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A point particle of small mass m moves in free fall through a background vacuum spacetime metric g^0_{ab} and creates a first-order metric perturbation $h^{1\text{ret}}_{ab}$ that diverges at the particle. Elementary expressions are known for the singular m/r part of $h^{1\text{ret}}_{ab}$ and for its tidal distortion determined by the Riemann tensor in a neighborhood of m. Subtracting this singular part $h^{1\text{S}}_{ab}$ from $h^{1\text{ret}}_{ab}$ leaves a regular remainder $h^{1\text{R}}_{ab}$. The self-force on the particle from its own gravitational field adjusts the world line at $O(m)$ to be a geodesic of $g^0_{ab} + h^{1\text{R}}_{ab}$. The generalization of this description to second-order perturbations is developed and results in a wave equation governing the second-order $h^{2\text{ret}}_{ab}$ with a source that that has an $O(m^2)$ contribution from the stress-energy tensor of m added to a term quadratic in $h^{1\text{ret}}_{ab}$. Second-order self-force analysis is similar to that at first order: The second-order singular field $h^{2\text{S}}_{ab}$ subtracted from $h^{2\text{ret}}_{ab}$ yields the regular remainder $h^{2\text{R}}_{ab}$, and the second-order self-force is then revealed as geodesic motion of m in the metric $g^0_{ab} + h^{1\text{R}} + h^{2\text{R}}$.

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I. OVERVIEW

Recent, impressive fully relativistic numerical analysis has been brought to bear on a black hole binary system with a mass ratio of 100 to 1 [1, 2], and the evolution is followed for two full orbits before coalescence. The two disparate length scales of an extreme or intermediate mass-ratio binary pose a challenge for numerical relativists to resolve the geometry in the vicinity of the small object while efficiently analyzing the remainder of spacetime and providing gravitational wave trains for a number of orbits. Second-order perturbation theory in general relativity might more efficiently meet the challenge of the difficult numerical problems of extreme and intermediate mass-ratio binaries.

Early descriptions of second-order perturbation theory [3–10] have focused on perturbations with no matter sources and are typically limited to metrics with a substantial amount of symmetry. However, Habisohn [11] presents a fully general description of matter-free second-order perturbation theory for a background vacuum spacetime metric g^0_{ab}.

Rosenthal [12–14] was first to describe a formal approach to second order perturbation theory which includes a small-mass δ-function point source. However, an actual application of his approach does not appear to be straightforward.

The heart of this manuscript extends Habisohn's [11] second-order analysis to allow for a perturbing δ-function point mass. Our formalism is closely related to the traditional description of linear perturbation theory.

We begin in Section II with the formal expansion of the Einstein tensor, for a metric $g_{ab} + h_{ab}$, in powers of h_{ab}. First-order perturbation theory is summarized in Section III for the case that the source is a δ-function object of small mass m. In the test mass limit m moves along a geodesic γ_0 of the background metric g^0_{ab}. With a finite mass m the metric is perturbed by the retarded field $h^{1\text{ret}}_{ab}$ at first order in m, and m's worldline deviates from γ_0 by an amount of $O(m)$ as m itself interacts with $h^{1\text{ret}}_{ab}$ as a consequence of the first-order *gravitational self-force* as described in Section IV. Throughout this manuscript we assume that the effects of m's spin and multipole structure on its motion are insignificant when compared with the self-force effects.

The extension of Habisohn's [11] second-order analysis to allow a δ-function point source demands careful consideration of the singular behavior of the metric in a neighborhood of m as described in Section V. Ultimately the wave equation for the second-order $h^{2\text{ret}}_{ab}$ appears in Eq. (26) as one might have expected, and the self-force analysis at second-order is seen to be similar in style to the analysis at first-order.

The application of second order perturbation theory for a small mass still requires an effort which is strongly dependent upon the details of the actual problem of interest. Practical considerations are emphasized in Section VI.

Notation and conventions

In a neighborhood of a geodesic γ_0 of the background metric g^0_{ab} we use *locally inertial and Cartesian* (LIC) coordinates [15] where the timelike coordinate is t, the spatial indices i, j, k and l run from 1 to 3, the spatial coordinates are x^i and $r^2 \equiv x^i x^j \eta_{ij}$. In addition LIC coordinates have special properties on γ_0: the coordinate t is the proper time, the spatial coordinates are all zero $x^i = 0$, the metric is the flat Minkowski metric η_{ab}, and all first coordinate derivatives of g^0_{ab} vanish. Second derivatives of g^0_{ab} on γ_0 determine a curvature length and time scale $\mathcal{R}$, and the components of the Riemann tensor then scale as $1/\mathcal{R}^2$ and their time derivatives along γ_0 scale as $1/\mathcal{R}^3$. After some fine-tuning of the coordinates [15–17], the metric in a neighborhood of γ_0 may be put

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into the form

$$g^0_{ab}\,dx^a\,dx^b = \eta_{ab}\,dx^a\,dx^b - x^i x^j R^0_{titj}(dt^2 + \delta_{kl}\,dx^k\,dx^l) - \frac{4}{3}x^i x^j R^0_{ikjt}\,dt\,dx^k + O(r^3/\mathcal{R}^3)\,, \qquad (1)$$

where the superscript ^{0} on the components of the Riemann tensor implies that it is to be evaluated on γ_0. Also, both R^0_{titj} and R^0_{ikjt} are symmetric and tracefree in the indices i and j as consequences of the vacuum Einstein equations.

Much of our analysis takes place in the *buffer zone* [16], a region spatially-surrounding γ_0 where $m \ll r \ll \mathcal{R}$. In the buffer zone r is small enough compared to the curvature length scale, $r \ll \mathcal{R}$, that the curvature of g^0_{ab} is barely apparent, and we have the luxury of being able to expand the actual metric $g^0_{ab} + h^{\rm ret}_{ab}$ away from flat spacetime in powers of two simultaneously small numbers, m/r and $r/\mathcal{R}$.

II. EXPANSION OF THE EINSTEIN TENSOR

We consider a perturbation h_{ab} of a given metric g_{ab}, and expand the Einstein tensor of the sum $G_{ab}(g+h)$ in terms of increasing powers of h_{ab} so that formally

$$G_{ab}(g+h) = G_{ab}(g) + G^{(1)}_{ab}(g,h) + G^{(2)}_{ab}(g,h) + \dots \qquad (2)$$

where Habisohn [11] describes an individual term in this expansion by

$$G^{(n)}_{ab}(g,h) = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}G_{ab}(g+\lambda h)\right]_{\lambda=0}. \qquad (3)$$

This notation implies that the operator $G^{(n)}_{ab}(g,h)$ returns an expression that scales as $(h_{ab})^n$. For $n=1$ and g_{ab} being a vacuum solution of the Einstein equation,

$$2G^{(1)}_{ab}(g,h) = -\nabla^c\nabla_c h_{ab} - \nabla_a\nabla_b h^c{}_c + 2\nabla_{(a}\nabla^c h_{b)c} - 2R_a{}^c{}_b{}^d h_{cd} + g_{ab}(\nabla^c\nabla_c h^d{}_d - \nabla^c\nabla^d h_{cd}), \qquad (4)$$

where ∇_a is the derivative operator compatible with the metric g_{ab}. Habisohn [11] provides the following expression for $G^{(2)}_{ab}(g,h)$ in his Eq. (3.1),

$$\begin{aligned} G^{(2)}_{ab}(g,h) = {} & \frac{1}{2}h^{cd}\nabla_a\nabla_b h_{cd} + \frac{1}{4}\left(\nabla_a h^{cd}\right)\nabla_b h_{cd} \\ & + \left(\nabla^{[c}h^{d]}{}_a\right)\nabla_c h_{db} - \frac{1}{4}C^d\left(2\nabla_{(a}h_{b)d} - \nabla_d h_{ab}\right) \\ & - h^{cd}\left(\nabla_c\nabla_{(a}h_{b)d} - \frac{1}{2}\nabla_c\nabla_d h_{ab}\right) \\ & + \left\{\frac{1}{8}C^d C_d - \frac{1}{4}h^{cd}\nabla^e\nabla_e h_{cd} - \frac{1}{8}\left(\nabla^e h^{cd}\right)\nabla_e h_{cd} \right. \\ & \left. + \frac{1}{4}h^{cd}\nabla_c C_d + \frac{1}{4}\left(\nabla^d h^{ce}\right)\nabla_c h_{de}\right\} g_{ab} \end{aligned} \qquad (5)$$

where

$$C_d \equiv 2\nabla^c h_{cd} - \nabla_d h_c{}^c. \qquad (6)$$

III. FIRST-ORDER PERTURBATION THEORY FOR A POINT MASS

We next consider the consequences of adding an object of small size and small mass m, with $m \ll \mathcal{R}$, to the vacuum spacetime whose metric is g^0_{ab}.

With a global coordinate system (T, X^i), the stress-energy tensor for m moving on a geodesic γ_0 of g^0_{ab} is

$$T_{ab}(\gamma_0) = m\frac{u_a u_b}{\sqrt{-g^0}}\frac{d\tau}{dT}\delta^3(X^i - \gamma^i_0(T)), \qquad (7)$$

where $\gamma^i_0(T)$ gives the spatial position of the geodesic as a function of T, and the four-velocity u_a, $\sqrt{-g^0}$, and proper time τ are all functions of T along the worldline.

The dominant effect of $T_{ab}(\gamma_0)$ on the spacetime metric results in the retarded metric perturbation $h^{\rm 1ret}_{ab}$ proportional to m which solves

$$G_{ab}(g^0 + h^{\rm 1ret}) = 8\pi T_{ab}(\gamma_0) + O(m^2)\,, \qquad (8)$$

with appropriate boundary conditions. The superscript 1 on any metric perturbation implies that $h^{\rm 1ret}_{ab}$ is $O(m)$, for example. Later we use $h^{\rm 2ret}_{ab}$ for an $O(m^2)$ metric perturbation and also use $h^{\rm ret}_{ab} \equiv h^{\rm 1ret}_{ab} + h^{\rm 2ret}_{ab} + O(m^3)$.

For this linear perturbation problem, we expand the Einstein tensor in Eq. (8) using Eq. (2) and isolate the terms linear in m to obtain the first-order perturbation equation,

$$G^{(1)}_{ab}(g^0, h^{\rm 1ret}) = 8\pi T_{ab}(\gamma_0). \qquad (9)$$

The Bianchi identity implies for arbitrary h_{ab} that if g_{ab} is a vacuum solution of the Einstein equation, then

$$\nabla^a G^{(1)}_{ab}(g,h) = 0, \qquad (10)$$

perhaps as a distribution. An integrability condition for Eq. (9) thus requires that $T_{ab}(\gamma_0)$ be divergence free. The assumption that the worldline of m is a geodesic γ_0 of g^0_{ab} guarantees that $\nabla^a T_{ab}(\gamma_0) = 0$ and that the integrability condition is satisfied.

IV. FIRST-ORDER GRAVITATIONAL SELF-FORCE

After $h^{\rm 1ret}_{ab}$ is found using Eq. (9) there are several ways of calculating, understanding and interpreting the gravitational self-force [17–23]. Our favorite is to note that $h^{\rm 1ret}_{ab}$ is naturally decomposed within a neighborhood of γ_0 into two complementary parts,

$$h^{\rm 1ret}_{ab} = h^{\rm 1S}_{ab} + h^{\rm 1R}_{ab}. \qquad (11)$$

The first part $h^{\rm 1S}_{ab}$ is the linear piece of the *singular field* $h^{\rm S}_{ab}$ which is a special solution of

$$G_{ab}(g^0 + h^{\rm S}) = 8\pi T_{ab}(\gamma_0) \qquad (12)$$

with the notable features that h^{S}_{ab}: (1) may be expanded in powers of m, (2) is local to m and does not depend upon boundary conditions, (3) is accessible via an asymptotic expansion [17–21] each term of which is singular or of limited differentiability on γ_0, and (4) does not exert a force on m itself, just as the Coulomb field of an electron at rest exerts no net force on the electron.

The substitution $h^{\mathrm{S}}_{ab} = h^{1\mathrm{S}}_{ab} + h^{2\mathrm{S}}_{ab} + O(m^3)$, with $h^{2\mathrm{S}}_{ab} = O(m^2)$, into Eq. (12) and the expansion of the Einstein tensor results in two equations, the first linear in m and the second quadratic,

$$G^{(1)}_{ab}(g^0, h^{1\mathrm{S}}) = 8\pi T_{ab}(\gamma_0) \tag{13}$$

$$G^{(1)}_{ab}(g^0, h^{2\mathrm{S}}) = -G^{(2)}_{ab}(g^0, h^{1\mathrm{S}}). \tag{14}$$

The inhomogeneous, linear singular field $h^{1\mathrm{S}}_{ab}$ looks like a Coulomb m/r field being tidally distorted by the Riemann tensor of g^0_{ab}. We qualitatively describe $h^{1\mathrm{S}}_{ab}$, using LIC coordinates associated with γ_0, as

$$h^{1\mathrm{S}}_{ab} \sim \frac{m}{r}\left(1 + \frac{x^2}{\mathcal{R}^2} + \ldots\right) ; \tag{15}$$

only the scaling of the leading terms are shown, and this scaling is valid in the buffer zone, where $m \ll r \ll \mathcal{R}$. We distinguish x from r to emphasize that x/r is generally finite but discontinuous C^{-1} in the limit $r \to 0$. The dominant term, scaling as just m/r, represents the linear in m terms in an m/r expansion of the Schwarzschild metric, as given in Eq. (A6) in Appendix A. The second term in the parentheses reflects the quadrupole distortion of the m/r field that is induced by the external Riemann tensor's tidal effects which scale as $x^2/\mathcal{R}^2$, as given by the terms proportional to m in Eq. (A8).

The complement of $h^{1\mathrm{S}}_{ab}$ is the homogeneous *regular* field $h^{1\mathrm{R}}_{ab} = h^{1\mathrm{ret}}_{ab} - h^{1\mathrm{S}}_{ab}$, from Eq. (11), which solves

$$G^{(1)}_{ab}(g^0, h^{1\mathrm{R}}) = 0\,. \tag{16}$$

The regular field $h^{1\mathrm{R}}_{ab}$ is smooth on γ_0 and, thus, qualitatively described in a neighborhood of γ_0 by

$$h^{1\mathrm{R}}_{ab} \sim \frac{m}{\mathcal{R}} + \frac{mx}{\mathcal{R}^2} + \frac{mx^2}{\mathcal{R}^3} + \ldots\ , \tag{17}$$

with the LIC coordinates associated with γ_0. Each term takes the form of an external multipole moment proportional to m.

The regular field $h^{1\mathrm{R}}_{ab}$ is added to g^0_{ab} to create the *external* metric

$$g^{\mathrm{ext}}_{ab} \equiv g^0_{ab} + h^{1\mathrm{R}}_{ab} \tag{18}$$

which governs the geodesic motion of m. After all, $h^{1\mathrm{R}}_{ab}$ is a homogeneous solution of Eq. (16) with no variation over a length scale comparable to m. An observer in a neighborhood of m, with no a priori knowledge of the global spacetime, could measure the actual metric $g^0_{ab} + h^{1\mathrm{R}}_{ab} + h^{1\mathrm{S}}_{ab}$ at $O(m)$ and could distinguish the singular behavior of $h^{1\mathrm{S}}_{ab}$ from the remainder $g^0_{ab} + h^{1\mathrm{R}}_{ab}$. However, the observer would be unable to distinguish $h^{1\mathrm{R}}_{ab}$ from g^0_{ab} in the combination $g^0_{ab} + h^{1\mathrm{R}}_{ab}$ at linear order via *local measurements only* because $g^0_{ab}+h^{1\mathrm{R}}_{ab}$ is a smooth solution of the vacuum Einstein equations at linear order. The observer would then naturally note that the worldline of m is a geodesic $\gamma_0 + \gamma_{1\mathrm{R}}$ of the metric $g^0_{ab} + h^{1\mathrm{R}}_{ab}$. The difference between the two worldlines is denoted $\gamma_{1\mathrm{R}}$ and reflects the effects of what is often called the gravitational self-force, even though there is neither a force on m nor an acceleration of its worldline within the external metric $g^0_{ab} + h^{1\mathrm{R}}_{ab}$.

It is apparent that an $O(m)$ coordinate transformation of the original LIC coordinates for γ_0 would remove the dipole term in Eq. (17) and put the sum $g^0_{ab} + h^{1\mathrm{R}}_{ab}$ into the same form as displayed in Eq. (1), with $O(m)$ changes in the components of the *external* Riemann tensor.

In an application $h^{1\mathrm{ret}}_{ab}$ is typically found numerically while $h^{1\mathrm{S}}_{ab}$ (or its approximation, *cf.* Section VI) is found analytically, then $h^{1\mathrm{R}}_{ab} = h^{1\mathrm{ret}}_{ab} - h^{1\mathrm{S}}_{ab}$ gives the regular remainder (or its approximation) which is used to determine the self-force and the appropriate geodesic $\gamma_0 + \gamma_{1\mathrm{R}}$ of $g^0_{ab} + h^{1\mathrm{R}}_{ab}$.

V. SECOND-ORDER PERTURBATION THEORY

We assume that we have solved a first-order self-force problem of interest and have, in hand, $h^{1\mathrm{ret}}_{ab}$, $h^{1\mathrm{S}}_{ab}$, $h^{1\mathrm{R}}_{ab}$, the initial geodesic γ_0 of g^0_{ab} and the self-force modified geodesic $\gamma_0 + \gamma_{1\mathrm{R}}$ of $g^0_{ab} + h^{1\mathrm{R}}_{ab}$.

For the second order problem we also require $h^{2\mathrm{S}}_{ab}$ which can be determined via an asymptotic expansion of Eq. (14), and scales as

$$h^{2\mathrm{S}}_{ab} \sim \frac{m^2}{r^2}\left(1 + \frac{x^2}{\mathcal{R}^2} + \ldots\right) \tag{19}$$

with LIC coordinates. The dominant term, scaling as m^2/r^2, is the term quadratic in m in an m/r expansion of the Schwarzschild metric, as given in Eq. (A7) for $n = 2$. The second term in the parentheses reflects the quadrupole distortion of the m^2/r^2 field that is induced by the external Riemann tensor's tidal effects which scale as $x^2/\mathcal{R}^2$, as given by the $O(m^2)$ terms in Eq. (A8).

To understand second-order perturbation theory requires understanding two distinct and critical roles played by the first order regular field $h^{1\mathrm{R}}_{ab}$. First, the stress-energy tensor of m is $T_{ab}(\gamma_0 + \gamma_{1\mathrm{R}})$, where the argument implies that the worldline of m is now a geodesic of $g^0_{ab} + h^{1\mathrm{R}}_{ab}$. The change in the stress-energy tensor resulting from the first-order self-force is

$$\begin{aligned} T_{ab}(\gamma_0{+}\gamma_{1\mathrm{R}}) - T_{ab}(\gamma_0) \\ &= m\Delta\left(\frac{u_a u_b}{\sqrt{-g}}\frac{d\tau}{dT}\right)\delta^3[X^i - \gamma_0^i(T)] \\ &\quad - m\frac{u_a u_b}{\sqrt{-g^0}}\frac{d\tau}{dT}\,\gamma_{1R}^j\frac{\partial}{\partial X^j}\delta^3[X^i-\gamma_0^i(T)] \end{aligned} \quad (20)$$

where the Δ operation reflects the $O(m)$ change in the quantity in parentheses which follows from changing the metric to $g^0_{ab} + h^{1\mathrm{R}}_{ab}$ from g^0_{ab}. Thus the difference between the two stress-energy tensors is a distribution of $O(m^2)$ with support on γ_0 and consists of terms with a δ-function and with a gradient of a δ-function.

A second effect of $h^{1\mathrm{R}}_{ab}$ on the second-order problem is the modification of the tidal environment of m by $h^{1\mathrm{R}}_{ab}$ which becomes an $O(m)$ part of the *external* metric as in Eq. (18). This creates $O(m)$ changes in the the external Riemann tensor's multipole moments. These changes are responsible for $O(m^2)$ corrections to $h^{1\mathrm{S}}_{ab}$ which we label $h^{2\mathrm{S}\dagger}_{ab}$. Thus the singular field is *not* derived solely from the initial geodesic and the background metric g^0_{ab}, rather it specifically includes effects from the self-force modification of the geodesic and from the additional $O(m^2)$ tidal distortion of $h^{1\mathrm{S}}_{ab}$ caused by $h^{1\mathrm{R}}_{ab}$, and these $O(m^2)$ contributions to the singular field constitute $h^{2\mathrm{S}\dagger}_{ab}$.

The presence of $h^{1\mathrm{R}}_{ab}$ in the external metric $g^0_{ab} + h^{1\mathrm{R}}_{ab}$ modifies the tidal effects of the external Riemann tensor on the singular field and Eq. (15) becomes

$$h^{1\mathrm{S}}_{ab} + h^{2\mathrm{S}\dagger}_{ab} \sim \frac{m}{r}\left[1 + \frac{x^2}{\mathcal{R}^2}\left(1+\frac{m}{\mathcal{R}}\right) + \ldots\right], \quad (21)$$

where we are now using LIC coordinates for the geodesic $\gamma_0 + \gamma_{1\mathrm{R}}$ of $g^0_{ab} + h^{1\mathrm{R}}_{ab}$. The $m/\mathcal{R}$ term in the parentheses adds an $O(m^2)$ contribution to h^{S}_{ab}; however, the $O(m^2)$ $h^{2\mathrm{S}\dagger}$ is naturally grouped with $h^{1\mathrm{S}}_{ab}$ because its presence in Eq. (21) algebraically resembles part of $h^{1\mathrm{S}}_{ab}$ in Eq. (15) much more than any part of $h^{2\mathrm{S}}_{ab}$ in Eq. (19).

Through second order the singular field is thus represented by

$$h^{\mathrm{S}}_{ab} = h^{1\mathrm{S}}_{ab} + h^{2\mathrm{S}\dagger}_{ab} + h^{2\mathrm{S}}_{ab} + O(m^3). \quad (22)$$

An immediate application of this notation is in the recognition that

$$G^{(1)}_{ab}(g^0 + h^{1\mathrm{R}}, h^{1\mathrm{S}} + h^{2\mathrm{S}\dagger}) = 8\pi T_{ab}(\gamma_0 + \gamma_{1\mathrm{R}}) + O(m^3), \quad (23)$$

which is the natural extension of Eq. (13) to second-order. The presence of $h^{1\mathrm{R}}_{ab}$ as part of the external metric in the first argument of $G^{(1)}_{ab}$ requires the addition of $h^{2\mathrm{S}\dagger}_{ab}$ to the second argument. We have already described $h^{1\mathrm{R}}_{ab}$ in Eq. (11), and it is natural then to define $h^{2\mathrm{R}}_{ab}$ via

$$h^{2\mathrm{ret}}_{ab} = h^{2\mathrm{R}}_{ab} + h^{2\mathrm{S}\dagger}_{ab} + h^{2\mathrm{S}}_{ab}. \quad (24)$$

We now confront the second-order problem which requires a solution for $h^{2\mathrm{ret}}_{ab}$ from

$$G_{ab}(g^0 + h^{1\mathrm{ret}} + h^{2\mathrm{ret}}) = 8\pi T_{ab}(\gamma_0 + \gamma_{1\mathrm{R}}) + O(m^3), \quad (25)$$

when we are given the metric perturbations $h^{1\mathrm{ret}}_{ab}$, $h^{1\mathrm{R}}_{ab}$, $h^{1\mathrm{S}}_{ab}$, $h^{2\mathrm{S}\dagger}_{ab}$, $h^{2\mathrm{S}}_{ab}$, and the worldlines γ_0 and $\gamma_0{+}\gamma_{1\mathrm{R}}$. We expand the left hand side about g^0_{ab}, rearrange some terms, and substitute for $G^{(1)}_{ab}(g^0, h^{1\mathrm{ret}})$ from Eq. (9) to obtain

$$\begin{aligned} G^{(1)}_{ab}(g^0, h^{2\mathrm{ret}}) &= 8\pi T_{ab}(\gamma_0 + \gamma_{1\mathrm{R}}) - 8\pi T_{ab}(\gamma_0) \\ &\quad - G^{(2)}_{ab}(g^0, h^{1\mathrm{ret}}) \,. \end{aligned} \quad (26)$$

This wave equation for $h^{2\mathrm{ret}}_{ab}$ is the primary formal result of this manuscript. At the source each stress-energy term is $O(m)$; however, their difference is a distribution with support on γ_0 and is of $O(m^2)$ as given in Eq. (20).

The integrability condition for Eq. (26) is easily satisfied away from γ_0 because there $G^{(1)}_{ab}(g^0, h^{1\mathrm{ret}}) = 0$ and the fact that for any h_{ab} if $G^{(1)}_{ab}(g^0, h) = 0$ then it follows that $\nabla^a G^{(2)}_{ab}(g^0, h) = 0$, as shown by Habisohn [11] in his Eq. (3.7). Thus the divergence of the right hand side is zero away from γ_0. The discussion of the integrability condition in a neighborhood of γ_0 is deferred until just after Eq. (31) below.

Eq. (26) becomes surprisingly transparent after some analysis (while cavalierly dropping terms of $O(m^3)$ along the way) when h^{ret}_{ab} is re-expressed with the substitutions $h^{1\mathrm{ret}}_{ab} = h^{1\mathrm{R}}_{ab} + h^{1\mathrm{S}}_{ab}$ and $h^{2\mathrm{ret}}_{ab} = h^{2\mathrm{R}}_{ab} + h^{2\mathrm{S}}_{ab} + h^{2\mathrm{S}\dagger}_{ab}$. Then the substitutions for the stress-energy tensors from Eqs. (13) and (23) lead to

$$G^{(1)}_{ab}(g^0, h^{2R} + h^{2\mathrm{S}\dagger} + h^{2\mathrm{S}}) = G^{(1)}_{ab}(g^0 + h^{1R}, h^{1\mathrm{S}} + h^{2\mathrm{S}\dagger}) - G^{(1)}_{ab}(g^0, h^{1\mathrm{S}}) - G^{(2)}_{ab}(g^0, h^{1R} + h^{1\mathrm{S}}) \,. \quad (27)$$

Use of the identity in Eq. (B3) modifies the RHS with the result that

$$G^{(1)}_{ab}(g^0, h^{2R} + h^{2\mathrm{S}\dagger} + h^{2\mathrm{S}}) = G^{(1)}_{ab}(g^0 + h^{1R}, h^{1\mathrm{S}} + h^{2\mathrm{S}\dagger}) - G^{(2)}_{ab}(g^0, h^{1\mathrm{S}}) - G^{(2)}_{ab}(g^0, h^{1R}) - G^{(1)}_{ab}(g^0 + h^{1R}, h^{1\mathrm{S}}). \quad (28)$$

On the RHS, the fourth term cancels that part of the first term which is linear in $h^{1\mathrm{S}}_{ab}$. The terms linear in $h^{2\mathrm{S}}$ on the LHS and quadratic in h^{1S} on the RHS cancel from Eq. (14). The terms linear in $h^{2S\dagger}$ on each side of the equation cancel up to a term of $O(m^3)$, which is ignored. When the dust has settled what remains is

$$G^{(1)}_{ab}(g^0, h^{2R}) = -G^{(2)}_{ab}(g^0, h^{1R}), \tag{29}$$

which reveals obvious consistency for this second order perturbation formalism: When h^{1S}_{ab}, $h^{2S\dagger}_{ab}$ and h^{2S}_{ab} correctly capture their respective parts of the singular behavior of the retarded field, the regular remainder $h^{1R}_{ab}+h^{2R}_{ab}$ appears as a source-free metric perturbation at second order in m as described by Habisohn [11]. The integrability condition for Eq. (29) is satisfied in a manner similar to that for Eq. (26) away from γ_0.

The second-order self-force is similar to the first-order self-force. In a neighborhood of m, $h^{2\text{ret}}_{ab}$ is naturally decomposed into two complementary parts, $h^{2\text{ret}}_{ab} = h^{2R}_{ab} + (h^{2S\dagger}_{ab} + h^{2S}_{ab})$, where $h^{2S\dagger}_{ab} + h^{2S}_{ab}$ exerts no force on m itself. The second-order self-force then moves m along a geodesic of $g^0_{ab} + h^{1R}_{ab} + h^{2R}_{ab}$.

The sanguine simplicity of Eq. (29) hides the complexity of its application. It might appear as though h^{2R}_{ab} may be solved only in terms of h^{1R}_{ab} in a neighborhood of γ_0, but what is lacking is the description of the boundary condition which is typically given as a condition on the retarded field h^{ret}_{ab}. To find h^{2R}_{ab} it is necessary first to find $h^{1\text{ret}}_{ab}$ and to evaluate h^{1S}_{ab} as an asymptotic expansion in a neighborhood of γ_0; these lead to $h^{1R}_{ab} = h^{1\text{ret}}_{ab} - h^{1S}_{ab}$. With h^{1R}_{ab} the self-force modification of the worldline may be determined. At this point $h^{2S\dagger}_{ab}$ and h^{2S}_{ab} are accessible via asymptotic expansions and $h^{2\text{ret}}_{ab}$ could be evaluated via Eq. (26). Only then is h^{2R}_{ab} able to be determined.

VI. PRACTICAL CONCERNS

In most situations, only an asymptotic approximation h^{s}_{ab} to the exact h^{S}_{ab} is likely to be known, and as a consequence an actual application of the formalism described above is not as elementary as it might appear. In this case, $h^{r}_{ab} \equiv h^{\text{ret}}_{ab} - h^{\text{s}}_{ab}$ is an approximation to the actual regular field h^{R}_{ab}. With these approximations some concerns appear in a neighborhood of the δ-function point source m. The proper evaluation of the self-force, via h^{r}_{ab}, requires that h^{r}_{ab} match both the value and first coordinate derivatives of h^{R}_{ab} on γ_0. In turn, this requires that the difference $h^{S}_{ab} - h^{\text{s}}_{ab}$ be zero on γ_0, and also, with LIC coordinates, that all first coordinate derivatives of this difference also be zero on γ_0.

Experience [24–30] has shown that in numerical work if the difference $h^{S}_{ab} - h^{\text{s}}_{ab}$ of these two singular fields is increasingly more differentiable, then the numerical analysis will be increasingly more accurate.

In some self-force analyses [31]

$$\begin{aligned} h^{1S}_{ab} &= h^{1\text{s}}_{ab} + O(mx^4/r\mathcal{R}^4) \;\text{ and} \\ h^{2S\dagger}_{ab} + h^{2S}_{ab} &= h^{2\text{s}\dagger}_{ab} + h^{2\text{s}}_{ab} + O(m^2x^4/r^2\mathcal{R}^4). \end{aligned} \tag{30}$$

We assume henceforth that we have such a precisely described approximation h^{s}_{ab} to h^{S}_{ab}.

For first order analyses, the integrability condition required for using Eq. (9) to solve for $h^{1\text{ret}}_{ab}$ is easily satisfied. The approximation for $h^{1\text{s}}_{ab}$ is then accurate enough that $h^{1\text{r}}$ is C^2 on γ_0, and the accuracy of the computed self-force effects are not limited by this approximation.

To derive a second-order equation for $h^{2\text{r}}_{ab}$ follow the same instructions as for Eq. (29) while using h^{r}_{ab} and h^{s}_{ab} instead of h^{R}_{ab} and h^{S}_{ab}, and do not use Eqs. (13), (14) or (23) for substitutions. The result is

$$\begin{aligned} G^{(1)}_{ab}(g^0, h^{2\text{r}}) = &-G^{(2)}_{ab}(g^0, h^{1\text{r}}) \\ &- [G^{(2)}_{ab}(g^0, h^{1\text{s}}) + G^{(1)}_{ab}(g^0, h^{2\text{s}})] \\ &+ [8\pi T_{ab}(\gamma_0 + \gamma_{1\text{r}}) - G^{(1)}_{ab}(g^0 + h^{1\text{r}}, h^{1\text{s}} + h^{2\text{s}\dagger})] \\ &- [8\pi T_{ab}(\gamma_0) - G^{(1)}_{ab}(g^0, h^{1\text{s}})]. \end{aligned} \tag{31}$$

The integrability condition for using Eq. (31) to solve for $h^{2\text{r}}$ is satisfied everywhere except, perhaps, precisely on γ_0 where the analysis entails some modest difficulty. The order terms associated with $h^{1\text{s}}_{ab}$ and $h^{2\text{s}\dagger}_{ab} + h^{2\text{s}}_{ab}$ (given above) provide an estimate for the behavior of the source on the righthand side in a neighborhood of γ_0. Most of the terms on the righthand side are either distributions or differentiable and well behaved on γ_0. The uncertainty involving the source is dominated by the $G^{(2)}_{ab}(g^0, h^{1\text{s}})$ and $G^{(1)}_{ab}(g^0, h^{2\text{s}\dagger} + h^{2\text{s}})$ terms; each of these scales as two spatial derivatives of $m^2x^4/r^2\mathcal{R}^4$, which is $O(m^2x^2/r^2\mathcal{R}^4)$ and finite but discontinuous on γ_0. The divergence of this term is then $O(m^2x/r^2\mathcal{R}^4)$ which diverges on γ_0. However, the integral of this divergence (contracted with a smooth test vector field of order unity) over a small volume of radius r_* about m is then $O(m^2r_*^2/\mathcal{R}^4)$. If we choose r_* such that m, r_*, and $\mathcal{R}$ are related by

$$r_*^2/\mathcal{R} \lesssim m \ll r_* \ll \mathcal{R}\ , \tag{32}$$

then it follows that the integrated divergence over the volume of radius r_* is $O(m^3/\mathcal{R}^3)$. For $r > r_*$ the integrability condition is satisfied. Thus, the integrability condition fails only at $O(m^3)$ which does not hinder the analysis at $O(m^2)$. No fundamental difficulty prevents solving Eq. (31) for $h^{2\text{r}}_{ab}$. The resultant $h^{2\text{r}}_{ab}$ is C^1 on γ_0 and is sufficient to find second order self-force effects.

VII. SUMMARY AND CONCLUSIONS

Upon reflection, Eq. (26) describes the second-order perturbation problem for a δ-function point mass in a quite satisfactory manner and is the primary result of this manuscript. The metric perturbation $h^{2\text{ret}}_{ab}$ may be determined directly, and the h^{S}_{ab}, h^{R}_{ab} decomposition of h^{ret}_{ab} is only required for determining the effects of the self-force.

It is notable that the representation of a small mass m by a δ-function point source works as well at second-order as it does at first order.

VIII. ACKNOWLEDGEMENT

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Appendix A: Nonlinear perturbation theory and tidal distortion of a small black hole

The simplest example of non-linear perturbation theory in General Relativity involves perturbing flat spacetime by putting a small, spherical object of mass m down on the origin of Minkowski space. Outside the object the geometry must be the Schwarzschild metric from Birkhoff's theorem.

The usual coordinates of Minkowski space form an LIC coordinate system because the spatial origin $x^i = 0$ is a geodesic, and the other LIC conditions are clearly satisfied. We define a covariant vector in the radial direction via $n_i = \nabla_i r$. With a Schwarzschild black hole of mass m present at the spatial origin, the metric takes the unfamiliar form

$$g^{\mathrm{schw}}_{ab}\mathrm{d}x^a\mathrm{d}x^b = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \frac{r}{r-2m}n_kn_l\mathrm{d}x^k\mathrm{d}x^l + (\delta_{kl}-n_kn_l)\mathrm{d}x^k dx^l\,. \tag{A1}$$

An alternative description of this form of the Schwarzschild metric is

$$g^{\mathrm{schw}}_{ab} = \eta_{ab} + {}_0h^{\mathrm{S}}_{ab}\,, \tag{A2}$$

where ${}_0h^{\mathrm{S}}_{ab}$ is to be identified as the *singular field* from self-force analysis, and the leading subscript 0 implies that this monopole part of the singular field is spherically symmetric. From Eq. (A1) it follows that

$$\begin{aligned}{}_0h^{\mathrm{S}}_{ab}\mathrm{d}x^a\mathrm{d}x^b &= (g^{\mathrm{schw}}_{ab}-\eta_{ab})\mathrm{d}x^a\mathrm{d}x^b \\ &= \frac{2m}{r}\mathrm{d}t^2 + \frac{2m}{r-2m}n_kn_l\mathrm{d}x^k\mathrm{d}x^l\,. \end{aligned}\tag{A3}$$

The nth order part of ${}_0h^{\mathrm{S}}_{ab}$ scales as m^n and may be isolated with

$${}_0h^{n\mathrm{S}}_{ab} \equiv \frac{m^n}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}m^n}h^{\mathrm{S}}_{ab}\right]_{m=0}\,. \tag{A4}$$

This provides the formal representation

$$h^{\mathrm{S}}_{ab} = \sum_{n=1}^{\infty} h^{n\mathrm{S}}_{ab}\,. \tag{A5}$$

For our elementary example, the first term in this sum is

$${}_0h^{1\mathrm{S}}_{ab}\mathrm{d}x^a\mathrm{d}x^b = \frac{2m}{r}\mathrm{d}t^2 + \frac{2m}{r}n_kn_l\mathrm{d}x^k\mathrm{d}x^l, \tag{A6}$$

and for $n>1$

$${}_0h^{n\mathrm{S}}_{ab}\mathrm{d}x^a\mathrm{d}x^b = \left(\frac{2m}{r}\right)^n n_kn_l\mathrm{d}x^k\mathrm{d}x^l. \tag{A7}$$

In this treatment of the Schwarzschild metric the singular features of ${}_0h^{n\mathrm{S}}_{ab}$ are identified, and the absence of a regular field h^{R}_{ab} is assured by the flat nature of the initial Minkowski metric.

A more subtle example places a Schwarzschild black hole in a region of spacetime that is empty but has slowly changing curvature from some distant source. In that case the metric of a black hole placed on the origin of the LIC coordinate system of Eq. (1) would be perturbed by the background curvature and could be analyzed by use of the Regge-Wheeler [32] formalism. The boundary condition at large r requires that the perturbed metric approach the form given in Eq. (1). The boundary condition as $r\to 2m$ requires that the perturbation be well behaved on the future event horizon of the small black hole. In the time independent limit the wave equations for the metric perturbations admit analytic solutions which satisfy the boundary conditions [17].

The dominant tidal effects present in both $h^{1\mathrm{S}}_{ab}$ and $h^{2\mathrm{S}}_{ab}$ are seen in the quadrupole $l=2$ terms of Eq. (9) of [17], which we reproduce here as

$$\begin{aligned}{}_2h^{\mathrm{S}}_{ab}dx^adx^b \;=\; & R^0_{titj}x^ix^j\left[(4m/r-4m^2/r^2)\,\mathrm{d}t^2\right.\\ & \left.+\,2m^2/r^2(\delta_{kl}-n_kn_l)\,\mathrm{d}x^k\mathrm{d}x^l\right]\\ & +\,\frac{8m}{3r}x^ix^jR^0_{ikjt}\,\mathrm{d}x^k\mathrm{d}t + O(mx^3/r\mathcal{R}^3)\\ & + O(m^2x^2/r\mathcal{R}^3) + O(m^3x^2/r^2\mathcal{R}^3)\end{aligned}\tag{A8}$$

The order terms here result from the possible slow time dependence of the tidal field and are all much smaller in the buffer zone than the explicit terms provided.

A more extensive analysis of h^{S}_{ab} in a similar style is given in [20]. An alternative treatment in a dramatically different style is given in [21].

Appendix B: Useful Identity

An identity used in deriving Eqs. (29) and (31) results from considering two different expansions of the same expression $G(g^0+h^{1\mathrm{R}}+h^{1\mathrm{S}})$. On the one hand, treating $h^{1\mathrm{R}}_{ab}+h^{1\mathrm{S}}_{ab}$ as a single quantity, it expands to be

$$G^{(1)}_{ab}(g^0,h^{1\mathrm{R}}+h^{1\mathrm{S}}) + G^{(2)}_{ab}(g^0,h^{1\mathrm{R}}+h^{1\mathrm{S}}) + O(m^3). \tag{B1}$$

On the other hand, first grouping $h^{1\mathrm{R}}_{ab}$ with g^0_{ab} while expanding in powers of $h^{1\mathrm{S}}_{ab}$, and subsequently expanding in powers of $h^{1\mathrm{R}}_{ab}$, it becomes

$$\begin{aligned}&G^{(1)}_{ab}(g^0,h^{1\mathrm{R}}) + G^{(2)}_{ab}(g^0,h^{1\mathrm{R}}) + G^{(2)}_{ab}(g^0,h^{1\mathrm{S}})\\ &\qquad + G^{(1)}_{ab}(g^0+h^{1\mathrm{R}},h^{1\mathrm{S}}) + O(m^3).\end{aligned}\tag{B2}$$

Equating these two expressions reveals that

$$
\begin{aligned}
G^{(2)}_{ab}(g^0, h^{1\mathrm{ret}}) = {} & G^{(2)}_{ab}(g^0, h^{1\mathrm{S}}) + G^{(2)}_{ab}(g^0, h^{1\mathrm{R}}) \\
& + G^{(1)}_{ab}(g^0 + h^{1\mathrm{R}}, h^{1\mathrm{S}}) - G^{(1)}_{ab}(g^0, h^{1\mathrm{S}}) \\
& + O(m^3). \qquad \text{(B3)}
\end{aligned}
$$

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