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Phys. Rev. D **85**, 063512 — Published 15 March 2012

DOI: 10.1103/PhysRevD.85.063512

Newtonian and Relativistic Cosmologies

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Abstract

Cosmological N-body simulations are now being performed using Newtonian gravity on scales larger than the Hubble radius. It is well known that a uniformly expanding, homogeneous ball of dust in Newtonian gravity satisfies the same equations as arise in relativistic FLRW cosmology, and it also is known that a correspondence between Newtonian and relativistic dust cosmologies continues to hold in linearized perturbation theory in the marginally bound/spatially flat case. Nevertheless, it is far from obvious that Newtonian gravity can provide a good *global* description of an inhomogeneous cosmology when there is significant nonlinear dynamical behavior at small scales. We investigate this issue in the light of a perturbative framework that we have recently developed [1], which allows for such nonlinearity at small scales. We propose a relatively straightforward "dictionary"—which is exact at the linearized level—that maps Newtonian dust cosmologies into general relativistic dust cosmologies, and we use our "ordering scheme" to determine the degree to which the resulting metric and matter distribution solve Einstein's equation. We find that, within our ordering scheme, Einstein's equation fails to hold at "order 1" at small scales and at "order ϵ" at large scales. We then find the additional corrections to the metric and matter distribution needed to satisfy Einstein's equation to these orders. While these corrections are of some interest in their own right, our main purpose in calculating them is that their smallness should provide a criterion for the validity of the original "dictionary" (as well as simplified versions of this dictionary). We expect that, in realistic Newtonian cosmologies, these additional corrections will be very small; if so, this should provide strong justification for the use of Newtonian simulations to describe relativistic cosmologies, even on scales larger than the Hubble radius.

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I. INTRODUCTION

With the improvements in computational abilities that have taken place in recent years, it is now feasible to do numerical simulations of structure formation in cosmology on scales comparable to—or even larger than—the Hubble radius. Such simulations are being carried out by a number of groups [2–5]. However, these simulations are being carried out using Newtonian gravity. Although it would appear reasonable to expect Newtonian gravity to yield a good description of phenomena on scales much smaller than the Hubble radius—except, of course, in the immediate vicinity of strong field objects—at first thought, it might seem absurd that it could be expected to yield a reliable description of phenomena on scales comparable to, or larger than, the Hubble radius. After all, Newtonian gravity posits forces that act instantaneously over arbitrarily large distances, whereas the dynamical evolution laws of general relativity assert that all influences propagate causally and that the distribution of matter outside of one's past light cone is irrelevant. Similarly, the Newtonian gravity description of the Hubble expansion involves relative motion of bodies, whereas the general relativistic description involves the expansion of space. Why should Newtonian gravity give an accurate description of behavior on scales comparable to—or greater than—the Hubble radius, when the relative velocity of bodies is comparable to—or greater than—the speed of light?

Nevertheless, as we shall review in the next section, it is well known (see, e.g., [6]) that under the assumptions of spatial homogeneity and isotropy, the equations for a uniformly expanding pressureless fluid ("dust") in Newtonian gravity are identical to the dynamical equations for a dust filled Friedmann-Lemaître-Robinson-Walker (FLRW) universe in general relativity—even in the case of nonvanishing spatial curvature. An explanation for this remarkable correspondence can be found from the fact that in both Newtonian gravity and general relativity, in the presence of spherical symmetry, the behavior of a co-moving ball of dust does not depend upon the distribution of matter outside of the ball[1]. Thus, in both Newtonian gravity and general relativity, the dynamical behavior of a co-moving ball of dust in a homogeneous, isotropic universe is the same as it would be if that ball were placed in an empty, asymptotically flat spacetime. However, for a sufficiently small ball of dust in an otherwise empty spacetime, Newtonian gravity should be an excellent

[1] This fact is closely related to the fact that there is no gravitational field/curvature inside a spherical shell of matter in Newtonian gravity (by Newton's theorem) and general relativity (by Birkhoff's theorem).

approximation to general relativity. Thus, for a sufficiently small ball, the density and comoving radius of the ball must satisfy the same dynamical equations in Newtonian gravity and general relativity. This implies that the equations for the density and comoving radius in a homogeneous, isotropic Newtonian dust cosmology must coincide with the equations for the density and scale factor in a FLRW dust cosmology. This correspondence continues to hold in the presence of a cosmological constant term in the Newtonian and general relativistic equations.

The above argument relies crucially on exact spherical symmetry. Thus, one might expect that no such correspondence between Newtonian and relativistic cosmologies would hold if one perturbs the homogeneous, isotropic solutions away from spherical symmetry. Remarkably, however, the correspondence between Newtonian and relativistic cosmologies extends into the regime of linearized perturbation theory in the case of perturbations off of a spatially flat FLRW dust cosmology. More precisely, as pointed out by Bardeen [7] and will be reviewed in the next section, the scalar gauge-invariant variables of linearized relativistic perturbation theory obey exactly the same equations as the variables describing linearized irrotational dust perturbations of the corresponding Newtonian cosmology. Furthermore, it is not difficult to see that this correspondence extends to the vector case as well, i.e., vector gauge-invariant variables of linearized relativistic perturbation theory obey exactly the same equations as the corresponding Newtonian variables describing vorticity perturbations. Thus, the scalar and vector sectors[2] of linearized relativistic perturbation theory off of a spatially flat FLRW dust model are in exact correspondence with arbitrary Newtonian perturbations off of the corresponding Newtonian dust cosmology.

Further justification for the validity of the Newtonian approximation in cosmology is provided by the work of Oliynyk [8, 9] (see also Futamase [10]). Oliynyk rigorously proved that for a given 3-torus Newtonian cosmology, there exists a one-parameter family of general relativistic solutions that limits to this Newtonian cosmology, thus showing that there are general relativistic solutions that are arbitrarily close to the Newtonian solution. However, for Oliynyk's one-parameter families, the ratio of the size of the universe to the Hubble radius goes to zero in the limit[3]. Thus, the general relativistic solutions proven by Oliynyk to be

[2] The tensor modes correspond to additional degrees of freedom present only in general relativity, and they have no Newtonian correspondence.

[3] Oliynyk formulated his limit as one in which the 3-torus remains of fixed size, but the speed of light—and the Hubble radius—goes to infinity. By re-scaling the spatial coordinates, his limit can be reformulated

very close to a Newtonian solution have size small compared with the Hubble radius, and thus have no "long wavelength part". Thus, Oliynyk's results do not directly address the issue of whether Newtonian simulations on scales comparable to the Hubble radius correspond closely to a general relativistic solution, but it can be viewed as providing additional justification for the validity of Newtonian gravity on scales small compared with the Hubble radius.

Taken together, the above considerations strongly suggest that for a universe that is sufficiently close to a spatially flat FLRW dust model, Newtonian gravity should provide a good description of structure formation on all scales. However, the situation is far from straightforward for the following reasons: (i) Although, as described above, there is a correspondence at linearized order between Newtonian theory and general relativity, the "dictionary" needed to translate a linearized Newtonian solution into metric and matter perturbations in any particular gauge is nontrivial, and it is not obvious how this dictionary compares with standard dictionaries used for the Newtonian and post-Newtonian approximations to general relativity on small scales. Thus, it is not obvious how to produce a "global dictionary" that works on all scales. (ii) If one has a candidate global dictionary, it is not obvious how to formulate criteria to determine whether the resulting general relativistic spacetime is "sufficiently close" to a solution to Einstein's equation to trust its predictions. The main complication here is that the failure to take post-Newtonian corrections into account on small scales will cause the general relativistic spacetime to fail to satisfy Einstein's equation by a larger amount than the failure to properly account in any way for the long wavelength perturbations. For most applications in cosmology, the tiny post-Newtonian corrections to the metric and matter motion on small scales are of no interest, but the leading order deviation of the metric and matter density from a FLRW model on large scales is of great interest. Thus, the proper criteria for being "sufficiently close" to a solution to Einstein's equation must take into account the distinction between small scales and large scales. (iii) One would like to know explicitly what the dominant corrections to Newtonian cosmology are, both to be able to quantitatively judge its reliability and to be able to make its predictions more accurate.

The difficulties in addressing the above issues stem from the fact that the approximations of Newtonian gravity (which, *a priori*, is expected to be good on small scales) and linearized

as one in which the speed of light remains constant and the Hubble radius goes to a well defined limit, but the size of the 3-torus then approaches zero in the limit.

perturbation theory (which, *a priori*, is expected to be good on large scales) are incompatible. Specifically, in the Newtonian gravity approximation certain nonlinear terms in the equations are kept (as they must be at small scales), but it is essential that time derivatives of quantities be small compared with space derivatives [8, 9, 11, 12]. By contrast, linearized perturbation theory allows time derivatives of quantities to be comparable to their space derivatives (as they must be at large scales), but it is essential that all nonlinear terms be negligible. In order to properly treat phenomena on all scales, one needs an approximation scheme that can accommodate nonlinear phenomena on small scales but treats time derivatives on the same footing as space derivatives on large scales. We recently proposed an approach that accomplishes this [1], and we will apply this approach here[4].

The main questions we wish to address in this paper can now be stated concretely as follows. Suppose that a Newtonian cosmological simulation has been performed on a 3-torus (i.e., periodic boundary conditions), where the size of the 3-torus may be larger than the Hubble radius. For convenience, we assume that the Newtonian solution has been presented as a continuum solution—i.e., that suitable smoothing has been done if the solution was produced from an N-body simulation. We would like to know the following: (1) What general relativistic spacetime and dust matter distribution should we associate to this Newtonian cosmology, i.e., what "global dictionary" should we use? (2) To what extent is this spacetime a solution to Einstein's equation, i.e., what are the leading order terms in Einstein's equation that fail to be satisfied? (3) What are the leading order corrections to the metric and dust distribution that improve the accuracy of this solution, and how large are these corrections?

Our approach will be to use the framework of [1] to provide a "counting scheme" for the sizes of terms in Einstein's equation. We will start with a candidate "global dictionary," which is suggested by the known correspondences between Newtonian gravity and general relativity in the exactly homogeneous and isotropic case and at the linearized level. We will then see that in our counting scheme, the resulting general relativistic spacetime fails to satisfy Einstein's equation to $O(1)$ at small scales and to $O(\epsilon)$ at large scales. The main effort in our paper will then be to find the corrections to the metric and dust distribution that, within our counting scheme, improve the accuracy of the solution to $O(1)$ at small scales and to $O(\epsilon)$ at large scales. It should be emphasized that we shall not *prove* existence of

[4] Our approach is closely related to [13]; see also [14].

a one-parameter family of solutions to Einstein's equation with the properties we desire—a far more difficult task than solving for leading order corrections. Nevertheless, if the leading order corrections we obtain are small compared with terms appearing in the original global dictionary, we believe that this provides a strong indication that there is a general relativistic solution that corresponds closely to the Newtonian cosmology. Conversely, if these corrections are not negligibly small, then either the Newtonian cosmology is not providing a sufficiently accurate representation of the general relativistic spacetime or the dictionary being used will have to be significantly modified.

Our analysis also addresses concerns that have been expressed with regard to the use of a "Newtonianly perturbed FLRW metric," which corresponds to the using the "abridged dictionary" given by (2.46)–(2.48) below. Ishibashi and Wald [15] have argued that this metric should provide an excellent description of our universe. However, several authors [16, 17] have objected to the use of this metric on the grounds that, if taken literally, and dust peculiar velocities are ignored, then strong constraints relating to exact solutions of Einstein's equation apply, and the metric is only able to describe a spatially homogeneous continuum. Other concerns have been raised by Rasanen [18]. The spacetime metric and dust matter distribution that we produce in this paper—as summarized in section IV—solves Einstein's equation to a much higher degree of accuracy than the Newtonianly perturbed FLRW metric does, and, in particular, fully takes into account peculiar velocities and leading nonlinear terms in the Einstein equation. No inconsistencies of any kind are encountered in obtaining this much more accurate solution. Thus, the approximate solution considered in [15] should be fully justified provided only that the corrections to (2.46)–(2.48) given in section IV are negligibly small, as we argue is the case.

We remark that if one has an equation $\mathcal{E}(F) = 0$, one must draw a clear distinction between having a quantity f that approximately solves this equation (i.e., $\mathcal{E}(f) \approx 0$) as compared with having a quantity f that is an approximate solution (i.e., $f \approx F$ for some exact solution $\mathcal{E}(F) = 0$). If the equation is suitably well posed, if $\mathcal{E}(f) \approx 0$, and if F is the exact solution with the same initial data as f, then f and F will remain close to each other for sufficiently early times. However, f may fail to remain close to F at late times because of the build-up of secular effects. For example, the Newtonian solution for the motion of Mercury solves the general relativistic equations of motion to an excellent approximation at all times, but provides a very poor approximation to the general relativistic solution for the position

of Mercury after $\sim 10^6$ years. In this paper, we are concerned with the issue of obtaining general relativistic spacetimes that solve Einstein's equation to an excellent approximation at all times, but we will not be concerned with the issue of whether these spacetimes provide good global-in-time approximations to exact solutions of Einstein's equation.

In the next section, we shall review the correspondence between homogeneous, isotropic Newtonian cosmology with dust matter and FLRW models in general relativity, as well as the correspondence at the linearized level between these models in the marginally bound/spatially flat case. On the basis of this correspondence, we will propose a dictionary (2.40)–(2.44) to translate Newtonian cosmologies into general relativistic spacetimes. In section III, we will apply our counting scheme [1] to analyze how well Einstein's equation is being satisfied, and we will obtain corrections needed to satisfy Einstein's equation to $O(1)$. We will then obtain the further modifications to the metric and dust distribution needed to obtain a solution to Einstein's equation to $O(\epsilon)$ at large scales. These corrections are of some interest in their own right. For example, as we shall see in Appendix B, there are small modifications of some global properties of the cosmology, such as a slight modification of the expansion rate and the introduction an (even smaller) anisotropic expansion. However, our main purpose in determining these corrections is to provide a criterion for the validity of the Newtonian cosmology as translated into a general relativistic spacetime via the dictionary (2.40)–(2.44) and/or its abridgment (2.46)–(2.48) or its simplification (2.49)–(2.51): The full set of metric and matter corrections to our original dictionary are given by eqs. (4.1)–(4.13) and these corrections can be computed straightforwardly for any given Newtonian cosmology. The smallness of these corrections should provide a reliable criterion for judging the validity of using a Newtonian simulation with the dictionary (2.40)–(2.44) (or its abridgment or simplification) to describe a relativistic cosmology.

II. BACKGROUND AND LINEARIZED CORRESPONDENCE

In this section we shall review the correspondence between homogeneous, isotropic Newtonian dust cosmology and FLRW models, as well as the correspondence between linearized perturbations of these models. We shall then propose a dictionary—valid to linearized order—the accuracy of which will be evaluated and improved upon in the following section.

In Newtonian gravity, the gravitational field is described by a Newtonian potential ϕ,

and the dust matter is described by a mass density ρ and a velocity field v^i. The Newtonian field potential is related to the mass density by the Poisson equation

$$\partial^i \partial_i \phi + \Lambda = 4\pi\rho\,, \tag{2.1}$$

which we have generalized to allow for the presence of a cosmological constant Λ. In addition, the matter variables must satisfy mass conservation and Euler equations, which, for dust matter, take the form

$$\partial_t \rho + \partial_i(\rho v^i) = 0\,, \tag{2.2}$$

$$\partial_t(\rho v^i) + \partial_j(\rho v^i v^j) = -\rho\partial^i\phi\,. \tag{2.3}$$

In these equations, the flat Euclidean metric of space is used to contract indices.

A. Background Correspondence

As a cosmological ansatz, we seek a solution to the above equations of Newtonian gravity in which the density is spatially uniform, $\rho = \rho_0(t)$, and the velocity field is uniformly expanding, $v^i = H(t)x^i$. Note that H is related to the radius, a, of any comoving ball by

$$H = \frac{1}{a}\frac{da}{dt}\,. \tag{2.4}$$

Since $\partial_i v^i \equiv 3H(t)$, (2.2) implies that

$$\partial_t \rho_0 + 3H\rho_0 = 0\,, \tag{2.5}$$

from which it follows that

$$\rho_0 = \rho_{0,\mathrm{init}} a^{-3}\,. \tag{2.6}$$

Using mass conservation, we can eliminate ρ from the Euler equation,

$$\partial_t v^i + v^j\partial_j v^i = -\partial^i\phi\,. \tag{2.7}$$

The Poisson equation, (2.1), has the non-singular solution

$$\phi_0 = \frac{2\pi}{3}\rho_0 r^2 - \frac{\Lambda}{6}r^2 + A(t)\,. \tag{2.8}$$

Substituting for v and ϕ, we obtain

$$\frac{dH}{dt} + H^2 = -\frac{4\pi}{3}\rho_0 + \frac{\Lambda}{3}\,, \tag{2.9}$$

which is one of the Friedmann equations. To obtain the other Friedmann equation, we rewrite this equation as

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi}{3}\frac{\rho_{0,\mathrm{init}}}{a^3} + \frac{\Lambda}{3}\,. \tag{2.10}$$

Integrating once, we obtain

$$H^2 = \frac{8\pi}{3}\rho_0 + \frac{\Lambda}{3} - \frac{k}{a^2}\,, \tag{2.11}$$

where k is a constant of integration. By choosing the size of the comoving ball appropriately, we may choose k to take the values $0, \pm 1$. When $\Lambda = 0$, the value of k determines whether the universe is unbound and expands forever ($k = -1$), is marginally bound and expands forever but with expansion velocity approaching zero ($k = 0$), or is bound and will recollapse within finite time ($k = +1$). Of course, in Newtonian gravity, k does not have any interpretation in terms of spatial curvature; space is always Euclidean.

Equations (2.9) and (2.11) are precisely the equations satisfied by dust FLRW models in general relativity. The underlying reason for this exact correspondence was discussed in the Introduction.

B. Linearized Correspondence

We first re-write the exact Newtonian equations relative to some (arbitrarily chosen) background solution of the previous section. We introduce comoving coordinates

$$x'^i = \frac{x^i}{a}\,. \tag{2.12}$$

We then define the velocity variable v'^i by

$$v'^i \equiv a\frac{dx'^i}{dt'} = \frac{dx^i}{dt} - \frac{1}{a}\frac{da}{dt}x^i = v^i - Hx^i\,, \tag{2.13}$$

so v'^i measures the velocity relative to the Hubble flow of the background solution. We also define density and potential deviations from the background, δ and ψ, by

$$\rho = \rho_0(1+\delta)\,, \tag{2.14}$$

$$\phi = \phi_0 + \psi\,. \tag{2.15}$$

In terms of these quantities, the Newtonian equations are

$$\partial^{i'}\partial_{i'}\psi = 4\pi a^2\rho_0\delta\,, \tag{2.16}$$

$$\partial_{t'}\delta + \frac{1}{a}\partial_{i'}\left((1+\delta)v'^{i'}\right) = 0\,, \tag{2.17}$$

$$\partial_{t'}v'^{i'} + \frac{1}{a}v'^{j'}\partial_{j'}v'^{i'} + Hv'^{i'} = -\frac{1}{a}\partial^{i'}\psi\,. \tag{2.18}$$

From now on, since we will always work in comoving coordinates, we shall drop the primes.

Next, we re-write these equations using "conformal time" τ defined by

$$\frac{d\tau}{dt} = \frac{1}{a}\,. \tag{2.19}$$

We also denote derivatives with respect to conformal time with overdots. In terms of the conformal time variable, the Newtonian background equations are

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}a^2\rho_0 + \frac{\Lambda}{3}a^2 - k\,, \tag{2.20}$$

$$\frac{d}{d\tau}\left(\frac{\dot{a}}{a}\right) = -\frac{4\pi}{3}a^2\rho_0 + \frac{\Lambda}{3}a^2\,, \tag{2.21}$$

$$\dot{\rho}_0 + 3\frac{\dot{a}}{a}\rho_0 = 0\,, \tag{2.22}$$

and the Newtonian equations for the quantities describing the deviations from the background are

$$\partial^i\partial_i\psi_N = 4\pi a^2\rho_0\delta_N\,, \tag{2.23}$$

$$\dot{\delta}_N + \partial_i\left((1+\delta_N)v_N^i\right) = 0\,, \tag{2.24}$$

$$\dot{v}_N^i + v_N^j\partial_j v_N^i + \frac{\dot{a}}{a}v_N^i = -\partial^i\psi_N\,, \tag{2.25}$$

where we have now added a subscript N so that these Newtonian quantities can be easily distinguished from the corresponding general relativistic quantities that we will introduce later. We emphasize that (2.23)–(2.25) are exact. We shall assume below that these equations are solved on a 3-torus, i.e., a "box" at fixed comoving coordinates (of the background solution) with periodic boundary conditions.

Linearizing (2.23)–(2.25) about the background solution, we obtain

$$\partial^i\partial_i\psi_N^{(1)} = 4\pi a^2\rho_0\delta_N^{(1)}\,, \tag{2.26}$$

$$\dot{\delta}_N^{(1)} + \partial_i v_N^{(1)i} = 0\,, \tag{2.27}$$

$$\dot{v}_N^{(1)i} + \frac{\dot{a}}{a}v_N^{(1)i} = -\partial^i\psi_N^{(1)}\,. \tag{2.28}$$

We now compare the linearized Newtonian equations with the linearized general relativistic equations about a dust FLRW model. In [7], Bardeen decomposed linearized metric and stress-energy perturbations into their scalar, vector, and tensor parts, which evolve independently. He then introduced gauge invariant quantities to describe these perturbations. In the case of a perfect fluid, the two scalar gauge invariant variables describing metric perturbations are related—in his notation, $\Phi_H = -\Phi_A$—so the scalar perturbations are fully described by Φ_H (or Φ_A), the scalar part of the velocity perturbation, v_s^i, and a density perturbation variable ϵ_m (defined by eqs. (3.9)–(3.11) and (3.13) of [7]). We can similarly decompose a linearized Newtonian perturbation: $\psi_N^{(1)}$ and $\delta_N^{(1)}$ are scalar quantities, and the velocity perturbation can be decomposed as

$$v_N^{(1)i} = v_{N\mathrm{s}}^{(1)i} + v_{N\mathrm{v}}^{(1)i}\,, \tag{2.29}$$

where $v_{N\mathrm{s}}^{(1)i}$ can be written as a gradient and $\partial_i v_{N\mathrm{v}}^{(1)i} = 0$. Newtonian perturbations have no tensor part. It is then straightforward to see that, as pointed out by Bardeen [7], under the correspondence

$$\psi_N^{(1)} \longleftrightarrow \Phi_A = -\Phi_H\,, \tag{2.30}$$

$$v_{N\mathrm{s}}^{(1)i} \longleftrightarrow v_s^i\,, \tag{2.31}$$

$$\delta_N^{(1)} \longleftrightarrow \epsilon_m\,, \tag{2.32}$$

the linearized Newtonian equations become identical to the equations describing scalar perturbations of a spatially flat dust cosmology as given by eqs. (4.3), (4.5) and (4.8) of [7]. Note that this correspondence holds only for perturbations of spatially flat models, i.e., there are additional terms in the linearized Einstein equation when the background solution has nonvanishing spatial curvature.

It is not difficult to see that the exact correspondence between linearized Newtonian gravity and general relativistic perturbations of spatially flat models extends to vector perturbations as well with

$$v_{N\mathrm{v}}^{(1)i} \longleftrightarrow v_c^i\,, \tag{2.33}$$

where the gauge invariant quantity v_c^i is defined by Bardeen's eq. (3.23). Specifically, the vector part of the linearized Euler equation (2.28) for $v_{N\mathrm{v}}^{(1)i}$ is identical to Bardeen's eq. (4.13) for v_c^i. There are no additional Newtonian equations for vector perturbations. However, there

is an additional general relativistic equation [Bardeen's eq. (4.12)],

$$\partial^j \partial_j \Psi^i = -16\pi a^2 \rho_0 v_c^i \,, \tag{2.34}$$

which is a Poisson equation for a quantity Ψ^i not present in Newtonian theory. On a torus, this equation has a solution if and only if there is no spatially homogeneous part of v_c^i, i.e.,

$$\int d^3x \, v_{N\mathrm{v}}^{(1)i} = 0 \,. \tag{2.35}$$

This equation must hold for all for all times τ, but it is easily checked that this equation is preserved under time evolution, so it suffices to impose it at any one time. Thus, (2.35) is a constraint that must be imposed upon a linearized Newtonian solution in order that it correspond to a linearized solution of Einstein's equation under the correspondence (2.33)

In summary, provided only that the constraint (2.35) is satisfied, there is an exact correspondence, given by (2.30)–(2.34), between the complete linearized Newtonian equations for dust matter off of a homogeneous and isotropic background and the scalar and vector parts of the linearized Einstein equation off of a spatially flat dust FLRW background. As previously noted, there are no counterparts to tensor perturbations in Newtonian gravity, i.e., general relativity has these additional degrees of freedom not present in Newtonian gravity.

C. A Proposed Dictionary

Based upon the results of the previous subsections, we now shall propose a dictionary that translates a solution $(\psi_N, \delta_N, v_N^i)$ of the exact Newtonian equations (2.23)–(2.25) into a general relativistic spacetime metric g_{ab} and dust matter stress-energy tensor $T_{ab} = \rho u_a u_b$. In the next section, we shall investigate the extent to which (g_{ab}, T_{ab}) satisfies Einstein's equation as well as what further corrections need to be made to (g_{ab}, T_{ab}) to make it solve Einstein's equation to higher accuracy.

First, the Newtonian equations (2.23)–(2.25) were written relative to a "background solution" of (2.9) and (2.11) with "scale factor" a and mass density ρ_0. Since these Newtonian background equations are identical to the equations for a dust filled FLRW general relativistic spacetime, we define our dictionary so that it associates the corresponding FLRW spacetime to this background solution. Thus, we have defined our dictionary for the case $\psi_N = \delta_N = v_N^i = 0$. In the following, we shall restrict consideration to the case where $k = 0$

for the background solution, since this is the only case where we expect a good dictionary to exist when deviations from homogeneity and isotropy occur. We shall assume that periodic boundary conditions have been imposed on the Newtonian background solution, so that the corresponding FLRW background solution has 3-torus spatial slices. For convenience, we assume that the co-moving spatial coordinates of the Newtonian and FLRW background solutions range between 0 and 1.

In order for our dictionary to produce a definite general relativistic spacetime, we must make a choice of gauge for the metric. In the context of linearized perturbation theory, a natural and very useful gauge choice is the longitudinal gauge, in which the metric takes the form

$$ds^2 = a^2(\tau)\left[-(1+2A)d\tau^2 - 2B_i dx^i d\tau + ((1+2H_L)\delta_{ij} + h_{ij})\, dx^i dx^j\right], \tag{2.36}$$

where $\partial^i B_i = 0$, $\partial^j h_{ij} = 0$ and $h^i{}_i = 0$, and spatial indices $i, j, k, \ldots$ are raised and lowered with the background flat Euclidean metric δ_{ij}. In the context of linearized perturbation theory, the quantities A, B_i, H_L, and h_{ij} represent the metric perturbation, and it can be shown that an arbitrary metric perturbation can be put in the form (2.36) by an infinitesimal gauge transformation. It also can be shown that this gauge is essentially unique, i.e., there is essentially no additional gauge freedom that maintains the form (2.36). However, linearized perturbation theory is not adequate for our purposes, since our dictionary is required to map Newtonian solutions that differ by a finite amount from the Newtonian background solution into metrics that differ by a finite amount from an FLRW model. Nevertheless, it should be possible to show via the implicit function theorem that for metrics that differ from an FLRW model by a sufficiently small but finite amount, the metric form (2.36)—with $\partial^i B_i = 0$, $\partial^j h_{ij} = 0$ and $h^i{}_i = 0$—always can be imposed by a (nonlinear) gauge transformation. We shall not attempt to prove such a result here, and will merely take (2.36) as an ansatz for the metric in constructing our dictionary. However, we believe that imposition of the metric form (2.36) does not involve any loss of generality if the metric is sufficiently close to an FLRW model.

The stress-energy tensor of dust in the general relativistic spacetime takes the form

$$T_{ab} = \rho u_a u_b \,. \tag{2.37}$$

We define the three-velocity, v^i, of the dust to be such that the components, u^μ, of the four-velocity in our gauge are proportional to $(1, v^i)$. Normalizing using the metric form

(2.36), we obtain

$$u^\mu = \frac{1}{a\sqrt{1+2A+2B_j v^j - ((1+2H_L)\delta_{jk}+h_{jk})\, v^j v^k}}(1, v^i)\,. \tag{2.38}$$

Thus, this equation gives the formula for the 4-velocity u^a appearing in (2.37) in terms of the 3-velocity v^i that will be specified by our dictionary below. We define the fractional density perturbation δ in the general relativistic model via

$$\rho = \rho_0(1+\delta)\,. \tag{2.39}$$

As already stated above, the Newtonian solution is specified by $(\psi_N, \delta_N, v_N^i)$. With the above gauge choice, the general relativistic spacetime and matter distribution is specified by $(A, B_i, H_L, h_{ij}, \delta, v^i)$. Our proposed dictionary will therefore be defined by providing formulas for A, B_i, H_L, h_{ij}, δ, and v^i in terms of the Newtonian variables. To obtain this dictionary, we start by taking the formulas that hold at linearized order under the correspondence of the previous section, which we obtain by expressing the Bardeen variables appearing in (2.30)–(2.33) in terms of $(A, B_i, H_L, h_{ij}, \delta, v^i)$. Then we improve our definition for B_i (and correspondingly for v^i) by requiring consistency with the nonlinear momentum constraint at small scales, leading to the replacement of $\rho_0 v^i$ by $\rho_0(1+\delta_N)v^i$. We thereby propose the following dictionary:

$$A = -H_L = \psi_N\,, \tag{2.40}$$

$$(1+\delta_N)v^i = (1+\delta_N)(v_N^i + B^i) - \overline{(1+\delta_N)v_N^i}\Big|_{\rm v}\,, \tag{2.41}$$

$$\delta = \delta_N - \frac{3}{4\pi\rho_0 a^2}\left[\left(\frac{\dot a}{a}\right)^2 \psi_N + \frac{\dot a}{a}\dot\psi_N\right]\,, \tag{2.42}$$

$$h_{ij} = 0\,, \tag{2.43}$$

and B^i is the solution to the equation

$$\partial^j\partial_j B^i = -16\pi\rho_0 a^2\left((1+\delta_N)v_N^i - \overline{(1+\delta_N)v_N^i}\right)\Big|_{\rm v}\,, \tag{2.44}$$

with[5] $\overline{B}_i = 0$, where the overbar denotes spatial average, i.e.,

$$\overline{f} \equiv \int d^3x f\,. \tag{2.45}$$

[5] The condition $\overline{B}_i = 0$ can be imposed by using the coordinate freedom $x^i \to x^i + F^i(t)$.

(Recall that the comoving spatial coordinates are assumed to range from 0 to 1.) In (2.41) and (2.44), the notation $|_{\rm v}$ denotes the "vector part" of a quantity in a decomposition of the type (2.29).

At large scales, one would expect the vector part of v^i to be negligible because in linear perturbation theory, vector modes are known to decay [7]. In addition, comparing the Poisson equation for B^i with the Poisson equation for ψ_N, one would expect B^i to be smaller than ψ_N by order v/c at small scales, and thus B^i should be negligible compared with A and H_L at all scales. Thus, (2.41) should yield a negligibly small correction to the equation $v^i = v^i_N$. Thus, under normal circumstances, it should be acceptable to replace our proposed global dictionary with the following *abridged version* of the dictionary:

$$A = -H_L = \psi_N \,, \tag{2.46}$$

$$v^i = v^i_N \,, \tag{2.47}$$

$$\delta = \delta_N - \frac{3}{4\pi\rho_0 a^2}\left[\left(\frac{\dot{a}}{a}\right)^2 \psi_N + \frac{\dot{a}}{a}\dot{\psi}_N\right] \,, \tag{2.48}$$

together with $B^i = h_{ij} = 0$. This abridged dictionary corresponds to a continuum version of the dictionary[6] given by Chisari and Zaldarriaga [19].

Finally, on small scales δ_N should dominate the other terms appearing on the right side of (2.42). Thus, on scales much smaller than the Hubble radius, it should be possible to use the following *simplified version* of the dictionary:

$$A = -H_L = \psi_N \,, \tag{2.49}$$

$$v^i = v^i_N \,, \tag{2.50}$$

$$\delta = \delta_N \,, \tag{2.51}$$

together with $B^i = h_{ij} = 0$. This simplified dictionary is very commonly assumed. However, on scales comparable to the Hubble radius, all terms in (2.42) should be of comparable size, so if one is interested in investigating behavior on large scales, the full dictionary or abridged dictionary should be used.

As explained above, our dictionary (2.40)–(2.44) has been constructed so as to produce a solution of Einstein's equation to linearized order in $(A, B_i, H_L, h_{ij}, \delta, v^i)$. A Newtonian cosmology that corresponds to our universe should have $\psi_N \ll 1$ and $|v^i_N| \ll 1$, but will

[6] Their definition of ρ differs from ours by a term involving the perturbed volume element.

normally have $\delta_N \gg 1$ on small scales. Therefore, it is not obvious, *a priori* how accurate our dictionary is in producing a solution to Einstein's equation. In fact, it is clear that there may be difficulties in this regard because the dictionary should produce a spacetime that nearly satisfies the linearized Einstein equation on small scales, but the linearized Einstein equation is incompatible (via the linearized Bianchi identity) with the nonlinear dynamical behavior of matter[7] that occurs on small scales. We now investigate how close (2.40)–(2.44) comes to producing a solution to Einstein's equation.

III. COUNTING SCHEME AND IMPROVED DICTIONARY

As stated at the end of the previous section, we wish to determine how close our dictionary comes to producing a solution of Einstein's equation when the Newtonian cosmology has $\psi_N \ll 1$ and $|v_N^i| \ll 1$, but may have $\delta_N \gg 1$ at small scales. To analyze this issue, we need a consistent approximation scheme that can take advantage of the fact that the deviation of the metric from an FLRW model is small on all scales, but permits very large deviations of the stress-energy tensor from an FLRW model to occur on small scales. Such an approximation scheme was recently developed by us in [1] and used to analyze the backreaction effects of small scale inhomogeneities on large scale dynamics. We refer the reader to that reference for the precise mathematical formulation of the approximation. For our purposes here, it suffices to observe that, as in ordinary perturbation theory, in our approximation scheme there is a "small parameter" ϵ (denoted λ in [1]) that measures the deviation, $\gamma_{ab} = g_{ab} - g^{(0)}_{ab}$, of the metric g_{ab} from a background metric $g^{(0)}_{ab}$, so $\gamma_{ab} = O(\epsilon)$. However, unlike ordinary perturbation theory, first spacetime derivatives of γ_{ab} are allowed to be $O(1)$, and second spacetime derivatives of γ_{ab}—and, hence, the deviations of the stress-energy tensor from the background stress-energy—are allowed to be $O(1/\epsilon)$. In particular, the quadratic products $\nabla_c\gamma_{ab}\nabla_f\gamma_{de}$ and $\gamma_{ab}\nabla_c\nabla_f\gamma_{de}$ that appear in Einstein's equation are $O(1)$, so our approximation scheme allows small scale inhomogeneities to affect the dynamics of the background metric. One of the main results of [1] is that, in fact, the only possible effect that these nonlinear terms can have on the dynamics of the background metric is to contribute an effective stress-energy that is traceless and has positive energy, corresponding

[7] In many references (see, e.g., [19, 20]), the linearized Einstein equation is written down together with the nonlinear dynamical equations for matter. This combined system of equations is mathematically inconsistent.

to the presence of gravitational radiation. For the present work, we assume that the universe contains a negligible amount of gravitational radiation, so that this effective stress-energy tensor can be set to zero, and the background metric (which has FLRW symmetry) therefore obeys the ordinary Einstein equation with dust stress energy tensor.

In addition to analyzing the effects of small scale inhomogeneities on the dynamics of the background metric, in [1] perturbation theory was generalized to allow for significant nonlinearity at small scales, while at the same time maintaining a linearized description at large scales (see also [13]). In order to ascribe different behavior to perturbations at different scales these notions must of course be defined. In [1], the notion of the "long wavelength part" of quantities was defined in a mathematically precise manner by considering the weak limit of these quantities as $\epsilon \to 0$. As explained in [1], at sufficiently small but finite ϵ, this should correspond closely to taking an average over a spatial scale[8] L that is small compared with the background curvature (i.e., the Hubble radius) but sufficiently large that at this scale and beyond we have $|\delta| \ll 1$. For the present work, we shall identify the long wavelength part, $A^{(L)}_{a_1\cdots a_n}$, of a tensor field, $A_{a_1\cdots a_n}$, with the spatial average[9] of its components

$$A^{(L)}_{\mu_1\cdots\mu_n}(x) = \langle A_{\mu_1\cdots\mu_n}\rangle(x) \equiv \int d^3x'\, W_L(x-x')A_{\mu_1\cdots\mu_n}(x')\,, \tag{3.1}$$

using a suitable "window function," $W_L(x-x')$, of size L, i.e., a smooth function which is equal to 1 for $a^2|x-x'|^2 < L$, and which smoothly falls to 0 outside of this region[10]. The requirement that L be much smaller than the background curvature scale ensures that this averaging process is well-defined, whereas the requirement that L be sufficiently large that $|\delta| \ll 1$ should ensure that the long wavelength parts of perturbations behave linearly.

We define the "short wavelength part" of $A_{a_1\cdots a_n}$ by

$$A^{(S)}_{a_1\cdots a_n} \equiv A_{a_1\cdots a_n} - A^{(L)}_{a_1\cdots a_n}\,, \tag{3.2}$$

thereby providing a decomposition of any quantity into its long and short wavelength parts. The framework of [1] allows one to make different assumptions in a mathematically consistent manner about the long and short wavelength parts of the various quantities. In particular,

[8] For the present universe, $L \approx 100$ Mpc should meet these criteria.

[9] In [1] averages over (short) time intervals were also performed. The purely spatial averaging we perform here with a suitable "window function" corresponds precisely to the averaging done in [13].

[10] Equivalently, one could work in Fourier transform space and multiply the Fourier transform, $\hat{A}^{(L)}_{a_1\cdots a_n}(k)$, by $\hat{W}_L(k)$, where $\hat{W}_L(k)$ interpolates between 1 (for $k < 1/L$) and 0 for $k \gg 1/L$.

derivatives of short wavelength parts can pick up the factors of $1/\epsilon$ described above, but derivatives do not increase the size of long wavelength parts.

Our framework can be straightforwardly applied to cosmological Newtonian gravity. It is natural in this case also to impose the additional conditions that velocities are suitably "small" and time derivatives of quantities are correspondingly small compared with space derivatives at small scales. Specifically, the sizes we assign[11] to the short wavelength part of the quantities $(\psi_N, v^i_N, \delta_N)$ of the previous section are given in Table I. On the other hand,

Quantity	Order
$\psi_N^{(S)}$	ϵ
$v_N^{(S)i}$	$\epsilon^{1/2}$
$\delta_N^{(S)}$	ϵ^{-1}
∂_0	$\epsilon^{-1/2}$
∂_i	ϵ^{-1}

TABLE I. Small scale order counting for Newtonian quantities

the long wavelength part of all of these quantities and their space and time derivatives are assumed to be $O(\epsilon)$. It should be noted that certain products of short wavelength quantities can have $O(\epsilon)$ large scale average. In particular, long wavelength averages of nonlinear quantities corresponding Newtonian potential energy[12], kinetic energy, and linear momentum enter the perturbation equation for $\gamma^{(L)}_{ab}$.

Our aim here is simply to use the framework of [1] as a "counting scheme" in powers of ϵ to see how close our dictionary comes to producing a solution to Einstein's equation. Specifically, we assume that we have been provided with a Newtonian cosmological solution where the "sizes" of quantities correspond to Table I. To complete our "counting scheme," we must also assign an ϵ-order to $B_i^{(S)}$. Since B_i is obtained by solving the Poisson equation (2.44) and, according to Table I, the source term is of order $\epsilon^{-1/2}$, we assign $B_i^{(S)}$ the order $\epsilon^{3/2}$.

Having assigned ϵ-orders to all quantities, we may ask the following question: If we substitute the Newtonian solution into our dictionary (2.40)–(2.44) to produce a spacetime

[11] The orders we assign to the quantities in Table I correspond to the post-Newtonian orderings of Futamase and Schutz [11] up to a rescaling of the spatial coordinates, but they differ from the post-Newtonian orderings of Oliynyk [8, 9].

[12] A priori, $\langle \delta_N \psi_N \rangle = O(1)$, but due to the fact that δ_N is bounded below by -1, in fact $\langle \delta_N \psi_N \rangle = O(\epsilon)$; see the lemma of section II of [1].

metric g_{ab} and dust stress-energy tensor T_{ab}, how close does (g_{ab}, T_{ab}) come to satisfying Einstein's equation?

If we are to have confidence that the dictionary is producing a good approximation to a solution to Einstein's equation, we would want Einstein's equation to be solved to at least $O(1)$ in ϵ. This is a nontrivial requirement, since there are individual terms, such as δ, in Einstein's equation that are $O(1/\epsilon)$ in our counting scheme. As we shall see below, the dictionary (2.40)–(2.44) solves Einstein's equation to $O(1/\epsilon)$ but fails to yield a solution to Einstein's equation at $O(1)$ in ϵ. Nevertheless, we will then show that we can make further small corrections to the metric so that Einstein's equation does hold to $O(1)$. As we shall see, these metric corrections should be $O(\epsilon^2)$ at small scales and therefore should be negligible. If so, our original dictionary (2.40)–(2.44) should be producing an accurate relativistic cosmology in terms of its description of the metric and matter distribution on small scales.

In addition, if the dictionary is to be trusted for its description of large scale structure—including on scales comparable to (or larger than) the Hubble radius—we would want Einstein's equation to hold to at least $O(\epsilon)$ at large scales. As we shall see below, even after we have made the necessary corrections to the metric so that Einstein's equation is satisfied to $O(1)$ at small scales, Einstein's equation will fail to hold to $O(\epsilon)$ at large scales in our counting scheme. We will therefore make further large scale corrections to the dictionary so that Einstein's equation holds to $O(\epsilon)$ at large scales. As we shall see, although these corrections are formally of order ϵ, they would be expected to make negligible corrections to ordinary linearized perturbation theory at long wavelengths. If so, our original dictionary (2.40)–(2.44) should be producing an accurate relativistic cosmology in terms of its description of the metric and matter distribution on large scales.

The above corrections provide us with an improved dictionary that incorporates the dominant general relativistic corrections to (2.40)–(2.44). Although the improved dictionary is undoubtedly far more precise than would be needed for most applications, it is important as a matter of principle to know that corrections can be made so that Einstein's equation holds to $O(1)$ at all scales and at $O(\epsilon)$ on large scales. Furthermore, for any given Newtonian cosmology, the correction terms appearing in the improved dictionary can be calculated straightforwardly, and their size should give a reliable indication of the accuracy of the original dictionary (2.40)–(2.44). If, as indicated above, these correction terms are

negligibly small, then the Newtonian cosmology should provide—via the original dictionary (2.40)–(2.44) and/or its abridgment (2.46)–(2.48) or simplification (2.49)–(2.51)—an excellent description of what is predicted by general relativity.

A. Solving Einstein's equation to $O(1)$

1. *How well are Einstein's equation solved by the original dictionary?*

Appendix A presents the calculation of Einstein's equation for the metric (2.36) and stress-energy tensor (2.37)–(2.39), keeping all terms that could potentially contribute to $O(1)$ as well as all terms that could potentially contribute to $O(\epsilon)$ at large scales. Given a Newtonian cosmological solution $(\psi_N, v_N^i, \delta_N)$, we substitute it into the dictionary (2.40)–(2.44), and substitute the result into Einstein's equation, freely using the Newtonian equations to simplify the resulting expressions. Equation (A2) yields

$$\begin{aligned} & G^0{}_0(g) + \Lambda - 8\pi T^0{}_0 \\ &= \frac{3}{a^2}\left\{-\frac{2}{3}\partial^i\partial_i\psi_N - \frac{8}{3}\psi_N\partial^i\partial_i\psi_N - \partial^i\psi_N\partial_i\psi_N\right\} - 8\pi\rho_0\left\{-\delta_N - (1+\delta_N)v_N^i v_{Ni}\right\} + o(1) \\ &= \frac{3}{a^2}\left\{-\frac{8}{3}\psi_N\partial^i\partial_i\psi_N - \partial^i\psi_N\partial_i\psi_N\right\} + 8\pi\rho_0(1+\delta_N)v_N^i v_{Ni} + o(1)\,, \end{aligned} \tag{3.3}$$

where we used the Poisson equation (2.23) for ψ_N in the second equality. Since the quantities $\psi_N\partial^i\partial_i\psi_N$, $\partial^i\psi_N\partial_i\psi_N$ and $\delta_N v_N^i v_{Ni}$ are each $O(1)$ in our counting scheme (and these terms do not cancel), we see that this component of Einstein's equation is *not* satisfied to $O(1)$.

Equation (A3) yields

$$G^0{}_i(g) - 8\pi T^0{}_i = \frac{2}{a^2}\left\{-\partial_i\dot{\psi}_N - \frac{\dot{a}}{a}\partial_i\psi_N\right\} - 8\pi\rho_0\left[(1+\delta_N)v_{Ni}\right]\big|_{\rm s} + o(1)\,. \tag{3.4}$$

However, from the Newtonian equations of motion (2.23) and (2.24), it follows that

$$\partial^i\partial_i\dot{\psi}_N + \frac{\dot{a}}{a}\partial^i\partial_i\psi_N = -4\pi\rho_0 a^2\partial_i\left[(1+\delta_N)v_N^i\right]\,. \tag{3.5}$$

On the torus this may be integrated, giving

$$\partial_i\dot{\psi}_N + \frac{\dot{a}}{a}\partial_i\psi_N = -4\pi\rho_0 a^2\left[(1+\delta_N)v_{Ni}\right]\big|_{\rm s}\,. \tag{3.6}$$

Thus, we obtain

$$G^0{}_i(g) - 8\pi T^0{}_i = o(1)\,, \tag{3.7}$$

i.e., these components of Einstein's equation *are* satisfied[13] to $O(1)$.

Finally, from eq. (A4) we obtain the space-space components,

$$\begin{aligned} & G^i{}_j(g) + \Lambda\delta^i{}_j - 8\pi T^i{}_j \\ &= \frac{1}{a^2}\left\{2\ddot{\psi}_N - 4\psi_N\partial^k\partial_k\psi_N - 3\partial_k\psi_N\partial^k\psi_N\right\}\delta^i{}_j + \frac{1}{a^2}\left\{4\psi_N\partial^i\partial_j\psi_N + 2\partial^i\psi_N\partial_j\psi_N\right\} \\ &\quad + \frac{1}{2a^2}\left\{\partial^i\dot{B}_j + \partial_j\dot{B}^i\right\} - 8\pi\rho_0(1+\delta_N)v^i_N v_{Nj} + o(1)\,. \end{aligned} \tag{3.8}$$

Thus, these components of Einstein's equation are *not* satisfied to $O(1)$.

2. Corrections to the dictionary needed to solve Einstein's equation to $O(1)$

We will now show that all components of Einstein's equation can be satisfied to $O(1)$ by making the additional corrections χ, ξ, and $\jmath_{ij}$ to the spacetime metric as follows:

$$A = \psi_N + \chi + \xi\,, \tag{3.9}$$

$$H_L = -\psi_N - \chi\,, \tag{3.10}$$

$$h_{ij} = \jmath_{ij}\,, \tag{3.11}$$

with $\xi^{(S)}$, $\chi^{(S)}$, and $\jmath^{(S)}_{ij}$ all $O(\epsilon^2)$. However, we do not make any modifications to the original dictionary expressions for v^i, δ, and B^i, i.e., we continue to use

$$(1+\delta_N)v^i = (1+\delta_N)(v^i_N + B^i) - \overline{(1+\delta_N)v^i_N}\Big|_{\rm v}\,, \tag{3.12}$$

$$\delta = \delta_N - \frac{3}{4\pi\rho_0 a^2}\left[\left(\frac{\dot{a}}{a}\right)^2\psi_N + \frac{\dot{a}}{a}\dot{\psi}_N\right]\,, \tag{3.13}$$

$$\partial^j\partial_j B^i = -16\pi\rho_0 a^2\left((1+\delta_N)v^i_N - \overline{(1+\delta_N)v^i_N}\right)\Big|_{\rm v}\,. \tag{3.14}$$

In particular, it should be emphasized that no additional corrections are made to the matter distribution variables δ and v^i.

We have already seen that the original dictionary solved the time-space components of Einstein's equation to $O(1)$ and it is not difficult to see that these equations continue to hold with the above revisions. Thus, to solve Einstein's equation to $O(1)$, we need only consider the space-space components (A4) and the time-time component (A2). To solve (A4), we

[13] In fact, the precise forms of (2.41) and (2.44) were chosen so that no further corrections to v^i and B^i would be needed to satisfy Einstein's equation to $O(1)$.

note that we can uniquely decompose any symmetric tensor field E_{ij} on a 3-torus with flat metric δ_{ij} and flat derivative operator ∂_i as

$$E_{ij} = U\delta_{ij} + \partial_i\partial_j V - \frac{1}{3}\partial^k\partial_k V + 2\partial_{(i}W_{j)} + X_{ij}\,, \tag{3.15}$$

with $\partial^i W_i = 0$, $\partial^i X_{ij} = 0$, and $X^i{}_i = 0$. This defines the decomposition of E_{ij} into its scalar (U, V), vector (W_i), and tensor (X_{ij}) parts. Thus, we can solve an equation of the form $E_{ij} = 0$ by separately solving its scalar, vector, and tensor parts. To begin, we take the double divergence of the traceless part of (A4). We obtain[14]

$$\begin{aligned}
&-\frac{2}{3a^2}\partial^i\partial_i\partial^j\partial_j\xi + \frac{1}{a^2}\partial_i\partial^j\left\{4\psi_N\partial^i\partial_j\psi_N + 2\partial^i\psi_N\partial_j\psi_N\right\}\\
&-\frac{1}{3a^2}\partial^i\partial_i\left\{4\psi_N\partial^j\partial_j\psi_N + 2\partial^j\psi_N\partial_j\psi_N\right\}\\
&= 8\pi\rho_0\partial_i\partial^j\left[(1+\delta_N)v^i_N v_{Nj}\right] - \frac{8\pi}{3}\rho_0\partial^i\partial_i\left[(1+\delta_N)v^k_N v_{Nk}\right] + o\left(\frac{1}{\epsilon^2}\right)\,.
\end{aligned} \tag{3.16}$$

Here we have dropped terms which are $o(1/\epsilon^2)$, since we have taken two spatial derivatives of an equation that we wish to satisfy to $O(1)$. We can solve (3.16) to the desired order by defining ξ to be the solution to the following double Poisson equation:

$$\begin{aligned}
\partial^i\partial_i\partial^j\partial_j\xi = {}& 3\partial_i\partial^j\left\{2\psi_N\partial^i\partial_j\psi_N + \partial^i\psi_N\partial_j\psi_N\right\} - \partial^i\partial_i\left\{2\psi_N\partial^j\partial_j\psi_N + \partial^j\psi_N\partial_j\psi_N\right\}\\
&-12\pi\rho_0 a^2\partial_i\partial^j\left[(1+\delta_N)v^i_N v_{Nj}\right] + 4\pi\rho_0 a^2\partial^i\partial_i\left[(1+\delta_N)v^k_N v_{Nk}\right]\,.
\end{aligned} \tag{3.17}$$

A solution for ξ exists on a torus because the source term is a divergence and therefore has no spatially constant piece. This solution is unique up to a spatially constant function of time, which we fix by requiring that its spatial average, $\overline{\xi}$, vanishes. Since the double divergence of the traceless part of (A4) has now been solved to $O(1/\epsilon^2)$, the scalar part of the traceless part of (A4) should now be solved to $O(1)$, as desired. Note that since the four spatial derivatives applied to ξ yields a quantity that is $O(1/\epsilon^2)$, the short wavelength part, $\xi^{(S)}$, of ξ should be $O(\epsilon^2)$, so our assumption that $\xi^{(S)}$ is $O(\epsilon^2)$ is self-consistent.

Next, we show that, with this choice of ξ, the trace of (A4) also is satisfied to $O(1)$. Substituting the revised dictionary (3.9)–(3.14) into the trace of (A4), we find that we must satisfy

$$\begin{aligned}
&\frac{3}{a^2}\left\{\frac{2}{3}\partial^i\partial_i\xi + 2\ddot{\psi}_N\right\} + \frac{1}{a^2}\left\{-8\psi_N\partial^i\partial_i\psi_N - 7\partial_i\psi_N\partial^i\psi_N\right\}\\
&= 8\pi\rho_0(1+\delta_N)v^i_N v_{Ni} + o(1)\,.
\end{aligned} \tag{3.18}$$

[14] In substituting for v^i we have neglected some terms proportional to $1/(1+\delta_N)$, which can in fact be quite large in low density regions. When one makes a uniform momentum correction $-\overline{(1+\delta_N)v^i_N}\Big|_{\rm v}$, this corresponds, in a low density region, to a very large velocity correction which is unphysical. If such a situation were to occur, then a fix would be to transfer some of this momentum to a higher density region.

To see if this equation holds, we take its Laplacian. The double Laplacian of ξ will then appear, and we can substitute for this quantity using (3.17). Since we want to solve (3.18) to $O(1)$, and each spatial derivative increases the small scale order by a factor of $1/\epsilon$, we wish to solve the Laplacian of (3.18) to $O(1/\epsilon^2)$, so the equation we wish to solve is

$$\frac{3}{a^2}\left\{2\partial^i\partial_i\ddot{\psi}_N - 2\partial^i\left(\partial_i\psi_N\partial^j\partial_j\psi_N\right)\right\} = 24\pi\rho_0\partial^j\partial_i\left[(1+\delta_N)v_N^iv_{Nj}\right] + o\left(\frac{1}{\epsilon^2}\right). \tag{3.19}$$

However, using the Newtonian equations (2.23)–(2.25), as well as the Friedmann equations for the Newtonian background, one can show that

$$\begin{aligned}&\partial^i\partial_i\left\{2\ddot{\psi}_N + 6\frac{\dot{a}}{a}\dot{\psi}_N + \left[4\partial_\tau\left(\frac{\dot{a}}{a}\right) + 2\left(\frac{\dot{a}}{a}\right)^2\right]\psi_N\right\}\\ &= 8\pi\rho_0a^2\partial_i\partial_j\left[(1+\delta_N)v_N^iv_N^j\right] + 2\partial_i\left(\partial^j\partial_j\psi_N\partial^i\psi_N\right),\end{aligned} \tag{3.20}$$

so (3.19) is indeed solved to the desired order. We have thus fully solved the scalar parts of the space-space part of Einstein's equation to $O(1)$.

Next, we consider the vector part of (A4) by taking its divergence, using the fact that the scalar parts have already been solved. We obtain

$$\begin{aligned}&\frac{1}{2a^2}\partial^j\partial_j\dot{B}^i + \frac{1}{a^2}\partial^j\left\{4\psi_N\partial^i\partial_j\psi_N + 2\partial^i\psi_N\partial_j\psi_N\right\}\Big|_{\rm v}\\ &= 8\pi\rho_0\partial^j\left[(1+\delta_N)v_N^iv_{Nj}\right]\Big|_{\rm v} + o\left(\frac{1}{\epsilon}\right).\end{aligned} \tag{3.21}$$

Using the Newtonian Euler and mass conservation equations, along with the definition of B_i, one can show that

$$\begin{aligned}&\frac{1}{2a^2}\left\{\partial^j\partial_j\dot{B}^i + 2\frac{\dot{a}}{a}\partial^j\partial_jB^i\right\} + \frac{1}{a^2}\partial^j\left\{4\psi_N\partial^i\partial_j\psi_N + 2\partial^i\psi_N\partial_j\psi_N\right\}\Big|_{\rm v}\\ &= 8\pi\rho_0\partial^j\left[(1+\delta_N)v_N^iv_{Nj}\right]\Big|_{\rm v},\end{aligned} \tag{3.22}$$

so equality does hold for the terms explicitly written in (3.21). Thus, the vector part of (A4) is satisfied to $O(1)$.

The tensor part of (A4) is all that remains of this equation. We obtain

$$-\frac{1}{a^2}\partial^k\partial_kj^i{}_j + \frac{1}{a^2}\left\{4\psi_N\partial^i\partial_j\psi_N + 2\partial^i\psi_N\partial_j\psi_N\right\}\Big|_{\rm t} = 8\pi\rho_0(1+\delta_N)v_N^iv_{Nj}\Big|_{\rm t} + o(1), \tag{3.23}$$

where $|_t$ denotes the tensor part of a quantity in its decomposition (3.15). To solve this equation to the desired order, we define $\jmath_{ij}$ to be the solution of

$$\begin{aligned}\partial^k\partial_kj^i{}_j = \Big\{&8\pi\rho_0a^2(1+\delta_N)v_N^iv_{Nj} + 4\psi_N\partial^i\partial_j\psi_N + 2\partial^i\psi_N\partial_j\psi_N\\ &-8\pi\rho_0a^2\overline{(1+\delta_N)v_N^iv_{Nj}} + 2\overline{\partial^i\psi_N\partial_j\psi_N}\Big\}\Big|_{\rm t},\end{aligned} \tag{3.24}$$

where the overline denotes spatial average (see (2.45)). The terms with the overline in (3.24) have been added in so that the source has vanishing integral, as is necessary in order to be able to solve the Poisson equation. Since these terms are[15] $O(\epsilon)$, we will satisfy (3.23) to the desired order by choosing $\jmath_{ij}$ to solve (3.24). We fix the ambiguity in $\jmath_{ij}$ by requiring $\overline{\jmath}_{ij} = 0$. Since two spatial derivatives applied to $\jmath_{ij}$ yields a quantity that is $O(1)$, the short wavelength part, $\jmath^{(S)}_{ij}$, of $\jmath_{ij}$ should be $O(\epsilon^2)$, so our assumption that $\jmath^{(S)}_{ij}$, is $O(\epsilon^2)$ is self-consistent. We have now solved (A4) to $O(1)$.

Finally, we consider the time-time component of Einstein's equation. Substitution of the dictionary into (A2) yields

$$\begin{aligned}&\frac{3}{a^2}\left\{-\frac{2}{3}\partial^i\partial_i\psi_N - \frac{2}{3}\partial^i\partial_i\chi - \frac{8}{3}\psi_N\partial^i\partial_i\psi_N - \partial^i\psi_N\partial_i\psi_N\right\}\\ &= 8\pi\rho_0\left[-\delta_N - (1+\delta_N)v^i_N v_{Ni}\right] + o(1)\,.\end{aligned} \tag{3.25}$$

Using the Newtonian field equation (2.23), we obtain

$$\frac{3}{a^2}\left\{-\frac{2}{3}\partial^i\partial_i\chi - \frac{8}{3}\psi_N\partial^i\partial_i\psi_N - \partial^i\psi_N\partial_i\psi_N\right\} = -8\pi\rho_0(1+\delta_N)v^i_N v_{Ni} + o(1)\,. \tag{3.26}$$

We define χ to be the solution to the Poisson equation

$$\begin{aligned}\partial^i\partial_i\chi = &-4\psi_N\partial^i\partial_i\psi_N - \frac{3}{2}\partial^i\psi_N\partial_i\psi_N + 4\pi a^2\rho_0(1+\delta_N)v^i_N v_{Ni}\\ &-\frac{5}{2}\overline{\partial^i\psi_N\partial_i\psi_N} - 4\pi a^2\rho_0\overline{(1+\delta_N)v^i_N v_{Ni}}\,,\end{aligned} \tag{3.27}$$

with $\overline{\chi} = 0$. Since two spatial derivatives applied to χ yields a quantity that is $O(1)$, the short wavelength part, $\chi^{(S)}$, of χ should be $O(\epsilon^2)$.

Thus, we have shown that Einstein's equation can be solved to $O(1)$ by making the corrections (3.9)–(3.11) to the original dictionary, where χ, ξ, and $\jmath_{ij}$ are given, respectively, by (3.27), (3.16), and (3.24). Although it is extremely important as a matter of principle that such corrections can be made so as to obtain a solution to $O(1)$, we expect that these corrections will be negligibly small compared with ψ_N.

B. Improving the solution to $O(\epsilon)$ at large scales

In the previous subsection we obtained a solution to $O(1)$. However, as previously stated, if our dictionary is to be trusted for its description of large scale structure—including on

[15] See footnote 12.

scales comparable to (or larger than) the Hubble radius—we want Einstein's equation to hold to at least $O(\epsilon)$ at large scales. Within the context of ordinary perturbation theory, this corresponds to solving the linearized perturbation equation. Within our generalized perturbative framework, this corresponds to solving the generalized linearized perturbation equation (87) of [1]. The difference between these, as noted earlier, is that long wavelength averages of products of small scale quantities enter into the generalized linearized equation.

It is easy to check that, even with the corrections (3.9)–(3.11), our dictionary does *not* produce a solution to $O(\epsilon)$ at long wavelengths. Therefore, we will need to make the following additional *long wavelength* corrections to our metric and matter variables:

$$A = \psi_N + \chi + \xi + X + \Xi\,, \tag{3.28}$$

$$H_L = -\psi_N - \chi - X\,, \tag{3.29}$$

$$(1+\delta_N)v_i = (1+\delta_N)(v_{Ni} + B_i) - \overline{(1+\delta_N)v_{Ni}}\Big|_{\rm v} + P_i\,, \tag{3.30}$$

$$\delta = \delta_N - \frac{3}{4\pi\rho_0 a^2}\left[\left(\frac{\dot a}{a}\right)^2\psi_N + \frac{\dot a}{a}\dot\psi_N\right] + \Delta\,, \tag{3.31}$$

$$h_{ij} = \gamma_{ij} + J_{ij}\,. \tag{3.32}$$

No additional long wavelength correction is needed for B_i. Here, the quantities Ξ, X, P_i, Δ, and J_{ij} are assumed to be $O(\epsilon)$ and to have vanishing short wavelength part. Hence, they do not contribute to Einstein's equation to $O(1)$ and thus do not spoil the solution obtained in the previous subsection.

Our strategy is to apply the averaging operator $\langle\cdot\rangle$ (see eq. (3.1)) to Einstein's equation, and to choose the above new correction terms in order to obtain a solution to $O(\epsilon)$. For our the calculations below, it is useful to note that the averaging operator $\langle\cdot\rangle$ commutes with differentiation. Note also that since $\psi_N = O(\epsilon)$, we clearly have $\langle\psi_N^2\rangle = O(\epsilon^2)$, and, consequently, we have

$$\langle\partial^i\psi_N\partial_i\psi_N\rangle + \langle\psi_N\partial^i\partial_i\psi_N\rangle = \frac{1}{2}\partial^i\partial_i\langle\psi_N^2\rangle = O(\epsilon^2)\,. \tag{3.33}$$

Thus, we may freely "integrate by parts" to set $\langle\psi_N\partial^i\partial_i\psi_N\rangle = -\langle\partial^i\psi_N\partial_i\psi_N\rangle$ in our calculations.

As before, we begin with the double divergence of the trace-free part of the space-space components (A4) of Einstein's equation. Substituting our new dictionary (3.28)–(3.32), applying the averaging operator $\langle\cdot\rangle$, and using the equation obtained by applying the averaging

operator to (3.17) to simplify the resulting expression, we obtain

$$-\frac{2}{3a^2}\partial^i\partial_i\partial^j\partial_j\Xi = o(\epsilon)\,. \tag{3.34}$$

Thus Ξ can only have a spatially constant part, i.e.,

$$\Xi = \overline{\Xi}\,, \tag{3.35}$$

where $\overline{\Xi}$ may be an arbitrary function of τ. Examining the scalar homogeneous parts of the metric,

$$\overline{ds}^2 = a^2(\tau)\left[-(1+2\overline{X}+2\overline{\Xi})d\tau^2 + (1-2\overline{X})\delta_{ij}dx^i dx^j\right]\,, \tag{3.36}$$

we see that $\overline{\Xi}$ corresponds to gauge freedom in the choice of time coordinate. We fix this freedom by setting

$$\Xi = -2\overline{X}\,, \tag{3.37}$$

corresponding to using conformal time.

Next, we consider the trace of (A4). Substituting from the dictionary, applying $\langle\cdot\rangle$, and using $\Xi = -2\overline{X}$, we obtain

$$\begin{aligned}
&\frac{3}{a^2}\left\{\frac{2}{3}\partial^i\partial_i\langle\xi\rangle + 2\left(\langle\ddot{\psi}_N\rangle + \langle\ddot{\chi}\rangle + \ddot{X}\right) + 2\frac{\dot{a}}{a}\left(3\langle\dot{\psi}_N\rangle + 3\langle\dot{\chi}\rangle + 3\dot{X} + \langle\dot{\xi}\rangle - 2\dot{\bar{X}}\right)\right.\\
&\left.+\left[4\partial_\tau\left(\frac{\dot{a}}{a}\right) + 2\left(\frac{\dot{a}}{a}\right)^2\right]\left(\langle\psi_N\rangle + \langle\chi\rangle + X + \langle\xi\rangle - 2\bar{X}\right)\right\} + \frac{1}{a^2}\langle\partial^i\psi_N\partial_i\psi_N\rangle\\
&= 8\pi\rho_0\langle(1+\delta_N)\,v_N^i v_{Ni}\rangle + o(\epsilon)\,.
\end{aligned} \tag{3.38}$$

As before, we take the Laplacian of this equation and substitute the average of (3.17), obtaining

$$\begin{aligned}
&\frac{3}{a^2}\partial^j\partial_j\left\{2\left(\langle\ddot{\psi}_N\rangle + \langle\ddot{\chi}\rangle + \ddot{X}\right) + 2\frac{\dot{a}}{a}\left(3\langle\dot{\psi}_N\rangle + 3\langle\dot{\chi}\rangle + 3\dot{X} + \langle\dot{\xi}\rangle\right)\right.\\
&\left.+\left[4\partial_\tau\left(\frac{\dot{a}}{a}\right) + 2\left(\frac{\dot{a}}{a}\right)^2\right]\left(\langle\psi_N\rangle + \langle\chi\rangle + X + \langle\xi\rangle\right)\right\} - \frac{6}{a^2}\partial_i\langle\partial^i\psi_N\partial^j\partial_j\psi_N\rangle\\
&= 24\pi\rho_0\partial_i\partial^j\langle(1+\delta_N)\,v_N^i v_{Nj}\rangle + o(\epsilon)\,.
\end{aligned} \tag{3.39}$$

Simplifying further using the average of (3.20) and then inverting the Laplacian, we obtain

$$\begin{aligned}
&2\ddot{X} + 6\frac{\dot{a}}{a}\dot{X} - 4\frac{\dot{a}}{a}\dot{\bar{X}} + \left[4\partial_\tau\left(\frac{\dot{a}}{a}\right) + 2\left(\frac{\dot{a}}{a}\right)^2\right]\left(X - \overline{X}\right)\\
&= -2\langle\ddot{\chi}\rangle - 6\frac{\dot{a}}{a}\langle\dot{\chi}\rangle - 2\frac{\dot{a}}{a}\langle\dot{\xi}\rangle - \left[4\partial_\tau\left(\frac{\dot{a}}{a}\right) + 2\left(\frac{\dot{a}}{a}\right)^2\right]\left(\langle\chi\rangle + \langle\xi\rangle\right)\\
&\quad -\frac{1}{3}\overline{\partial^i\psi_N\partial_i\psi_N} + \frac{8\pi\rho_0 a^2}{3}\overline{(1+\delta_N)v_N^i v_{Ni}} + o(\epsilon)\,.
\end{aligned} \tag{3.40}$$

Here, the constant of integration was determined by requiring consistency with (3.38). Thus, the trace of (A4) is satisfied to $O(\epsilon)$ at long wavelengths provided that X satisfies this second order ordinary differential equation in time. The "scalar parts" of the long wavelength part of (A4) have now been satisfied to $O(\epsilon)$

Using (3.22), it is not difficult to see that the long wavelength part of the divergence of (A4) is solved to $O(\epsilon)$ at large scales, without any need for further corrections. Thus, the "vector part" of (A4) has been satisfied to the desired order at long wavelengths. Only the "tensor part" of (A4) remains. Substituting the dictionary and applying the averaging operator $\langle\cdot\rangle$, we obtain

$$\begin{aligned}
&\frac{1}{a^2}\left\{\ddot{J}^i{}_j + \frac{\dot{a}}{a}\dot{J}^i{}_j - \partial^k\partial_k J^i{}_j\right\} \\
&= -\frac{1}{a^2}\left\{\langle\ddot{j}^i{}_j\rangle + \frac{\dot{a}}{a}\langle j^i{}_j\rangle - \partial^k\partial_k\langle j^i{}_j\rangle\right\} + \frac{2}{a^2}\left\langle\partial^i\psi_N\partial_j\psi_N\right\rangle\Big|_{\rm t} + 8\pi\rho_0\langle(1+\delta_N)v_N^i v_{Nj}\rangle\Big|_{\rm t} + o(\epsilon) \\
&= -\frac{1}{a^2}\left\{\langle\ddot{j}^i{}_j\rangle + \frac{\dot{a}}{a}\langle j^i{}_j\rangle\right\} + \frac{2}{a^2}\overline{\partial^i\psi_N\partial_j\psi_N}\Big|_{\rm t} + 8\pi\rho_0\overline{(1+\delta_N)v_N^i v_{Nj}}\Big|_{\rm t} + o(\epsilon)\,,
\end{aligned} \tag{3.41}$$

where we used the average of (3.24) in the second line. Thus the tensor part of (A4) is solved to $O(\epsilon)$ at large scales provided that J_{ij} solves this wave equation. This completes the solution of the long wavelength part of (A4) to $O(\epsilon)$.

Next, we consider (A3). Substituting from the dictionary, applying $\langle\cdot\rangle$, and taking the divergence, we obtain

$$\begin{aligned}
&-\frac{2}{a^2}\partial^i\partial_i\left\{\langle\dot{\psi}_N\rangle + \dot{X} + \langle\dot{\chi}\rangle + \frac{\dot{a}}{a}\left(\langle\psi_N\rangle + X + \langle\chi\rangle + \langle\xi\rangle\right)\right\} \\
&= 8\pi\rho_0\partial^i\left((1+\delta_N)v_{Ni} + P_i\right) + o(\epsilon)\,.
\end{aligned} \tag{3.42}$$

Using the average of (3.5) to simplify this expression, we obtain

$$-\frac{2}{a^2}\partial^i\partial_i\left\{\dot{X} + \langle\dot{\chi}\rangle + \frac{\dot{a}}{a}\left(X + \langle\chi\rangle + \langle\xi\rangle\right)\right\} = 8\pi\rho_0\partial^i P_i + o(\epsilon)\,. \tag{3.43}$$

We solve this equation by setting P^i to be

$$P^i = -\frac{1}{4\pi\rho_0 a^2}\partial^i\left(\dot{X} + \langle\dot{\chi}\rangle + \frac{\dot{a}}{a}\left(X + \langle\chi\rangle + \langle\xi\rangle\right)\right)\,. \tag{3.44}$$

This satisfies the "scalar part" of (A3) to the desired order at long wavelengths. It is easy to check that the vector part of (A3) is also satisfied without the need for any further corrections.

Finally, we consider the remaining component of Einstein's equation, the time-time component (A2). Substituting and averaging, we find that this equation is satisfied to the required order by making the density correction

$$\begin{aligned}\Delta &= -\langle (1+\delta_N) v_N^i v_{Ni} \rangle - \frac{5}{8\pi a^2 \rho_0} \langle \partial^i \psi_N \partial_i \psi_N \rangle - \frac{3}{8\pi \rho_0 a^2} \left\{ -\frac{2}{3} \partial^i \partial_i \left(X + \langle \chi \rangle \right) \right. \\ &\quad \left. + 2\frac{\dot{a}}{a} \left(\dot{X} + \langle \dot{\chi} \rangle \right) + 2 \left(\frac{\dot{a}}{a} \right)^2 \left(X - 2\overline{X} + \langle \chi \rangle + \langle \xi \rangle \right) \right\} \\ &= -\overline{(1+\delta_N) v_N^i v_{Ni}} - \frac{5}{8\pi a^2 \rho_0} \overline{\partial^i \psi_N \partial_i \psi_N} - \frac{3}{8\pi \rho_0 a^2} \left\{ -\frac{2}{3} \partial^i \partial_i X \right. \\ &\quad \left. + 2\frac{\dot{a}}{a} \left(\dot{X} + \langle \dot{\chi} \rangle \right) + 2 \left(\frac{\dot{a}}{a} \right)^2 \left(X - 2\overline{X} + \langle \chi \rangle + \langle \xi \rangle \right) \right\} . \end{aligned} \tag{3.45}$$

Here, the average of (3.27) was used to get the second line.

Einstein's equation has now been fully solved to $O(1)$ everywhere, and to $O(\epsilon)$ at large scales. All of the quantities appearing in our dictionary are uniquely determined by the Newtonian solution, except for X and J_{ij}, which obey second order differential equations in time. The degrees of freedom associated with X correspond to the long wavelength degrees of freedom present in the dust matter sector in ordinary linearized perturbation theory. It would be natural to fix X by requiring that Δ and P_i vanish at an initial time[16]. The degrees of freedom associated with J_{ij} correspond to the presence of long wavelength gravitational radiation.

Finally, we consider the magnitude of the additional long wavelength quantities $\Xi = -2\overline{X}$, X, P_i, Δ, and J_{ij} that we have just obtained. The equations for these quantities involve terms of the form $\langle \partial_i \psi_N \partial_j \psi_N \rangle$ and $\langle (1+\delta_N) v_{Ni} v_{Nj} \rangle$ as well as $\langle \xi \rangle$, $\langle \chi \rangle$, and $\langle \jmath_{ij} \rangle$, which themselves are sourced by terms of the form $\langle \partial_i \psi_N \partial_j \psi_N \rangle$ and $\langle (1+\delta_N) v_{Ni} v_{Nj} \rangle$. Thus, the additional long wavelength quantities appearing in our new dictionary (3.28)–(3.32) should have a magnitude of the order of the Newtonian potential energy and kinetic energy of the dust matter. Although very small, the homogeneous (i.e., spatially constant) part of these quantities provides the dominant correction to the background FLRW dust cosmology. We compute these corrections explicitly in Appendix B. However, the long wavelength corrections at finite wavelength are sourced by large scale inhomogeneities in $\langle \partial_i \psi_N \partial_j \psi_N \rangle$ and $\langle (1+\delta_N) v_{Ni} v_{Nj} \rangle$. These terms should be *extremely* small as compared with, say, $\langle \delta_N \rangle$.

[16] Since $\overline{P}_i$ vanishes identically, this does not fix the spatially homogeneous part, $\overline{X}$, of X. An additional condition on $\overline{X}$ will be imposed in Appendix B .

Thus, the long wavelength corrections we have obtained in this subsection should make entirely negligible contributions to Newtonian large scale structure.

IV. SUMMARY

Combining all of the results of the previous section, we have the following *Oxford dictionary* for translating a Newtonian cosmological solution $(\psi_N, v_N^i, \delta_N)$ to a general relativistic spacetime metric (2.36) and dust stress-energy (2.37):

$$A = \psi_N + \chi + \xi + X - 2\overline{X}\,, \tag{4.1}$$

$$H_L = -\psi_N - \chi - X\,, \tag{4.2}$$

$$(1+\delta_N)v_i = (1+\delta_N)(v_{Ni} + B_i) - \overline{(1+\delta_N)v_{Ni}}\Big|_{\rm v} + P_i\,, \tag{4.3}$$

$$\delta = \delta_N - \frac{3}{4\pi\rho_0 a^2}\left[\left(\frac{\dot a}{a}\right)^2\psi_N + \frac{\dot a}{a}\dot\psi_N\right] + \Delta\,, \tag{4.4}$$

$$h_{ij} = \jmath_{ij} + J_{ij}\,. \tag{4.5}$$

Here, the quantities B_i, ξ, χ, $\jmath_{ij}$, P^i, and Δ are given by

$$\partial^j\partial_j B^i = -16\pi\rho_0 a^2\left((1+\delta_N)v_N^i - \overline{(1+\delta_N)v_N^i}\right)\Big|_{\rm v}\,, \tag{4.6}$$

$$\begin{aligned}\partial^i\partial_i\partial^j\partial_j\xi =\ & 3\partial_i\partial^j\left\{2\psi_N\partial^i\partial_j\psi_N + \partial^i\psi_N\partial_j\psi_N\right\} - \partial^i\partial_i\left\{2\psi_N\partial^j\partial_j\psi_N + \partial^j\psi_N\partial_j\psi_N\right\}\\ & -12\pi\rho_0 a^2\partial_i\partial^j\left[(1+\delta_N)v_N^i v_{Nj}\right] + 4\pi\rho_0 a^2\partial^i\partial_i\left[(1+\delta_N)v_N^k v_{Nk}\right]\,,\end{aligned} \tag{4.7}$$

$$\begin{aligned}\partial^i\partial_i\chi =\ & -4\psi_N\partial^i\partial_i\psi_N - \frac{3}{2}\partial^i\psi_N\partial_i\psi_N + 4\pi a^2\rho_0(1+\delta_N)v_N^i v_{Ni}\\ & -\frac{5}{2}\overline{\partial^i\psi_N\partial_i\psi_N} - 4\pi a^2\rho_0\overline{(1+\delta_N)v_N^i v_{Ni}}\,,\end{aligned} \tag{4.8}$$

$$\begin{aligned}\partial^k\partial_k\jmath^i{}_j =\ & \Big\{8\pi\rho_0 a^2(1+\delta_N)v_N^i v_{Nj} + 4\psi_N\partial^i\partial_j\psi_N + 2\partial^i\psi_N\partial_j\psi_N\\ & -8\pi\rho_0 a^2\overline{(1+\delta_N)v_N^i v_{Nj}} + 2\overline{\partial^i\psi_N\partial_j\psi_N}\Big\}\Big|_{\rm t}\,,\end{aligned} \tag{4.9}$$

$$P^i = -\frac{1}{4\pi\rho_0 a^2}\partial^i\left(\dot X + \langle\dot\chi\rangle + \frac{\dot a}{a}\left(X + \langle\chi\rangle + \langle\xi\rangle\right)\right)\,, \tag{4.10}$$

$$\begin{aligned}\Delta =\ & -\overline{(1+\delta_N)v_N^i v_{Ni}} - \frac{5}{8\pi a^2\rho_0}\overline{\partial^i\psi_N\partial_i\psi_N} - \frac{3}{8\pi\rho_0 a^2}\left\{-\frac{2}{3}\partial^i\partial_i X\right.\\ & \left. +2\frac{\dot a}{a}\left(\dot X + \langle\dot\chi\rangle\right) + 2\left(\frac{\dot a}{a}\right)^2\left(X - 2\overline{X} + \langle\chi\rangle + \langle\xi\rangle\right)\right\}\,,\end{aligned} \tag{4.11}$$

with $\overline{B}_i = \overline{\xi} = \overline{\chi} = \overline{\jmath}_{ij} = 0$. The quantities J_{ij} and X satisfy the differential equations

$$\begin{aligned}
&\frac{1}{a^2}\left\{\ddot{J}^i{}_j + \frac{\dot{a}}{a}\dot{J}^i{}_j - \partial^k\partial_k J^i{}_j\right\} \\
&= -\frac{1}{a^2}\left\{\langle\ddot{\jmath}^i{}_j\rangle + \frac{\dot{a}}{a}\langle\dot{\jmath}^i{}_j\rangle\right\} + \frac{2}{a^2}\,\overline{\partial^i\psi_N\partial_j\psi_N}\Big|_{\rm t} + 8\pi\rho_0\overline{(1+\delta_N)v_N^i v_{Nj}}\Big|_{\rm t}\,. \qquad (4.12)
\end{aligned}$$

and

$$\begin{aligned}
&2\ddot{X} + 6\frac{\dot{a}}{a}\dot{X} - 4\frac{\dot{a}}{a}\dot{\overline{X}} + \left[4\partial_\tau\left(\frac{\dot{a}}{a}\right) + 2\left(\frac{\dot{a}}{a}\right)^2\right]\left(X - \overline{X}\right) \\
&= -2\langle\ddot{\chi}\rangle - 6\frac{\dot{a}}{a}\langle\dot{\chi}\rangle - 2\frac{\dot{a}}{a}\langle\dot{\xi}\rangle - \left[4\partial_\tau\left(\frac{\dot{a}}{a}\right) + 2\left(\frac{\dot{a}}{a}\right)^2\right]\left(\langle\chi\rangle + \langle\xi\rangle\right) \\
&\quad -\frac{1}{3}\overline{\partial^i\psi_N\partial_i\psi_N} + \frac{8\pi\rho_0 a^2}{3}\overline{(1+\delta_N)v_N^i v_{Ni}}\,. \qquad (4.13)
\end{aligned}$$

Like the Oxford English Dictionary, the above dictionary should be far more detailed and precise than needed for everyday use. Nevertheless, it may be comforting to have it on one's bookshelf in case the need does arise. Furthermore, as a matter of principle, it is of importance to know that a dictionary of this accuracy—namely, solving Einstein's equation to $O(1)$ on all scales and to $O(\epsilon)$ on large scales—can be constructed without running into inconsistencies.

Our main purpose in obtaining the complete dictionary (4.1)–(4.5) was to evaluate the accuracy of the original dictionary (2.40)–(2.44) (as well as its abridgement (2.46)–(2.48) and simplification (2.49)–(2.51)). We have argued that for a Newtonian cosmology that satisfies $\psi_N \ll 1$ and $|v_N^i| \ll 1$ but may have $\delta_N \gg 1$ at small scales, all of the additional terms appearing in (4.1)–(4.5) as compared with (2.40)–(2.44) should be negligibly small. Whether or not this is actually the case for any given Newtonian cosmology can be determined by computing the quantities ξ, χ, $\jmath_{ij}$, P^i, Δ, X, and J_{ij} given by (4.7)–(4.13). If these quantities are indeed negligibly small, then one can have confidence that the Newtonian cosmology is accurately representing a general relativistic spacetime via the original dictionary (2.40)–(2.44). If, in addition, B^i is negligibly small (see (4.6)), then one is similarly justified in using the abridged dictionary (2.46)–(2.48). These statements remain valid even in cases where the Newtonian cosmology is describing phenomena on scales comparable to or larger than the Hubble radius.

ACKNOWLEDGMENTS

We thank Todd Oliynyk for helpful discussions. This research was supported in part by NSF Grant No. PHY08-54807 to the University of Chicago.

Appendix A: Einstein's equation

For the metric ansatz (2.36), we write down the various components of the perturbed Einstein equation,

$$G^{\mu}{}_{\nu}(g) - G^{\mu}{}_{\nu}(g^{(0)}) = 8\pi \left(T^{\mu}{}_{\nu} - T^{(0)\mu}{}_{\nu}\right) , \tag{A1}$$

keeping all terms which are $O(1)$ at small scales or $O(\epsilon)$ at large scales in our counting scheme. We introduce the notation $o(1;\epsilon)$ to denote a quantity which is $o(1)$ at small scales and $o(\epsilon)$ at large scales. The ${}^{\mu}{}_{\nu} = {}^{0}{}_{0}$ equation reads

$$\begin{aligned} &\frac{3}{a^2}\left\{-2\frac{\dot{a}}{a}\dot{H}_L + \frac{2}{3}\partial^i\partial_i H_L + 2\left(\frac{\dot{a}}{a}\right)^2 A - \frac{8}{3}H_L\partial^i\partial_i H_L - \partial_i H_L\partial^i H_L\right\} \\ &= 8\pi\rho_0\left\{-\delta - (1+\delta)v^i(v_i - B_i)\right\} + o(1;\epsilon)\,, \end{aligned} \tag{A2}$$

the ${}^{\mu}{}_{\nu} = {}^{0}{}_{i}$ equation is

$$\frac{2}{a^2}\left\{\partial_i\dot{H}_L - \frac{\dot{a}}{a}\partial_i A\right\} - \frac{1}{2a^2}\partial^j\partial_j B_i = 8\pi\rho_0(1+\delta)(v_i - B_i) + o(1;\epsilon)\,, \tag{A3}$$

and the ${}^{\mu}{}_{\nu} = {}^{i}{}_{j}$ equation is

$$\begin{aligned} &\frac{1}{a^2}\left\{\partial^k\partial_k(H_L + A) - 2\ddot{H}_L - 4\frac{\dot{a}}{a}\dot{H}_L + 2\frac{\dot{a}}{a}\dot{A} + 4\partial_\tau\left(\frac{\dot{a}}{a}\right)A + 2\left(\frac{\dot{a}}{a}\right)^2 A\right\}\delta^i{}_j \\ &+\frac{1}{a^2}\left\{-4H_L\partial^k\partial_k H_L - 2A\partial^k\partial_k A - 2H_L\partial^k\partial_k A - 2\partial_k H_L\partial^k H_L - \partial_k A\partial^k A\right\}\delta^i{}_j \\ &-\frac{1}{a^2}\partial^i\partial_j(H_L + A) + \frac{1}{a^2}\left\{4H_L\partial^i\partial_j H_L + 2H_L\partial^i\partial_j A + 2A\partial^i\partial_j A + \partial^i A\partial_j A\right. \\ &\left.+\partial^i A\partial_j H_L + \partial^i H_L\partial_j A + 3\partial^i H_L\partial_j H_L\right\} \\ &+\frac{1}{2a^2}\left\{\partial^i\dot{B}_j + \partial_j\dot{B}^i + 2\frac{\dot{a}}{a}\partial^i B_j + 2\frac{\dot{a}}{a}\partial_j B^i\right\} + \frac{1}{a^2}\left\{\ddot{h}^i{}_j + 2\frac{\dot{a}}{a}\dot{h}^i{}_j - \partial^k\partial_k h^i{}_j\right\} \\ &= 8\pi\rho_0(1+\delta)v^i(v_j - B_j) + o(1;\epsilon)\,. \end{aligned} \tag{A4}$$

Appendix B: Modified background metric

In this Appendix we compute the homogeneous part of the metric and matter distribution as given by our final dictionary (4.1)–(4.5). These can be viewed as providing the dominant

corrections to the background cosmology produced by small scale inhomogeneities. The relevant equations can be obtained by taking spatial integrals of the equations of section IV. We find that the spatially homogeneous parts of the metric components are given by

$$\overline{A} = -\overline{X}\,, \tag{B1}$$

$$\overline{H}_L = -\overline{X}\,, \tag{B2}$$

$$\overline{h}_{ij} = \overline{J}_{ij}\,, \tag{B3}$$

as well as $\overline{B}^i = 0$ (see footnote 5). Thus, the homogeneous part of the metric takes the form

$$\overline{ds}^2 = a^2(\tau)\left[-(1-2\overline{X})d\tau^2 + ((1-2\overline{X})\delta_{ij} + \overline{J}_{ij})dx^i dx^j\right]\,. \tag{B4}$$

We also have

$$\overline{\delta} = \overline{\Delta}\,, \tag{B5}$$

$$\overline{P}^i = 0\,, \tag{B6}$$

and

$$\overline{(1+\delta)(v_i - B_i)} = 0\,. \tag{B7}$$

In addition, the quantities $\overline{X}$, $\overline{\Delta}$, and $\overline{J}_{ij}$ satisfy

$$2\ddot{\overline{X}} + 2\frac{\dot{a}}{a}\dot{\overline{X}} - \left[4\partial_\tau\left(\frac{\dot{a}}{a}\right) + 2\left(\frac{\dot{a}}{a}\right)^2\right]\overline{X} = -\frac{1}{3}\overline{\partial^i\psi_N\partial_i\psi_N} + \frac{8\pi}{3}a^2\rho_0\overline{(1+\delta_N)v_N^i v_{Ni}}\,, \tag{B8}$$

$$\overline{\Delta} = -\overline{(1+\delta_N)v_N^i v_{Ni}} - \frac{5}{8\pi a^2\rho_0}\overline{\partial^i\psi_N\partial_i\psi_N} - \frac{3}{8\pi\rho_0 a^2}\left\{2\frac{\dot{a}}{a}\dot{\overline{X}} - 2\left(\frac{\dot{a}}{a}\right)^2\overline{X}\right\}\,, \tag{B9}$$

and

$$\frac{1}{a^2}\left\{\ddot{\overline{J}}_j^{\,i} + \frac{\dot{a}}{a}\dot{\overline{J}}_j^{\,i}\right\} = \left\{\frac{2}{a^2}\overline{\partial^i\psi_N\partial_j\psi_N} + 8\pi\rho_0\overline{(1+\delta_N)v_N^i v_{Nj}}\right\}\Bigg|_{\rm t}\,. \tag{B10}$$

It is clear that the metric perturbation given by $\overline{X}$ can be interpreted as taking one to a new FLRW spacetime, with scale factor

$$\hat{a}(\tau) = a(\tau)(1-\overline{X})\,. \tag{B11}$$

We now derive modified Friedmann equations for $\hat{a}$. To linear order in barred quantities, we have

$$\frac{1}{\hat{a}}\frac{d\hat{a}}{d\tau} = \frac{1}{a}\frac{da}{d\tau} - \dot{\overline{X}}\,, \tag{B12}$$

so

$$\left(\frac{1}{\hat{a}}\frac{d\hat{a}}{d\tau}\right)^2 = \left(\frac{1}{a}\frac{da}{d\tau}\right)^2 - 2\frac{\dot{a}}{a}\dot{\overline{X}}$$
$$= \frac{8\pi\rho_0\hat{a}^2}{3}\left(1+\overline{\Delta}+\overline{(1+\delta_N)v_N^i v_{Ni}}\right) + \frac{5}{3}\overline{\partial^i\psi_N\partial_i\psi_N} + \frac{\Lambda\hat{a}^2}{3}\,. \qquad \text{(B13)}$$

Here, we made use of the Friedmann equation for a as well as eq. (B9). Similarly, we have

$$\frac{d}{d\tau}\left(\frac{1}{\hat{a}}\frac{d\hat{a}}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{1}{a}\frac{da}{d\tau}\right) - \ddot{\overline{X}}$$
$$= -\frac{4\pi\rho_0\hat{a}^2}{3}\left(1+\overline{\Delta}+2\overline{(1+\delta_N)v_N^i v_{Ni}}\right) - \frac{2}{3}\overline{\partial^i\psi_N\partial_i\psi_N} + \frac{\Lambda\hat{a}^2}{3}\,. \qquad \text{(B14)}$$

To put these equations in a more recognizable form, we note that for dust matter, $\nabla_a(\rho u^a) = 0$, so the integrated flux of ρu^a over a Cauchy surface Σ

$$\mathcal{N} = -\int_\Sigma \rho u^d \epsilon_{dabc} \qquad \text{(B15)}$$

is a constant, i.e., independent of Σ. $\mathcal{N}$ is often referred to as the "total number of baryons"; in an N-body simulation, it would correspond to the total number of particles in the simulation. Evaluating the right side of (B15), we obtain

$$\mathcal{N} = \rho_0 a^3\left(1+\overline{\Delta}+\frac{1}{2}\overline{(1+\delta_N)v_N^i v_{Ni}} + \frac{3}{4\pi\rho_0 a^2}\overline{\partial^i\psi_N\partial_i\psi_N} - 3\overline{X}\right)\,. \qquad \text{(B16)}$$

It is natural to use our freedom in choosing initial conditions for a solution to (B8) to require $\mathcal{N} = \mathcal{N}_0 = \rho_0 a^3$, so that the total number of particles is the same as in the background spacetime. This condition yields

$$\overline{\Delta} = -\frac{1}{2}\overline{(1+\delta_N)v_N^i v_{Ni}} - \frac{3}{4\pi\rho_0 a^2}\overline{\partial^i\psi_N\partial_i\psi_N} + 3\overline{X}\,. \qquad \text{(B17)}$$

Combining this equation with (B9), we obtain

$$0 = -\frac{1}{2}\overline{(1+\delta_N)v_N^i v_{Ni}} + \frac{1}{8\pi a^2\rho_0}\overline{\partial^i\psi_N\partial_i\psi_N} - 3\overline{X} - \frac{3}{8\pi a^2\rho_0}\left\{2\frac{\dot{a}}{a}\dot{\overline{X}} - 2\left(\frac{\dot{a}}{a}\right)^2\overline{X}\right\}\,, \qquad \text{(B18)}$$

whose time derivative is (B8).

We define the average particle number density $\hat{\rho}$ relative to $\hat{a}$ by

$$\hat{\rho}\hat{a}^3 = \mathcal{N} = \mathcal{N}_0 = \rho_0 a^3\,. \qquad \text{(B19)}$$

In terms of $\hat{\rho}$, the Friedmann equations become

$$\left(\frac{1}{\hat{a}}\frac{d\hat{a}}{d\tau}\right)^2 = \frac{8\pi\hat{\rho}\hat{a}^2}{3}\left(1+\frac{1}{2}\overline{(1+\delta_N)v_N^i v_{Ni}} - \frac{1}{8\pi\hat{\rho}\hat{a}^2}\overline{\partial^i\psi_N\partial_i\psi_N}\right) + \frac{\Lambda\hat{a}^2}{3}\,, \tag{B20}$$

$$\frac{d}{d\tau}\left(\frac{1}{\hat{a}}\frac{d\hat{a}}{d\tau}\right) = -\frac{4\pi\hat{\rho}\hat{a}^2}{3}\left(1+\frac{3}{2}\overline{(1+\delta_N)v_N^i v_{Ni}} - \frac{1}{4\pi\hat{\rho}\hat{a}^2}\overline{\partial^i\psi_N\partial_i\psi_N}\right) + \frac{\Lambda\hat{a}^2}{3}\,. \tag{B21}$$

From these equations, one can read off the effective energy density and pressure, including the contributions from small scale inhomogeneities,

$$\rho_{\rm eff} = \hat{\rho}\left(1+\frac{1}{2}\overline{(1+\delta_N)v_N^i v_{Ni}} - \frac{1}{8\pi\hat{\rho}\hat{a}^2}\overline{\partial^i\psi_N\partial_i\psi_N}\right)\,, \tag{B22}$$

$$P_{\rm eff} = \hat{\rho}\left(\frac{1}{3}\overline{(1+\delta_N)v_N^i v_{Ni}} - \frac{1}{24\pi\hat{\rho}\hat{a}^2}\overline{\partial^i\psi_N\partial_i\psi_N}\right)\,. \tag{B23}$$

The correction terms in $\rho_{\rm eff}$ correspond precisely to averaged gravitational potential energy and kinetic energy, as expected [1, 10, 13]. They can be interpreted as renormalizing the proper mass density $\hat{\rho}$ to an "ADM mass density" $\rho_{\rm eff}$. For virialized systems, the correction terms in $P_{\rm eff}$ cancel, as pointed out in [13]. Thus, we see that the corrections resulting from $\overline{X}$ and $\overline{\Delta}$ correspond to modifying the FLRW background to a new FLRW spacetime with small corrections to the average effective mass density and pressure that arise from small scale Newtonian gravitational potential energy and stresses as well as small scale kinetic motions.

The remaining corrections due to $\overline{J_{ij}}$ perturb one to an anisotropically expanding Bianchi model. It can be seen from (B10) that anisotropies in the spatial average of the Newtonian stresses and/or kinetic motions must necessarily induce an anisotropic expansion of the universe. However, we would expect these effects to be extremely small.

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