have used the generalized Abel-Plana formula for the summation over the eigenvalues of λ. This allowed us (i) to extract explicitly the boundary-free Wightman function and (ii) to present the sphere-induced part in terms of an integral rapidly convergent in the coincidence limit (for points away from the boundary). The local geometry away from the sphere is the same as in the boundary-free dS spacetime and the renormalization for the VEVs of the field squared and the energy-momentum tensor is reduced to that for the boundary-free dS spacetime. The latter is well investigated in the literature. In the same way we can evaluate the Wightman function in a general vacuum state corresponding to the mode functions (2.7) with a linear combination of Hankel functions. The corresponding expression for the boundary-induced part has the form similar to Eq. (2.24) with a more general bilinear combination of the modified Bessel functions of order ν.

By using the representation of the Wightman function, the VEVs of the field squared and the energy-momentum tensor are decomposed into the boundary-free and boundary-induced parts. For the region inside the sphere the latter are given by Eqs. (3.3), (4.4), and (4.9), for the field squared and the energy-momentum tensor, respectively. An important feature is that the vacuum energy-momentum tensor has an off-diagonal component (4.9) which describes energy flux along the radial direction. With dependence on the boundary condition and the mass of the field, this flux can be either positive or negative. The boundary induced VEVs for both field squared and the energy-momentum tensor depend on time through the proper radius of the sphere and the proper distance from the sphere's center. This property is a consequence of the maximal symmetry of

the Bunch-Davies vacuum state.

We have explicitly checked that the boundary-induced part in the VEV of the energy-momentum tensor obeys the trace relation (4.11) and the covariant conservation equation. In particular, the energy-momentum tensor is traceless for a conformally coupled massless field. The trace anomaly is present in the boundary-free part only. For a conformally coupled massless scalar field the flux vanishes. In this case we have simple relations, Eqs. (3.6) and (4.13), between the boundary-induced VEVs for spherical boundaries in dS and Minkowski spacetimes. The latter are consequences of the conformal relation between the problems in Minkowski and dS spacetimes. Divergences are found in the VEVs as the surface of the sphere is approached. The leading terms in the corresponding asymptotic expansions in terms of the distance from the boundary are given by Eqs. (3.13), (4.20) and (4.21). Written in terms of the proper distance from the boundary, for a non-conformally coupled field these leading terms coincide with the corresponding terms for a spherical boundary in Minkowski spacetime. This is a consequence of the fact that for points near the boundary the dominant contribution to the VEVs comes from the modes with small wavelengths which are not influenced by the gravitational field.

In the region outside a spherical shell, the eigenvalues for the quantum number λ are continuous and the modes of the field realizing the Bunch-Davies vacuum state are given by Eq. (5.1). We have explicitly extracted from the Wightman function the part corresponding to dS spacetime without boundaries. The boundary-induced part is given by Eq. (5.7) for the Wightman function and by Eqs. (6.1) and (6.8), (6.9) for the field squared and the energy-momentum tensor, respectively. General formulas are simplified in the asymptotic regions. For points near the boundary the leading terms in the expansions over the distance from the sphere are given by expressions (6.11) and (6.12). In this region the energy density and the azimuthal stresses have the same sign for the exterior and interior regions, whereas the radial stress and the energy flux have opposite signs. In particular, for a minimally coupled scalar field the energy flows away from the boundary for Dirichlet boundary condition and toward the boundary for non-Dirichlet boundary conditions.

Most interesting is the behavior far from the sphere; qualitatively different behavior occurs depending on the sign of $\nu^2 = D^2/2 - \xi D(D+1) - m^2\alpha^2$, where D is the number of spatial dimensions, ξ is the conformal parameter, m is the mass of the scalar field, and α is the curvature scale. When ν is positive, the mean field squared and the energy-momentum tensor fall off as a power, while when ν is imaginary, the large distance behavior is damped oscillatory with the amplitude decaying as $(\eta/r)^{2(D-1)}$ for the field squared and the diagonal components of the energy-momentum tensor. For a scalar field with Neumann boundary condition the VEVs at large distances are suppressed by an additional factor $(\eta/r)^2$ compared with the case of non-Neumann boundary conditions. Note that the behavior of the VEVs at distances larger than the curvatures scale of the background spacetime is completely different from the case of a spherical boundary in Minkowski spacetime. In the latter case and for a massive field the boundary-induced VEVs at distances larger than the Compton wavelength of the scalar particle are exponentially suppressed by the factor $e^{-m(r-a)}$. For the problem in dS spacetime, under the condition $m \lesssim \alpha^{-1}$, the decay of the VEVs is power-law. Exponential damping can occur only in an intermediate region $\alpha \gg r \gg 1/m$.

In Section VII we have generalized the expressions of the Casimir densities in the exterior region for a class of spherically-symmetric metrics in the region $r < a$ described by the line element (7.1). The geometry in the exterior region is given by the dS line element (2.2). A special case with $u(r) = 0$ is considered when the time and radial variables in the field equation are separated. The VEVs in the exterior region are decomposed as the sum of pure dS part and the part induced by the geometry in the interior region. We have shown that the expressions for the latter are obtained from the corresponding expressions outside the Robin sphere, investigated in Sections V and VI, taking the ratio of the Robin coefficients in the form (7.18), where $R_l(r, \lambda)$ is the solution of the interior radial equation regular at the origin.

At the end, we would like to emphasize that the main subject of the present paper is the investigation of the local characteristics of the vacuum, the VEVs for the field squared and the energy-momentum tensor, at the points away from the boundaries. They do not contain surface divergences and are completely determined within the framework of standard renormalization procedure in quantum field theory without boundaries. We expect that similar results would be obtained in the model in which instead of externally imposed boundary condition the fluctuating field is coupled to a smooth background potential that implements the boundary condition in a certain limit [52].

Acknowledgments

Appendix A: Boundary-free part of the Wightman function

For the further evaluation of the boundary-free part, given by Eq. (2.23), firstly we apply Gegenbauer's addition theorem for the Bessel functions (see, for instance, Ref. [38]) to the series over l. This gives the following expression for the Wightman function:

$$
W_{\mathrm{dS}}(x, x') = \frac{\pi e^{i(\nu-\nu^*)\pi/2}}{\Gamma(n/2)nS_D} \frac{2^{-n/2-1}\alpha^{1-D}(\eta\eta')^{D/2}}{(r^2 + r'^2 - 2rr'\cos\theta)^{n/4}}
$$
$$
\times \int_0^\infty dz\, z^{n/2+1} H_\nu^{(1)}(z\eta) H_{\nu^*}^{(2)}(z\eta')
$$
$$
\times J_{n/2}(z\sqrt{r^2 + r'^2 - 2rr'\cos\theta}). \quad (A1)
$$

As the next step, we write the product of the Hankel functions in terms of the Macdonald function:

$$
e^{i(\nu-\nu^*)\pi/2} H_\nu^{(1)}(z\eta) H_{\nu^*}^{(2)}(z\eta') = \frac{4}{\pi^2} K_\nu(-iz\eta) K_\nu(iz\eta'),
$$
$$(A2)$$

and use the integral representation [41]

$$
K_\nu(Z) K_\nu(z) = \frac{1}{4} \int_{-\infty}^{+\infty} dy \int_0^\infty \frac{dw}{w} e^{-\nu y - Zzw^{-1}\cosh y}
$$
$$
\times \exp\left(-\frac{w}{2} - \frac{Z^2 + z^2}{2w}\right) \quad (A3)
$$

for the product of the Macdonald functions. Substituting Eqs. (A2) and (A3) into Eq. (A1), we first integrate over x and then with respect to w, with the result

$$
W_{\mathrm{dS}}(x, x') = \frac{\alpha^{1-D}}{2\pi S_D} \int_0^\infty dz \frac{z^{\nu+D/2-1}}{[z^2 - 2u(x, x')z + 1]^{D/2}},
$$
$$(A4)$$

where

$$
u(x, x') = 1 + \frac{(\eta - \eta')^2 - r^2 - r'^2 + 2rr'\cos\theta}{2\eta\eta'}. \quad (A5)
$$

In deriving Eq. (A4) we have assumed that $|u| < 1$. The integral in Eq. (A4) is expressed in terms of the associated Legendre function $P_{\nu-1/2}^{(1-D)/2}(u(x, x'))$ (see Ref. [53]). Expressing this function through the hypergeometric function, after some transformations, we get the final expression for the Wightman function in dS spacetime (for two-point functions in boundary-free dS spacetime see Ref. [47]):

$$
W_{\mathrm{dS}}(x, x') = \frac{\alpha^{1-D}}{(4\pi)^{(D+1)/2}} \frac{\Gamma(D/2+\nu)\Gamma(D/2-\nu)}{\Gamma((D+1)/2)}
$$
$$
\times {}_2F_1\left(\frac{D}{2}+\nu, \frac{D}{2}-v; \frac{D+1}{2}; \frac{1+u(x, x')}{2}\right). (A6)
$$

Note that, if we denote by $X(x)$ the coordinates in the higher-dimensional embedding space for dS spacetime, then one can write $u(x, x') = 1 + [X(x) - X(x')]^2/(2\alpha^2)$. The property that the Wightman function depends on spacetime points through $[X(x) - X(x')]^2$ is related to the maximal symmetry of the Bunch-Davies vacuum state.

Appendix B: Minkowski spacetime limit

In this appendix, for the boundary-induced part in the Wightman function, we explicitly demonstrate the limiting transition to the geometry of a spherical boundary in Minkowski spacetime. In the Minkowski spacetime limit one has $\alpha \to \infty$ and the modulus of the order of the modified Bessel functions in Eq. (2.22) is large, $\nu \approx i\sigma$, $\sigma = m\alpha \gg 1$. In addition, we have $\eta \approx \alpha - t$. We make use of the uniform asymptotic expansions for the modified Bessel functions for imaginary values of the order with large modulus. For $z < 1$, the leading terms in these expansions have the form (see, for example, Ref. [54])

$$
K_{i\sigma}(\sigma z) \sim \sqrt{\frac{2\pi}{\sigma}} e^{-\sigma\pi/2} \cos[\sigma f(z) - \frac{\pi}{4}],
$$
$$
I_{i\sigma}(\sigma z) + I_{-i\sigma}(\sigma z) \sim -\frac{2e^{\sigma\pi/2}}{\sqrt{2\pi\sigma}} \sin[\sigma f(z) - \frac{\pi}{4}](B1)
$$

where

$$
f(z) = \ln\left(\frac{1 + \sqrt{1 - z^2}}{z}\right) - \sqrt{1 - z^2}. \quad (B2)
$$

In the case $z > 1$ one has the asymptotics

$$
K_{i\sigma}(\sigma z) \sim \sqrt{\frac{\pi}{2\sigma}} \frac{e^{-\sigma\pi/2-\sigma g(z)}}{(z^2 - 1)^{1/4}},
$$
$$
I_{i\sigma}(\sigma z) + I_{-i\sigma}(\sigma z) \sim \frac{2}{\sqrt{2\pi\sigma}} \frac{e^{\sigma\pi/2+\sigma g(z)}}{(z^2 - 1)^{1/4}}, \quad (B3)
$$

with

$$
g(z) = -\operatorname{arcsec} z + \sqrt{z^2 - 1}, \quad g'(z) = \frac{1}{z}\sqrt{z^2 - 1}. \ (B4)
$$

From Eqs. (B1) and (B3) it follows that the dominant contribution to the boundary-induced part of the Wightman function in Eq. (2.24) comes from the integration range $x > m$. In this range we have

$$
I_\nu(x\eta') K_\nu(x\eta) + I_{-\nu}(x\eta) K_\nu(x\eta')
$$
$$
\approx \frac{\cosh\{m\alpha[g(z) - g(z')]\}}{m\alpha(z'^2 - 1)^{1/4}(z^2 - 1)^{1/4}}, \quad (B5)
$$

where $z = x\eta/m\alpha$ and $z' = x\eta'/m\alpha$. By taking into account that $\eta/\alpha \approx 1 - t/\alpha$ and $\eta'/\alpha \approx 1 - t'/\alpha$, it can be seen that

$$
g(z) - g(z') \approx \sqrt{(x/m)^2 - 1}\,(t' - t)/\alpha. \quad (B6)
$$

Now, combining this with Eq. (B5), we get

$$
I_\nu(x\eta') K_\nu(x\eta) + I_{-\nu}(x\eta) K_\nu(x\eta') \approx \frac{\cosh(\Delta t\sqrt{x^2 - m^2})}{\alpha\sqrt{x^2 - m^2}},
$$
$$(B7)$$

with $\Delta t = t' - t$. Substituting this into the expression for the boundary-induced part of the Wightman function, Eq. (2.24), to the leading order one finds

$$
\begin{aligned}
W_{\mathrm{b}}(x, x') \approx\ & -\frac{(rr')^{-n/2}}{\pi n S_D} \sum_{l=0}^{\infty}(2l + n)C_l^{n/2}(\cos\theta) \\
& \times \int_m^{\infty} dx\, x\, \frac{\bar{K}_\mu(xa)}{\bar{I}_\mu(xa)} I_\mu(xr)I_\mu(xr') \\
& \times \frac{\cosh(\Delta t\sqrt{x^2 - m^2})}{\sqrt{x^2 - m^2}}.
\end{aligned}
\tag{B8}
$$

The expression in the right-hand side coincides with the Wightman function inside a spherical boundary in the Minkowski bulk [26]. The Minkowski space limit for the Wightman function in the exterior region is considered in a similar way.

Appendix C: Green's function

In the text, we computed the Wightman function in order to calculate the VEVs of the field-squared and the energy-momentum tensor. Of course, these quantities can equally well be computed from the Green's function for this problem. The calculation of the latter is similar to that given for the Wightman function, but there are some points of interest, so we sketch the derivation here.

Because the nontrivial structure in the dS geometry lies in the time dependence, it is natural to isolate that by constructing a reduced Green's function in the variable η, η' ($n = D - 2$):

$$
\begin{aligned}
G(x, x') =\ & \frac{\alpha^{1-D}}{n S_D (rr')^{n/2}} \sum_{l=0}^{\infty}(2l + n)C_l^{n/2}(\cos\theta) \\
& \times \sum_\lambda \frac{2\lambda}{a} T_\mu(\lambda a) J_\mu(\lambda r) J_\mu(\lambda r') f(\eta, \eta').
\end{aligned}
\tag{C1}
$$

Here the notation is as in Sec. II, and in particular, the eigenvalues λ are the roots of Eq. (2.10). The reduced Green's functions satisfies

$$
\begin{aligned}
& \left[\frac{\partial^2}{\partial\eta^2} + \frac{1-D}{\eta}\frac{\partial}{\partial\eta} + \frac{\alpha^2 m^2 + \xi D(D-1)}{\eta^2} + \lambda^2 \right] f(\eta, \eta') \\
& = -\left(\frac{\eta}{\alpha}\right)^{D-1} \delta(\eta - \eta').
\end{aligned}
\tag{C2}
$$

The solution of this equation corresponding to the Bunch-Davies vacuum (the analog of outgoing wave solutions) is

$$
f(\eta, \eta') = \frac{\pi}{4i} e^{i\pi(\nu - \nu^*)/2}(\eta\eta')^{D/2} H_\nu^{(1)}(\lambda\eta_>)H_{\nu^*}^{(2)}(\kappa\eta_<),
\tag{C3}
$$

which uses the Wronskian (2.14) and $\eta_>$ ($\eta_<$) is the greater (lesser) of η, η'.

To resolve the difficulties with oscillatory integrals and an implicit equation for the eigenvalues, we can again use the generalized Abel-Plana formula (2.19), which leads to the breakup of the Green's function into a free part referring only to the dS background, and a term which exhibits the effect of the sphere:

$$
G(x, x') = G_{\mathrm{dS}}(x, x') + G_{\mathrm{b}}(x, x'),
\tag{C4}
$$

where

$$
\begin{aligned}
G_{\mathrm{dS}}(x, x') =\ & -\frac{\pi}{4i}\frac{e^{i\pi(\nu - \nu^*)/2}}{\alpha^{D-1}n S_D}\frac{(\eta\eta')^{D/2}}{(rr')^{n/2}} \sum_{l=0}^{\infty}(2l + n) \\
& \times C_l^{n/2}(\cos\theta) \int_0^{\infty} d\lambda\, \lambda\, J_\mu(\lambda r)J_\mu(\lambda r') \\
& \times H_\nu^{(1)}(\kappa r_>)H_{\nu^*}^{(2)}(\lambda r_<),
\end{aligned}
\tag{C5}
$$

and

$$
\begin{aligned}
G_{\mathrm{b}}(x, x') =\ & \frac{1}{\pi i}\frac{\alpha^{1-D}}{n S_D}\frac{(\eta\eta')^{D/2}}{(rr')^{n/2}} \sum_{l=0}^{\infty}(2l + n)C_l^{n/2}(\cos\theta) \\
& \times \int_0^{\infty} d\lambda\, \lambda\, \frac{\bar{K}_\mu(\lambda a)}{\bar{I}_\mu(\lambda a)} I_\mu(\lambda r)I_\mu(\lambda r') \\
& \times \left[K_\nu(\lambda\eta_>)I_{-\nu}(\lambda\eta_<) + K_\nu(\lambda\eta_<)I_\nu(\lambda\eta_>) \right].
\end{aligned}
\tag{C6}
$$

There is, in fact, no discontinuity associated with the last factor, since the quantity in square brackets here is

$$
\frac{\pi/2}{\sin\pi\nu}\left[I_{-\nu}(\lambda\eta)I_{-\nu}(\lambda\eta') - I_\nu(\lambda\eta)I_\nu(\lambda\eta') \right].
\tag{C7}
$$

Of course, $G_{\mathrm{b}}(x, x')$ is simply $iW_{\mathrm{b}}(x, x')$ given in Eq. (2.24).

The calculation of the Green's function for the exterior region proceeds similarly. Again, if we impose the boundary condition at the smallest value of r in the region, here at $r = a$, we have the radial function g_μ given in Eq. (5.2), which have continuum normalization

$$
\int_a^{\infty} dr\, r\, g_\mu(\lambda r)g_\mu(\lambda r') = \frac{1}{\lambda}\delta(\lambda - \lambda')[\bar{J}_\mu^2(\lambda a) + \bar{Y}_\mu^2(\lambda a)].
\tag{C8}
$$

Thus the exterior Green's function is

$$
\begin{aligned}
G(x, x') =\ & \frac{i\pi}{4}\frac{e^{i\pi(\nu - \nu^*)/2}}{\alpha^{D-1}n S_D} \sum_{l=0}^{\infty}(2l + n)C_l^{n/2}(\cos\theta) \\
& \times \int_0^{\infty} d\lambda\, \frac{\lambda g_\mu(\lambda r)g_\mu(\lambda r')}{\bar{J}_\mu^2(\lambda a) + \bar{Y}_\mu^2(\lambda a)} \\
& \times H_\nu^{(1)}(\lambda\eta_>)H_{\nu^*}^{(2)}(\lambda\eta_<).
\end{aligned}
\tag{C9}
$$

This evidently leads to the boundary-dependent part given by Eq. (5.7) multiplied by i.

Incidentally, we might record here the imaginary rotations of the Hankel functions, since these are not given in the standard tables:

$$
\begin{aligned}
H_{\nu^*}^{(2)}(e^{\pi i/2}x) &= 2\left[e^{i\nu^*\pi/2}I_{\nu^*}(x) + \frac{i}{\pi}e^{-i\nu^*\pi/2}K_\nu(x) \right], \\
H_\nu^{(1)}(e^{-\pi i/2}x) &= 2\left[e^{-i\nu\pi/2}I_{\nu^*}(x) - \frac{i}{\pi}e^{i\nu\pi/2}K_\nu(x) \right].
\end{aligned}
$$

for real or imaginary ν, which terms are related by evident complex conjugation.

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