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Phys. Rev. D **85**, 045004 — Published 6 February 2012

DOI: 10.1103/PhysRevD.85.045004

FTPI-MINN-11/28, UMN-TH-3020/11

$\mathcal{N} = (0,2)$ Deformation of CP(1) Model: Two-dimensional Analog of $\mathcal{N} = 1$ Yang-Mills Theory in Four Dimensions

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Abstract

We consider two-dimensional $\mathcal{N} = (0,2)$ sigma models with the $\mathbf{CP}(1)$ target space. A minimal model of this type has one left-handed fermion. Nonminimal extensions contain, in addition, N_f right-handed fermions. Our task is to derive expressions for the β functions valid to all orders. To this end we use a variety of methods: (i) perturbative analysis; (ii) instanton calculus; (iii) analysis of the supercurrent supermultiplet (the so-called hypercurrent) and its anomalies, and some other arguments. All these arguments, combined, indicate a direct parallel between the heterotic $\mathcal{N} = (0,2)$ $\mathbf{CP}(1)$ models and four-dimensional super-Yang–Mills theories. In particular, the minimal $\mathcal{N} = (0,2)$ $\mathbf{CP}(1)$ model is similar to $\mathcal{N} = 1$ supersymmetric gluodynamics. Its exact β function can be found; it has the structure of the Novikov–Shifman–Vainshtein–Zakharov (NSVZ) β function of supersymmetric gluodynamics. The passage to nonminimal $\mathcal{N} = (0,2)$ sigma models is equivalent to adding matter. In this case an NSVZ-type exact relation between the β function and the anomalous dimensions γ of the "matter" fields is established. We derive an analog of the Konishi anomaly. At large N_f our β function develops an infrared fixed point at small values of the coupling constant (analogous to the Banks–Zaks fixed point). Thus, we reliably predict the existence of a conformal window. At $N_f = 1$ the model under consideration reduces to the well-known $\mathcal{N} = (2,2)$ $\mathbf{CP}(1)$ model.

1 Introduction

This paper could have been called "Perturbative and nonperturbative aspects of $\mathcal{N} = (0, 2)$ sigma models: the β function, Konishi anomaly, conformal window and all that in $\mathbf{CP}(1)$." 2D-4D correspondence is a popular topic in the current literature. Its discussion has a long history, see e.g. [1, 2, 3, 4, 5, 6, 7]. Most theoretical efforts were focused on a relation between four-dimensional $\mathcal{N} = 2$ SQCD and two-dimensional $\mathcal{N} = (2, 2)$ sigma models. The former support non-Abelian strings [1, 3]. The latter appear as low-energy effective theories on the non-Abelian string world sheet. It is not surprising then that the BPS-protected sectors of the 4D parents and 2D daughter theories are related. For more details on this, the readers are referred to [8].

Later on, the bulk theories supporting non-Abelian strings were deformed to break $\mathcal{N} = 2$ in 4D down to $\mathcal{N} = 1$. It was found [9, 10] that the low-energy theories on the string world sheet are no longer $\mathcal{N} = (2, 2)$ supersymmetric. Instead, one gets $\mathcal{N} = (0, 2)$ heterotic sigma models with the $\mathbf{CP}(N - 1)$ target space. This finding gave a strong impetus to explorations of these heterotic models which had been previously discussed only in general terms [11, 12, 13, 14, 15].

In this paper we will study two-dimensional $\mathcal{N} = (0, 2)$ sigma models with the $\mathbf{CP}(1)$ target space. A minimal model of this type has one left-handed fermion which, together with a complex scalar field, enters an $\mathcal{N} = (0, 2)$ chiral superfield. This minimal model can be readily extended. Nonminimal extensions contain, in addition, N_f right-handed fermions. In particular, if $N_f = 1$, the nonminimal model under consideration reduces to the conventional $\mathcal{N} = (2, 2)$ $\mathbf{CP}(1)$ model.

In this paper we will focus on various derivations of exact expressions for the β functions (valid to all orders in the $\mathbf{CP}(1)$ coupling). Remarkably, our results will exhibit a direct parallel between the heterotic $\mathcal{N} = (0, 2)$ $\mathbf{CP}(1)$ models and four-dimensional super-Yang–Mills theories. In particular, the minimal $\mathcal{N} = (0, 2)$ $\mathbf{CP}(1)$ model is similar to $\mathcal{N} = 1$ supersymmetric gluodynamics. Its exact β function can be found; it has the structure of the Novikov–Shifman–Vainshtein–Zakharov (NSVZ) β function [16, 17] in supersymmetric Yang–Mills theory without matter. Then we pass to nonminimal $\mathcal{N} = (0, 2)$ sigma models. It turns out that this passage corresponds to adding (adjoint) matter in four-dimensional super-Yang–Mills theory. Thus, in the nonminimal $\mathcal{N} = (0, 2)$ $\mathbf{CP}(1)$ models we will obtain an NSVZ-type exact relation between the β function and the anomalous dimensions γ of the "matter" fields.

Our arguments will be based on a number of methods. First, we will carry out a perturbative (super)graph analysis. This will allow us to obtain the β functions at two-loop level. Comparison with the $N_f = 1$ case which is in fact $\mathcal{N} = (2, 2)$ will give us the first indication on the emergence of the NSVZ-type β function.

Then we will study the instanton measure, using parallels with the analogous NSVZ derivation. We will obtain a version of nonrenormalization theorem in the instanton background. Essentially we will demonstrate that the instanton measure is exhausted by a one-loop calculation, in much the same way as it was the case in 4D super-Yang–Mills theories [16] and in 2D $\mathcal{N} = (2,2)$ sigma models [18]. From this result one can readily deduce a β function of the NSVZ type.

Our third argument is based on the analysis of the supercurrent supermultiplet (the so-called hypercurrent) and its anomalies. Not only will the NSVZ β function be confirmed, but, in addition we will understand the difference between the holomorphic and canonic couplings, which is exactly the same as in the 4D super-Yang–Mills [19]. *En route* we will derive a 2D analog of the Konishi anomaly. This is a necessary element of the β function derivation through the hypercurrent anomaly. The exact formula that we obtain relates the β function of the nonminimal models with the anomalous dimension of the "matter fields." The latter is known as an expansion in perturbation theory.

At large N_f our β function develops an infrared fixed point at small values of the coupling constant (analogous to the Banks–Zaks fixed point [20] [1]). Since the position of this fixed point is at $g^2 \sim 1/N_f$, we can use the leading-order result for the anomalous dimension to *prove* the existence of the fixed point. In other words, in the nonminimal models a conformal window exists starting from some critical value N_f^*. Near the lower edge of the conformal window the theory is presumably strongly coupled.

One can ask a natural question: why we consider only the **CP**(1) model and do not generalize to CP$(N-1)$ with arbitrary N? This is due to an anomaly in heterotic models pointed out in [21]. This anomaly prevents us from considering the models we study in this paper for arbitrary N. However, some other nonminimal generalization of the $\mathcal{N} = (0,2)$ CP$(N-1)$ models will be studied in our forthcoming work [22].

The structure of this paper is as follows. We formulate the minimal $\mathcal{N} = (0,2)$ models in Sec. 2. In Sec. 3 we carry out perturbative calculations of the β function up to two-loop order, in superfield formalism, as outlined in [23]. In Sec. 4 we start studying nonperturbative effects in the minimal model (instanton and its measure). We construct exact instanton measure. In this construction we take into account zero modes, one-loop effects in the instanton background, and then, following NSVZ [24], use a nonrenormalization theorem for two and more loops. The instanton background gives us a particularly clear way to see the cancelation of higher loops. The all

[1] More exactly, it should have been referred to as the Belavin–Migdal–Banks–Zaks.

loop exact β function is presented in Sec. 4.1. In Sec. 5 we calculate explicitly the supercurrent supermultiplet for this model. In Sec. 6 we extend the minimal model by adding “matter”, i.e. the right-handed fermion fields. Following the same road as in the minimal model, we calculate the two-loop β function perturbatively in the nonminimal model. Then we exploit the instanton analysis to obtain an exact relation between the β function and the anomalous dimension γ of the “matter” fields. In Sec. 7 we calculate the supercurrent supermultiplets for the extended (nonminimal) models. Section 8 is devoted to a 2D analog of the Konishi anomaly in the extended models. Finally, Sec. 9 demonstrates the appearance of a conformal window. Main conclusions and prospects for future explorations are summarized in Sec. 10.

2 Formulation of the minimal heterotic CP(1) model

In this section we will formulate the minimal $\mathcal{N}=(0,2)$ **CP**(1) sigma model (previously it was studied e.g. in [12, 14]). We will use $\mathcal{N}=(0,2)$ superfield formalism. Note that due to anomaly [21] it is impossible to generalize this model to $\mathbf{CP}(N-1)$.

The Lagrangian of the model under consideration is

$$\mathcal{L}_A=\frac{1}{g^2}\int d^2\theta_R\frac{A^\dagger i\overset{\leftrightarrow}{\partial}_{RR}A}{1+A^\dagger A}\,, \tag{1}$$

where A is a bosonic chiral superfield:

$$A(x,\theta_R^\dagger,\theta_R)=\phi(x)+\sqrt{2}\theta_R\psi_L(x)+i\theta_R^\dagger\theta_R\partial_{LL}\phi\,, \tag{2}$$

ϕ is a complex scalar, and ψ_L is a left-handed Weyl fermion. The superfield A can be understood as taking values on the **CP**(1) manifold, and, thus, can be endowed with the following nonlinear transformations:

$$A\to A+\epsilon+\bar\epsilon A^2\,,\quad A^\dagger\to A^\dagger+\bar\epsilon+\epsilon(A^\dagger)^2\,, \tag{3}$$

plus a U(1) rotation.

In components, we can write the Lagrangian as

$$G\left\{\partial^\mu\phi\partial_\mu\phi^\dagger+i\psi_L^\dagger\overset{\leftrightarrow}{\partial}_{RR}\psi_L-2i\,\frac{1}{\chi}\,\psi_L^\dagger\psi_L\,\phi^\dagger\overset{\leftrightarrow}{\partial}_{RR}\phi\right\}\,. \tag{4}$$

The derivatives ∂_{RR} and ∂_{LL} are defined in Appendix A, see Eq. (A.3). Here we denote by G the Kähler metric on the target space (S^2 in the case at hand), in the Fubini–Study form,

$$G = \frac{2}{g^2\,\chi^2}\,, \tag{5}$$

where

$$\chi \equiv 1 + \phi\,\phi^\dagger\,. \tag{6}$$

Moreover, R is the Ricci tensor,

$$R = \frac{2}{\chi^2}\,, \tag{7}$$

while g^2 is the coupling constant.

The coupling constant g can be complexified. In what follows we will deal with the holomorphic coupling g_h defined as

$$\frac{2}{g_h^2} = \frac{2}{g^2} + i\frac{\omega}{2\pi}\,. \tag{8}$$

In terms of the holomorphic coupling the Lagrangian of the minimal model has the form

$$\begin{aligned}\mathcal{L}_A &= \int d^2\theta_R \frac{i}{2g_h^2}\frac{A^\dagger \partial_{RR} A}{1+A^\dagger A} + \text{H.c.} \\ &= -\frac{i}{2g_h^2}\int d\theta_R \frac{\bar{D}_L A^\dagger \partial_{RR} A}{(1+A^\dagger A)^2} + \frac{i}{2\bar{g}_h^2}\int d\theta_R^\dagger \frac{D_L A \partial_{RR} A^\dagger}{(1+A^\dagger A)^2}\,.\end{aligned} \tag{9}$$

The target space invariance of the integrand is maintained in the second line. In perturbative loop calculations and in instanton analysis we will use the canonical coupling g. To differentiate between the bare and renormalized couplings we will use subscripts 0 and r where appropriate.

In Sect. 6 we will extend this minimal model by adding N_f "matter" fields.

3 Perturbative superfield calculation of the β function

Fermions do not contribute to the β function at one loop (see e.g. [24]). Therefore, the first coefficient of the β function in the minimal heterotic model is the same as

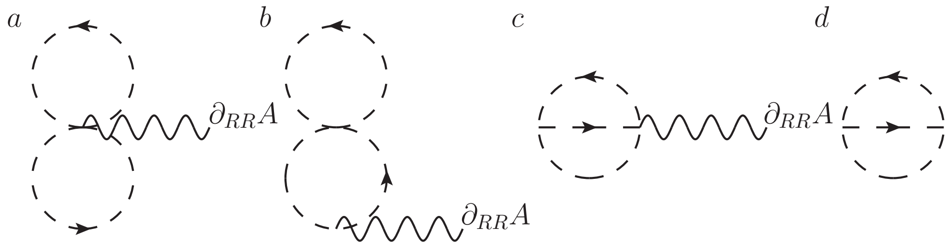

Figure 1: Two-loop correction to the coupling g by A-loops. The dashed line denotes the propagator of the quantum part of A in the chosen background, while the wave line denotes the background field.

in the nonsupersymmetric $\mathbf{CP}(1)$ model (see [24, 25]). The first nontrivial task to address is the calculation of the second coefficient.

In this section we will use the superfield method to calculate the two-loop β function in the minimal model. We will use a linear background field method, setting the background field

$$A_{bk} = f e^{-ix\cdot k}\,.$$

The basic method is roughly the same as that in [24]. The superfield calculation was outlined in our previous paper [23]. We expand the action around the chosen background, splitting the superfield A in two parts, classical (background) and quantum. Then we calculate relevant diagrams with quantum fields in loops.

If we limit ourselves to the origin in the target space (i.e. $\phi = 0$) and forgo the check of the target space invariance , at two-loop order the β function is determined by the diagrams in Fig. 1. As previously we use the ϵ regularization, where

$$\epsilon = 2 - d\,.$$

The last diagram does not explicitly exhibit $\partial_{RR}A$ as the external line, but it does produce a contribution due to momentum insertion. A quick evaluation tells us that the first three diagrams contribute only double poles, and they cancel among themselves, as they should. The graph-by-graph results are listed in Table 1, where the following notation is used

$$I \equiv \int \frac{d^{2-\epsilon}p}{(2\pi)^d}\,\frac{1}{p^2 - m^2}\,. \tag{10}$$

In logarithmically divergent graphs the following correspondence takes place at one loop:

$$\frac{1}{\epsilon} \leftrightarrow \ln\frac{M}{m}\,, \tag{11}$$

Diagram	Double pole	Single pole
a	$-\frac{3}{2}\frac{g^2}{2}\frac{A^\dagger i\partial_{RR}A}{1+A^\dagger A}I^2$+H.c.	0
b	$\frac{g^2}{2}\frac{A^\dagger i\partial_{RR}A}{1+A^\dagger A}I^2$+H.c.	0
c	$\frac{1}{2}\frac{g^2}{2}\frac{A^\dagger i\partial_{RR}A}{1+A^\dagger A}I^2$+H.c.	0
d	0	$-\frac{g^2}{2}\frac{i}{4\pi}\frac{A^\dagger i\partial_{RR}A}{1+A^\dagger A}I$+H.c.

Table 1: Two-loop calculation of g^{-2} renormalization in the ϵ expansion follows that in Figure 1. $I = \int \frac{d^{2-\epsilon}p}{(2\pi)^d}\frac{1}{p^2-m^2}$, see [25].

where the left-hand side represents dimensional regularization, while the right-hand side the Pauli–Villars regularization; M is the mass of the Pauli–Villars regulator. The remaining contribution due to the last diagram results in the following two-loop β function:

$$\frac{1}{g_r^2} = \frac{1}{g_0^2}\left(1 - ig_0^2 I - \frac{i}{4\pi}g_0^4 I\right), \tag{12}$$

or

$$\beta(g^2) = -\frac{g^4}{2\pi}\left(1 + \frac{g^2}{4\pi}\right). \tag{13}$$

Below we will argue that higher loops iterate the two-loop expression in a geometrical progression, so that the full result for the β function in the minimal heterotic model is

$$\beta(g^2) = -\frac{g^4}{2\pi}\left(1 - \frac{g^2}{4\pi}\right)^{-1}. \tag{14}$$

A parallel with the NSVZ β function in supersymmetric gluodynamics [16, 17] is evident.

4 Non-perturbative calculation through the instanton measure

Bosonic $\mathbf{CP}(N-1)$ models exhibit instanton solutions [26, 27]. Hence, this is also the case for the $\mathcal{N} = (0, 2)$ models. For $\mathbf{CP}(1)$, the bosonic (anti-)instanton solution with the unit topological charge is

$$\phi = \frac{y}{z - z_0}\,, \quad \phi^\dagger = \frac{\bar{y}}{\bar{z} - \bar{z}_0}\,, \tag{15}$$

where y and z_0 are the collective coordinates: z_0 is the instanton center while a complex number y parametrizes its size and a U(1) phase. Our notation in Euclidean space-time is explained in Appendix B to which the reader is referred to for further details. We easily get the bosonic zero modes, by taking derivatives of the instanton solution with respect to the above collective coordinates. There are four (real) zero modes, or, two complex [24].

The fermion zero modes can be obtained by applying supersymmetry and superconformal symmetry. From the supersymmetry transformation induced by $Q^\dagger$, one obtains the following fermion zero mode:

$$\psi^\dagger_{\bar{z}} = \frac{\bar{y}\alpha}{(\bar{z} - \bar{z}_0)^2}\,. \tag{16}$$

From the superconformal transformation, we get another zero mode,

$$\psi^\dagger_{\bar{z}} = \frac{\bar{y}\beta^\dagger}{\bar{z} - \bar{z}_0}\,. \tag{17}$$

Note that $\mathcal{N} = (0,2)$ theory we deal with *two* fermion zero modes rather than four, which appear in $\mathcal{N} = (2,2)$ **CP**(1) model. The reason is that, involution is lost upon transition to Euclidean space. No zero mode arises from the background $\phi = \frac{y}{z - z_0}$ (see also [14]). This means that the superinstanton under consideration has no collective coordinates $\alpha^\dagger$ and β. The fact that we deal with two rather than four fermion zero modes agrees with the coefficient in the chiral anomaly (see Sec. 5) which is twice smaller in $\mathcal{N} = (0,2)$ compared to $\mathcal{N} = (2,2)$.

Assembling everything together, we obtain the instanton superfield in the form

$$A_{\rm inst} = \frac{y}{z - z_0}\,, \quad A^\dagger_{\rm inst} = \frac{\bar{y}(1 + 4i\theta^\dagger\beta^\dagger)}{\bar{z}_{\rm ch} - \bar{z}_0 - 4i\theta^\dagger\alpha}\,, \tag{18}$$

where [2]

$$\bar{z}_{\rm ch} = \bar{z} - 2i\theta^\dagger\theta\,.$$

[2]Note that in Sect. 4 we will use θ and $\theta^\dagger$ to denote the Grassmannian variables in Euclidean superspace. We intentionally drop the subscript "R" to distinguish from those in Minkowski superspace.

To derive the instanton measure, we need to define the integral over the collective coordinates. To this end, as usual, we proceed from the mode expansion to the collective coordinates of the zero modes (moduli). In particular, as explained in [24], we need to calculate the normalization of the zero modes given by

$$\int dz d\bar{z}\; G_{ij}\delta\phi_i\delta\phi_i^\dagger\,. \tag{19}$$

As a technical point, we note that the two of the bosonic (real) modes (conformal) and the fermionic superconformal mode are actually logarithmically divergent in the infrared under the normalization. However, these divergences are canceled by similar divergences coming from the one-loop contribution due to the nonzero modes. This was explicitly verified in the case of nonsupersymmetric $\mathbf{CP}(1)$ models in [28]; the argument readily extends to the supersymmetric case too.

As it follows from the norm of the modes, each (complex) boson zero mode is accompanied by the factor $2/g^2$ and each (complex) fermion zero mode is accompanied by the factor $g^2/2$. The dependence on the instanton size $|y|$ will be omitted temporarily and recovered at a later stage on the basis of dimension arguments. Hereafter, we will drop the constant numerical factors, since they contribute only to an overall constant. As a result, at this stage we arrive at the following instanton measure

$$d\mu = \;\text{const.}\;\left(\frac{1}{g^2}\right)^{n_b}\left(g^2\right)^{n_f}\; e^{-\frac{4\pi}{g^2}}\; dyd\bar{y}\;\; dz_0 d\bar{z}_0\, d\alpha d\beta^\dagger\,, \tag{20}$$

where $n_b = 2$ and $n_f = 1$. (We hasten to add that this is *not* the final result.)

So far quantum corrections have not yet been discussed. In the $\mathcal{N} = (2, 2)$ model, the one-loop corrections due to the nonzero modes in the instanton background cancel each other completely [24, 18]. In the $\mathcal{N} = (0, 2)$ model this is not quite the case. Let us consider the one-loop effects in more detail. For the nonzero bosonic modes, we will expand the field ϕ as

$$\phi = \phi_{\text{inst}} + \frac{g}{\sqrt{2}}\delta\phi = \phi_{\text{inst}} + \frac{g}{\sqrt{2}}\sum_n \phi_n a_n\,. \tag{21}$$

Note that the part ϕ_{inst} contains the boson zero modes. The functions ϕ_n in the expansion (21) are the eigenfunctions of the operator

$$-\frac{\partial}{\partial z}\frac{1}{\chi^2_{\text{inst}}}\frac{\partial}{\partial\bar{z}}\phi_n = E_n^2\;\frac{\phi_n}{\chi^2_{inst}} \tag{22}$$

normalized by the condition

$$\int\frac{\phi_n^\dagger\phi_n}{\chi^2_{inst}}d^2x = 1\,, \tag{23}$$

where $\chi_{\rm inst} = 1 + \phi_{\rm inst}^\dagger \phi_{\rm inst}$, and E_n^2 is the n-th eigenvalue. At the one-loop level we can rewrite the action as

$$
-\frac{4\pi}{g^2} - \int d^2x \sum_n E_n^2 a_n^\dagger a_n \frac{\phi_n^\dagger \phi_n}{\chi_{\rm inst}^2}\,, \tag{24}
$$

which, according to the standard rule of functional integration, gives

$$
\int [\mathscr{D}\delta\phi][\mathscr{D}\delta\phi^\dagger] \to \left\{ \det\left[-\frac{\partial}{\partial z}\frac{1}{\chi_{inst}^2}\frac{\partial}{\partial \bar z}\right]\right\}^{-1} = \prod_n \frac{1}{E_n^2}\,. \tag{25}
$$

As for the fermion nonzero modes, we perform a similar expansion. Note that after the Wick rotation, the left-handed fermion ψ_L is no longer related to $\psi_L^\dagger$ by Hermitian conjugation. Therefore in this section we will use $\psi_{\bar z}$ and $\psi_z^\dagger$, respectively, to denote them.

Consider the expansion for the fermion fields

$$
\psi_{\bar z} = \psi_{\bar z,{\rm inst}} + \frac{g}{\sqrt{2}}\sum_n b_n u_n\,, \quad \psi_z^\dagger = \psi_{z,{\rm inst}}^\dagger + \frac{g}{\sqrt{2}}\sum_n c_n \bar v_n\,, \tag{26}
$$

where b_n and c_n are complex Grassmannian parameters. The functions u_n and v_n are defined via

$$
i\partial_z \frac{1}{\chi_{\rm inst}^2} u_n = \frac{\mathcal{E}_n}{\chi_{\rm inst}^2} v_n\,, \quad i\partial_{\bar z} v_n = \mathcal{E}_n u_n\,, \tag{27}
$$

subject to the normalization conditions similar to that of ϕ_n. The part of the action that contains ψ_L now becomes

$$
\int d^2x\; i\psi_z^\dagger \partial_z \frac{2}{g^2\chi_{\rm inst}^2}\psi_{\bar z} = \int d^2x\; i\sum_n \mathcal{E}_n c_n b_n \frac{\bar v_n v_n}{\chi_{\rm inst}^2}\,. \tag{28}
$$

Therefore, integration over the Grassmannian parameters yields $\prod_n \mathcal{E}_n$. Note that in solving Eq. (27) we obtain

$$
-\partial_z \frac{1}{\chi_{\rm inst}^2}\partial_{\bar z} v_n = \frac{\mathcal{E}_n^2}{\chi_{\rm inst}^2} v_n\,, \tag{29}
$$

which is exactly the same as the equation that defines ϕ_n. Hence the boson-fermion degeneracy follows,

$$
\mathcal{E}_n^2 = E_n^2\,.
$$

In principle the eigenvalue $\mathcal{E}_n$ could be both positive and negative. Let us elucidate this subtle point. In fact, here we are double counting the eigenstates in calculating the fermion determinant. It is easy to see that this is the case if we turn first to the $\mathcal{N}=(2,2)$ theory. There, the relevant action is given by

$$\int d^2x \left(i\sum_n \mathcal{E}_n c_n b_n \frac{\bar{v}_n v_n}{\chi^2_{\rm inst}} + i\sum_n \mathcal{E}_n \bar{b}_n \bar{c}_n \frac{\bar{u}_n u_n}{\chi^2_{\rm inst}} \right). \tag{30}$$

Integration runs over four Grassmann parameters (at each level),

$$\prod_n db_n d\bar{b}_n dc_n d\bar{c}_n \,, \tag{31}$$

In the $\mathcal{N}=(0,2)$ case we have to identify

$$b_n \leftrightarrow \bar{c}_n \,, \quad c_n \leftrightarrow \bar{b}_n \,. \tag{32}$$

In other words, we should count only the field configurations that correspond either to $\mathcal{E}_n$ or to $-\mathcal{E}_n$ (assuming $E_n>0$).

As a result, in our $\mathcal{N}=(0,2)$ theory, $\prod \mathcal{E}_n$ should be understood as $(\prod E_n^2)^{\frac{1}{2}}$, and, hence, symbolically we can write

$$\int [\mathscr{D}\delta\psi_{\bar{z}}][\mathscr{D}\delta\psi^\dagger_z] \to \left\{ \det \left[\begin{array}{cc} 0 & i\partial_z \frac{1}{\chi^2_{\rm inst}} \\ i\frac{1}{\chi^2_{\rm inst}}\partial_{\bar{z}} & 0 \end{array} \right] \right\}^{\frac{1}{2}} = \left(\prod_n E_n^2 \right)^{\frac{1}{2}} . \tag{33}$$

The product runs over the nonzero modes.

As a result, due to the lack of balance between the numbers of the modes (bosonic versus fermionic), we do not have complete cancelation of the one-loop correction coming from the boson nonzero modes by that coming from the fermion nonzero modes. This is in contradistinction with the situation in the $\mathcal{N}=(2,2)$ theory.

We have to evaluate the one-loop contribution from the nonzero modes in the instanton background. In four dimensions this kind of calculation was carried out in [29], and in pure bosonic **CP**(1) model it has been done in [28]. All we need to know is the general form of the one-loop correction due to nonzero modes in the instanton measure, $\exp\left(\text{const.} \log \frac{M}{|y|}\right)$, with no explicit g^2 dependence.[3] Here M is mass of the ultraviolet (UV) regulator of the theory. Thus, the one-loop effect will bring us a prefactor M^κ. We will determine it using our knowledge of the bosonic **CP**(1) model.

Explicitly, we can write down the instanton measure for the bosonic **CP**(1) model to one-loop order, which is given in [28] and also entirely fixed by the β function at

[3] See, however, a remark after Eq. (19).

two-loop level. We will postpone the second derivation till Sec. 4.1, and just show the final result. The measure is

$$d\mu \sim \left(\frac{M^2}{g^2}\right)^{n_b} M^{-2}\, dy d\bar{y}\; dz_0 d\bar{z}_0\,, \qquad n_b = 2\,, \tag{34}$$

where the factor M^{-2} comes from the one-loop correction due to the nonzero modes, and, hence, $\prod_n E_n^{-2} = M^{-2}$. Given Eq. (33), we immediately conclude that the one-loop correction to the instanton measure in our $\mathcal{N} = (0,2)$ model is M^{-1}.

With this knowledge in hand we can return to Eq. (20) which contains only zero modes. After inserting nonzero mode one-loop effects we find the instanton measure in the form

$$d\mu \sim \left(\frac{M^2}{g^2}\right)^{n_b} \left(\frac{g^2}{M}\right)^{n_f} M^{-1} e^{-\frac{4\pi}{g^2}}\; dy d\bar{y}\; dz_0 d\bar{z}_0 d\;\alpha d\beta^\dagger\,, \tag{35}$$

with $n_b = 2$ and $n_f = 1$. As we will argue in Sect. 4.1, this is the exact formula.

Finally, note that the instanton measure is dimensionless. Therefore, we need to reinstate an appropriate dimensional parameter. There is a unique choice, the instanton size, which is, simultaneously, the infrared cutoff in the instanton calculation. It is given by $|y|$.

This leaves us with the following master formula for the measure in the $\mathcal{N} = (0,2)$ $\mathbf{CP}(1)$ model:

$$\begin{aligned} d\mu &= \left(\frac{M^2}{g^2}\right)^{n_b} \left(\frac{g^2}{M}\right)^{n_f} (M)^{-1}\, e^{-\frac{4\pi}{g^2}}\; d\log(y) d\log(\bar{y})\; dz_0 d\bar{z}_0\; d\alpha d\beta^\dagger\,, \\ n_b &= 2\,, \quad n_f = 1\,. \end{aligned} \tag{36}$$

4.1 A nonrenormalization theorem

This is not the end of the story, however. We have to address the question of two- and higher-loop corrections in the instanton background. In this subsection we will argue that they vanish. Are arguments are intended to show that the instanton measure in (36) is all loop exact, i.e., it does not receive higher loop corrections. The proof is a version of the nonrenormalization theorem [16, 23].

Let us recall that in the instanton background superfield $A_{\rm inst}$ and $A^\dagger_{\rm inst}$, we can apply supersymmetry transformation given by

$$\theta \to \theta + \epsilon\,, \quad \theta^\dagger \to \theta^\dagger + \epsilon^\dagger\,, \quad \bar{z}_{\rm ch} \to \bar{z}_{\rm ch} + 4i\epsilon\theta^\dagger\,. \tag{37}$$

Under such a transformation the superinstanton transforms as

$$A_{\rm inst} \;=\; \frac{y}{z-z_0} \to \frac{y}{z-z_0}\,, \tag{38}$$

$$A^\dagger_{\rm inst} \;=\; \frac{\bar y(1+4i\theta^\dagger\beta^\dagger)}{\bar z_{\rm ch}-\bar z_0-4i\theta^\dagger\alpha} \to \frac{\bar y(1+4i\theta^\dagger\beta^\dagger+4i\epsilon^\dagger\beta^\dagger)}{\bar z_{\rm ch}-\bar z_0+4i\epsilon\theta^\dagger-4i\theta^\dagger\alpha-4i\epsilon^\dagger\alpha}\,. \tag{39}$$

To make the background invariant under such transformations, one can assign appropriate transformation laws to the collective coordinates (moduli), namely,

$$\bar y \to \bar y(1+4i\epsilon^\dagger\beta^\dagger)\,,\quad \bar z_0 \to \bar z_0 + 4i\epsilon^\dagger\alpha\,,\quad \alpha\to\alpha+\epsilon\,,\quad \beta^\dagger\to\beta^\dagger\,. \tag{40}$$

Combining (37) and (40) it is not difficult to see that the instanton field configuration remains intact when we apply the supersymmetry transformations. Moreover, our expression for the integration measure over the collective coordinates, (36), is invariant too. This implies the following. Supersymmetry understood as a combined action of (37) and (40) is preserved classically by our chosen instanton background, and, hence, it will be preserved in loops. In particular, this forbids any correction to (36) of the form [4] $1+cg^2\,\log(M^2|y|^2)$, since such a term would change the power of $\bar y$, that would be in contradiction with (40).

Moreover, nonlogarithmic corrections of the type $1+cg^2+c'g^4+\cdots$ also do not show up in multiloop calculations. This follows from a nonrenormalization argument similar to [23]. Consider a correction that could possibly come from two or more loops. In this case we can always write the loop integration in the form

$$\int d^2z d\theta d\theta^\dagger f(x,\theta,\theta^\dagger,z_0,\bar z_0,y,\bar y,\alpha,\beta^\dagger)\,, \tag{41}$$

where the function f must be be invariant under supersymmetry transformations supplemented by (40). This tells us that f can only be a function of the following (invariant) arguments:

$$y\,,\quad \bar y(1+4i\beta^\dagger\theta^\dagger)\,,\quad z-z_0\,,\quad \bar z_{\rm ch}-\bar z_0+4i\theta^\dagger\alpha\,,\quad \theta-\alpha\,,\quad \beta^\dagger\,. \tag{42}$$

In the subsequent analysis we will only indicate the explicit dependence of f on $\bar y(1+4i\beta^\dagger\theta^\dagger)$, $\bar z_{\rm ch}-\bar z_0+4i\theta^\dagger\alpha$, $\theta-\alpha$ and $\beta^\dagger$. Only these variables will be of importance. Due to the Grassmannian nature of $\theta-\alpha$ and $\beta^\dagger$, the function f can be represented

[4] The scale $|y|$ serves as a natural infrared cutoff. Note that the infrared cutoff is provided by the absolute value of y, while the dependence on the phase angle is trivial.

as a sum of two terms,

$$
\begin{aligned}
& f\left(\bar{y}(1+4i\beta^\dagger\theta^\dagger), \bar{z}_{ch}-\bar{z}_0+4i\theta^\dagger\alpha, \theta-\alpha, \beta^\dagger\right) \\
= \; & f_0\left(\bar{y}(1+4i\beta^\dagger\theta^\dagger), \bar{z}_{ch}-\bar{z}_0+4i\theta^\dagger\alpha\right) \\
& +(\theta-\alpha)\beta^\dagger f_1\left(\bar{y}(1+4i\beta^\dagger\theta^\dagger), \bar{z}_{ch}-\bar{z}_0+4i\theta^\dagger\alpha\right), \qquad (43)
\end{aligned}
$$

where $f_{0,1}$ are some other functions. It is obvious that upon integration over θ, only f_1 can survive, and the integration takes the form

$$
\int d^2z \; d\theta^\dagger \, \beta^\dagger \, f_1\left(\bar{y}, \bar{z}-\bar{z}_0+4i\theta^\dagger\alpha\right). \qquad (44)
$$

Next, we shift $\bar{z}$, and then the remaining integral has to vanish. It vanishes, indeed! Note that the integration is finite and local, hence the shift in $\bar{z}$ must be valid.

4.2 The full β function

Now we know that our expression for the instanton measure is all-loop exact. It depends on the Pauli–Villars regulator mass M explicitly, through M^2, and implicitly, through $g^2(M)$. The overall dependence on M must cancel, i.e,

$$
\frac{d}{d\log(M)}\left(-\frac{4\pi}{g^2}-\log g^2+\log M^2\right)=0\,. \qquad (45)
$$

This gives us the all loop exact β function for the coupling constant g,

$$
\beta(g^2) = -\frac{g^4}{2\pi}\frac{1}{1-\frac{g^2}{4\pi}}\,. \qquad (46)
$$

The two-loop coefficient is in agreement with (13) determined by a direct perturbation calculation.

5 Supercurrent supermultiplet (hypercurrent)

In this section we will analyze the hypercurrent (see [30, 31]) of the minimal model. This will set the stage for an alternative derivation of the β function which will be completed in Sect. 7. Our consideration will run parallel to that of [17].

Classically, the model under consideration has a conserved U(1) current corresponding to rotations of the chiral fermion ψ_L,

$$j_{LL} = G\psi_L^\dagger \psi_L \,. \tag{47}$$

The supercurrent of $\mathcal{N} = (0,2)$ supersymmetry is

$$S_{LLL} = i\sqrt{2}G\partial_{LL}\phi^\dagger \psi_L \,, \qquad S_{LRR} = 0 \,. \tag{48}$$

Finally, the energy momentum tensor of our model has the form

$$\begin{aligned} T_{LLLL} &= -2G\partial_{LL}\phi^\dagger \partial_{LL}\phi - iG\psi_L^\dagger \mathcal{D}_{LL}\psi_L + iG(\mathcal{D}_{LL}\psi_L^\dagger)\psi_L \,, \\ T_{RRRR} &= -2G\partial_{RR}\phi^\dagger \partial_{RR}\phi \,, \\ T_{LLRR} &= 0 \,. \end{aligned} \tag{49}$$

It is easy to see that the three currents, j_{LL}, S_{LLL} and T_{LLLL} form an $\mathcal{N} = (0,2)$ (non-chiral) supermultiplet, which we will denote by $\mathcal{J}_{LL}$ and refer to it as the hypercurrent,

$$\mathcal{J}_{LL} = j_{LL}(x) + i\theta_R S_{LLL}(x) + i\theta_R^\dagger S_{LLL}^\dagger(x) - \theta_R \theta_R^\dagger T_{LLLL}(x) \,. \tag{50}$$

In fact, the above multiplet has a concise superfield expression, namely,

$$\mathcal{J}_{LL} = \frac{1}{2}G(\bar{D}_L A^\dagger)D_L A \,, \tag{51}$$

with the left-handed fermion current as its lowest component.

As was mentioned, the Lagrangian (4) is invariant under U(1) chiral rotations. Therefore, the current j_{LL} is conserved classically, $\partial_{RR} j_{LL} = 0$. This also tells us that $j_{RR} = 0$. Both relations are certainly true at the classical level. In fact, it is obvious that at the classical level the hypercurrent $\mathcal{J}_{LL}$ is conserved as a whole, $\partial_{RR}\mathcal{J}_{LL} = 0$.

The supercurrent conservation is

$$\partial_{LL} S_{RRL} + \partial_{RR} S_{LLL} = 0 \,. \tag{52}$$

Classically we have $S_{RRL} = 0$, and, therefore, the conservation law simplifies, $\partial_{RR} S_{LLL} = 0$. As for the energy-momentum tensor, its conservation tells us that

$$\begin{aligned} \partial_{LL} T_{RRRR} + \partial_{RR} T_{LLRR} &= 0 \,, \\ \partial_{LL} T_{RRLL} + \partial_{RR} T_{LLLL} &= 0 \,. \end{aligned} \tag{53}$$

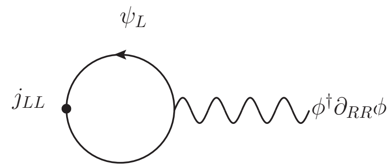

Figure 2: One loop diagram for j_{LL} anomaly.

The condition of tracelessness is

$$T_{LLRR} + T_{RRLL} = 0\,. \tag{54}$$

Moreover, imposing the "symmetrycity" condition on T,

$$T_{LLRR} = T_{RRLL}\,, \tag{55}$$

we obtain that $T_{LLRR} = 0$.

Quantum mechanically (i.e. with loops included) the current j_{LL} is anomalous. It is easy to see that the diagram in Fig. 2 does not vanish,

$$\partial_{RR} j_{LL} = \frac{i}{2\pi}\frac{\partial_{LL}\phi^\dagger\partial_{RR}\phi - \partial_{RR}\phi^\dagger\partial_{LL}\phi}{(1+\phi^\dagger\phi)^2}\,. \tag{56}$$

Much in the same way as in the Adler–Bell–Jackiw anomaly [32] and in [17], the chiral current nonconservation is exhausted by one loop in the *Wilsonian* sense. Invoking the superfield formalism, we can translate (56) in the superfield language,

$$\partial_{RR}\mathcal{J}_{LL} = \frac{1}{4\pi}\left[D_L\frac{\partial_{RR}A\bar{D}_LA^\dagger}{(1+A^\dagger A)^2} - \bar{D}_L\frac{\partial_{RR}A^\dagger\bar{D}_LA}{(1+A^\dagger A)^2}\right]\,, \tag{57}$$

where the right-hand side is *exact* in the Wilsonian sense.

Following the general arguments of [30, 31] it is convenient to rewrite Eq. (57) in a general form

$$\partial_{RR}\mathcal{J}_{LL} = -\frac{1}{2}D_L\mathcal{W}_R + \frac{1}{2}\bar{D}_L\bar{\mathcal{W}}_R\,, \tag{58}$$

where

$$\mathcal{W}_R = -\frac{1}{2\pi}\frac{\partial_{RR}A\bar{D}_LA^\dagger}{(1+A^\dagger A)^2}\,. \tag{59}$$

The superfield $\mathcal{W}_R$ on the right-hand side was absent at the classical level. The expression for $\mathcal{W}_R$ contains S_{LRR} as its lowest component [30], namely,

$$\mathcal{W}_R = -S^\dagger_{LRR} + i\theta_R(T_{LLRR} + i\partial_{RR} j_{5,LL}) + i\theta_R\theta^\dagger_R\partial_{LL}S^\dagger_{LRR}\,. \tag{60}$$

Note that the coefficient in front of $i\theta_R$ contains the real and imaginary parts. The former is the trace of the energy-momentum tensor, the latter is the divergence of the U(1) current.

With this information in hand we finally we arrive at

$$S_{LRR} = \frac{1}{\sqrt{2}\,\pi}\frac{\partial_{RR}\phi^\dagger\psi_L}{(1+\phi^\dagger\phi)^2}\,,$$

$$T_{LLRR} = -\frac{1}{2\pi}\left[\frac{\partial_{LL}\phi^\dagger\partial_{RR}\phi + \partial_{RR}\phi^\dagger\partial_{LL}\phi}{(1+\phi^\dagger\phi)^2} + \frac{2\psi^\dagger_L i\mathcal{D}_R\psi_L}{(1+\phi^\dagger\phi)^2}\right]. \tag{61}$$

The first line in (61) presents the superconformal anomaly while the second line presents the scale anomaly. We see that in the Wilsonian sense these anomalies are exhausted by one loop. In particular, the for the trace of the energy-momentum tensor we obtain

$$T_{LLRR} = -T^\mu_\mu = \frac{\beta(g^2)}{g^2}\mathcal{L}_A\,. \tag{62}$$

Comparing the right-hand sides of (61) and (62) we conclude that

$$\beta_{\rm Wilsonian} = -\frac{g^4}{2\pi}\,. \tag{63}$$

The denominator in Eq. (46) is of an infrared origin. One can say that it comes at the stage of taking the matrix element of the right-hand side of (61). Alternatively, one can say that it appears in passing from the holomorphic coupling to the canonically normalized coupling [33]. One should compare this with exactly the same situation in four-dimensional supersymmetric gluodynamics [16, 17, 33].

6 Adding fermions

In this section we consider a more general (nonminimal) version of $\mathcal{N} = (0,2)$ $\mathbf{CP}(1)$ nonlinear model, which is analogous to four-dimensional $\mathcal{N} = 1$ super-Yang–Mills theory with adjoint matter. This will turn out beneficial for two reasons: first, we will strengthen the case for our all-loop exact β function. Second, we will find a conformal window in multiflavor heterotic $\mathbf{CP}(1)$ models.

To add "matter" we have to use the $\mathcal{N}=(0,2)$ superfield B_i ($i=1,2,...,N_f$) (see e.g. [25]) with the following structure

$$B_i(x,\theta_R,\theta_R^\dagger)=\psi_{R,i}(x)+\sqrt{2}\theta_R F_i(x)+i\theta_R^\dagger\theta_R\partial_{LL}\psi_{R,i}(x)\,. \tag{64}$$

As usual, the F terms are auxiliary. Thus, the superfield B contains only a single right-handed fermion degree of freedom (per flavor). The latter has no bosonic counterpart. Then the heterotic $\mathbf{CP}(1)$ model with matter acquires the following Lagrangian:

$$\mathcal{L}=\mathcal{L}_A+\mathcal{L}_B=\int d^2\theta_R\left[\frac{1}{g^2}\frac{A^\dagger i\overleftrightarrow{\partial}_{RR}A}{1+A^\dagger A}+\frac{1}{2}\frac{\vec{B}^\dagger\cdot\vec{B}}{(1+A^\dagger A)^2}\right], \tag{65}$$

where $\vec{B}$ is the vector made of fermionic chiral superfields,

$$\vec{B}=\left\{B_i(x,\theta_R,\theta_R^\dagger)\right\}. \tag{66}$$

It is easy to see that if $N_f=1$, the model (65) reduces to the $\mathcal{N}=(2,2)$ $\mathbf{CP}(1)$ model. This circumstance will be used below. The B_i fields live on the tangent space of $\mathbf{CP}(1)$, and, hence, are endowed with the following target space symmetry transformation:

$$\vec{B}\to\vec{B}+2\bar{\epsilon}A\vec{B}\,,\quad \vec{B}^\dagger\to\vec{B}^\dagger+2\epsilon A^\dagger\vec{B}^\dagger\,. \tag{67}$$

We will find an analog of the NSVZ β function which replaces that of the minimal model (see Sec. 2). To this end we will exploit (a) instanton calculus and (b) hypercurrent analysis. We recall that adding fermions in the way described above is only possible for $\mathbf{CP}(1)$. The only exception is the case $N_f=1$. In this case we deal with the nonchiral $\mathbf{CP}(1)$ model which can be readily generalized to $\mathbf{CP}(N-1)$.

6.1 Two-loop result: direct calculation

First we will show that the perturbative calculation at two-loop level gives exactly the answer we expect. We will collect two-loop corrections to the renormalization of g^{-2} by considering the quantum correction to the kinetic term of the superfield A. As compared with the previous case, all we have to change is that now we need to take the B_i loop into account, see Fig. 3.

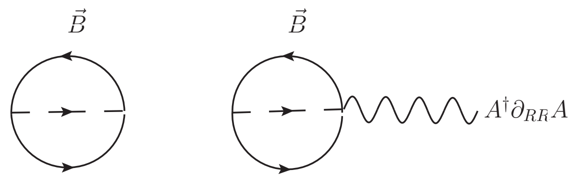

Figure 3: Two-loop correction to $\beta(g^2)$ due to the B_i loops.

For each flavor there will be a single diagram that contributes. At the two-loop level distinct flavors do not interfere with each other, they enter additively. Hence, we can guess the result from what happens in the $\mathcal{N}=(2,2)$ **CP**(1) model, in which case the single-pole contribution due to B cancels that due to the A loop. Indeed, as was mentioned above, if $N_f=1$ we deal with the $\mathcal{N}=(2,2)$ model, in which all loops in the charge renormalization higher than the first loop must cancel.

So without actually having to do the calculation (which is not difficult, though), we can write down the final result

$$\delta\mathcal{L}_{B\,\text{two loop}} = \int d^2\theta_R N_f \frac{g^2}{2}\frac{i}{4\pi}\frac{A^\dagger i\partial_{RR}A}{1+A^\dagger A} I + \text{H.c.}\,, \tag{68}$$

where I is defined in Eq. (10). As a result we have, at two-loop level

$$\beta(g^2) \equiv \frac{\partial}{\partial \log M} g^2 = -\frac{g^4}{2\pi}\left(1-\frac{N_f-1}{4\pi}g^2\right)\,, \tag{69}$$

where M is the mass of the ultraviolet regulator (e.g. the Pauli–Villars mass). Shortly we will see that Eq. (69) can be rewritten as

$$\begin{aligned}\beta(g^2) &= -\frac{g^4}{2\pi}\frac{1-\frac{N_f g^2}{4\pi}}{1-\frac{g^2}{4\pi}}\\ &= -\frac{g^4}{2\pi}\frac{1+\frac{N_f\gamma}{2}}{1-\frac{g^2}{4\pi}}\,,\end{aligned} \tag{70}$$

where γ is the anomalous dimension [5] of the B fields (which is one and the same for all matter fields due to the flavor symmetry of the model (65)). Needless to say, at $N_f=1$ the β function degenerates into a one-loop expression. This is welcome since

[5] The anomalous dimension is defined below in (74).

at $N_f = 1$ we, in fact, deal with the $\mathcal{N} = (2,2)$ model, whose β function is exhausted by one loop [18]. At two-loop level Eqs. (69) and (70) are identical. In higher orders (70) is exact, as we will argue below.

How does the anomalous dimension γ appear? In the Lagrangian (65), as we evolve it from M down to a current normalization point μ, we should take care of the wave-function renormalization of the B_i fields, in addition to the g^2 renormalization. The fields B_i live on the tangent space of the target manifold, and the covariant structure is uniquely fixed.

$$\mathcal{L}_{B,UV} = \int d^2\theta_R \frac{1}{2}\frac{\vec{B}_0^\dagger \vec{B}_0}{(1+A^\dagger A)^2}\,, \quad \mathcal{L}_{B,IR} = \int d^2\theta_R \frac{Z}{2}\frac{\vec{B}_0^\dagger \vec{B}_0}{(1+A^\dagger A)^2} \equiv \int d^2\theta_R \frac{1}{2}\frac{\vec{B}_r^\dagger \vec{B}_r}{(1+A^\dagger A)^2}\,. \tag{71}$$

The Z factor can be absorbed into the redefinition of B_i. It leaves a remnant, however, in the form of the Konishi anomaly, in much the same way as in four-dimensional super-Yang–Mills [17].

Alternatively, we could introduce Z-factor in the ultraviolet as follows. We denote it by Z_0,

$$\mathcal{L}_{B,UV} = \int d^2\theta_R \frac{1}{2} Z_0 \frac{\vec{B}^\dagger \vec{B}}{(1+A^\dagger A)^2}\,, \quad \mathcal{L}_{B,IR} = \int d^2\theta_R \frac{1}{2}\frac{\vec{B}^\dagger \vec{B}}{(1+A^\dagger A)^2}\,. \tag{72}$$

Note that $Z_0 = Z^{-1}$. These two renormalization schemes are consistent. The ultraviolet factor Z_0 is used in instanton calculus, see Sec. 6.2. We introduce it here to make easier the comparison with the previous instanton calculations, for example [16, 18, 34]

For each flavor, we have one and the same diagram for the Z factor which, after a simple and straightforward calculation, yields

$$Z = \frac{1}{Z_0} \equiv \left(\frac{B_{i,r}}{B_{i,0}}\right)^2 = 1 - ig^2 I\,, \tag{73}$$

and

$$\gamma(\vec{B}) \equiv -\frac{\partial}{\partial \log \mu} \log Z \equiv -\frac{\partial}{\partial \log M} \log Z_0 = -\frac{g^2}{2\pi} + O(g^6)\,. \tag{74}$$

6.2 Instanton calculus

In this section we will apply the instanton analysis in the multiflavor model to substantiate Eq. (70). In fact, the difference between the N_f-flavor model and the minimal model in essence reduces to a different number of the fermion zero modes in the given one-instanton background. The target space symmetry ensures that there are zero modes associated to the right-handed fermions $\vec{\psi}_z$. There are two of those for each "matter" field,

$$\psi_{z,i} = \frac{y\alpha_i^\dagger}{(z-z_0)^2}\,, \quad \psi_{z,i} = \frac{y\beta_i}{z-z_0}\,. \tag{75}$$

Note that in the instanton measure we no longer have the factor g^2 for each pair of fermion zero modes of $\vec{\psi}_z$. This is in contradistinction with what we had for the modes of the fermion component of the superfield A. The normalization of the B terms in (65) is canonical.

At the loop level we get corrections to the collective coordinates $\alpha_i^\dagger$ and β_i due to the corresponding wave-function renormalization. Each fermion superfield acquires $\sqrt{Z}$, and so do $\alpha_i^\dagger$ and β_i. Correspondingly, each $d\alpha_i^\dagger$ and $d\beta_i$ will introduce a factor $Z^{-\frac{1}{2}}$, where Z is defined as in (73) and (74). Therefore, summarizing, for each "matter" fermion field, we have the accompanying factor $(ZM)^{-1}$.

The second question we must address is the one-loop correction due to nonzero modes. Following the same road as in Sec. 4, we can build the expansion in the nonzero modes using the eigenfunctions u_n and v_n defined in (27). Indeed, each flavor will give

$$\int [\mathscr{D}\delta\psi_{z,i}][\mathscr{D}\delta\psi^\dagger_{\bar{z},i}] \to \left\{ \det \begin{bmatrix} 0 & i\partial_z \frac{1}{\chi^2_{\rm inst}} \\ i\frac{1}{\chi^2_{\rm inst}}\partial_{\bar{z}} & 0 \end{bmatrix} \right\}^{\frac{1}{2}} = \left(\prod_n E_n^2 \right)^{\frac{1}{2}}, \tag{76}$$

and, hence, each extra flavor will contribute M in the instanton measure after an appropriate regularization of the infinite product. One can easily see that when $N_f = 1$, we recover the fact that the one-loop determinant from the boson and fermion nonzero modes, respectively, cancel each other. This is certainly what we expect [18, 24] As a result, at the end of the day, we get the following expression for the measure:

$$\begin{aligned} d\mu &= \left(\frac{M^2}{g_0^2}\right)^2 \left(\frac{g_0^2}{M}\right)^1 \left(\frac{1}{Z_0 M}\right)^{N_f} M^{-1+N_f} e^{-\frac{4\pi}{g_0^2}} \\ &\times \; d\log(y) d\log(\bar{y}) \; dz_0 d\bar{z}_0 \; d\alpha d\beta^\dagger \prod_{i=1}^{N_f} d\alpha_i^\dagger d\beta_i\,. \end{aligned} \tag{77}$$

Next, we note that our nonrenormalization theorem in the instanton background derived in the minimal model (Sect. 4.1) holds in the nonminimal model too. The

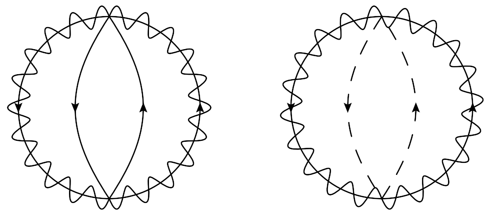

Figure 4: An illustration of how the cancelation at higher loop level happens. The dashed lines are the ϕ propagators, the solid lines are those of $\psi_{\bar{z}}$, and the solid lines with the wavy lines superimposed denote the propagators of $\psi_{z,i}$.

general argument telling us that in the instanton background, all one-particle irreducible diagrams with two loops or more do not contribute is essentially the same as in Sect. 4.1. We can illustrate how it happens in the component language for three-loop graphs shown in Fig. 4. The diagram displayed on the left and on the right cancel each other.

Recall that the Z factors of the B_i fields get renormalized. These are one-particle reducible graphs in the instanton background not seen in the above consideration (in the instanton background the $\psi_{z,i}$ kinetic terms vanish due to equations of motion). They have to be included in the measure additionally, as was done in (77).

Asserting that the overall dependence of the instanton measure $d\mu$ on the ultraviolet cut-off M should cancel, we arrive at the exact relation between the β function and the anomalous dimension $\gamma(B_i)$,

$$\beta(g^2) = -\frac{g^4}{2\pi}\frac{1+\frac{N_f}{2}\gamma(B_i)}{1-\frac{1}{4\pi}g^2}\,, \tag{78}$$

exactly as in (70).

In the multiflavor model neither $\beta(g^2)$ nor $\gamma(B_i)$ are all-loop exact. But the relation between them is exact. This is similar to the situation in $\mathcal{N}=1$ super-Yang–Mills theory with matter in four dimensions. As in the NSVZ β function, the knowledge of Z's at one-loop order gives $\beta(g^2)$ at two-loop order, and so on.

7 Hypercurrent for N_f flavors

Now we can generalize the hypercurrent, passing from the minimal model (Sect. 5) to the multiflavor model. At the classical level the operator $\mathcal{J}_{LL}$ is defined exactly in the same way as in the minimal $\mathcal{N}=(0,2)$ model.

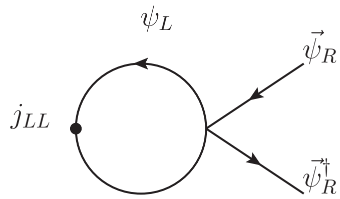

Figure 5: One-loop diagram for the j_{LL} anomaly in the $\mathcal{N}=(0,2)$ $\mathbf{CP}(1)$ models with matter.

The current j_{LL} is corrected at the quantum level through the anomalous diagram depicted in Fig. 2 and, in addition, through a new diagram shown in Fig. 5. Now the anomaly can be expressed in superfields as

$$\partial_{RR}\mathcal{J}_{LL} = \frac{1}{4\pi}\left[D_L\frac{\partial_{RR}A\bar{D}_LA^\dagger}{(1+A^\dagger A)^2} - \bar{D}_L\frac{\partial_{RR}A^\dagger D_LA}{(1+A^\dagger A)^2} + \frac{g^2}{2}\frac{\{D_L,\bar{D}_L\}}{2i}\frac{2\vec{B}^\dagger\vec{B}}{(1+A^\dagger A)^2}\right]. \tag{79}$$

It is not difficult to understand that the last term on the right-hand side is just the leading term of expansion of the exact (Wilsonian) formula,

$$\partial_{RR}\mathcal{J}_{LL} = \frac{1}{4\pi}\left[D_L\frac{\partial_{RR}A\bar{D}_LA^\dagger}{(1+A^\dagger A)^2} - \bar{D}_L\frac{\partial_{RR}A^\dagger D_LA}{(1+A^\dagger A)^2}\right] - \frac{1}{4}\gamma\frac{\{D_L,\bar{D}_L\}}{2i}\frac{2\vec{B}^\dagger\vec{B}}{(1+A^\dagger A)^2}. \tag{80}$$

To substantiate this point let us consider the renormalization-group evolution of the bare Lagrangian (65). The exact Wilsonian effective Lagrangian has the form

$$\mathcal{L}_{\text{Wilsonian}} = \int d^2\theta_R\left[\frac{i}{2g_h^2}\frac{A^\dagger\partial_{RR}A}{1+A^\dagger A} - \frac{i}{2\bar{g}_h^2}\frac{A\partial_{RR}A^\dagger}{1+A^\dagger A} + \frac{1}{2}\frac{\vec{B}_r^\dagger\cdot\vec{B}_r}{(1+A^\dagger A)^2}\right], \tag{81}$$

where g_h^2 stands for the holomorphic running coupling whose renormalization is exhausted by one loop. The response of this Lagrangian to scale transformations reduces to

$$\delta\mathcal{L}_{\text{Wilsonian}} \propto \int d^2\theta_R\left[-\frac{i\beta_{\text{Wilsonian}}}{2g_h^4}\,\frac{A^\dagger\partial_{RR}A}{1+A^\dagger A} + \frac{i\bar{\beta}_{\text{Wilsonian}}}{2\bar{g}_h^4}\,\frac{A\partial_{RR}A^\dagger}{1+A^\dagger A} - \frac{\gamma}{2}\frac{\vec{B}_r^\dagger\cdot\vec{B}_r}{(1+A^\dagger A)^2}\right]. \tag{82}$$

The expression inside the square brackets in (82), up to a minus sign, is just another component of (49). Thus, Eq. (82) confirms (80). In components Eq. (80) is equivalent to

$$S_{LRR} = \frac{1}{\sqrt{2}\pi}\frac{\partial_{RR}\phi^\dagger\psi_L}{(1+\phi^\dagger\phi)^2}\,,$$

$$T_{LLRR} = -\frac{1}{2\pi}\left[\frac{\partial_{LL}\phi^\dagger\partial_{RR}\phi+\partial_{RR}\phi^\dagger\partial_{LL}\phi}{(1+\phi^\dagger\phi)^2}+\frac{2\psi_L^\dagger i\mathcal{D}_{RR}\psi_L}{(1+\phi^\dagger\phi)^2}\right]$$

$$+\gamma\left[\frac{\vec{\psi}_R^\dagger i\overleftrightarrow{\mathcal{D}}_{LL}\vec{\psi}_R}{(1+\phi^\dagger\phi)^2}-\frac{2\psi_L^\dagger\psi_L\vec{\psi}_R^\dagger\vec{\psi}_R}{(1+\phi^\dagger\phi)^4}\right]\,. \tag{83}$$

Returning to Eq. (80) we observe that the last term on the right-hand side will convert itself into the first term through the Konishi anomaly (Sect. 8). This will produce the numerator of the β function, cf. Eq. (70). The denominator will appear upon taking the matrix element of the operator $A^\dagger\partial A/(1+A^\dagger A)$, or, alternatively, upon the transition from the holomorphic coupling to the canonical coupling.

8 "Konishi" anomaly

As was mentioned in the Introduction, adding "flavor" fields B_i in the tangent space is similar to adding adjoint matter in the $\mathcal{N}=1$ four-dimensional super-Yang–Mills theory. This similarity extends rather far. In particular, in this section we will derive an analog of the Konishi anomaly [35].

For each matter field that we introduced, we have an extra (classical) U(1) symmetry, corresponding to individual rotations of the B_i fields, see Eq. (65). It is obvious that the corresponding classically conserved U(1) currents are

$$j_{RR,i} = \frac{g^2}{2}G\psi_{R,i}^\dagger\psi_{R,i}\,. \tag{84}$$

These currents are the lowest components of the superfield operators

$$\mathcal{J}_{RR,i} \equiv \frac{1}{(1+A^\dagger A)^2}B_i^\dagger B_i\,,\qquad i=1,2,...,N_f\,. \tag{85}$$

At the quantum level, due to the anomaly, these matter U(1) currents cease to be conserved. Instead, evaluating the diagrams in Fig. 6, we find

$$\partial_{LL}\mathcal{J}_{RR,i} = -\frac{1}{4\pi}\left[D_L\frac{\partial_{RR}A\bar{D}_LA^\dagger}{(1+A^\dagger A)^2}-\bar{D}_L\frac{\partial_{RR}A^\dagger D_LA}{(1+A^\dagger A)^2}\right]\,, \tag{86}$$

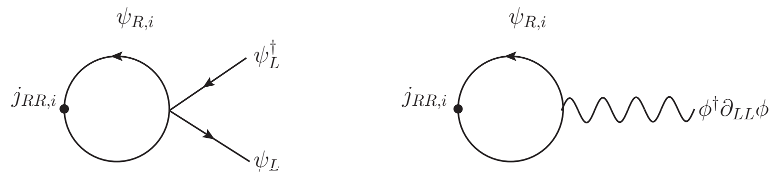

Figure 6: One-loop correction to the $U(1)$ current $j_{RR,i}$.

for each (fixed) value of i. Comparing this expression with the last term in Eq. (80) (in its imaginary part) we see that Eq. (80) can be rewritten as follows:

$$\partial_{RR}\mathcal{J}_{LL} = \frac{1+(N_f\gamma/2)}{4\pi}\left[D_L\frac{\partial_{RR}A\bar{D}_LA^\dagger}{(1+A^\dagger A)^2} - \bar{D}_L\frac{\partial_{RR}A^\dagger D_LA}{(1+A^\dagger A)^2}\right]. \tag{87}$$

The real part of the superfield relation takes the form

$$T_{LLRR} = -\frac{1+(N_f\gamma/2)}{2\pi}\left[\frac{\partial_{LL}\phi^\dagger\partial_{RR}\phi+\partial_{RR}\phi^\dagger\partial_{LL}\phi}{(1+\phi^\dagger\phi)^2} + \frac{2\psi_L^\dagger i\mathcal{D}_{RR}\psi_L}{(1+\phi^\dagger\phi)^2}\right]. \tag{88}$$

These are still Wilsonian operator formulas which present a direct parallel with, say, Eq. (2.111) in [34].

It is clear that the passage to the generator of the 1-particle irreducible vertices (or, alternatively, from the holomorphic coupling to the canonic coupling [33]) proceeds exactly in the same manner as in the minimal model, resulting in the replacement

$$1+(N_f\gamma/2) \to \frac{1+(N_f\gamma/2)}{1-g^2/4\pi}. \tag{89}$$

9 Conformal window

The β function that we have just derived, see Eq. (70), has a remarkable property. Assume that $N_f \gg 1$. Then it develops an infrared fixed point at parametrically small values of g^2, such that one can still trust the one-loop result for the anomalous dimension of the matter fields,

$$\frac{g_*^2}{2\pi} = \frac{2}{N_f} \ll 1, \tag{90}$$

where the asterisk labels the fixed point. If we choose the bare coupling constant in the interval $(0, 2/N_f)$ then the theory under consideration is asymptotically free in the ultraviolet and conformal in the infrared. At large N_f it is weakly coupled at all distances. Perturbative calculations of the anomalous dimensions of various operators make sense.

It is clear that on the side of small N_f the conformal window should extend to some $N_f^* > 1$. At $N_f = 1$ supersymmetry of the model under consideration is enhanced up to (2,2), and this model is certainly nonconformal. Rather, it develops a mass gap.

10 Conclusion

In this paper we studied a class of two-dimensional $\mathcal{N} = (0, 2)$ nonlinear sigma models with $\mathbf{CP}(1)$ as the target space. We presented a number of arguments indicating the similarity of these models with with four-dimensional super-Yang–Mills with adjoint matter. Two further questions could be asked here.

- Can we generalize our nonrenormalization theorem to other models?
- Can we see further implications of the 2D/4D correspondence?

These two questions are interrelated. As for the first one, the answer is positive, at least in part. The original NSVZ argument can be applied to a large class of models with flag manifolds as the target manifolds. We do expect our analysis to go through.

The fermion anomaly [21] does not allow us to extend our multiflavor models in the direction of $\mathbf{CP}(N-1)$. This is due to the fact that the anomaly free condition $p_1 = 0$ (p_1 is the first Pontryagin class) rules out $\mathbf{CP}(N-1)$ target spaces except for $N_f = 1$. If $N_f = 1$, the model becomes nonchiral.

There are two obvious "technical" questions to be explored. First, the (bi)fermion condensates. It is well-known that such a condensate develops [24] in the $\mathcal{N} = (2, 2)$ $\mathbf{CP}(1)$ model. Moreover, it plays the role of the order parameter distinguishing between two distinct vacua of this theory. In the minimal $\mathcal{N} = (0, 2)$ $\mathbf{CP}(1)$ model such a phenomenon seems to be impossible since it is impossible to build a Lorentz scalar from the ψ_L field alone. We checked that one instanton in the minimal model does not give rise to the fermion condensate. Whether or not they develop at $N_f > 2$ remains to be seen.

Next, we will explore [22] the NSVZ-like β functions derived in this paper in models including additional chiral fields, such as [25]. At the moment we can formulate a conjecture that the relation (78) will stay intact; all dependence on the additional coupling constants will be hidden in the anomalous dimensions.

Another issue of interest is the occurrence of conformality. The nonsupersymmetric (bosonic) **CP**(1) model is known to be conformal when the vacuum angle ω (see Eq. (8)) equals to π. Is it a hint that the conformal window of the $\mathcal{N} = (0, 2)$ models extends all the way down to $N_f = 2$?

Acknowledgments

We thank J. Chen, A. Vainshtein and J. Yagi for illuminating discussions. XC is grateful to the 2011 Simons workshop in Mathematics and Physics and the Simons Center for Geometry and Physics for hospitality, where part of the work has been done.

The work of XC and MS was supported in part by DOE grant DE-FG02-94ER40823.

Appendix A: Notations in Minkowski spacetime

In this appendix we give a description of $\mathcal{N} = (0, 2)$ $D = 1 + 1$ superspace and fix the notations.

The space-time coordinate $x^\mu = \{t, z\}$ can be promoted to superspace by adding a complex Grassmann variable θ_R and its complex conjugate $\theta_R^\dagger$. Where-ever our expressions are dependent on the representation of Clifford algebra, we use the following convention.

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^3 = \gamma^0\gamma^1 \,. \tag{A.1}$$

Under this representation Dirac fermion is expressed as

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} . \tag{A.2}$$

We define the left moving and right moving derivatives as

$$\partial_{LL} = \partial_t + \partial_z \,, \qquad \partial_{RR} = \partial_t - \partial_z \,, \tag{A.3}$$

and use the following definition for the superderivatives:

$$D_L = \frac{\partial}{\partial\theta_R} - i\theta_R^\dagger\partial_{LL} \,, \qquad \bar{D}_L = -\frac{\partial}{\partial\theta_R^\dagger} + i\theta_R\partial_{LL} \,. \tag{A.4}$$

Their commutator gives $\{D_L, \bar{D}_L\} = 2i\partial_{LL}$.

To change between ordinary coordinates and the lightcone coordinates, we have, for supercurrent:

$$S_L^0 = S_{RRL} + S_{LLL}\,, \quad S_L^1 = S_{LLL} - S_{RRL}\,. \tag{A.5}$$

And for $T^{\mu\nu}$:

$$T_{LLLL} = T_{00} + T_{10} + T_{11} + T_{01}\,, \tag{A.6}$$
$$T_{LLRR} = T_{00} + T_{10} - T_{11} - T_{01}\,, \tag{A.7}$$
$$T_{RRLL} = T_{00} - T_{10} - T_{11} + T_{01}\,, \tag{A.8}$$
$$T_{RRRR} = T_{00} - T_{10} + T_{11} - T_{01}\,. \tag{A.9}$$

The shifted space-time coordinates that satisfy the chiral condition are

$$y^0 = t + i\theta_R^\dagger\theta_R\,, \qquad y^1 = z + i\theta_R^\dagger\theta_R\,. \tag{A.10}$$

The antichiral counterparts are

$$\tilde{y}^0 = t - i\theta_R^\dagger\theta_R\,, \qquad \tilde{y}^1 = z - i\theta_R^\dagger\theta_R\,. \tag{A.11}$$

Under supersymmetric transformation $\delta_\epsilon + \delta_{\bar{\epsilon}}$

$$\theta_R \to \theta_R + \epsilon\,, \qquad \theta_R^\dagger \to \theta_R^\dagger + \bar{\epsilon}\,,$$
$$y^\mu \to y^\mu + 2i\bar{\epsilon}\theta_R\,, \qquad \tilde{y}^\mu \to \tilde{y}^\mu - 2i\theta_R^\dagger\epsilon\,, \tag{A.12}$$

where $\mu = 0, 1$.

We can now define the chiral $\mathcal{N} = (0, 2)$ bosonic and fermionic superfields in our model,

$$A(y^\mu, \theta_R) = \phi(y^\mu) + \sqrt{2}\theta_R\psi_L(y^\mu)\,,$$
$$B(y^\mu, \theta_R) = \psi_R(y^\mu) + \sqrt{2}\theta_R F(y^\mu)\,. \tag{A.13}$$

Here ϕ, ψ_L and ψ_R describe physical degrees of freedom, while F will enter without derivatives and, thus, can be eliminated by virtue of equations of motion.

Appendix B: Notations in Euclidean spacetime

The Wick rotation is defined by

$$x^1 = x\,, \quad x^2 = it\,. \tag{B.1}$$

Gamma matrices:

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^3 = i\gamma^1\gamma^2\,. \tag{B.2}$$

We define

$$\partial_z = \partial_1 - i\partial_2\,, \quad \partial_{\bar z} = \partial_1 + i\partial_2\,, \tag{B.3}$$

and the supercharges are given by

$$Q = \frac{\partial}{\partial\theta} + i\theta^\dagger\partial_{\bar z}\,, \quad \bar{Q} = -\frac{\partial}{\partial\theta^\dagger} - i\theta\partial_{\bar z}\,, \tag{B.4}$$

together with the commutation relation

$$\{Q, \bar{Q}\} = -2i\partial_{\bar z}\,. \tag{B.5}$$

Correspondingly, superderivatives are given by

$$D = \frac{\partial}{\partial\theta} - i\theta^\dagger\partial_{\bar z}\,, \quad \bar{D} = -\frac{\partial}{\partial\theta^\dagger} + i\theta\partial_{\bar z}\,. \tag{B.6}$$

It is easy to verify that

$$D(z) = \bar{D}(z) = D(\bar{z}_{ch}) = 0\,, \tag{B.7}$$

where $\bar{z}_{ch} = \bar{z} - 2i\theta^\dagger\theta$.

We define the Dirac fermion to be

$$\psi = \begin{pmatrix} \psi_z \\ \psi_{\bar z} \end{pmatrix}. \tag{B.8}$$

Now bosonic chiral superfield is defined as

$$A = \phi(\bar{z}_{ch}, z) + \sqrt{2}\theta\psi_{\bar z}(\bar{z}_{ch}, z)\,. \tag{B.9}$$

Fermionic chiral superfield can be written down in a similar manner.

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