Subtraction scheme for next-to-next-to-leading order computations

Radja Boughezal, Kirill Melnikov, and Frank Petriello

Phys. Rev. D **85**, 034025 — Published 14 February 2012

DOI: 10.1103/PhysRevD.85.034025

ANL-HEP-PR-11-67
NUHEP-TH/11-24

A subtraction scheme for NNLO computations

Radja Boughezal*
High Energy Physics Division, Argonne National Laboratory, Argonne, IL 60439, USA

Kirill Melnikov†
Department of Physics and Astronomy, Johns Hopkins University, Baltimore, MD 21218, USA

Frank Petriello‡
Department of Physics & Astronomy, Northwestern University, Evanston, IL 60208, USA
High Energy Physics Division, Argonne National Laboratory, Argonne, IL 60439, USA

We use the known soft and collinear limits of tree- and one-loop scattering amplitudes – computed over a decade ago – to explicitly construct a subtraction scheme for next-to-next-to-leading order (NNLO) computations. Our approach combines partitioning of the final-state phase space together with the technique of sector decomposition, following recent suggestions in Ref. [1]. We apply this scheme to a toy example: the NNLO QED corrections to the decay of the Z boson to a pair of massless leptons. We argue that the main features of this subtraction scheme remain valid for computations of processes of arbitrary complexity with NNLO accuracy.

I. INTRODUCTION

Asymptotic freedom and factorization of short- and long-distance effects in the hard scattering of hadrons together allow us to employ perturbative computations in QCD to extract information about the physics of parton-parton interactions. Perturbative QCD computations are organized by "loop order" in an expansion in the strong coupling constant. At both leading- and next-to-leading order in α_s, the conceptual framework for perturbative QCD computations is well-established. While this does not necessarily make such calculations easy, or even feasible in some cases, the existence of a general framework is needed to claim a full understanding of the structure of perturbative QCD at these orders.

The crucial element of such an understanding is the concept of infrared and collinear safety. It leads to computational algorithms [2] that permit the calculation of arbitrary hadron-collider observables to next-to-leading order (NLO) accuracy. These methods rely upon approximating matrix elements containing an additional massless particle by appropriate limits in singular regions of phase space. These limits are simple enough to be integrated over the unresolved phase space of the radiated particle, without any reference to a particular observable. Unfortunately, the situation is much more confusing at next-to-next-to-leading order (NNLO) and beyond. To illustrate this point, we note that no NNLO computation of a cross section for a $2 \to 2$ scattering process with strongly interacting particles exists, in spite of the fact that a large number of two-loop virtual amplitudes [3–7] and *all* singular limits of tree- and one-loop amplitudes [8–15] have been known for over ten years. The reason is that a working algorithm that combines these ingredients to obtain physical cross sections has not been formulated. Consequently, a significant number of existing fully differential NNLO computations [16–30] were performed using unorthodox approaches, that are only remotely related to mainstream NLO subtractions ideas considered generalizable to higher orders [2].

An alternative NLO subtraction scheme due to Frixione, Kunszt and Signer (FKS) [31], was not the scheme of choice when NLO computations were initially being undertaken, but has received renewed interest recently [32]. The main idea of the FKS subtraction is to partition the phase-space such that in every sector only one definite external particle i can become soft, or two definite external particles i and j can become collinear to each other. If such a partitioning exists, it is clear that all singularities in a given sector are easily extracted if the unresolved phase space is parameterized in terms of the energy of particle i and the relative angle between directions of particles i and j.

In the case of NNLO computations the "elementary building block" is the double-unresolved phase space, where two given particles can become soft or collinear to a third particle. Extraction of singularities in the triple collinear limit is non-trivial, but can be accomplished by applying the technique of sector decomposition [16, 33, 34] to the three-parton unresolved phase-space. However, early applications of sector decomposition at NNLO [16, 20, 21] did not perform an initial par-

*rboughezal@hep.anl.gov
†melnikov@pha.jhu.edu
‡f-petriello@northwestern.edu

tition of the phase space to separate collinear singularities, and instead attempted to find a suitable phase-space parameterization for an *entire* process at NNLO. The complexity of this endeavor slowed down the progress of NNLO computations for hadron-collider observables using sector decomposition after initial success with such computations for the hadro-production of the Higgs and electroweak bosons.

It was recently pointed out by Czakon [1, 35] that by combining the idea of phase-space partitioning from FKS with the idea that sector decomposition can be applied to real emission integrals [16, 34], a powerful framework for NNLO computations is obtained. The purpose of this paper is to elaborate on this observation and show explicitly how this framework can be used to obtain physics results. We extend the results of Refs. [1, 35] in several ways.

- We demonstrate how to obtain the real-virtual corrections in a similar fashion as the double-real corrections. This allows the known NLO results for the process with an extra parton to be recycled into the NNLO calculation, using an FKS partitioning of the phase space combined with known results for the eikonal current and the one-loop splitting amplitudes [14].

- We address the scheme dependence of the NNLO result by comparing the conventional dimensional regularization calculation with one performed in the four-dimensional helicity scheme.

- We show that the initial-state triple collinear parameterization introduced in Ref. [1] also furnishes a suitable framework in which to extract singularities from the final-state triple-collinear region of phase space (we explain in detail in the text the exact definitions of the double-collinear and triple-collinear regions).

- We introduce a convenient parameterization for handling the double-collinear regions of phase space, based on an iterated application of the Catani-Seymour phase-space mapping at NLO [2].

For the sake of simplicity, we study two-loop QED corrections to the decay rate of the Z-boson to an electron-positron pair. Dealing with QED corrections offers significant simplifications, yet is far from trivial, and is a good place in which to develop and test ideas.

In the following sections we explain in detail our approach. In Section II, we discuss the extraction of singularities from double-real radiation corrections. In Section III, we explain how one-loop corrections to the one-photon real-emission process are treated. In Section IV we describe the dependence of the results on the choice of the regularization scheme. In Section V we present our conclusions.

II. DOUBLE REAL RADIATION

We consider the decay of a Z-boson to an $e^+e^-\gamma\gamma$ final state, $Z(p_Z) \to e^+(p_+) + e^-(p_-) + \gamma(p_1) + \gamma(p_2)$. We must design a strategy to integrate the squared matrix element for this process over the phase space of the final-state particles. This is non-trivial because such an integration eventually leads to phase-space regions that contain collinear[1] and soft singularities. The main problem is not that such singularities exist, but that in different regions of phase space, different subsets of particles lead to singularities in the matrix elements. As a first step, we partition the phase-space such that we know which final-state particles can develop singularities in each partition.

We begin by describing the collinear partitioning. We introduce four functions

$$\Delta_i^{\pm} = 1 - \boldsymbol{n}^{\pm} \cdot \boldsymbol{n}_i, \tag{1}$$

where $\boldsymbol{n}_i$ is a unit vector in the direction of the photon i and $\boldsymbol{n}^{\pm}$ are the unit vectors in the directions of the positively- and negatively-charged leptons. For each photon, we introduce a partition of unity

$$1 = \frac{\Delta_i^+}{\Delta_i^+ + \Delta_i^-} + \frac{\Delta_i^-}{\Delta_i^+ + \Delta_i^-} = \rho_i^+ + \rho_i^-, \tag{2}$$

and obtain

$$1 = \prod_{i=1}^{2} (\rho_i^+ + \rho_i^-) = \rho_1^+\rho_2^+ + \rho_1^-\rho_2^- + \rho_1^+\rho_2^- + \rho_1^-\rho_2^+. \tag{3}$$

We introduce $\rho_i^a \rho_j^b = \delta_{ij}^{-a,-b}$, and re-write the previous equation as

$$1 = \delta_{12}^{--} + \delta_{12}^{++} + \delta_{12}^{-+} + \delta_{12}^{+-}. \tag{4}$$

Each of the contributions on the right-hand side of Eq. (4) defines a primary sector whose phase space we must parameterize separately. The superscripts of each δ indicate the potentially singular collinear directions for each of the two photons. For example, in the primary sector labeled by δ_{12}^{--} both photons can become collinear to the electron. In the sector labeled by δ_{12}^{+-} only one photon can be collinear to the electron, while the other can become collinear to the positron. Using Eq. (4), we decompose the phase-space as

$$\mathrm{dLips}_{e+e-\gamma_1\gamma_2} = \sum_{a,b=\pm} \mathrm{dLips}_{e+e-\gamma_1\gamma_2}^{ab}, \tag{5}$$

where

$$\begin{aligned} \mathrm{dLips}_{e+e-\gamma_1\gamma_2}^{ab} = \frac{1}{2!} \int [\mathrm{d}p_-][\mathrm{d}p_+][\mathrm{d}p_1][\mathrm{d}p_2] \\ \times (2\pi)^d \delta^d(p_Z - p_- - p_+ - p_1 - p_2)\delta_{12}^{ab}, \end{aligned} \tag{6}$$

[1] We treat all final state particles as massless.

For convenience, we introduce the short-hand notation $[dp] = \mathrm{d}^{d-1}\boldsymbol{p}/(2p_0(2\pi)^{d-1})$, where $d = 4-2\epsilon$ is the dimensionality of space-time. The overall 1/2! is the symmetry factor for the two final-state photons.

Out of the four primary sectors, two sectors contain triple-collinear singularities where both photon momenta are collinear to either the electron or positron momentum. The other two sectors contain double-collinear singularities, where the momentum of one photon is collinear to the electron momentum while the other photon is collinear to the positron. It is sufficient to understand one triple-collinear and one double-collinear partition. The remaining primary sectors are obtained by a simple re-labeling of the final-state particles. When discussing phase-space parameterizations in the relevant sectors, we make use of the fact that the electron and positron cannot develop soft singularities, and that in the process $Z \to e^+e^-\gamma_1\gamma_2$, the kinematic configuration where the electron and positron momenta are collinear is non-singular. If such singularities could occur, only an additional partitioning would be required to handle them.

A. The triple collinear sector

1. *Phase-space parameterization*

We consider the δ_{12}^{--} primary sector, where the photons and the electron can develop collinear singularities. We must also consider soft singularities that appear when the energies of one or both photons vanish. Our discussion of the phase-space parameterization closely follows Refs. [1, 35].

We first explain how the energies of the two photons are parameterized. We denote the sum of the four-momenta of the electron and positron by $Q = p_+ + p_-$. Momentum conservation implies $p_Z - p_1 - p_2 = Q$ and $0 < Q^2 < m_Z^2$. We write $m_Z^2 - Q^2 = \Delta m_Z^2$, $0 < \Delta < 1$. Squaring the momentum conservation equation, we find

$$m_Z^2 - 2m_Z(E_1+E_2) + 2E_1E_1(1-\boldsymbol{n}_1\cdot\boldsymbol{n}_2) = Q^2, \quad (7)$$

where $E_{1,2}$ are the energies of the two photons and $\boldsymbol{n}_{1,2}$ are three-dimensional unit vectors along their momenta. We parameterize the photon energies by $E_i = \xi_i m_Z/2$, and the relative angle between them by $\eta_{12} = (1-\boldsymbol{n}_1\cdot\boldsymbol{n}_2)/2$. Solving for ξ_1 or ξ_2 in Eq. (7) then yields

$$\xi_1 = \frac{\Delta-\xi_2}{1-\xi_2\eta_{12}} \quad \text{or} \quad \xi_2 = \frac{\Delta-\xi_1}{1-\xi_1\eta_{12}}. \quad (8)$$

We can remove the symmetry factor in Eq. (6) by requiring that $E_1 > E_2$ and by using the fact that the matrix element is symmetric under the interchange of γ_1 and γ_2. We obtain

$$\mathrm{dLips}_{e^+e^-\gamma_1\gamma_2}^{--} = (2\pi)^d \int [\mathrm{d}p_-][\mathrm{d}p_+][\mathrm{d}p_1][\mathrm{d}p_2]\delta_{12}^{--}\times \delta^d(p_Z-p_--p_+-p_1-p_2)\theta(\xi_1\xi_{\max}(\xi_1)-\xi_2), \quad (9)$$

where

$$\xi_{\max}(\xi_1) = \min\left[1, \frac{1-\xi_1}{\xi_1(1-\xi_1\eta_{12})}\right]. \quad (10)$$

We decompose the four-particle phase-space into "regular" and "singular" phase-spaces:

$$\mathrm{dLips}_{e^+e^-\gamma_1\gamma_2}^{--} = \mathrm{dLips}_{\mathrm{reg}} \times \mathrm{dLips}_{\mathrm{sing}}^{--}, \quad (11)$$

where

$$\mathrm{dLips}_{\mathrm{sing}}^{--} = [\mathrm{d}p_1][\mathrm{d}p_2]\,\delta_{12}^{--}\,\theta(\xi_1\xi_{\max}(\xi_1)-\xi_2), \quad (12)$$

and

$$\mathrm{dLips}_{\mathrm{reg}} = [\mathrm{d}p_-][\mathrm{d}p_+](2\pi)^d\delta^{(d)}(Q-p_+-p_-). \quad (13)$$

We begin with a discussion of the singular phase-space. We note that since the Z-boson decays at rest and there are only four particles in the final state, we can choose the momenta of any three particles to be in the four-dimensional space, without any $(d-4)$-dimensional components. The three-momentum of the fourth particle is determined by momentum conservation, and is also in the four-dimensional space. To have a simple parametrization, we choose the direction of the electron momentum to be the z-axis. Then, $p_- = E_-(1,\boldsymbol{n}_-)$, $\boldsymbol{n}_- = (0,0,1)$ and $p_{1,2} = E_{1,2}\,(1,\boldsymbol{n}_1)$, where $\boldsymbol{n}_1 = (\sin\theta_1, 0, \cos\theta_1)$ and $\boldsymbol{n}_2 = (\sin\theta_2\cos\varphi, \sin\theta_2\sin\varphi, \cos\theta_2)$. We also introduce the following notation for the scalar product of $\boldsymbol{n}_-$ with $\boldsymbol{n}_{1,2}$:

$$\eta_{1,2} = \frac{1-\boldsymbol{n}_-\cdot\boldsymbol{n}_{1,2}}{2}. \quad (14)$$

We then find the scalar products

$$2p_i\cdot p_- = 2E_-m_Z\xi_i\eta_i, \quad 2p_1\cdot p_2 = m_Z^2\xi_1\xi_2\eta_{12}. \quad (15)$$

We now write the parametrization of the singular phase-space using the angular variables just introduced:

$$\begin{aligned}\mathrm{dLips}_{\mathrm{sing}}^{--} &= \frac{\delta_{12}^{--}\theta(\xi_1-\xi_2)}{64\pi^2}\left(\frac{m_Z}{2\pi}\right)^{2d-4} \\ &\times \mathrm{d}\Omega_1^{(d-2)}\mathrm{d}\Omega_2^{(d-3)}\,\mathrm{d}\xi_1\,\xi_1^{1-2\epsilon}\,\mathrm{d}\xi_2\,\xi_2^{1-2\epsilon} \\ &\times \mathrm{d}\eta_1\,[\eta_1(1-\eta_1)]^{-\epsilon}\,\mathrm{d}\eta_2\,[\eta_2(1-\eta_2)]^{-\epsilon} \\ &\times \mathrm{d}\cos\varphi\,\left(1-\cos^2\varphi\right)^{-1/2-\epsilon}.\end{aligned} \quad (16)$$

The goal is to rewrite the integration over $\cos\varphi$ in a way that makes the factorization of singularities manifest. To this end, we introduce the variable κ:

$$\kappa = \frac{(1-\cos(\theta_1-\theta_2))\,(1+\cos\varphi)}{2\,(1-\cos(\theta_1-\theta_2)+(1-\cos\varphi)\sin\theta_1\sin\theta_2)}. \quad (17)$$

Because $\cos\varphi$ can be used to parameterize η_{12}, we will need a relationship between η_{12} and κ. It can be easily derived by solving the above equation for $\cos\varphi$ and then using the solution in the expression for η_{12}. We find

$$\eta_{12} = \frac{(\eta_1-\eta_2)^2}{\tilde{N}(\eta_1,\eta_2,\kappa)}, \tag{18}$$

where

$$\begin{aligned}\tilde{N}(\eta_1,\eta_2,\kappa) &= \eta_1+\eta_2-2\eta_1\eta_2 \\ &\quad -2(1-2\kappa)\sqrt{\eta_1\eta_2(1-\eta_1)(1-\eta_2)}.\end{aligned} \tag{19}$$

Finally, we need the Jacobian for the $\varphi \to \kappa$ variable transformation, and a simple expression for $\sin^2\varphi = 1-\cos^2\varphi$. The relevant equations are

$$\frac{\mathrm{d}\cos\varphi}{\mathrm{d}\kappa} = \frac{2\eta_{12}^2}{(\eta_1-\eta_2)^2}, \quad 1-\cos^2\varphi = \frac{4\kappa(1-\kappa)\eta_{12}^2}{(\eta_1-\eta_2)^2}. \tag{20}$$

We can now change variables $\varphi \to \kappa$ in Eq. (16) for the singular phase-space. We find

$$\begin{aligned}\mathrm{dLips}_{\mathrm{sing}}^{--} &= \frac{\delta_{12}^{--}\theta(\xi_1-\xi_2)m_Z^{2d-4}}{2^{4+2\epsilon}(2\pi)^{2d-2}} \\ &\times \mathrm{d}\Omega_1^{(d-2)}\mathrm{d}\Omega_2^{(d-3)}\,\mathrm{d}\xi_1\,\xi_1^{1-2\epsilon}\,\mathrm{d}\xi_2\,\xi_2^{1-2\epsilon} \\ &\times \mathrm{d}\eta_1\,[\eta_1(1-\eta_1)]^{-\epsilon}\,\mathrm{d}\eta_2\,[\eta_2(1-\eta_2)]^{-\epsilon} \\ &\times \mathrm{d}\kappa\,(\kappa(1-\kappa))^{-1/2-\epsilon}\,\frac{\eta_{12}^{1-2\epsilon}}{|\eta_1-\eta_2|^{1-2\epsilon}}.\end{aligned} \tag{21}$$

As we will see later, Eq. (21) gives us the singular phase space in a form that is convenient for the extraction of singularities.

Next, we discuss the regular phase space. We write it as

$$\begin{aligned}\mathrm{dLips}_{\mathrm{reg}} &= [\mathrm{d}p_-][\mathrm{d}p_+](2\pi)^d\delta^{(d)}(Q-p_+-p_-) \\ &= \frac{\mathrm{d}\Omega_{e-}^{(d-1)}}{2(2\pi)^{d-2}}\frac{E_-^{1-2\epsilon}}{2(Q_0-\boldsymbol{Q}\cdot\boldsymbol{n}_-)},\end{aligned} \tag{22}$$

where $E_- = Q^2/[2(Q_0-\boldsymbol{Q}\cdot\boldsymbol{n}_-)]$ is the electron energy. It is important to understand which elements of the calculation can be simplified by setting the number of space-time dimensions to four, $d\to 4$. In the context of the phase-space discussion, we factor out the leading order phase-space for $Z\to e^+e^-$ in Eq. (22) and treat is as four-dimensional. Everything else in Eq. (22) is treated with exact ϵ-dependence. The leading-order phase-space that we use in what follows reads

$$\mathrm{dLips}_{Z\to e^+e^-} = \frac{\mathrm{d}\Omega^{(d-1)}}{8(2\pi)^{d-2}}\left(\frac{m_Z}{2}\right)^{-2\epsilon} \to \frac{\mathrm{d}\cos\theta\mathrm{d}\varphi}{32\pi^2}. \tag{23}$$

The regular phase-space becomes

$$\mathrm{dLips}_{\mathrm{reg}} = \mathrm{dLips}_{Z\to e^+e^-}\frac{2E_-}{(Q_0-\boldsymbol{Q}\cdot\boldsymbol{n}_-)}\left(\frac{2E_-}{m_Z}\right)^{-2\epsilon}, \tag{24}$$

where $\mathrm{dLips}_{Z\to e^+e^-}$ is taken in four dimensions, as in Eq. (23).

With the explicit parametrization of the phase space at hand, we are ready to discuss how the momenta of the final-state particles are generated. We follow the simple procedure described below:

1. first, we use $\xi_{1,2},\eta_{1,2},\kappa$ to generate the energies and momenta of the two photons;
2. second, we calculate $Q = p_Z - p_1 - p_2$;
3. third, we obtain $E_- = Q^2/(2(Q_0-\boldsymbol{Q}\cdot\boldsymbol{n}_-))$, assuming $\boldsymbol{n}_-$ is along the z-axis;
4. finally, the four-momentum of an electron is taken as $p_- = E_-(1,0,0,1)$, and the four-momentum of the positron is calculated by $p_+ = p_Z - p_1 - p_2 - p_-$.

The remaining issue is the extraction of singularities from the matrix element of $Z\to e^+e^-\gamma_1\gamma_2$ in the δ_{12}^{--} sector. The matrix element is singular if either photon is soft, and also if either photon momentum is collinear to the electron momentum. By analyzing the potentially singular denominators, it is straightforward to find that there are three sectors to consider. These sectors are identified by changes of variables $(\xi,\eta,\kappa)\to\{x_{i=1..5}\}$ that we make in order to factor out all singularities from the matrix element. The three sectors are[2]:

1. S_1^{--}, where $\xi_1 = x_1$, $\xi_2 = x_{\max}x_2x_1$, $\eta_1 = x_3$, $\eta_2 = x_4x_3$, $\kappa = x_5$;
2. S_2^{--}, where $\xi_1 = x_1$, $\xi_2 = x_{\max}x_2x_4x_1$, $\eta_1 = x_3x_4$, $\eta_2 = x_3$, $\kappa = x_5$;
3. S_3^{--}, where $\xi_1 = x_1$, $\xi_2 = x_{\max}x_2x_1$, $\eta_1 = x_2x_3x_4$, $\eta_2 = x_3$, $\kappa = x_5$.

For each of the sectors S_i we must express the phase-space through the new variables and find the singular limits of the amplitudes. We illustrate how this is accomplished for the sector S_1^{--}. The remaining two sectors are handled in a similar fashion. We note that we obtain three sectors, rather than the five obtained in Ref. [1], because of our restriction to QED, which does not contain the analog of a triple-gluon vertex and therefore has a slightly simplified singularity structure.

2. *The sector S_1^{--}*

For the sector S_1^{--}, we write the phase-space in the following form

$$\begin{aligned}&\mathrm{dLips}_{S_1}^{--} = \underline{\mathrm{dLips}}_{S_1}^{--}\left[x_1^4x_2^2x_3^2x_4m_Z^2\delta_{12}^{--}\right], \\ &\underline{\mathrm{dLips}}_{S_1}^{--} = \mathrm{dNorm}\;\mathrm{PS}_w\;(\mathrm{PS})^{-\epsilon} \\ &\times\frac{\mathrm{d}x_1}{x_1^{1+4\epsilon}}\frac{\mathrm{d}x_2}{x_2^{1+2\epsilon}}\frac{\mathrm{d}x_3}{x_3^{1+2\epsilon}}\frac{\mathrm{d}x_4}{x_4^{1+\epsilon}}\frac{\mathrm{d}\kappa}{\pi(\kappa(1-\kappa))^{1/2}},\end{aligned} \tag{25}$$

---

[2] We use the notation $x_{\max}$ for the *function* $\xi_{\max}(x_1)$

where

$$\mathrm{dNorm} = \mathrm{dLips}_{Z\to e^+e^-}\frac{\Gamma(1+\epsilon)^2 m_Z^{2d-6}}{(4\pi)^d}\mathcal{B}_\epsilon^{\mathrm{RR}},$$
$$\mathcal{B}_\epsilon^{\mathrm{RR}} = 1 - \frac{\pi^2}{2}\epsilon^2 - 2\zeta_3\epsilon^3 + \frac{3\pi^4}{40}\epsilon^4 + \mathcal{O}(\epsilon^5), \tag{26}$$

and the normalization factors are given by

$$\mathrm{PS}_w = \frac{x_{\max}^2(1-x_4)}{N(x_3,x_4,x_5)}\frac{2E_-}{(Q_0 - \boldsymbol{Q}\cdot\boldsymbol{n}_-)},$$
$$\mathrm{PS} = 16\left[\frac{(1-x_4)}{N(x_3,x_4,x_5)}\right]^2(1-x_3)(1-x_4x_3) \tag{27}$$
$$\times[\kappa(1-\kappa)]x_{\max}^2\left(\frac{2E_-}{m_Z}\right)^2.$$

Note that the normalization of the various pieces is chosen so that the m_Z-dependent factor in dNorm can be factored out entirely. This will be true for both the two-loop virtual and real-virtual contributions. The factor of m_Z^2 in the square brackets in Eq. (25) is present to make the product of the square bracket and the amplitude squared for $Z\to e^+e^-\gamma\gamma$ dimensionless. Note also that dNorm in Eq. (26) is the same for all sectors, and that we will be using it in equations for other sectors below. The function $N(x_3,x_4,x_5)$ appears in the equation for η_{12} after it is expressed through the new variables. It reads

$$N(x_3,x_4,x_5) = 1 + x_4 - 2x_3x_4 - 2(1-2x_5)\sqrt{x_4(1-x_3)(1-x_3x_4)}. \tag{28}$$

The momenta of all the particles are written through x-variables in the following way:

$$E_1 = \frac{m_Z}{2}x_1, \quad E_2 = \frac{m_Z}{2}x_1x_2x_{\max},$$
$$\eta_{12} = \frac{x_3(1-x_4)^2}{N(x_3,x_4,x_5)}, \tag{29}$$
$$\cos\theta_1 = 1-2x_3, \quad \cos\theta_2 = 1-2x_3x_4.$$

Angles $\theta_{1,2}$ are the polar angles for the two photons. We take $\sin\theta_{1,2}$ to be positive-definite. We choose photon γ_1 to be in the $x-z$ plane, so that $\varphi_1 = 0$. The azimuthal angle of the photon φ_2 is calculated to be

$$\sin\varphi_2 = \frac{\sqrt{4x_5(1-x_5)}(1-x_4)}{N(x_3,x_4,x_5)},\ \cos\varphi_2 = \lambda\sqrt{1-\sin\varphi_2^2}, \tag{30}$$

where $\lambda = \mathrm{sgn}((1-\cos\theta_1\cos\theta_2) - 2\eta_{12})$. Finally, the momenta of the two photons are given by

$$p_1 = E_1(1, \sin\theta_1, 0, \cos\theta_1),$$
$$p_2 = E_2(1, \sin\theta_2\cos\varphi, \sin\theta_2\sin\varphi, \cos\theta_2). \tag{31}$$

The momentum of the electron is $p_- = E_-(1,0,0,1)$, and the positron four-momentum is obtained from momentum conservation.

With the parameterization of the phase-space at hand, we can discuss extraction of singularities from the matrix element. To this end, the factor in square brackets in Eq. (25) is combined with the amplitude squared. This should give a finite expression in all singular limits. We introduce the *regular* function

$$F_1(\{x_{i=1..5}\}) = \left[x_1^4x_2^2x_3^2x_4m_Z^2\delta_{12}^{--}\right]|\mathcal{M}_{Z\to e^+e^-\gamma\gamma}|^2, \tag{32}$$

A calculation of $Z\to e^+e^-\gamma\gamma$ contribution to Z-decay rate involves integration of the function F_1 over the phase-space in Eq. (25):

$$\int \mathrm{d}\underline{\mathrm{Lips}}_{S_1}^{--}F_1(x_1,x_2,x_3,x_4,x_5). \tag{33}$$

Because of the structure of $\mathrm{d}\underline{\mathrm{Lips}}_{S_1}^{--}$ in Eq. (25), the integrand is singular if one of the integration variables x_i, $i=1,..4$ vanishes. Such singularities can be extracted by writing

$$x_i^{-1-n_i\epsilon} = -\frac{1}{n_i\epsilon}\delta(x) + \left[\frac{1}{x_i}\right]_+ - n\epsilon\left[\frac{\ln x_i}{x_i}\right]_+ + \mathcal{O}(\epsilon^2) \tag{34}$$

for all of the singular variables. The plus-distributions are defined in a standard way:

$$\int_0^1 \mathrm{d}x\left[\frac{1}{x}\right]_+ f(x) = \int_0^1 \mathrm{d}x\,\frac{f(x)-f(0)}{x}. \tag{35}$$

Resolving the plus-distributions as in Eq. (35), we find that the integration in Eq. (33) requires knowing F_1 when one or more of its arguments vanishes. The calculation of these limits requires care. Although the function F_1 is regular everywhere, the matrix element $\mathcal{M}_{Z\to e^+e^-\gamma\gamma}$ is singular when certain x-variables vanish. The physical limits that correspond to a set of particular x-variables vanishing can be deduced from the expressions for the momenta given in Eqs. (29, 31). We describe the relevant limits below.

- If $x_1 = 0$, the energies of both photons vanish. This corresponds to the double-soft limit. In the double-soft limit, the QED eikonal currents completely factorize and we obtain

  $$|\mathcal{M}_{Z\to e^+e^-\gamma\gamma}|^2 \to e^4 J_1 J_2 |\mathcal{M}_{Z\to e^-e^+}|^2, \tag{36}$$

  where the square of the eikonal current for the photon i reads

  $$J_i = \frac{2p_-\cdot p_+}{(p_-\cdot p_i)(p_+\cdot p_i)}. \tag{37}$$

  The scalar products are computed using the explicit parametrization of the momenta given in Eqs.(29, 30, 31). We obtain for the function F_1

  $$F_1|_{x_1=0} = \frac{16e^4}{m_Z^2}|\mathcal{M}_{Z\to e^-e^+}|^2. \tag{38}$$

- If $x_1 \neq 0$ but $x_2 = 0$, the photon γ_1 is hard and the photon γ_2 is soft. The matrix element becomes

$$|\mathcal{M}_{Z\to e^+e^-\gamma_1\gamma_2}|^2 \to e^2 J_2 |\mathcal{M}_{Z\to e^+e^-\gamma_1}|^2 \qquad (39)$$

in this limit. Calculating the function F_1 for $x_1 \neq 0, x_2 = 0$, we obtain

$$F_1|_{x_2=0} = e^2 \frac{4(p_+\cdot p_-)x_1^2 x_3}{E_- E_+ x_{\max}^2} \frac{(1-\boldsymbol{n}_+\cdot\boldsymbol{n}_1)}{\Delta_{12}} \times |\mathcal{M}_{Z\to e^+e^-\gamma_1}|^2, \qquad (40)$$

where $\Delta_{12} = \prod_{i=1}^{2} (1 - \boldsymbol{n}_i\cdot\boldsymbol{n}_+ - \boldsymbol{n}_i\cdot\boldsymbol{n}_-)$.

- Equation (40) develops singularities when $x_3 \to 0$, in which case photon γ_1 becomes collinear to the electron. We do not show helicity labels in what follows because helicity is conserved along massless fermion lines. In the collinear limit we find

$$|\mathcal{M}_{Z\to e^+e^-\gamma_1}|^2 \approx \frac{2e^2}{s_{1e}} P_{e\gamma}(\epsilon,z) |\mathcal{M}_{Z\to e^+\tilde{e}^-}|^2, \qquad (41)$$

where the momentum of $\tilde{e}^-$ is given by the sum $p_- + p_1$, $P_{e\gamma}(\epsilon,z)$ is the $e \to e+\gamma$ splitting function given in the Appendix, and $s_{1e} = 2p_-\cdot p_1 = 2E_- m_Z x_1 x_3$. Upon evaluating F_1 in that limit, we obtain

$$F_1|_{x_2=0,x_3=0} = \frac{16e^4 x_1}{m_Z E_- x_{\max}^2 \Delta_{12}} P_{e\gamma}(\epsilon,z) \times |\mathcal{M}_{Z\to e^+\tilde{e}^-}|^2. \qquad (42)$$

The fraction of energy carried away by the electron in the $e \to e+\gamma$ splitting is expressed through the variable $z = 1/(1 + m_Z x_1/(2E_-))$.

- We next consider the $x_4 = 0$ limit, which corresponds to the photon momentum p_2 being collinear to the electron momentum p_-. We calculate

$$F|_{x_4=0} = \frac{e^2 x_1^3 x_2 x_3 \delta_{12}^{--} m_Z}{E_- x_{\max}} P_{e\gamma}(\epsilon,z) \times |\mathcal{M}_{Z\to e^+\tilde{e}^-\gamma_1}|^2, \qquad (43)$$

where the momentum of $\tilde{e}^-$ is $p_- + p_2$ and $z = 1/(1 + m_Z x_{\max} x_2 x_1/(2E_-))$.

- Finally, we consider $x_3 = 0$. This limit corresponds to the triple collinear limit, when the momenta of photons γ_1 and γ_2 are parallel to the electron momentum p_-. In the triple collinear limit the matrix element factorizes as

$$|\mathcal{M}_{Z\to e^+e^-\gamma_1\gamma_2}|^2 = \left(\frac{2e^2}{s_{12e}}\right)^2 \times P_{\epsilon,e\gamma_1\gamma_2}(\epsilon, z_e, z_1, z_2) |\mathcal{M}_{Z\to e^+\tilde{e}^-}|^2, \qquad (44)$$

where the momentum of $\tilde{e}^-$ is $p_- + p_1 + p_2$ and the triple splitting function $P_{e\gamma_1\gamma_2}$ can be found in the Appendix. The energy fractions in this case are given by

$$z_e = \left[1 + \frac{m_Z}{2E_-} x_1(1 + x_{\max} x_2)\right]^{-1}, \quad z_1 = \frac{m_Z x_1}{2E_-} z_e, \quad z_2 = \frac{m_Z x_1 x_2 x_{\max}}{2E_-} z_e. \qquad (45)$$

To compute F_1 at $x_3 = 0$, we introduce the notation

$$s_{12e} \approx 2E_- m_Z x_1 x_3 d_{12e} + \mathcal{O}(x_3^2), \quad d_{12e} = 1 + x_2 x_{\max} x_4 + \frac{m_Z}{2E_-} \frac{x_1 x_2 x_{\max}(1-x_4)^2}{N(0,x_4,x_5)}. \qquad (46)$$

Using this notation, the function F_1 is easy to write down. As an example, we present an explicit result for $F_1(x_1, x_2, 0, x_4, x_5)$ in the CDR regularization scheme:

$$F_1|_{x_3=0} = \frac{e^4}{E_-^2} |\mathcal{M}_{Z\to e^+\tilde{e}^-}|^2 \Bigg\{ \frac{x_1^2 x_2}{2x_{\max}} [P_1(\epsilon, z_e, z_1, z_2) + P_2(\epsilon, z_e, z_2, z_1)] + \frac{x_1^2 x_2^2 x_4}{d_{12e}} P_2(\epsilon, z_e, z_1, z_2) + \frac{x_1^2 x_2}{d_{12e} x_{\max}} P_2(\epsilon, z_e, z_2, z_1) - (1-\epsilon)^2 \frac{x_1^2 x_2^2 x_4}{d_{12e}^2} \times \left(x_{\max} x_2 x_4 + \frac{1}{x_{\max} x_2 x_4} \right) + 2\epsilon(1-\epsilon) \frac{x_1^2 x_2^2 x_4}{d_{12e}^2} \Bigg\}. \qquad (47)$$

The functions P_1 and P_2 are presented in the Appendix.

The above formulae describe all the QED singular limits in the sector S_1^{--}. They can be implemented in a computer code in a straightforward way.

B. The double collinear sector

We next consider the δ_{12}^{-+} primary sector. The singularities in this sector arise when the photon γ_1 is collinear to the electron, and the photon γ_2 is collinear to the positron. Soft singularities for both photons are possible.

1. *The phase-space parametrization*

To develop a suitable description of the four-particle phase space in this partition, we make use of the momentum parametrization developed by Catani and Seymour [2] for next-to-leading order calculations. Specifically, we adopt the momentum mapping for final-final dipoles, using the language of Ref. [2]. We will also use the nomenclature employed in Ref. [2] to describe different particles contributing to the dipoles, to make clear the connection to the discussion in that reference.

We parameterize the phase-space for $Z \to e^+e^-\gamma_1\gamma_2$ in two steps. In the first step, we treat photon γ_1 as "emitted", the electron as "emitter" and the positron as "spectator". The $3 \to 2$ momentum mapping in this case is given in Ref. [2]. It is determined by the momentum conservation equation

$$\gamma_1 + p_- + p_+ = \tilde{p}_{1-} + \tilde{p}_+, \tag{48}$$

and the relations between old and new momenta:

$$\begin{aligned} p_1 &= z_1\tilde{p}_{1-} + y_1(1-z_1)\tilde{p}_+ + p_{1,\perp}, \\ p_- &= (1-z_1)\tilde{p}_{1-} + y_1 z_1\tilde{p}_+ - p_{1,\perp}, \\ p_+ &= (1-y_1)\tilde{p}_+. \end{aligned} \tag{49}$$

The momenta $\tilde{p}_{1-}$ and $\tilde{p}_+$ are light-like: $\tilde{p}_{1-}^2 = \tilde{p}_+^2 = 0$. The momentum $p_{1,\perp}$ is orthogonal to both of them. For the momentum parametrization in Eq. (49), the phase-space reads

$$\begin{aligned} \mathrm{dLips}(p_-,p_+,p_1,p_2) &= \frac{\mathrm{dLips}(\tilde{p}_{1-},\tilde{p}_+,p_2)}{2!} \\ &\times \frac{\mathrm{d}y_1\mathrm{d}z_1\mathrm{d}\Omega^{(1)}_{d-2}}{4(2\pi)^{d-1}}\left(2\tilde{p}_{1-}\cdot\tilde{p}_+\right)^{1-\epsilon} \\ &\times (1-y_1)^{1-2\epsilon}\, y_1^{-\epsilon}\left(z_1(1-z_1)\right)^{-\epsilon}. \end{aligned} \tag{50}$$

In the second step, we apply a similar mapping for the momenta of the "reduced" reaction $Z \to \tilde{p}_{1-} + \tilde{p}_+ + p_2$, by considering γ_2 as "emitted", $\tilde{e}^+$ as "emitter" and $\tilde{e}_{1-}$ as " spectator". The momentum conservation equation becomes

$$\tilde{p}_{1-} + \tilde{p}_+ + p_2 = \tilde{p}_{+2} + \tilde{\tilde{p}}_{1-}, \tag{51}$$

and the new momentum parameterizations read

$$\begin{aligned} p_2 &= z_2\tilde{p}_{+2} + y_2(1-z_2)\tilde{\tilde{p}}_{1-} + p_{2,\perp}, \\ \tilde{p}_+ &= (1-z_2)\tilde{p}_{+2} + y_2 z_2\tilde{\tilde{p}}_{1-} - p_{2,\perp}, \\ \tilde{p}_{1-} &= (1-y_2)\tilde{\tilde{p}}_{1-}. \end{aligned} \tag{52}$$

Continuing with the parametrization of the phase-space shown in Eq. (50), we obtain

$$\begin{aligned} \mathrm{dLips}(p_-,p_+,p_1,p_2) &= \frac{\mathrm{dLips}(\tilde{\tilde{p}}_{1-},\tilde{p}_{2+})}{2!} \\ &\times (m_Z^2)^{2-2\epsilon}\frac{\mathrm{d}y_1\mathrm{d}z_1\mathrm{d}\Omega^{(1)}_{d-2}}{4(2\pi)^{d-1}}\,\frac{\mathrm{d}y_2\mathrm{d}z_2\mathrm{d}\Omega^{(2)}_{d-2}}{4(2\pi)^{d-1}} \\ &\times y_1^{-\epsilon}z_1^{-\epsilon}y_2^{-\epsilon}z_2^{-\epsilon}(1-y_1)^{1-2\epsilon} \\ &\times (1-y_2)^{2-3\epsilon}(1-z_2)^{1-2\epsilon}(1-z_1)^{-\epsilon}. \end{aligned} \tag{53}$$

We can express the momenta of all particles through $y_{1,2}, z_{1,2}$. We choose $\tilde{\tilde{p}}_{1-}, \tilde{p}_{2+}$ to be along the positive and negative z-axis respectively. The corresponding momenta read

$$\tilde{\tilde{p}}_{1-} = \frac{m_Z}{2}(1,0,0,1), \quad \tilde{p}_{2+} = \frac{m_Z}{2}(1,0,0,-1). \tag{54}$$

We choose $p_{2,\perp}$ along the x-axis and use Eq. (52) to obtain $\tilde{p}_+$ and p_2. We find

$$\begin{aligned} p_2 &= \frac{m_Z}{2}\Big(z_2 + y_2(1-z_2), 2\sqrt{z_2(1-z_2)y_2}, \\ &\qquad 0, y_2(1-z_2) - z_2\Big), \\ \tilde{p}_+ &= \frac{m_Z}{2}\Big((1-z_2+y_2z_2), -2\sqrt{z_2(1-z_2)y_2}, \\ &\qquad 0, y_2z_2 - (1-z_2)\Big). \end{aligned} \tag{55}$$

We can write momenta of the photon γ_1 and the electron using the above equations. The only semi-intricate step is the derivation of $p_{1,\perp}$. We find

$$\begin{aligned} p_{1,\perp} &= -m_Z\sqrt{z_1z_2(1-z_1)y_1(1-y_2)y_2}\,\boldsymbol{n}_+^{\varphi} \\ &+ m_Z\sqrt{z_1(1-z_1)y_1(1-y_2)(1-z_2)}\,\boldsymbol{n}_\perp^{\varphi}, \end{aligned} \tag{56}$$

where

$$\boldsymbol{n}_+^{\varphi} = (\cos\varphi, 0, 0, \cos\varphi), \quad \boldsymbol{n}_\perp^{\varphi} = (0, \cos\varphi, \sin\varphi, 0).$$

Using Eq. (56), we derive the following expression for the energy of the photon γ_1:

$$\begin{aligned} E_1 &= \frac{m_Z}{2}\Big[z_1(1-y_2) + y_1(1-z_1)(1-z_2+y_2z_2) \\ &- 2\sqrt{y_1z_1(1-z_1)z_2y_2(1-y_2)}\cos\varphi\Big]. \end{aligned} \tag{57}$$

The energy of the photon γ_2 can be read off from Eq. (55).

We now rewrite Eq. (53) by factoring out the Born phase-space and several other factors, similar to what has been done for the triple-collinear sector. From the momentum parametrization we see that, in addition to $y_{1,2}, z_{1,2}$, we need the azimuthal angle φ to describe the phase space. There are no singularities associated with this angle in the double-collinear sector, so we take it to be $\varphi = 2\pi x_5$, $0 \le x_5 \le 1$. In Eq. (53) we identify $\mathrm{dLips}(\tilde{\tilde{p}}_{1-}, \tilde{p}_{2+})$ with the leading order phase-space and obtain

$$\begin{aligned} \mathrm{dLips}(p_-,p_+,p_1,p_2) &= \mathrm{dNorm}\;\mathrm{Ps}_w\;\mathrm{PS}_{\mathrm{gen}}^{-\epsilon} \\ &\times \mathrm{d}y_1\mathrm{d}z_1\mathrm{d}y_2\mathrm{d}z_2\mathrm{d}x_5 y_1^{-\epsilon}z_1^{-\epsilon}y_2^{-\epsilon}z_2^{-\epsilon}m_Z^2, \end{aligned} \tag{58}$$

where dNorm is given in Eq. (26). The other factors read

$$\begin{aligned} \mathrm{PS}_w &= \frac{1}{2}(1-y_1)(1-y_2)^2(1-z_2), \\ \mathrm{PS}_{\mathrm{gen}} &= 4(1-y_1)^2(1-y_2)^3(1-z_2)^2 \\ &\times (1-z_1)(1-\cos^2\varphi). \end{aligned} \tag{59}$$

There are two scalar products that can become singular in the collinear limits:

$$\begin{aligned} 2p_+\cdot p_2 &= (1-y_1)y_2m_Z^2, \\ 2p_-\cdot p_1 &= y_1(1-y_2)(1-z_2)m_Z^2. \end{aligned} \tag{60}$$

Analyzing these scalar products and the expressions for the photon energies in Eqs.(55,57), we conclude that for the photon $i = 1, 2$, the soft singularity corresponds to $y_i = 0, z_i = 0$, while the collinear singularity corresponds to $y_i = 0$ and $z_i \neq 0$. We also note that the apparent vanishing of scalar products and photon energies at $y_i, z_i = 1$ also implies vanishing of the electron or positron energy, and for this reason does not lead to non-integrable singularities.

The two singular limits for each photon are factorized with the help of the two sectors $y_i < z_i$ and $z_i < y_i$. Since there are two photons, we get four sectors altogether. We show below the changes of variables needed to completely factorize singularities in each sector:

$$\begin{aligned}
&S_1^{-+}, \text{where } y_1 = x_1,\ \ z_1 = x_1 x_2,\ y_2 = x_3,\ z_2 = x_3 x_4,\\
&S_2^{-+}, \text{where } y_1 = x_2 x_1,\ \ z_1 = x_1,\ y_2 = x_3,\ z_2 = x_3 x_4,\\
&S_3^{-+}, \text{where } y_1 = x_1,\ \ z_1 = x_1 x_2,\ \ y_2 = x_3 x_4\ z_2 = x_3,\\
&S_4^{-+}, \text{where } y_1 = x_2 x_1,\ \ z_1 = x_1,\ y_2 = x_3 x_4,\ z_2 = x_3.
\end{aligned} \tag{61}$$

We note that in sector S_1^{-+}, there are only soft singularities. In sectors S_2^{-+}, S_3^{-+}, there is a soft singularity for one of the photons, and both soft and collinear singularities for the other. In sector S_4^{-+} there are soft and collinear singularities for both photons. The extraction of all the limits in all sectors is similar to the triple collinear limit that we already discussed. For illustrative purposes, we only discuss the most difficult sector S_4^{-+}.

2. Sector S_4^{-+}

To discuss singular limits in sector S_4^{-+} it is convenient to introduce the following short-hand notation for the photon energies:

$$E_1 = \frac{m_Z}{2} x_1 \Omega_1,\ E_2 = \frac{m_Z}{2} x_3 \Omega_2, \tag{62}$$

where

$$\begin{aligned}
\Omega_1 &= (1-y_2) + x_2(1-z_1)(1-z_2+y_2 z_2)\\
&\quad - 2\sqrt{x_2(1-z_1)z_2 y_2(1-y_2)}\cos\varphi,\\
\Omega_2 &= 1 + x_4\,(1-z_2)\,.
\end{aligned} \tag{63}$$

In this sector, singularities occur if any of the variables x_1, x_2, x_3, x_4 vanishes. To enable extraction of singularities, we write the phase space as

$$\begin{aligned}
&\mathrm{dLips}_{e^+e^-\gamma_1\gamma_2} = \mathrm{dNorm}\ \mathrm{Ps}_w\ \mathrm{PS}^{-\epsilon}\\
&\times \frac{\mathrm{d}x_1}{x_1^{1+2\epsilon}} \frac{\mathrm{d}x_2}{x_2^{1+\epsilon}} \frac{\mathrm{d}x_3}{x_3^{1+2\epsilon}} \frac{\mathrm{d}x_4}{x_4^{1+\epsilon}} \mathrm{d}x_5\\
&\times \left[x_1^2 x_2 x_3^2 x_4 m_Z^2 \delta_{12}^{-+}\right],
\end{aligned} \tag{64}$$

where dNorm is the same as in the triple collinear limit. The other factors read

$$\begin{aligned}
\mathrm{PS}_w &= \frac{1}{2}(1-y_1)(1-y_2)^2(1-z_2),\\
\mathrm{PS} &= 4(1-y_1)^2(1-y_2)^3(1-z_1)\\
&\times (1-z_2)^2(1-\cos^2\varphi).
\end{aligned} \tag{65}$$

The finite function in this case is given by the product of the term in brackets in Eq. (64) and the squared matrix element for $Z \to e^+e^-\gamma_1\gamma_2$:

$$F_4(\{x_{i=1,..5}\}) = x_1^2 x_2 x_3^2 x_4 \delta_{12}^{-+} |M_{Z\to e^+e^-\gamma_1\gamma_2}|^2. \tag{66}$$

We now describe several of the singular limits in this sector.

- The double-soft limit corresponds to $x_1 = 0$ and $x_3 = 0$. In this limit, the function F_4 evaluates to

$$F_4|_{x_1=0,x_3=0} = \frac{16e^4}{m_Z^2 \Omega_1 \Omega_2} |\mathcal{M}_{Z\to e^+e^-}|^2. \tag{67}$$

- If $x_1 = 0$ and $x_4 = 0$, the photon γ_1 is soft and the photon γ_2 is collinear to the positron. The function F_4 reads

$$\begin{aligned}
F_4|_{x_1=0,x_4=0} &= \frac{32 x_3 e^4}{(1-z_2) m_Z^2 \Omega_1 \Delta_{12}}\\
&\quad \times P_{e\gamma}(\epsilon, z) |\mathcal{M}_{Z\to\tilde{e}^+e^-}|^2,
\end{aligned} \tag{68}$$

  where $z = 1/(1+E_2/E_+)$ and the $\tilde{e}^+$ momentum is $p_+ + p_2$.

- If $x_1 = 0$, the photon γ_1 becomes soft and the function F_4 reads

$$\begin{aligned}
F_4|_{x_1=0} &= \frac{8 x_3^2 x_4 e^2 (p_-\cdot p_+)(1-\boldsymbol{n}_-\cdot\boldsymbol{n}_2)}{(1-y_2)(1-z_2) m_Z E_+ \Omega_1 \Delta_{12}}\\
&\quad \times |\mathcal{M}_{Z\to e^+e^-\gamma_2}|^2.
\end{aligned} \tag{69}$$

- A new type of singular limit corresponds to $x_2 = 0$ and $x_4 = 0$. This double-collinear limit corresponds to photon γ_1 collinear to the electron and photon γ_2 collinear to the positron. The function F_4 evaluates to

$$\begin{aligned}
F_4|_{x_2=0,x_4=0} &= \frac{16 e^4 x_1 x_3}{m_Z^2 (1-z_2)\Delta_{12}}\\
&\times P_{e\gamma}(\epsilon, z_1) P_{e\gamma}(\epsilon, z_2) |\mathcal{M}_{Z\to\tilde{e}^-\tilde{e}^+}|^2,
\end{aligned} \tag{70}$$

  where $z_1 = 1/(1+E_1/E_-)$ and $z_2 = 1/(1+E_2/E_+)$. The momentum of $\tilde{e}^-$ is the sum of the e^- and γ_1 momenta, and the momentum of $\tilde{e}^+$ is the sum of the e^+ and γ_2 momenta.

The calculation of other limits proceeds along similar lines. The resulting expressions are again straightforward to implement in a computer code.

III. VIRTUAL CORRECTIONS TO SINGLE PHOTON EMISSION

In this Section we discuss the one-loop corrections to the single photon emission process $Z \to e^+e^-\gamma$. We first explain the partitioning of the phase-space and its parametrization. We then discuss how to compute singular limits of scattering amplitudes.

Consider the process $Z \to e^-e^+\gamma_1$. Singularities can arise if the photon is either soft or collinear to the electron or positron. To account for the collinear divergences, we partition the phase space as

$$1 = \delta_1^- + \delta_1^+, \tag{71}$$

where $\delta_1^\pm = \rho_1^\mp$, and $\rho_1^\pm$ are defined in Section II. We write

$$\mathrm{dLips}_{e+e-\gamma_1} = \sum_{a=\pm} \mathrm{dLips}^a_{e+e-\gamma_1}, \tag{72}$$

where

$$\begin{aligned} \mathrm{dLips}^a_{e+e-\gamma_1} = &\int [\mathrm{d}p_-][\mathrm{d}p_+][\mathrm{d}p_1] \\ &\times (2\pi)^d \delta^d(p_Z - p_- - p_+ - p_1)\, \delta_1^a. \end{aligned} \tag{73}$$

In what follows, we consider the primary sector δ_1^-, where a collinear singularity can only occur when momenta of the electron and the photon become parallel. The other sector δ_1^+ gives a symmetric contribution that can be analyzed identically. We use a parametrization that is similar to the two-photon case. For $\mathrm{dLips}^-_{e+e-\gamma_1}$, we parameterize the photon energy as $E_1 = m_Z\xi_1/2$ and the relative angle between the photon and the electron as $\cos\theta_1 = 1-2\eta_1$. The reference frame is fixed by requiring that the electron momentum is along the z-axis and that the photon momentum is in the $x-z$ plane. Explicitly, we write

$$\begin{aligned} p_- &= E_- \left(1,0,0,1\right), \\ p_1 &= \frac{m_Z\xi_1}{2}\left(1, \sin\theta_1, 0, \cos\theta_1\right). \end{aligned} \tag{74}$$

The momentum of the positron is determined from momentum conservation, $p_+ = p_Z - p_- - p_1$. The energy of the electron is found by first computing the momentum $Q = p_Z - p_1$ and then calculating $E_- = Q^2/[2(Q_0 - \boldsymbol{Q}\cdot\boldsymbol{n}_e)]$. Finally, with this parametrization of the momenta, and borrowing notation that we already used when discussing the double-real emission, we write the phase space as

$$\begin{aligned} \mathrm{dLips}^-_{e^-e^+\gamma} = &\,\mathrm{Lips}_{Z\to e^+e^-} \frac{\mathrm{d}\Omega^{(d-2)}_{\gamma^-}}{(2\pi)^{d-1}} \\ &\times \frac{m_Z^{2-2\epsilon}E_-}{2(Q_0 - \boldsymbol{Q}\cdot\boldsymbol{n}_-)} \left(\frac{2E_-}{m_Z}\right)^{-2\epsilon} \\ &\times \frac{\mathrm{d}\xi_1}{\xi_1^{1+2\epsilon}} \frac{\mathrm{d}\eta_1}{\eta_1^{1+\epsilon}} (1-\eta_1)^{-\epsilon} \left[\xi_1^2\eta_1\delta_1^-\right]. \end{aligned} \tag{75}$$

We use Eq. (75) to construct a finite, integrable function when it is combined with the squared matrix element. To this end, we note that ξ_1 and η_1 are already suitable variables for the extraction of singularities, with ξ_1 controlling the soft limit and η_1 controlling the collinear limit. The squared matrix element in the real-virtual case is given by the interference of the tree and one-loop $Z \to e^+e^-\gamma$ amplitudes. We employ Passarino-Veltman reduction to express the one-loop scattering amplitude $Z \to e^+e^-\gamma$ in terms of one-loop integrals, and use the QCDloops program [36] to compute master integrals.

At first, it appears that we must simply repeat what we have done for the double-real emission corrections, extracting singularities of the matrix elements when ξ_1 or η_1 goes to zero. However, there is a subtlety here. One-loop amplitudes are not rational functions of ξ_1 and η_1, in contrast to their tree-level counterparts. To ameliorate this problem, we note that the master integrals which produce singularities in the δ_1^- sector can depend on $s_{e1} = 2p_-\cdot p_1$ raised to a non-integer power. Symbolically,

$$2\mathrm{Re}\left(\mathcal{M}^{(1)}_{Z\to e^+e^-\gamma}\mathcal{M}^{(0)*}_{Z\to e^+e^-\gamma}\right) = B_1 + B_2(s_{e1})^{-\epsilon}, \tag{76}$$

where $B_{1,2}$ are functions that can be Taylor expanded around the $s_{e1} = 0$ limit. Since $s_{e1} \sim \xi_1\eta_1$, the second term in the above equation provides additional $\mathcal{O}(\epsilon)$ contributions to the exponents of singular variables. Obtaining those exponents correctly is crucial for constructing a valid expansion of the real-virtual corrections in inverse powers of ϵ. Because there are two terms in Eq. (76), we must introduce two functions $F_{1,2}(\xi_1,\eta_1)$ to parameterize the matrix element. We write

$$\begin{aligned} \left[\xi_1^2\eta_1\delta_1^-\right] 2\mathrm{Re}&\left(\mathcal{M}^{(1)}_{Z\to e^+e^-\gamma}\mathcal{M}^{(0)*}_{Z\to e^+e^-\gamma}\right) = \\ &F_1(\xi_1,\eta_1) + \xi_1^{-\epsilon}\eta_1^{-\epsilon}F_2(\xi_1,\eta_1) = F(\xi_1,\eta_1). \end{aligned} \tag{77}$$

Away from the singular points $\xi_1 = 0$ and $\eta_1 = 0$, the function $F(\xi_1,\eta_1)$ is obtained by computing one-loop corrections to the radiative decay $Z \to e^+e^-\gamma$ using standard techniques. In the singular limits, we must distinguish between the two contributions. Inserting Eq. (77) into the phase space and performing the plus-distribution expansion for the ξ_1 and η_1 variables, we generate many terms with $\xi_1 = 0$ and/or $\eta_1 = 0$. We now discuss how to obtain $F_{1,2}$ at these singular points.

First, we consider the case $\xi_1 = 0$, which corresponds to the photon γ_1 becoming soft. In that limit, both tree and one-loop QED amplitudes factorize into the products of the eikonal current and the corresponding amplitudes with the photon removed [15]:

$$\mathcal{M}^{(0,1))}_{Z\to e^+e^-\gamma} \to e\left(\frac{p_-\cdot\epsilon_1}{p_-\cdot p_1} - \frac{p_+\cdot\epsilon_1}{p_+\cdot p_1}\right)\mathcal{M}^{(0,1)}_{Z\to e^+e^-}, \tag{78}$$

where ϵ_1 is the photon polarization vector. Using this result, it is easy to find the limit of the function $F(\xi_1,\eta_1)$:

$$
\begin{aligned}
F|_{\xi_1=0} &= \lim_{\xi_1=0}\Bigg[\xi_1^2\eta_1\delta_1^- \\
&\times 2\mathrm{Re}\Big(\mathcal{M}^{(1)}_{Z\to e^+e^-\gamma}\mathcal{M}^{(0)*}_{Z\to e^+e^-\gamma}\Big)\Bigg] \\
&= \frac{4e^2}{m_Z^2}2\mathrm{Re}\left(\mathcal{M}^{(1)}_{Z\to e^+e^-}\mathcal{M}^{(0)*}_{Z\to e^+e^-}\right).
\end{aligned} \tag{79}
$$

Because no terms that behave like $\xi^{-\epsilon}$ appear in this limit, we conclude that

$$
F_1(0,\eta_1) = F(0,\eta_1), \quad F_2(0,\eta_1) = 0. \tag{80}
$$

The next step is the calculation of the collinear $\eta_1 = 0$ limit. It is much more involved. The factorization in this case is given in terms of splitting amplitudes [14]:

$$
\begin{aligned}
&\mathcal{M}^{(0)}_{Z\to e^-e^+\gamma_1} \to \mathrm{Split}^{(0)}_{e^*_\lambda\to e_-\gamma}\mathcal{M}^{(0)}_{Z\to e^-e^+}, \\
&\mathcal{M}^{(1)}_{Z\to e^-e^+\gamma_1} \to \mathrm{Split}^{(0)}_{e^*_\lambda\to e_-\gamma}\mathcal{M}^{(1)}_{Z\to e^-e^+} \\
&\qquad + \mathrm{Split}^{(1)}_{e^*_\lambda\to e_-\gamma}\mathcal{M}^{(0)}_{Z\to e^-e^+}.
\end{aligned} \tag{81}
$$

Hence, the one-loop amplitude factorizes into the one-loop splitting amplitude times the tree photon-less amplitude, and the tree splitting amplitude times the one-loop photon-less amplitude. The relevant splitting functions were computed in Ref. [14]. They are given in terms of "standard matrix elements". For an $e^*\to e_a\gamma_b$ splitting, the situation here, the QED splitting amplitudes are

$$
\begin{aligned}
\mathrm{Split}^{(0)} &= -\frac{\bar{u}_a \not{\epsilon}_b u_{e^*}}{s_{ab}}, \\
\mathrm{Split}^{(1)} &= -2\Big(r_3(z)\mathrm{Split}^{(0)} - r_4(z)\mathrm{Split}^{(2)}\Big),
\end{aligned} \tag{82}
$$

where

$$
\mathrm{Split}^{(2)} = \frac{2\bar{u}_a \not{k}_b u_{e^*}(k_a\cdot\epsilon_b)}{s_{ab}^2}, \tag{83}
$$

and the two functions $r_{3,4}(z)$ parameterize loop contributions to the splitting functions. We must square the splitting functions and sum over the polarizations of the final-state particles. We find

$$
\begin{aligned}
\mathrm{Split}^{(0)}\times\mathrm{Split}^{(0)} &\to \frac{2}{s_{ab}}P_{e\gamma}(\epsilon,z), \\
\mathrm{Split}^{(0)}\times\mathrm{Split}^{(2)} &\to -\frac{2}{s_{ab}}\frac{z(1+z)}{1-z}.
\end{aligned} \tag{84}
$$

We can now use Eqs. (81, 82, 84), to derive the collinear $\eta_1\to 0$ limit of the matrix element:

$$
\begin{aligned}
&\mathrm{Re}\left(\mathcal{M}^{(0)*}_{Z\to e^+e^-\gamma}\mathcal{M}^{(1)}_{Z\to e^+e^-\gamma}\right) \to \\
&\frac{2P_{e\gamma}(\epsilon,z)}{s_{1e}}\mathrm{Re}\left(\mathcal{M}^{(0)*}_{Z\to e^+e^-}\mathcal{M}^{(1)}_{Z\to e^+e^-}\right) \\
&-\frac{4}{s_{1e}}\left(P_{e\gamma}(\epsilon,z)r_3(z) + \frac{z(1+z)}{1-z}r_4(z)\right). \\
&\times\mathrm{Re}\left(\mathcal{M}^{(0)*}_{Z\to e^+e^-}\mathcal{M}^{(0)}_{Z\to e^+e^-}\right),
\end{aligned} \tag{85}
$$

where $s_{1e} = 2p_-\cdot p_1$. The two functions $r_{3,4}(z)$ are proportional to $s_{1e}^{-\epsilon} = \big(m_Z^2\xi\eta(1-z)\big)^{-\epsilon}$ (see Ref. [14]). Consequently, the first term in the right hand side of Eq. (85) contributes to the function $F_1(\xi_1,0)$, and the second term contributes to function $F_2(\xi_1,0)$. We find

$$
F_1|_{\eta_1=0} = \frac{\xi_1 P_{e\gamma}(\epsilon,z)}{E_- m_Z}\mathrm{Re}\left(2\mathcal{M}^{(0)*}_{Z\to e^+e^-}\mathcal{M}^{(1)}_{Z\to e^+e^-}\right), \tag{86}
$$

where further simplifications are possible since $z = 1-\xi_1$. The limit of the function F_2 is more complicated. It reads

$$
\begin{aligned}
F_2(\xi_1,0) &= -2\mathrm{Re}\left[\mathcal{M}^{(0)}_{Z\to e^+e^-}\mathcal{M}^{(0)}_{Z\to e^+e^-}(-z)^{-\epsilon}\right] \\
&\times\frac{2\xi_1}{E_- m_Z}\left(P_{e\gamma}(\epsilon,z)\tilde{r}_3(z) + \frac{z(1+z)}{1-z}\tilde{r}_4(z)\right),
\end{aligned} \tag{87}
$$

where the two functions $\tilde{r}_{3,4}(z)$ are re-scaled versions of $r_{3,4}(z)$ in Ref. [14]. They are

$$
\begin{aligned}
\tilde{r}_3 &= -\frac{1}{2}\left(zf_1(z) - 2f_2 + \frac{(1-\delta_R\epsilon)\epsilon^2}{(1-\epsilon)(1-2\epsilon)}f_2\right), \\
\tilde{r}_4 &= \frac{1}{2}\frac{\epsilon^2(1-\delta_R\epsilon)}{(1-\epsilon)(1-2\epsilon)}f_2,
\end{aligned} \tag{88}
$$

where $\delta_R = 0$ in the FDH scheme [37] and $\delta_R = 1$ in conventional dimensional regularization (CDR). The two functions $f_{1,2}$ can be found in Ref. [14].

Finally, we comment on the construction of the expansion of the real-virtual contribution in plus distributions. The key point is that after the expansion is performed, we are able to get rid of $F_{1,2}(\xi_1,\eta_1)$ in favor of $F(\xi_1,\eta_1)$, for all values of ξ_1,η_1 except the singular ones. At the singular points, we have unambiguous expressions for $F_{1,2}$, as shown above.

IV. REGULARIZATION SCHEMES

We note that most of the formulae presented in the previous sections do not make reference to a particular regularization scheme. They are valid independently of the scheme. Only the splitting functions, the tree-level $Z\to e^+e^-\gamma$ amplitude, and the one-loop $Z\to e^+e^-\gamma$ and $Z\to e^+e^-$ amplitudes change upon switching the scheme. As an illustration, consider the various contributions to the triple-collinear primary sector in Section II A. The double-soft contribution to the function F_1 in Eq. (36) is given by the product of the square of eikonal currents and the tree-level matrix element for the $Z\to e^+e^-$ process. The eikonal current is scheme-independent, while if we choose to work with physical four-dimensional polarizations of the Z-boson, the matrix element for $Z\to e^+e^-$ becomes scheme-independent as well.

The real-virtual corrections proceed similarly. As explained in Section III we require the tree and one-loop matrix element for $Z \to e^+e^-\gamma$, the one-loop matrix element for $Z \to e^+e^-$, and the tree- and one-loop splitting functions for $e \to e + \gamma$. All of these objects are scheme-dependent, but this scheme-dependence is well-understood. In particular, the scheme-dependence of the splitting functions is given in Ref. [14], while the scheme-dependence of the one-loop non-singular amplitudes can be found in Ref. [38].

Hence, it appears that within the framework discussed here any regularization scheme is allowed. The only non-trivial, scheme-dependent contribution at NNLO that needs to be computed explicitly is the two-loop virtual corrections. We emphasize that in this framework, no $\mathcal{O}(\epsilon)$ terms of the double real emission amplitude $Z \to e^+e^-\gamma\gamma$ or real-virtual amplitude $Z \to e^+e^-\gamma$ need to be known through higher orders in ϵ, even in CDR. It seems at first glance that the $\mathcal{O}(\epsilon)$ contribution to the $Z \to e^+e^-\gamma$ amplitude is required, since it can hit a $1/\epsilon$ pole when the other photon has become collinear, leading to a finite contribution. However, it has been suggested recently that this term cancels when the double-real and real-virtual corrections are summed [39][3]. We have checked this statement by comparing the result from summing the double-real and real-virtual contributions in two different ways: with the full $\mathcal{O}(\epsilon)$ contribution retained, and with the $\mathcal{O}(\epsilon)$ term instead replaced by its collinear limit. The sum of double-real and real-virtual is identical in these two cases, indicating that this term does indeed not contribution to the final result for the cross section. We note that simply dropping the $\mathcal{O}(\epsilon)$ term would lead to a mismatch between the squared amplitude and the approximation we use in the collinear limits, causing a divergence in the integration. It is non-trivial to track exactly how the $\mathcal{O}(\epsilon)$ contribution cancels against similar terms in the collinear splitting functions, but since collinear limits are universal, the replacement that we advocate above appears to offer an easy, practical solution.

V. NUMERICAL CHECKS

To prove the utility of this method, we compute the contributions of double-real, real-virtual and virtual corrections to the decay rate of the Z-boson into leptons. For simplicity, we take the coupling of the Z-boson to leptons to be vector-like, and ignore all diagrams which contain photon vacuum-polarization contributions or its unitary cuts. We compare separately the double-real and real-virtual corrections to the four-and three-particle cuts of the vector-vector correlator, which we obtain using the optical theorem. We have presented these contributions separately because of the significant numerical cancellations between the double-real radiation, the real-virtual corrections, and the two-loop virtual terms that occur when summing them to obtain the total NNLO correction to the decay rate. The three- and four- particle cuts of three-loop master integrals required for such computation can be found in Ref. [40]. We present this comparison in the CDR scheme. Note, however, that the polarization vectors of the Z-boson are not continued to d-dimensions, making the tree decay rate $Z \to e^+e^-$ ϵ-independent. We also set $m_Z = 1$. We write

$$\Gamma_{Z\to e^+e^-} = \Gamma^{(0)}_{Z\to e^+e^-}\left(1 + \frac{3}{4}\frac{\alpha}{\pi} + \left(\frac{\alpha}{\pi}\right)^2 \delta^{(2)}\right), \qquad (89)$$

where $\delta^{(2)}$ is further split into three contributions :

$$\delta^{(2)} = \delta^{(2)}_{RR} + \delta^{(2)}_{RV} + \delta^{(2)}_{VV}. \qquad (90)$$

The result for the inclusive decay width can be obtained from the literature [41].

The two-loop virtual correction can trivially be obtained from the known result for the two-loop fermion form-factor. For this reason we do not present it here. From the analytic computation based on the optical theorem we find

$$\begin{aligned}
\delta^{(2)}_{RR} &= \frac{0.5}{\epsilon^4} + \frac{1.5}{\epsilon^3} - \frac{1.7246}{\epsilon^2} - \frac{14.074}{\epsilon} - 24.228;\\
\delta^{(2)}_{RV} &= -\frac{1}{\epsilon^4} - \frac{3}{\epsilon^3} + \frac{3.1794}{\epsilon^2} + \frac{22.88}{\epsilon} + 32.94.
\end{aligned} \qquad (91)$$

These results can be compared directly to our computations based on the soft and collinear limits of the relevant matrix elements. Before we present the corresponding results, we note one complication. There are interference contributions contained in $Z \to e^+e^-e^+e^-$ that can not be disregarded. Typically, these interference parts of the four-fermion final state correspond to certain cuts of non-planar diagrams and, hence, become part of our check. The four-fermion interference contribution only contains collinear singularities, and can be analyzed in the same way as the double-real emission contributions discussed in Section II. Because of the existence of this contribution, we split the double-real result into $e^+e^-\gamma\gamma$ and $e^+e^-e^+e^-$ final states. From our numerical calculation, we obtain

$$\begin{aligned}
\delta^{(2),4e}_{RR} &= -\frac{0.1799}{\epsilon} - 1.79,\\
\delta^{(2),\gamma}_{RR} &= \frac{0.5}{\epsilon^4} + \frac{1.5}{\epsilon^3} - \frac{1.726(5)}{\epsilon^2} - \frac{13.94(3)}{\epsilon} - 22.61(8),\\
\delta^{(2)}_{RV} &= -\frac{1}{\epsilon^4} - \frac{3}{\epsilon^3} + \frac{3.179}{\epsilon^2} + \frac{22.84}{\epsilon} + 32.97(3).
\end{aligned} \qquad (92)$$

The sum of the two double-real emission corrections agrees with Eq. (91), as does the real-virtual contribution, indicating the correctness of the numerical results.

[3] The reason for this cancellation is a well-understood independence of NLO QED corrections to $Z \to e^+e^-\gamma$ on the regularization scheme.

VI. CONCLUSIONS

In this paper we have described in detail a subtraction scheme which enables fully differential calculation at NNLO accuracy. By combining several ideas present in the literature, including an FKS partitioning of the final-state phase space and sector decomposition, the universal singular limits of amplitudes derived over a decade ago can finally be used to obtain actual physical cross sections. Our ideas are explained using the simple test case of $Z \to e^+e^-$ as an example. We discussed how to partition the phase space based on the collinear-singularity structure of the matrix element, and presented the explicit phase-space parameterizations from which the soft and collinear singularities can be extracted as poles in ϵ using sector decomposition. The treatment of the real-virtual corrections is described in a way that generalizes to more complicated processes. Numerical results that check our techniques were presented. We have chosen to work in the CDR regularization scheme, although the presented framework remains valid in other schemes such as FDH. It has been pointed out that difficulties exist when extending FDH to NNLO [42]. Although they can be fixed [43], with our current understanding the use of CDR imposes no additional technical difficulties, as discussed in Section IV.

One point we wish to emphasize about the result presented here is its generalization to more complicated processes. As mentioned earlier, one of the problems with earlier sector-decomposition based approaches to NNLO calculations was the need to completely reconsider the phase space and extraction of singularities upon changing the process. In particular, if one knew the NNLO corrections to $Z \to e^+e^-$, but wanted to study the NNLO corrections to $Z \to e^+e^-\gamma$, now would have to start from scratch. That is no longer the case for the framework described here. Differential Z decay serves as a building block for handling all final-state singularities, as we now describe.

We will consider the real-radiation correction $Z \to e^+(p_+) + e^-(p_-) + \gamma(p_1) + \gamma(p_2) + \gamma(p_3)$, the most difficult contribution, for definiteness. Introduce the following partition of phase space:

$$1 = \frac{1}{D} \sum_{(i,j)\in(1,2,3)} \left\{\delta_{ij,+} + \delta_{ij,-} + \delta_{ij,+-} + \delta_{ij,-+}\right\}. \quad (93)$$

Here, $\delta_{ij,k}$ allows p_i and p_j to be soft, but not any other particles; it also only allows collinear singularities when p_i, p_j, p_k are collinear. $\delta_{ij,kl}$ allows only p_i and p_j to be soft, and also allows only the collinear limits $p_i \parallel p_k$ and $p_j \parallel p_l$. D is the sum of all δ. It is simple to construct the appropriate δ functions, as discussed in Ref. [1]. Consider the partition with $\delta_{12,+}$ for concreteness. The contribution of this real-radiation correction to the differential cross section is schematically

$$\frac{d\sigma}{d\mathcal{O}_0} = \int \mathrm{dLips}_{e^+e^-\gamma_1\gamma_2\gamma_3} |\mathcal{M}|^2 \delta\left(\mathcal{O} - \mathcal{O}_0\right) \frac{\delta_{12,+}}{D} \quad (94)$$

where $\mathcal{O}$ is an observable being studied and $\mathcal{M}$ is the matrix element. In this partition, there is no soft or collinear singularity associated with p_3. We should therefore be able to use the phase-space parameterization and singularity extraction described in this paper, which handles the double-unresolved limit of photons p_1 and p_2. To make this manifest, rewrite the phase space of Eq. (94) as

$$\begin{aligned} \mathrm{dLips}_{e^+e^-\gamma_1\gamma_2\gamma_3} &= ds_{+-12}[dp_3][dp_{+-12}]\mathrm{dLips}_{e^+e^-\gamma_1\gamma_2} \\ &\times \delta^{(d)}(p_Z - p_3 - p_{+-12}), \end{aligned} \quad (95)$$

where s_{+-12} is the invariant mass of all final-state particles except the hard photon. The parameterization of momenta in $\mathrm{dLips}_{e^+e^-\gamma_1\gamma_2}$ is chosen to be the same as in Section II A. The form of p_3 in this parameterization is irrelevant; no soft or collinear singularities are associated with this momentum. We can simply reuse the NNLO results for $Z \to e^+e^-$ to obtain the corrections to $Z \to e^+e^-\gamma$. Adding additional photons to the final state only increases the number of partitions required. In this sense, $Z \to e^+e^-$ serves as a building block for extracting final-state singularities from any process. While we have demonstrated this for only one partition, it follows similarly for the others.

We are excited about the possible applications of these ideas to more phenomenologically interesting processes. We believe there is significant potential for applying these ideas to the calculation of $2 \to 2$ scattering processes at the LHC, and we look forward to their continued development.

Acknowledgments K.M. gratefully acknowledges useful conversations with Z. Kunszt and F. Caola, and would like to thank the KITP at UCSB for hospitality during the work on this paper. This research is supported by the US DOE under contract DE-AC02-06CH11357 and the grant DE-FG02-91ER40684, by the NSF under grants PHY-0855365 and PHY05-51164, and with funds provided by Northwestern University.

Appendix: Splitting functions

We collect here the splitting functions that we employed in this computation, in the CDR regularization scheme. For the $e \to e + \gamma$ splitting, we have

$$P_{e\gamma}(\epsilon, z) = \frac{2}{1-z} - (1+z) - \epsilon(1-z). \quad (96)$$

For the $e \to e\gamma_1\gamma_2$ splitting, we find [13]

$$\begin{aligned} P_{e\gamma\gamma}(\epsilon, z, z_1, z_2) &= \frac{s_{12e}^2}{2 s_{1e} s_{2e}} P_1(\epsilon, z_e, z_1, z_2) \\ &+ \frac{s_{12e}}{s_{1e}} P_2(\epsilon, z_e, z_1, z_2) \\ &+ (1-\epsilon)\left[\epsilon - (1-\epsilon)\frac{s_{2e}}{s_{1e}}\right] + (1 \leftrightarrow 2), \end{aligned} \quad (97)$$

where the functions $P_{1,2}$ read

$$P_1(\epsilon, z_e, z_1, z_2) = z_e \left(\frac{1+z_e^2}{z_1 z_2} - \epsilon \frac{z_1^2+z_2^2}{z_1 z_2} - \epsilon(1+\epsilon) \right),$$
$$P_2(\epsilon, z_e, z_1, z_2) = \frac{z_e(1-z_1)+(1-z_1)^3}{z_1 z_2} + \epsilon^2(1+z_e) - \epsilon(z_1^2+z_1 z_2+z_2^2)\frac{(1-z_2)}{z_1 z_2}. \quad (98)$$

[1] M. Czakon, Phys. Lett. **B693**, 259-268 (2010).
[2] S. Catani and M. H. Seymour, Nucl. Phys. B **485**, 291 (1997) [Erratum-ibid. B **510**, 503 (1998)].
[3] Z. Bern, L.J. Dixon and A. Ghinculov, Phys. Rev. D63, 053007 (2001).
[4] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, Nucl. Phys. B601, 318 (2001); Nucl. Phys. B601, 341 (2001).
[5] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, Nucl. Phys. B605, 486 (2001).
[6] E. W. N. Glover, C. Oleari and M.E. Tejeda-Yeomans B605, 467 (2001).
[7] L.W. Garland, T. Gehrmann, E.W.N. Glover, A. Koukoutsakis and E. Remiddi, Nucl. Phys. B642, 227 (2002).
[8] S. Catani, D. de Florian, M. Grazzini, Nucl. Phys. **B596**, 299-312 (2001).
[9] F. A. Berends and W. T. Giele, Nucl. Phys. B **313** (1989) 595.
[10] J. M. Campbell and E. W. N. Glover, Nucl. Phys. B **527** (1998) 264.
[11] S. Catani and M. Grazzini, Phys. Lett. B **446** (1999) 143.
[12] Z. Bern, V. Del Duca, W. B. Kilgore and C. R. Schmidt, Phys. Rev. D **60** (1999) 116001.
[13] S. Catani, M. Grazzini, Nucl. Phys. **B570**, 287-325 (2000).
[14] D. A. Kosower, P. Uwer, Nucl. Phys. **B563**, 477-505 (1999).
[15] S. Catani and M. Grazzini, Nucl. Phys. B **591** (2000) 435.
[16] C. Anastasiou, K. Melnikov and F. Petriello, Phys. Rev. D69, 076010 (2004).
[17] K. Melnikov and F. Petriello, Phys. Rev. D74, 114017 (2006).
[18] S. Catani and M. Grazzini, Phys. Rev. Lett. 98, 222002 (2007).
[19] S. Catani, L. Cieri, G. Ferrera, D. de Florian, M. Grazzini, Phys. Rev. Lett. 103, 082001 (2009).
[20] C. Anastasiou, K. Melnikov and F. Petriallo, Phys. Rev. Lett. 93, 262002 (2004).
[21] C. Anastasiou, K. Melnikov and F. Petriello, Nucl. Phys. B274, 197 (2005).
[22] C. Anastasiou, K. Melnikov and F. Petriello, Phys. Rev. Lett. 93, 032002 (2004).
[23] A. Gehrmann-De-Ridder, T. Gehrmann, E.W.N. Glover and G. Heinrich, JHEP 0712 094 (2007).
[24] A. Gehrmann-De-Ridder, T. Gehrmann, E.W.N. Glover and G. Heinrich, JHEP 0711 058 (2007).
[25] C. Anastasiou, K. Melnikov, F. Petriello, JHEP **0709**, 014 (2007).
[26] S. Biswas, K. Melnikov, JHEP **1002**, 089 (2010).
[27] K. Melnikov, Phys. Lett. **B666**, 336-339 (2008).
[28] G. Ferrera, M. Grazzini and F. Tramontano, Phys. Rev. Lett. **107**, 152003 (2011).
[29] S. Catani, L. Cieri, D. de Florian, G. Ferrera and M. Grazzini, arXiv:1110.2375 [hep-ph].
[30] C. Anastasiou, F. Herzog and A. Lazopoulos, arXiv:1110.2368 [hep-ph].
[31] S. Frixione, Z. Kunszt, A. Signer, Nucl. Phys. **B467**, 399-442 (1996).
[32] R. Frederix, S. Frixione, F. Maltoni and T. Stelzer, JHEP **0910**, 003 (2009).
[33] T. Binoth and G. Heinrich, Nucl. Phys. B585, 741 (2000).
[34] T. Binoth and G. Heinrich, Nucl. Phys. **B693**, 138 (2004).
[35] M. Czakon, Nucl. Phys. **B849**, 250-295 (2011).
[36] R.K. Ellis and G. Zanderighi, JHEP **0802**, 002 (2008).
[37] Z. Bern, A. De Freitas, L. J. Dixon, H. L. Wong, Phys. Rev. **D66**, 085002 (2002).
[38] A. Signer, D. Stockinger, Nucl. Phys. **B808**, 88-120 (2009).
[39] S. Weinzierl, Phys. Rev. D **84**, 074007 (2011).
[40] A. Gehrmann-De Ridder, T. Gehrmann, G. Heinrich, Nucl. Phys. **B682**, 265-288 (2004).
[41] A. L. Kataev, Phys. Lett. B **287**, 209 (1992).
[42] W. B. Kilgore, Phys. Rev. **D83**, 114005 (2011).
[43] R. Boughezal, K. Melnikov and F. Petriello, Phys. Rev. D **84**, 034044 (2011).