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Bubbles and Black Branes in Grand Canonical Ensemble

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Abstract

When the phase structure of the black brane in grand canonical ensemble is discussed, the bubble phase with the same boundary data should be included in this structure. As such, the phase transitions among bubbles, black branes and “hot flat space” are possible, therefore giving a much enriched phase structure. We also argue that under some conditions, either the grand canonical ensemble itself is unstable or there are some unknown new phases.

1 Introduction

Understanding the thermodynamic phase structure of black holes is helpful in learning the properties of space-time and quantum gravity. The phase structure of the AdS black holes can also be used to study the corresponding field theory within the context of AdS/CFT correspondence. The well-known Hawking-Page phase transition [1] between the AdS black hole and the “hot empty AdS space” corresponds to the confinement-deconfinement phase transition of the large- N_c $\mathcal{N} = 4$ super Yang-Mills at finite temperature [2]. The charged AdS black hole in the canonical ensemble also displays a van der Waals-Maxwell like phase transition which can also be understood from the dual field theory [3, 4].

The Hawking-Page or van der Waals-Maxwell like phase structure not only presents in AdS black holes but also exists in some other black holes. The Hawking-page like transition can also be found in the canonical ensemble between chargeless asymptotic flat black holes and “hot flat space” as discussed by York [5]. In the grand canonical ensemble, there can also be transitions between charged black holes and the “hot flat space” [6]. The van der Waals like phase transition was also seen in asymptotic flat and dS black hole [7, 8]. Unlike the black holes in AdS space, asymptotic flat and dS black holes are not thermodynamically stable objects due to Hawking radiation. So in discussing the phase structure of these objects, the black hole must be put in thermal contact with a heat reservoir with fixed thermodynamic data on the boundary so that they can form an equilibrium, and then the phase structure could be discussed.

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Black branes are the black hole like solutions of low energy effective theory of String or M-theory, which are also asymptotic flat. They also need to be put inside a heat reservoir in order to study their thermodynamic properties. In fact, they have similar phase transitions as those asymptotic flat black holes. In [9], the phase structure of black p -branes in D -dimension ($D = d + \tilde{d} + 2, p = d - 1$) spacetime in canonical ensemble is discussed, where the charge q inside the cavity is fixed. In this ensemble, for $\tilde{d} > 2$ charged cases, there is a critical charge q_c . When $q < q_c$, there is a first order van der Waals phase transition between the larger black brane solution and the smaller one at a certain transition temperature. However, the chargeless case and $\tilde{d} \leq 2$ charged cases are different from $\tilde{d} > 2$ charged cases. There is no such van der Waals phase transition and below a minimal temperature T_{\min} , there could not exist a black brane phase. For the chargeless case the “hot flat space” is the stable phase below T_{\min} , whereas in the charged cases we do not know what could exist at present stage, because of the absence of the charged “hot flat space”. One would wonder what could fill this part of the phase space.

It is well-known that there exists a regular “bubble of nothing” or bubble [10, 11] which carries the same flux as the black brane. One therefore wonders whether this bubble may play some role in the phase structure of the black brane. Precisely based on this consideration, bubble was considered in the canonical ensemble in [13] and the phase structure is enriched. There can be topological transitions among black branes, bubbles and “hot flat space” under certain conditions. The bubble really fills some part of the phase space though there is still some phase space left unoccupied.

The grand canonical ensemble of black brane system is a little different. There are no van der Waals like phase transitions as in the canonical ensemble. Since the charge is not fixed in this case, charged black brane can undergo Hawking-Page like phase transition into “hot flat space”. In [12], only black branes and “hot flat space” are taken into account in the grand canonical ensemble. In the present paper, we will discuss the role played by the bubble in the phase structure. The bubble can be generated from the black brane solution by a double wick rotation in Minkowski space-time, and in Euclidean space-time the effect is to interchange one world volume spatial coordinate and the time coordinate $x \leftrightarrow t$ in the metric, while leaving the form field and dilaton unchanged. So the bubble geometry is regular and has no horizon, hence no entropy. One expects that the period of the space direction L plays a similar role in the bubble case as β , the inverse temperature T , plays in the black brane system. As a result, the phase structure depends not only on the temperature and potential Φ , but also on L as in the canonical ensemble in [13]. We first follow closely the papers [12] and [13]. However, we will find out that in some conditions there could be phase transition processes where the Gibbs energy itself is not continuous and the system tends to the boundary of the cavity. Since our “zero-loop” approximation breaks down when the horizon tends to the boundary in these conditions, this may not be a right behavior of the system. Nevertheless, if we take these cases seriously, we will argue that there may be a new phase near the boundary developed either by quantum effects or other nonperturbative effects, or the grand canonical ensemble is not stable under certain circumstances.

In [13], extremal branes are also considered in the canonical ensemble and fill some part of the phase space of $\tilde{d} = 2, 1$ cases. Extremal branes are very different from the nonextremal ones, as discussed in [14, 15, 16]⁴. Their thermodynamic properties may not be obtained simply from the $r_+ = r_-$ limit of the nonextremal ones. For example, they can form equilibria with the environments at arbitrary temperature. This is because their topology near the horizon is different from the nonextremal ones. If we only consider the Euclidean time direction and the radius direction, the topology of the

⁴[14, 15, 16] argue that even when the area of the horizon for the extremal black hole is not zero, the entropy still vanishes. This seems to contradict with the string theory calculations. Although there are some efforts trying to resolve this discrepancy, for example [17] and the references therein, the reason for this discrepancy is still not fully understood. However we do not need this result in our paper since the area of the extremal black brane discussed in this paper is zero and hence it has no entropy.

nonextremal brane is R^2 near the horizon while the one for the extremal brane is $S^1 \times R$. So there is no need to avoid the conical singularity for the extremal brane and it can have arbitrary period in the Euclidean time direction hence arbitrary temperature. There is also other evidence that the extremal black branes can not be seen as the continuous extremal limit of the nonextremal ones, e.g. [17]. The extremal black branes also have zero horizon area hence no entropy [18]. Because of their BPS nature and zero entropy property, they may also split into smaller extremal branes with fewer charges without changing the entropy. However, if we do not care about this, we could still include the nondilatonic extremal brane in our discussion. We will see that only when the potential conjugate to the charge at the boundary is fixed at a certain value can it exist. In this case, it has zero Gibbs energy which means that it can coexist with the “hot flat space”. For dilatonic extremal branes, since the horizon is already singular, our “zero-loop” approximation is not appropriate near the horizon. Therefore we would not consider them in our discussion.

The organization of this paper is as follows: In section 2, we review the reduced action or the Gibbs free energy for black brane and obtain the one for the bubble. In section 3, we discuss the phase structure of this grand canonical ensemble including black branes, bubbles and the “hot flat space”. The final section contains some discussion on the extremal cases and the problems we meet.

2 The action

Let us recall the black p -brane solution in Euclidean signature in space-time dimension $D = d + \tilde{d} + 2$, ($d = p + 1$)

$$\begin{aligned}
ds^2 &= \Delta_+ \Delta_-^{-\frac{d}{D-2}} dt^2 + \Delta_-^{\frac{\tilde{d}}{D-2}} (dx^1)^2 + \Delta_-^{\frac{\tilde{d}}{D-2}} \sum_{i=2}^p (dx^i)^2 + \Delta_+^{-1} \Delta_-^{\frac{a^2}{2d}-1} d\rho^2 + \rho^2 \Delta_-^{\frac{a^2}{2d}} d\Omega_{\tilde{d}+1}^2 \\
A_{[p+1]} &= -ie^{a\phi_0/2} \left[\left(\frac{r_-}{r_+} \right)^{\tilde{d}/2} - \left(\frac{r_- r_+}{\rho^2} \right)^{\tilde{d}/2} \right] dt \wedge dx^1 \wedge \dots \wedge dx^p, \\
F_{[p+2]} &\equiv dA_{[p+1]} = -ie^{a\phi_0/2} \tilde{d} \frac{(r_- r_+)^{\tilde{d}/2}}{\rho^{\tilde{d}+1}} d\rho \wedge dt \wedge dx^1 \wedge \dots \wedge dx^p, \\
e^{2(\phi-\phi_0)} &= \Delta_-^a,
\end{aligned} \tag{1}$$

where a is the dilaton coupling and in supergravity theories with maximal supersymmetry, $a^2 = 4 - \frac{2d\tilde{d}}{D-2}$. Δ_{\pm} is defined as $\Delta_{\pm} = 1 - \left(\frac{r_{\pm}}{\rho} \right)^{\tilde{d}}$ for $r_+ > r_-$, with r_{\pm} being the two parameters characterizing the solution and related to the charge and the mass of the black brane. ϕ_0 is the asymptotic value of the dilaton at infinity which is related to the string coupling $g_s = e^{\phi_0}$. We have extracted one spatial direction x^1 from the sum of the spatial directions. See [19, 20] for details of the solution. From the metric, we see that the physical radius of the $\tilde{d} + 1$ sphere should be $\bar{\rho} \equiv \Delta_-^{\frac{a^2}{4d}} \rho$ and we also define $\bar{r}_{\pm} \equiv \Delta_-^{\frac{a^2}{4d}} r_{\pm}$. The charge is calculated to be $Q_d = \frac{\Omega_{\tilde{d}+1} \tilde{d}}{\sqrt{2\kappa}} e^{-a\phi_0/2} (r_+ r_-)^{\tilde{d}/2}$.

As mentioned in the introduction, the metric of the bubble is obtained from the black brane metric just by interchanging the time coordinate t and one of the world volume spatial coordinate x^1

$$ds^2 = \Delta_-^{\frac{\tilde{d}}{D-2}} dt^2 + \Delta_+ \Delta_-^{-\frac{d}{D-2}} (dx^1)^2 + \Delta_-^{\frac{\tilde{d}}{D-2}} \sum_{i=2}^p (dx^i)^2 + \Delta_+^{-1} \Delta_-^{\frac{a^2}{2d}-1} d\rho^2 + \rho^2 \Delta_-^{\frac{a^2}{2d}} d\Omega_{\tilde{d}+1}^2. \tag{2}$$

The form field $A_{[p+1]}$ and dilaton ϕ are kept unchanged. The space-time is restricted to the $\rho > r_+$ region. The singularity of the form field at $\rho = 0$ is excluded from the geometry and the charge is

the result of the flux around the noncontractible $S^{\tilde{d}+1}$. The period of the x^1 direction is chosen to avoid the conical singularity at $\rho = r_+$ and so the solution is regular. This solution can be obtained independently without referring to the black brane solution by directly solving the classical equation of motion. It can have the same boundary condition as the black brane when we put it in to a cavity. Since for an ensemble we only fix the boundary data, any classical solution that satisfies the boundary conditions can form an equilibrium with the cavity; therefore one can not exclude the bubble state from the black brane phase structure.

While there is an event horizon at $\rho = r_+$ for the black brane, there is no horizon for bubbles. For black p -branes, the inverse of the local temperature is fixed to be

$$\beta(\bar{\rho}) = \Delta_+^{1/2} \Delta_-^{-\frac{d}{2(d+\tilde{d})}} \beta^* = \Delta_+^{1/2} \Delta_-^{-1/\tilde{d}} \frac{4\pi\bar{r}_+}{\tilde{d}} \left(1 - \frac{\bar{r}_-^{\tilde{d}}}{\bar{r}_+^{\tilde{d}}}\right)^{\frac{1}{\tilde{d}} - \frac{1}{2}} \quad (3)$$

whereas the local radius of the x^1 direction is arbitrary. However, for bubbles, the inverse of the local temperature is arbitrary but the local period of x^1 is

$$L(\bar{\rho}) = \Delta_+^{1/2} \Delta_-^{-\frac{d}{2(d+\tilde{d})}} L^* = \Delta_+^{1/2} \Delta_-^{-1/\tilde{d}} \frac{4\pi\bar{r}_+}{\tilde{d}} \left(1 - \frac{\bar{r}_-^{\tilde{d}}}{\bar{r}_+^{\tilde{d}}}\right)^{\frac{1}{\tilde{d}} - \frac{1}{2}} \quad (4)$$

to avoid the conical singularity. β^* and L^* in above equations are the inverse temperature and the period seen from infinity respectively.

As in [9, 12], we put the black brane or bubble inside a cavity with a fixed radius $\bar{\rho} = \bar{\rho}_B$. To establish a grand canonical ensemble, we then fix all the local quantities at the wall of the cavity: the inverse temperature $\bar{\beta}$, local period \bar{L} in x^1 direction, local volume V_{p-1} of the x^i ($i = 2, \dots, p$) directions, dilaton ϕ_B , and the potential $\bar{\Phi}$ conjugate to the charge at the boundary $\bar{\rho}_B$. The charge/flux can be now expressed using the boundary data as $Q_d = \frac{\Omega_{\tilde{d}+1} \tilde{d}}{\sqrt{2}\kappa} e^{-a\phi_B/2} (\bar{r}_+ \bar{r}_-)^{\tilde{d}/2}$ with \bar{r}_{\pm} evaluated at the boundary. We can also define the potential Φ in the local inertial frame using the form field $A_{[p+1]} \equiv -i\sqrt{2}\kappa\Phi d\bar{t} \wedge d\bar{x}^1 \dots d\bar{x}^p$ where $(\bar{t}, \bar{x}^1, \dots, \bar{x}^p)$ are the coordinates in the local inertial frame. So Φ is the conjugate potential for Q and one can easily obtain

$$\bar{\Phi} = \Phi(\bar{\rho}_B) = \frac{1}{\sqrt{2}\kappa} e^{a\phi_B/2} \left(\frac{\bar{r}_-}{\bar{r}_+}\right)^{\frac{\tilde{d}}{2}} \left(\frac{\Delta_+}{\Delta_-}\right)^{\frac{1}{2}} \bigg|_{\bar{\rho}=\bar{\rho}_B}. \quad (5)$$

With this setup for the grand canonical ensemble, the classical Euclidean action for the black brane is obtained in [9]

$$\begin{aligned} I_E^{\text{brane}} = & - \frac{\bar{\beta}\bar{L}V_{p-1}\Omega_{\tilde{d}+1}}{2\kappa^2} \bar{\rho}_B^{\tilde{d}} \left[(\tilde{d}+2) \left(\frac{\Delta_+}{\Delta_-}\right)^{1/2} + \tilde{d}(\Delta_+\Delta_-)^{1/2} - 2(\tilde{d}+1) \right] \\ & - \frac{4\pi\bar{L}V_{p-1}\Omega_{\tilde{d}+1}}{2\kappa^2} \bar{r}_+^{\tilde{d}+1} \Delta_-^{-\frac{1}{2}-\frac{1}{\tilde{d}}} \left(1 - \frac{\bar{r}_-^{\tilde{d}}}{\bar{r}_+^{\tilde{d}}}\right)^{\frac{1}{2}+\frac{1}{\tilde{d}}} - \bar{\beta}\bar{L}V_{p-1}Q_d\bar{\Phi}, \end{aligned} \quad (6)$$

where we have expressed the volume $V_p = \bar{L}V_{p-1}$. According to the “zero-loop” approximation of the path integral, the Gibbs free energy G can be obtained from the classical action by $G = I_E/\bar{\beta}$ [21]. Since $G = E - TS - V_p Q_d \bar{\Phi}$, we can identify the energy E as the first term in (6) divided by $\bar{\beta}$, which is consistent with the ADM mass as $\bar{\rho}_B \rightarrow \infty$ [22], and the entropy S from the second term which is

the same as the one obtained in [18]. Since the bubble metric is just obtained by interchanging t and x^1 coordinate, we expect to obtain the bubble action by interchanging $\bar{\beta}$ and \bar{L} in the brane action:

$$I_E^{\text{bubble}} = - \frac{\bar{\beta}\bar{L}V_{p-1}\Omega_{\tilde{d}+1}}{2\kappa^2}\bar{\rho}_B^{\tilde{d}}\left[(\tilde{d}+2)\left(\frac{\Delta_+}{\Delta_-}\right)^{1/2} + \tilde{d}(\Delta_+\Delta_-)^{1/2} - 2(\tilde{d}+1)\right] \\ - \frac{4\pi\bar{\beta}V_{p-1}\Omega_{\tilde{d}+1}}{2\kappa^2}\bar{r}_+^{\tilde{d}+1}\Delta_-^{-\frac{1}{2}-\frac{1}{\tilde{d}}}\left(1 - \frac{\bar{r}_-^{\tilde{d}}}{\bar{r}_+^{\tilde{d}}}\right)^{\frac{1}{2}+\frac{1}{\tilde{d}}} - \bar{\beta}\bar{L}V_{p-1}Q_d\bar{\Phi} \quad (7)$$

As in the black brane case, one could check that by requiring the local minimum of the action with respect to charge Q_d and r_+ , we recover the equation of state $\bar{L} = L(\bar{\rho}_B)$ and $\bar{\Phi} = \Phi(\bar{\rho}_B)$ for equilibrium. This justifies the validity of this bubble action. Due to the absence of a horizon, the entropy of the bubble is zero and hence the sum of the first and the second term divided by $\bar{\beta}$ can be identified as the energy of the bubble, which can be checked to be consistent with the definition of ADM mass as $\bar{\rho}_B \rightarrow \infty$.

For future convenience, we define the dimensionless variables

$$x = \left(\frac{\bar{r}_+}{\bar{\rho}_B}\right)^{\tilde{d}}, \quad b = \frac{\beta}{4\pi\bar{\rho}_B}, \quad R = \frac{L}{4\pi\bar{\rho}_B}, \quad q = \left(\frac{Q_d^*}{\bar{\rho}_B}\right)^{\tilde{d}}, \quad \varphi = \sqrt{2}\kappa e^{-a\phi_B/2}\Phi. \quad (8)$$

where

$$Q_d^* \equiv \left(\frac{\sqrt{2}\kappa Q_d}{\Omega_{\tilde{d}+1}\tilde{d}}e^{a\phi_B/2}\right)^{\frac{1}{\tilde{d}}}. \quad (9)$$

We use the barred variables \bar{b} , $\bar{\varphi}$, \bar{R} to denote the quantities fixed on the boundary. With these dimensionless variables, one can rewrite the conditions for equilibrium as

$$\bar{b} = b(x, q), \quad b(x, q) \equiv \frac{1}{\tilde{d}} \frac{x^{1/\tilde{d}}(1-x)^{1/2}}{\left(1 - \frac{q^2}{x^2}\right)^{\frac{\tilde{d}-2}{2\tilde{d}}} \left(1 - \frac{q^2}{x}\right)^{\frac{1}{\tilde{d}}}} \quad (10)$$

for black branes, while for bubbles

$$\bar{R} = R(x, q), \quad R(x, q) \equiv \frac{1}{\tilde{d}} \frac{x^{1/\tilde{d}}(1-x)^{1/2}}{\left(1 - \frac{q^2}{x^2}\right)^{\frac{\tilde{d}-2}{2\tilde{d}}} \left(1 - \frac{q^2}{x}\right)^{\frac{1}{\tilde{d}}}}, \quad (11)$$

and

$$\bar{\varphi} = \varphi(x, q), \quad \varphi(x, q) \equiv \frac{q}{x} \left(\frac{1-x}{1 - \frac{q^2}{x}}\right)^{\frac{1}{2}} \quad (12)$$

for both branes and bubbles. We first constraint the range of x , q to $0 \leq q < x < 1$, and hence $0 \leq \varphi < 1$. In fact, this range can be extend to $x = q < 1$, $\varphi = 1$ continuously, which will be considered in the discussion section. The $x = q = 1$ is not well defined and in this limit φ could have arbitrary value. We will argue that this case can not be physically reached exactly and near this point our “zero-loop” approximation is not applicable.

We can also define the reduced action for black branes

$$\begin{aligned}
\tilde{I}_E^{\text{brane}}(\bar{b}, \bar{R}, \bar{\varphi}; q, x) &= \frac{2\kappa^2 I_E}{(4\pi)^2 \bar{\rho}_B^{\tilde{d}+2} V_{p-1} \Omega_{\tilde{d}+1}} \\
&= -\bar{b}\bar{R} \left[(\tilde{d}+2) \left(\frac{1-x}{1-\frac{q^2}{x}} \right)^{1/2} + \tilde{d}(1-x)^{1/2} \left(1 - \frac{q^2}{x} \right)^{1/2} - 2(\tilde{d}+1) + \tilde{d}q\bar{\varphi} \right] \\
&\quad - \bar{R}x^{1+1/\tilde{d}} \left(\frac{1-\frac{q^2}{x^2}}{1-\frac{q^2}{x}} \right)^{1/2+1/\tilde{d}}
\end{aligned} \tag{13}$$

and by exchanging \bar{b} and \bar{R} we obtain the reduced action for bubbles

$$\begin{aligned}
\tilde{I}_E^{\text{bubble}}(\bar{b}, \bar{R}, \bar{\varphi}; q, x) &= -\bar{b}\bar{R} \left[(\tilde{d}+2) \left(\frac{1-x}{1-\frac{q^2}{x}} \right)^{1/2} + \tilde{d}(1-x)^{1/2} \left(1 - \frac{q^2}{x} \right)^{1/2} - 2(\tilde{d}+1) + \tilde{d}q\bar{\varphi} \right] \\
&\quad - \bar{b}x^{1+1/\tilde{d}} \left(\frac{1-\frac{q^2}{x^2}}{1-\frac{q^2}{x}} \right)^{1/2+1/\tilde{d}}.
\end{aligned} \tag{14}$$

The grand potential or the Gibbs free energy is proportional to the reduced action divided by \bar{b} . Therefore, for a fixed inverse temperature \bar{b} , they differ only by a positive constant factor. So, to compare the Gibbs free energies of the bubble and the black brane at a fixed temperature, we can just use the reduced actions instead. Notice that even though we use the same x in the bubble and brane action, this does not mean that they are equal. They are independent variables since the physical meanings of r_+ for the black brane and the bubble are different. We also do not differentiate these two variables in the following discussion and this could be easily understood from the contexts.

3 Phase structure

If (10), (11) and (12) have solutions for fixed \bar{b} , \bar{R} , $\bar{\varphi}$, there could be locally stable black branes or bubbles in the grand canonical ensemble. To achieve this, first we can solve equation (12) to obtain

$$\frac{q^2}{x^2} = \frac{\bar{\varphi}^2}{1 - (1 - \bar{\varphi}^2)x} \tag{15}$$

and by substituting (15) into $b(x, q)$, $R(x, q)$ in (10) and (11), we find

$$b_{\bar{\varphi}}(x) \equiv b(x, q) = U_{\bar{\varphi}}(x), \quad \text{for black branes,} \tag{16}$$

$$\text{and } R_{\bar{\varphi}}(x) \equiv R(x, q) = U_{\bar{\varphi}}(x), \quad \text{for bubbles,} \tag{17}$$

where

$$U_{\bar{\varphi}}(x) \equiv \frac{x^{\frac{1}{\tilde{d}}}[1 - (1 - \bar{\varphi}^2)x]^{\frac{1}{2}}}{\tilde{d}(1 - \bar{\varphi}^2)^{\frac{1}{2} - \frac{1}{\tilde{d}}}}. \tag{18}$$

So to solve (10) and (11) is to solve $\bar{u} = U_{\bar{\varphi}}(x)$ for fixed \bar{u} and $\bar{\varphi}$, with $\bar{u} = \bar{b}$ for black brane and $\bar{u} = \bar{R}$ for bubble, and then from (15) we find q for the black brane or the bubble. Note that, in general, for $\bar{b} \neq \bar{R}$, the solution for the bubble and the black brane may not be the same. We use \bar{x} to denote the solution for the black brane and \bar{y} for the bubble. If $\bar{x} \neq \bar{y}$, the bubble and the black brane do not

have the same charge/flux from (15) and one can easily find out that the one with larger x solution carries more charge/flux.

The discussion on the solutions of $\bar{u} = U_{\bar{\varphi}}(x)$ under different conditions has already been done in [12] for the black brane, and can be directly used in the bubble case. The results can be summarized as follows,

1. For $\sqrt{\frac{\tilde{d}}{2+d}} < \bar{\varphi} < 1$, $U_{\bar{\varphi}}(x)$ is monotonically increasing for $0 < x < 1$. If $\bar{u} > U_{\bar{\varphi}}(1)$ there is no solution, and if $0 < \bar{u} < U_{\bar{\varphi}}(1)$, there is one unstable solution. The $U_{\bar{\varphi}}$ vs. x graph is shown in figure 1(a).
2. For $\bar{\varphi} < \sqrt{\frac{\tilde{d}}{2+d}}$, there is a local maximum for $U_{\bar{\varphi}}(x)$ with

$$\begin{aligned} u_{\max} &= \left(\frac{2}{2+\tilde{d}} \right)^{\frac{1}{\tilde{d}}} \left[\tilde{d}(\tilde{d}+2)(1-\bar{\varphi}^2) \right]^{-\frac{1}{2}} \\ &< \frac{1}{\sqrt{2\tilde{d}}} \left(\frac{2}{2+\tilde{d}} \right)^{\frac{1}{\tilde{d}}} \end{aligned} \quad (19)$$

at

$$x_{\max} = \frac{2}{(2+\tilde{d})(1-\bar{\varphi}^2)}. \quad (20)$$

- (a) When $\frac{\tilde{d}}{2+d} < \bar{\varphi} < \sqrt{\frac{\tilde{d}}{2+d}}$, the $U_{\bar{\varphi}}$ vs. x graph is shown in figure 1(b).
 - i. $0 < \bar{u} < U_{\bar{\varphi}}(1)$, there is one unstable solution.
 - ii. $U_{\bar{\varphi}}(1) < \bar{u} < u_{\max}$, there are two solutions: the smaller is unstable, and the larger is locally stable with $\tilde{I}_E > 0$.
- (b) For $\bar{\varphi} < \frac{\tilde{d}}{2+d}$, we define $\bar{x}_g = \frac{4(\tilde{d}+1)}{(\tilde{d}+2)^2(1-\bar{\varphi}^2)}$ where $u_g = U_{\bar{\varphi}}(\bar{x}_g) = \frac{(4(\tilde{d}+1))^{\frac{1}{\tilde{d}}}}{(\tilde{d}+2)^{1+\frac{2}{\tilde{d}}}\sqrt{1-\bar{\varphi}^2}}$ and $\tilde{I}(\bar{x}_g) =$
 0. The $U_{\bar{\varphi}}$ vs. x graph for this case is shown in figure 1(c).
 - i. $0 < \bar{u} < U_{\bar{\varphi}}(1)$, there is one unstable solution with $\tilde{I}_E > 0$.
 - ii. For $u_g < \bar{u} < u_{\max}$, there are two solutions: the smaller is unstable, and the larger is locally stable with $\tilde{I}_E > 0$.
 - iii. For $U_{\bar{\varphi}}(1) < \bar{u} < u_g$, there are two solutions: the smaller is unstable, and the larger is locally stable with $\tilde{I}_E < 0$.

By changing \bar{u} into \bar{b} or \bar{R} , we can obtain the property of the black brane or the bubble respectively. Whether the final state of the system is a black brane, a bubble or the “hot flat space” depends on which one has the smallest reduced action. There are four different cases. The first one, if both \bar{b} or \bar{R} are chosen in cases 1, 2(a)i, 2(a)ii, 2(b)i, 2(b)ii, both black brane and bubble are either unstable or locally stable with $\tilde{I}_E > 0$. Since the “hot flat space” is an equilibrium state with $\tilde{I}_E = 0$, and we suppose there is no other unknown equilibrium with $\tilde{I}_E < 0$ in these cases, the system will tend to the “hot flat space”, in line with the results of [12]. However, there is a problem in this claim, because the “hot flat space” may not be the global minimum of the Gibbs energy. We will come to this point in the discussion section. The second one, if \bar{b} is chosen in cases 2(b)i or 2(b)ii but \bar{R} is chosen in case 2(b)iii, the locally stable bubble will have $\tilde{I}_E < 0$ and the black brane is either unstable or has $\tilde{I}_E > 0$. Thus, the final state will be the bubble. There could also be a problem in this claim which will be discussed later. The third case is interchanging \bar{b} and \bar{R} , bubble and black brane in the previous case.

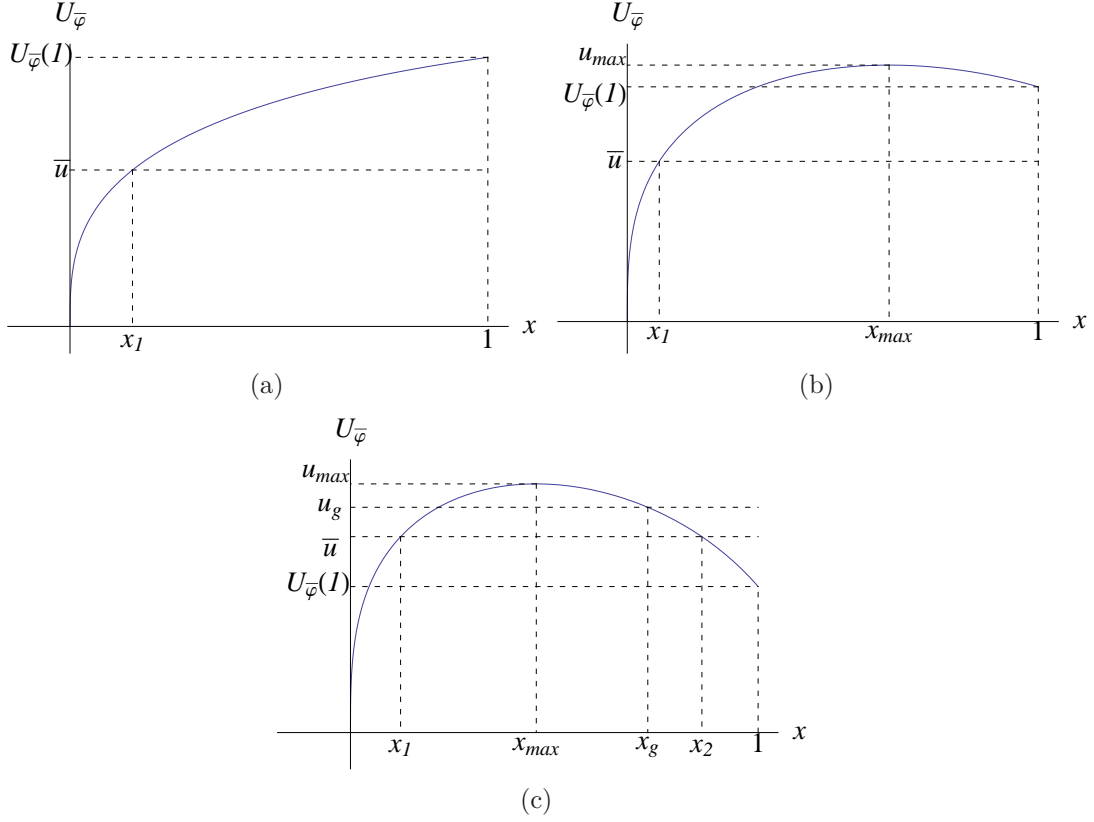


Figure 1: The $U_{\bar{\varphi}}$ vs. x diagrams for different $\bar{\varphi}$: (a) for $\sqrt{\frac{\tilde{d}}{2+d}} < \bar{\varphi} \leq 1$. (b) for $\frac{\tilde{d}}{2+d} < \bar{\varphi} < \sqrt{\frac{\tilde{d}}{2+d}}$. (c) for $\bar{\varphi} < \frac{\tilde{d}}{2+d}$.

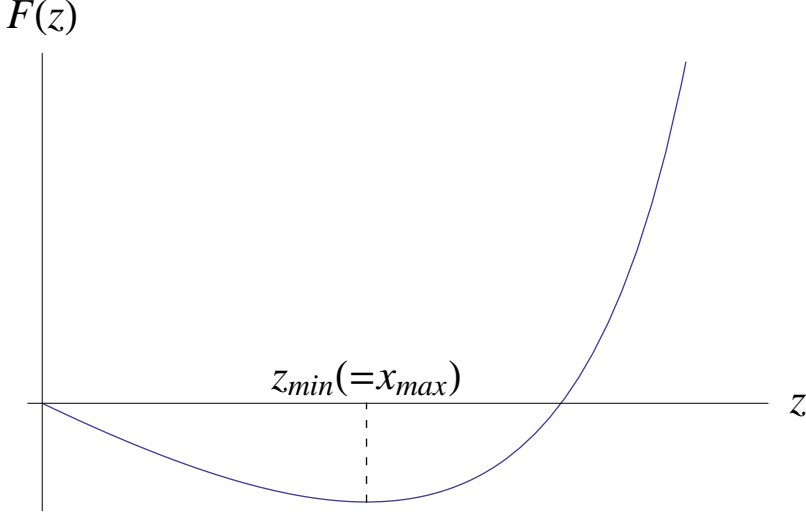


Figure 2: The $F(z)$ is increasing for $z > z_{\min} = x_{\max}$.

The final complicated case is when both \bar{b} and \bar{R} are chosen in case 2(b)iii where both bubble and black brane have negative reduced actions. In this case we have to compare these two negative ones.

For this purpose, we can express the reduced on-shell action for the black brane and the bubble in the same form: for the black brane

$$\tilde{I}_E^{\text{brane}} = -\bar{b}\bar{R}F_{\bar{\varphi}}(\bar{x}) \quad (21)$$

while for the bubble

$$\tilde{I}_E^{\text{bubble}} = -\bar{b}\bar{R}F_{\bar{\varphi}}(\bar{y}) \quad (22)$$

where

$$F_{\bar{\varphi}}(z) = (\tilde{d} + 2)\sqrt{[1 - (1 - \bar{\varphi}^2)z]} + \frac{\tilde{d}}{\sqrt{[1 - (1 - \bar{\varphi}^2)z]}} - 2(\tilde{d} + 1) \quad (23)$$

with $0 \leq \bar{\varphi} < \tilde{d}/(\tilde{d} + 2)$ and $0 < z < 1$, \bar{x} and \bar{y} being the solutions for (10) and (11) for the black brane and the bubble respectively. $F_{\bar{\varphi}}(z)$ has a minimum at $z_{\min} = \frac{2}{(2+\tilde{d})(1-\bar{\varphi}^2)}$, which is equal to x_{\max} in (20), and is increasing in the interval $z_{\min} < z < 1$ while decreasing in $0 < z < z_{\min}$ as indicated in figure 2. As a result, of the bubble and the black brane, the larger one will have smaller action. Since the monotonic property of b vs. x or R vs. x is depicted in the same diagram in figure 1(c) for fixed $\bar{\varphi}$, which is decreasing for $x > x_{\max} = z_{\min}$, the black brane will be larger, i.e. $\bar{x} > \bar{y}$, when $\bar{R} > \bar{b}$, and inversely, the bubble will be larger when $\bar{b} > \bar{R}$. Therefore, when $\bar{R} > \bar{b}$ the black brane will have smaller reduced action and when $\bar{R} < \bar{b}$, the reduced action for the bubble is smaller.

Finally, the possible phase structure of this grand canonical ensemble can be summarized as follows,

1. When $\frac{\tilde{d}}{\tilde{d}+2} < \bar{\varphi}$, the final state is the “hot flat space”.
2. When $0 < \bar{\varphi} < \frac{\tilde{d}}{\tilde{d}+2}$,
 - (a) When $\bar{b} \in (0, U_{\bar{\varphi}}(1)) \cup (u_g, +\infty)$ and $\bar{R} \in (0, U_{\bar{\varphi}}(1)) \cup (u_g, +\infty)$, the final state is the “hot flat space”.
 - (b) When $\bar{b} \in (U_{\bar{\varphi}}(1), u_g)$ and $\bar{R} \in (0, U_{\bar{\varphi}}(1)) \cup (u_g, +\infty)$, the final state is the black brane. Reciprocally, when $\bar{b} \in (0, U_{\bar{\varphi}}(1)) \cup (u_g, +\infty)$ and $\bar{R} \in (U_{\bar{\varphi}}(1), u_g)$, the final state is the bubble.

- (c) When $U_{\bar{\varphi}}(1) < \bar{b} < \bar{R} < u_g$ the final state is the black brane. Inversely, when $U_{\bar{\varphi}}(1) < \bar{R} < \bar{b} < u_g$, the final state is the bubble.
- (d) There could be some coexisting states:
- Three components coexisting point: $\bar{b} = \bar{R} = u_g$.
 - The bubble and black brane coexisting phase: $U_{\bar{\varphi}}(1) < \bar{b} = \bar{R} < u_g$.
 - The black brane and “hot flat space” coexisting phase: $\bar{b} = u_g$, for $\bar{R} > u_g$ or $\bar{R} \in (0, U_{\bar{\varphi}}(1))$. And the bubble and “hot flat space” coexisting phase: $\bar{R} = u_g$, for $\bar{b} > u_g$ or $\bar{b} \in (0, U_{\bar{\varphi}}(1))$.

4 Discussion

In the previous section we have discussed the possible phase structure of the grand canonical ensemble for the bubble, black brane, “hot flat space” system. Compared with the system with only black brane and “hot flat space”, the phase structure depends not only on the potential $\bar{\Phi}$ (or the dimensionless $\bar{\varphi}$), temperature T (or inverse temperature $\bar{\beta}$, dimensionless \bar{b}) but also on the size of one world volume spatial direction \bar{R} . However there could be some region of the phase space where our “zero-loop” approximation fails and we must exclude these region from our discussion. To see this, let us recount the phase structure here in another way. If we fix $\tilde{d}/(\tilde{d} + 2) < \bar{\varphi}$, there is nothing interesting but the “hot flat space”. If we fix $0 < \bar{\varphi} < \tilde{d}/(\tilde{d} + 2)$, there could be different phase transition processes:

1. If we fix the temperature such that $\bar{b} \in (0, U_{\bar{\varphi}}(1)) \cup (u_g, +\infty)$ and increase the size of the x^1 direction from small to large, at first the system will be the chargeless “hot flat space”. Then just above $\bar{R} = U_{\bar{\varphi}}(1)$ there is a phase transition to bubble with \bar{r}_+ close to the boundary of the cavity and the charge close to the extreme value. Put another way, the bubble suddenly appears from the boundary of the cavity. This is an instantly charging process with the Gibbs free energy jumping discontinuously from zero to a negative one, which characterizes a “zeroth-order” phase transition. Then as \bar{R} increases, \bar{r}_+ and the charge are decreasing gradually while the Gibbs energy is increasing. After \bar{R} increases to $\bar{R} = u_g$, where \bar{r}_+ and Q_d decrease to the corresponding values at x_g and the Gibbs energy increases to zero, the system begins to discharge and transforms to the “hot flat space” through a coexisting phase of the bubble and the “hot flat space”. This is a first order transition with continuous Gibbs energy. Then the system will stay in the “hot flat space”.
2. If we fix the temperature such that $\bar{b} \in (U_{\bar{\varphi}}(1), u_g)$ and do the same thing, at first the system is a black brane, of which the horizon size and the charge do not change as \bar{R} increases. There will be a phase transition to bubble at just above $\bar{R} = U_{\bar{\varphi}}(1)$ where the charge suddenly increases to near extremal one and the \bar{r}_+ also suddenly rises to near boundary. In other words, the same as in the previous case, the near extremal bubble appears from the boundary. Similar to the process in the previous case, the Gibbs energy decreases discontinuously which is a “zeroth-order” phase transition. Then, as we increase \bar{R} , the bubble becomes smaller and discharges gradually. When \bar{R} is raised to $\bar{R} = \bar{b}$, where the size of the bubble has shrunk to the same size as the original black brane, the bubble begins to transform back to the black brane through their coexisting phase. This is also a first order transition. After that, the system remains in the same black brane phase, i.e. the charge and the horizon size no longer change.

We can also fix \bar{R} and raise \bar{b} as in above two processes, and the consequence is that the black brane and the bubble will exchange their roles.

We have seen from above two cases that there are “zeroth-order” phase transition processes where the Gibbs free energies are not continuous. In fact, in these cases, for $\bar{R} < U_{\bar{\varphi}}(1)$, the Gibbs free energy

around $x = 1$ for the bubble, although not being at the stationary points, have already dropped below zero or below the one for the original black brane. Similarly, in the cases when we fix $\bar{\varphi} \in (\tilde{d}/(\tilde{d}+2), 1)$ where only the “hot flat space” exists, as the temperature is raised to a certain value, the reduced action near $x = 1$, still not being at the stationary points, will also drop below zero. It seems that, if the Gibbs energy here is correct for this thermodynamic system in these situations, the “hot flat space” is not a global minimum of the Gibbs free energy, and the system tends to $x = 1$ bubbles or black branes which are not the stationary points of the Gibbs free energy at first sight. Nevertheless, from (15), there seems to be an extremal solution for $x = q = 1$ with arbitrary $\bar{\varphi} < 1$. Since extremal branes can form equilibria with the environments having arbitrary temperature and \bar{R} [14, 15, 16], one may think that this extremal brane (from the extremal limit of (1), there is no difference between the bubble and the brane) will be the final steady state in all these cases. However, this is a spurious solution of (15). To see this, we define $r = (\bar{\rho}/\bar{\rho}_B)^{\tilde{d}}$ and look at φ at arbitrary $\bar{\rho}$ which is just (5) with $\bar{\rho}_B$ changed to $\bar{\rho}$, and using (15), we find:

$$\varphi(r, x) = \frac{\bar{\varphi}(r - x)^{1/2}}{(r(1 - x) + x\bar{\varphi}^2(r - 1))^{1/2}}. \quad (24)$$

From this equation, when $\bar{r}_+ = \bar{\rho}_B$ which is the extremal case with $x = 1$, the only solution has $\varphi(r \rightarrow 1, 1) = 1$ and at the equilibrium, $\bar{\varphi} = 1$, which means that the $x = q = 1$ extremal one can only have $\bar{\varphi} = 1$ boundary condition in equilibrium. This contradicts with the previous arbitrary $\bar{\varphi}$ solution. In consequence, the solution with $x = q = 1$ and arbitrary $\bar{\varphi}$ could not be physically realized exactly. If we first set $r = 1$ exactly (which is $\bar{\rho} = \bar{\rho}_B$) and $x < 1$, i.e. we set the boundary condition first and then choose $\bar{r}_+ < \bar{\rho}_B$, we find the boundary condition $\varphi(1, x) = \bar{\varphi}$ which could be arbitrary. This implies that the $x = q = 1$ solution with arbitrary $\bar{\varphi}$ can only be seen as a limiting case which can not be reached exactly. This argument is similar to the one used in [23]. So we should exclude this phase from the phase structure. Let us look at what happens near this extremal case but not reaching it exactly. For black branes, as x and q are approaching 1, the curvature singularity at r_- is coming closer to the horizon and also closer to the wall of the cavity. Since the quantum effect will be essential near the singularity, our “zero-loop” approximation is not applicable in these situations. The Gibbs free energy near $x = 1$ may be modified by quantum corrections. Similarly, for bubbles, when $x, q \rightarrow 1$ quantum effects must be important near $\rho = r_+$ and the wall. So, these “zeroth-order” phase transitions may not be the correct behavior of the system and are just indications of the failure of our method in describing the system under these circumstances. One possibility of the system near $x = 1$ is that there is a new stationary point near $x = 1$ developed by quantum effects or by some other unknown topological solutions, which may lift the Gibbs free energy near $x = 1$. If the new stationary point is a global minimum, there will be a new phase. The other possibility is that there is no new stationary points and the system itself is not well defined or is unstable, which means we can not set up a steady grand canonical ensemble under such circumstances.

It is worth mentioning that some solutions with both bubbles and black holes present were found in five dimensional pure gravity [24, 25] and can be uplift to string theory by embedding in eleven dimensions and a series of dualities [26, 27]. There could be configurations of branes connected by KK bubbles. It will be interesting to investigate whether the solutions obtained this way could play a role in the phase space of the black brane and may also be related to the issue raised above. We will leave this possibility as a future research direction.

After excluding the $x = q = 1$ extremal case, we can also study the extremal non-dilatonic brane with $x = q < 1$ whose geometry at the horizon is not singular, for example, $M2$ brane and $M5$ branes in M-theory, and $D3$ brane in string theory. From (13), we see that it has zero entropy automatically because of its zero horizon area which is already known in [18]. From (15) we obtain $\bar{\varphi} = 1$ and hence the Gibbs energy is zero for any \bar{b} from (13). Thus, it has the same Gibbs free energy as the

“hot flat space”. Therefore, only at $\bar{\varphi} = 1$ could the extremal brane exist and it can also coexist with the “hot flat space”. We should point out that this is a “zero-loop” result. If the higher order fluctuations are taken into account, the Gibbs energy of the extremal brane and the “hot flat space” may be different and these two states may not coexist. Since in our paper, we are considering only the leading order approximation, we will list the coexisting state as a possible phase. As is mentioned in the introduction section that the extremal brane can not be seen as a continuous extremal limit of the nonextremal one, it must appear as a result of some noncontinuous processes such as pair production or quantum tunnelling. But the dynamics behind these processes is not clear to us and is beyond the scope of this paper. As for the extremal dilatonic brane, since the space-time geometry at the horizon is singular, our method does not apply here and hence we exclude them from the phase structure.

So, if we take into account the discussion in this section, the phase structure listed at the end of the previous section would be modified. First, another extremal brane and “hot flat space” coexisting phase with $\bar{\varphi} = 1$ should be added to the phase structure. Second, in some regions of the phase space, the final states of the system is not clear to us due to the limitation of the method and we should exclude these regions from our discussion. In particular, when $\bar{\varphi} > \frac{\tilde{d}}{\tilde{d}+2}$ and either \bar{b} or \bar{R} less than some value about which the “zero-loop” approximation is not applicable, we are not sure what happens. We could estimate this value to be $(1 - \bar{\varphi}^2)^{1/2+1/\tilde{d}}/(2(\tilde{d}+1)(1 - \bar{\varphi}))$ by requiring the reduced action to be zero at $x = 1$. For $0 < \bar{\varphi} < \tilde{d}/(\tilde{d}+2)$, we are still not sure about the phases when either \bar{b} or \bar{R} is in $(0, U_{\bar{\varphi}}(1))$. We can not say too much about the final phases in these parts of phase space at present stage. This problem could be left for future study.

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