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Casimir-Polder repulsion: Polarizable atoms, cylinders, spheres, and ellipsoids

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Abstract

Recently, the topic of Casimir repulsion has received a great deal of attention, largely because of the possibility of technological application. The general subject has a long history, going back to the self-repulsion of a conducting spherical shell and the repulsion between a perfect electric conductor and a perfect magnetic conductor. Recently it has been observed that repulsion can be achieved between ordinary conducting bodies, provided sufficient anisotropy is present. For example, an anisotropic polarizable atom can be repelled near an aperture in a conducting plate. Here we provide new examples of this effect, including the repulsion on such an atom moving on a trajectory nonintersecting a conducting cylinder; in contrast, such repulsion does not occur outside a sphere. Classically, repulsion does occur between a conducting ellipsoid placed in a uniform electric field and an electric dipole. The Casimir-Polder force between an anisotropic atom and an anisotropic dielectric semispace does not exhibit repulsion. The general systematics of repulsion are becoming clear.

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I. INTRODUCTION

Although known since the time of Lifshitz's work on the subject [1], repulsive Casimir forces have recently received serious scrutiny [2]. Experimental confirmation of the repulsion that occurs when dielectric surfaces are separated by a liquid with an intermediate value of the dielectric constant has appeared [3], although this seems devoid of much practical application. The context of our work is the considerable interest in utilizing the quantum vacuum force or the Casimir effect in nanotechnology employing mesoscopic objects [4].

The first repulsive Casimir stress in vacuum was found by Boyer [5], who discovered the still surprising fact that the Casimir self-energy of a perfectly conducting spherical shell is positive. (This has become somewhat less mysterious, since the phenomenon is part of a general pattern [6–9].) Boyer later observed that a perfect electrical conductor and a perfect magnetic conductor repel [10], but this also seems beyond reach, since the unusual electrical properties must be exhibited over a wide frequency range. The analogous effect for metamaterials also seem impracticable [11].

Thus it was a significant advance when Levin et al. showed examples of repulsion between conducting objects, in particular between an elongated cylinder above a conducting plane with a circular aperture [2]. (See also Ref. [12].) They computed the quantum vacuum forces between conducting objects, by using impressive numerical finite-difference time-domain and boundary-element methods.

We subsequently showed [13] that repulsive Casimir-Polder forces between anisotropic atoms and a conducting half-plane, and even between such an atom and a conducting wedge of rather large opening angle, could be achieved. Of course, we must be careful to explain what we mean by repulsion: the total force on the atom is attractive, but the component of the force perpendicular to the symmetry axis of the conductor changes sign when the atom is sufficiently close to that axis. This is the only component that survives in the case of an aperture in a plane, so our analytic calculation provided a counterpart to the numerical work of Ref. [2].

In this paper we give some further examples. After demonstrating, in Sec. II, that Casimir-Polder repulsion between two atoms requires that both be sufficiently anisotropic, we show in Sec. III that the force between one such atom and a conducting cylinder is repulsive for motion confined to a perpendicular line not intersecting with the cylinder,

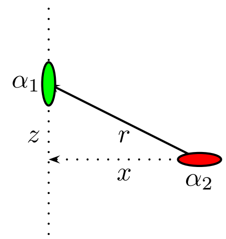

FIG. 1: Casimir-Polder interaction between two atoms of polarizability $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ separated by a distance r. Atom 1 is predominantly polarizable in the z direction, while atom 2 is predominantly polarizable in the x direction. The force on atom 1 in the z direction becomes repulsive sufficiently close to the polarization axis of atom 2 provided both atoms are sufficiently anisotropic.

provided the line is sufficiently far from the cylinder. The analogous effect does not occur for a spherical conductor (Sec. IV), as one might suspect since at large distances such a sphere looks like an isotropic atom. The classical interaction between a dipole and a conducting ellipsoid polarized by an external field is examined in Sec. V, which, as expected, yields a repulsive region. In contrast, in Sec. VI, we examine the Casimir-Polder interaction of an anisotropic atom with an anisotropic dielectric half-space, but this fails to reveal any repulsive regime.

In this paper we set $\hbar = c = 1$, and all results are expressed in Gaussian units except that Heaviside-Lorentz units are used for Green's dyadics.

II. CASIMIR-POLDER REPULSION BETWEEN ATOMS

The interaction between two polarizable atoms, described by general polarizabilities $\boldsymbol{\alpha}_{1,2}$, with the relative separation vector given by $\mathbf{r}$ is [14, 15]

$$U_{\rm CP} = -\frac{1}{4\pi r^7}\left[\frac{13}{2}\,\mathrm{Tr}\,\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2 - 28\,\mathrm{Tr}(\boldsymbol{\alpha}_1\cdot\hat{\mathbf{r}})(\boldsymbol{\alpha}_2\cdot\hat{\mathbf{r}}) + \frac{63}{2}(\hat{\mathbf{r}}\cdot\boldsymbol{\alpha}_1\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\cdot\boldsymbol{\alpha}_2\cdot\hat{\mathbf{r}})\right]. \tag{2.1}$$

This formula is easily rederived by the multiple scattering technique as explained in Ref. [16]. This reduces, in the isotropic case, $\boldsymbol{\alpha}_i = \alpha_i\mathbf{1}$, to the usual Casimir-Polder (CP) energy, $U_{\rm CP} = -\frac{23}{4\pi r^7}\alpha_1\alpha_2$. Suppose the two atoms are only polarizable in perpendicular directions, $\boldsymbol{\alpha}_1 = \alpha_1\hat{\mathbf{z}}\hat{\mathbf{z}}$, $\boldsymbol{\alpha}_2 = \alpha_2\hat{\mathbf{x}}\hat{\mathbf{x}}$. Choose atom 2 to be at the origin. The configuration is shown in Fig. 1. Then, in terms of the polar angle $\cos\theta = z/r$, the z-component of the force on atom

1 is

$$F_z = -\frac{63}{8\pi}\frac{\alpha_1\alpha_2}{x^8}\sin^{10}\theta\cos\theta(9-11\sin^2\theta). \tag{2.2}$$

In this paper, we are considering motion for fixed $x = r\sin\theta$, in the $y = 0$ plane. Evidently, the force is attractive at large distances, vanishing as $\theta \to 0$, but it must change sign at small values of z for fixed x, since the energy also vanishes as $\theta \to \pi/2$. The force component in the z direction vanishes when $\sin\theta = 3/\sqrt{11}$ or $\theta = 1.130$ or 25° from the x axis.[1]

No repulsion occurs if one of the atoms is isotropically polarizable. If both have cylindrically symmetric anisotropies, but with respect to perpendicular axes,

$$\boldsymbol{\alpha}_1 = (1-\gamma_1)\alpha_1\hat{\mathbf{z}}\hat{\mathbf{z}} + \gamma_1\alpha_1\mathbf{1}, \quad \boldsymbol{\alpha}_2 = (1-\gamma_2)\alpha_2\hat{\mathbf{x}}\hat{\mathbf{x}} + \gamma_2\alpha_2\mathbf{1}, \tag{2.3}$$

it is easy to check that if both are sufficiently anisotropic repulsion will occur. For example, if $\gamma_1 = \gamma_2$ repulsion in the z direction will take place close to the plane $z = 0$ if $\gamma \le 0.26$.

III. REPULSION OF AN ATOM BY A CONDUCTING CYLINDER

Now we turn to the Casimir-Polder (CP) interaction between a polarizable body ("atom") and a macroscopic body. That interaction is generally given by

$$E_{\rm CP} = -\int_{-\infty}^{\infty} d\zeta\, \mathrm{tr}\, \boldsymbol{\alpha}\cdot\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}), \tag{3.1}$$

where $\mathbf{r}$ is the position of the atom and ζ is the imaginary frequency, in terms of the polarizability of the atom $\boldsymbol{\alpha}$ and the Green's dyadic due to the macroscopic body, which for a body characterized by a permittivity ε satisfies the differential equation

$$\left(\frac{1}{\omega^2}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times -\mathbf{1}\varepsilon(\mathbf{r})\right)\cdot\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}') = \mathbf{1}\delta(\mathbf{r}-\mathbf{r}'). \tag{3.2}$$

In this paper, except for Sec. VI, we will consider perfect conducting boundaries S immersed in vacuum, in which case we need to solve this equation with $\varepsilon = 1$ for $\boldsymbol{\Gamma}$ subject to the boundary conditions $\hat{\mathbf{n}}\times\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}')\Big|_{\mathbf{r}\in S} = 0$, where $\hat{\mathbf{n}}$ is the normal to the surface of the

[1] After the first version of this paper was prepared, Ref. [17] appeared, which rederived these results, and then went on to extend the calculation to Casimir-Polder repulsion by an anisotropic dilute dielectric sheet with a circular aperture. The authors quite correctly point out that the statement in Ref. [13] that no repulsion is possible in the weak-coupling regime is erroneous.

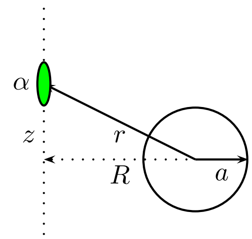

FIG. 2: Interaction between an anisotropically polarizable atom and a conducting cylinder of radius a. The force on the atom along a line which does not intersect the cylinder is considered. If the atom is only polarizable in that direction, and the line lies sufficiently far from the cylinder, the force component along the line changes sign near the point of closest approach.

conductor, which just states that the tangential components of the electric field must vanish on the conductor.

Let us henceforth assume that the polarizability has negligible frequency dependence (static approximation), and, in order to maximize the repulsive effect, the atom is only polarizable in the z direction, the direction of the trajectory (assumed not to intersect the cylinder), in which case the quantity we need to compute for a conducting cylinder of radius a is given by [18]

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}\Gamma_{zz}(r,\theta) &= \sum_{m=-\infty}^{\infty}\int_0^{\infty}\frac{d\kappa}{(2\pi)^3}\frac{\pi}{2a}\frac{1}{K_m(\kappa a)K_m'(\kappa a)}\left\{\frac{m^2}{r^2}K_m^2(\kappa r)+\kappa^2K_m'^2(\kappa r)\right.\\ &\left.\quad -\cos 2\theta\kappa a[I_m(\kappa a)K_m(\kappa a)]'\left(-\frac{m^2}{r^2}K_m^2(\kappa r)+\kappa^2K_m'^2(\kappa r)\right)\right\}. \end{aligned} \tag{3.3}$$

The geometry we are considering is illustrated in Fig. 2. It gives greater insight to give the transverse electric (TE) and transverse magnetic (TM) contributions to the CP energy:

$$E_{\rm CP}^{\rm TE} = -\frac{\alpha_{zz}}{4\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty}d\kappa\,\kappa\frac{I_m'(\kappa a)}{K_m'(\kappa a)}\left[\frac{\cos^2\theta}{r^2}m^2K_m^2(\kappa r)+\kappa^2\sin^2\theta K_m'^2(\kappa r)\right], \tag{3.4a}$$

$$E_{\rm CP}^{\rm TM} = \frac{\alpha_{zz}}{4\pi}\sum_{m=-\infty}^{\infty}\int_0^{\infty}d\kappa\,\kappa\frac{I_m(\kappa a)}{K_m(\kappa a)}\left[\frac{\sin^2\theta}{r^2}m^2K_m^2(\kappa r)+\kappa^2\cos^2\theta K_m'^2(\kappa r)\right]. \tag{3.4b}$$

The distance of the atom from the center of the cylinder is $r = R/\sin\theta$, where R is the distance of closest approach and θ is the polar angle, which ranges from 0 when the atom is at infinity to $\pi/2$ when the atom is closest to the cylinder.

At large distances, the CP force is dominated by the $m = 0$ term in the energy sum. Figure 3 shows that for $m = 0$ the TM mode dominates except near the position of closest

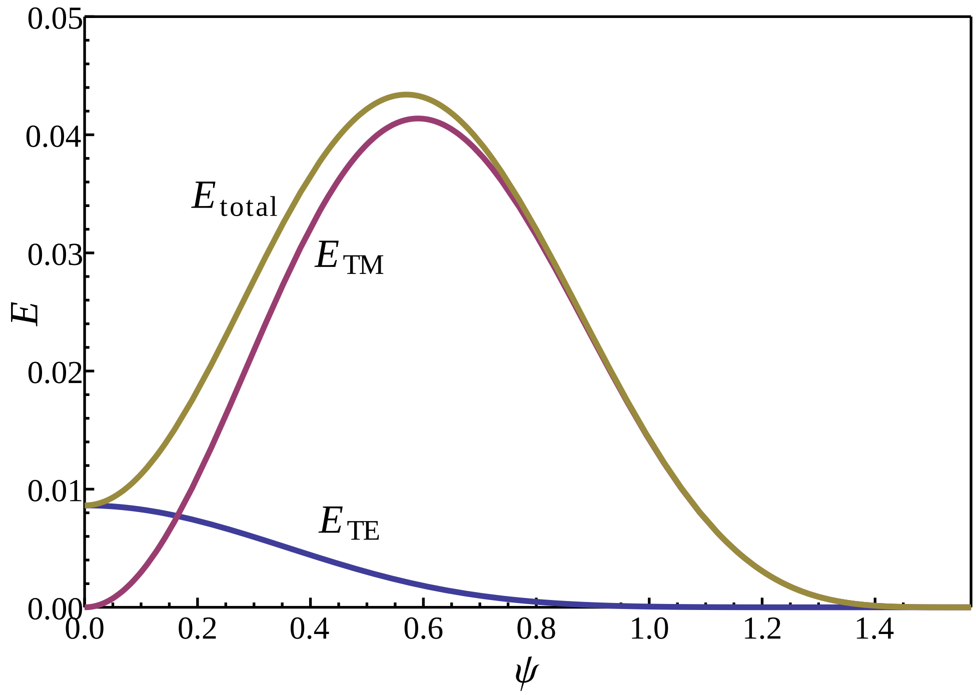

FIG. 3: $m = 0$ contributions to the Casimir-Polder energy between an anisotropic atom and a conducting cylinder. The (generally) lowest curve (blue) is the TE contribution, the second (magenta) is the TM contribution, and the top curve (yellow) is the total CP energy. In this case, the distance of closest approach of the atom is taken to be 10 times the radius of the cylinder. The energy E is plotted as a function of $\psi = \pi/2 - \theta$.

approach, where only the TE mode is nonzero. This indicates that there is a region of repulsion near $\theta = \pi/2$, since the total energy has a minimum for small $\psi = \pi/2 - \theta$. This effect is partially washed out by including higher m modes, as seen in Fig. 4, which shows the effect of including the first 5 m values. But the repulsion goes away if the line of motion passes too close to the cylinder. Numerically, we have found that to have repulsion close to the plane of closest approach requires that $a/R < 0.15$.

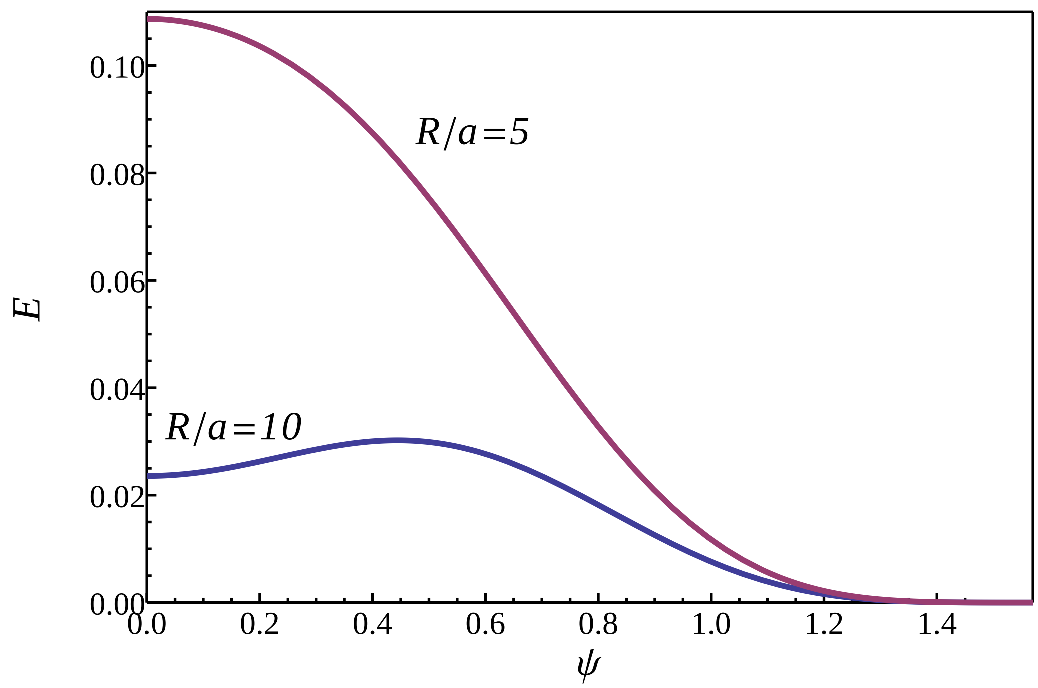

FIG. 4: The CP energy between an anisotropic atom and a conducting cylinder. Plotted is the total CP energy, the upper curve for the distance of closest approach R being 5 times the cylinder radius a, the lower curve for the distance of closest approach 10 times the radius. The curves move up slightly as more m terms are included, but have completely converged by the time $m = 3$ is included. Repulsion is clearly observed when $R/a = 10$, but not for $R/a = 5$.

IV. CP INTERACTION BETWEEN ATOM AND CONDUCTING SPHERE

It is straightforward to derive the TE and TM contributions for the interaction between a completely anisotropic atom and a conducting sphere as

$$E^{\mathrm{TM}} = \frac{\alpha_{zz}}{2\pi R^4}\cos^4\theta\sum_{l=1}^{\infty}(2l+1)\int_0^{\infty}dx\, g_l(x), \tag{4.1a}$$

$$E^{\mathrm{TE}} = \frac{\alpha_{zz}}{4\pi R^4}\cos^6\theta\sum_{l=1}^{\infty}(2l+1)\int_0^{\infty}dx\, f_l(x), \tag{4.1b}$$

where

$$g_l(x) = x\frac{s_l'(xa\cos\theta/R)}{e_l'(xa\cos\theta/R)}\left[\frac{1}{2}\cos^2\theta e_l'^2(x) + \frac{l(l+1)\sin^2\theta e_l^2(x)}{x^2}\right], \tag{4.2a}$$

$$f_l(x) = x\frac{s_l(xa\cos\theta/R)}{e_l(xa\cos\theta/R)}e_l^2(x), \tag{4.2b}$$

where the modified Riccati-Bessel functions are

$$s_l(x) = \sqrt{\frac{\pi x}{2}}I_{l+1/2}(x), \quad e_l(x) = \sqrt{\frac{2x}{\pi}}K_{l+1/2}(x). \tag{4.3}$$

We expect in the case of a sphere not to see Casimir repulsion at large distances. The reason is that far from the sphere it appears to be an isotropic atom, which, as we have seen above will not give a repulsive force on another completely anisotropic atom. Indeed, far from the sphere we can replace the Bessel functions of argument xa/r by their leading small argument approximations and we easily find

$$E^{\text{TM}} \sim \frac{\alpha_{zz}a^3}{4\pi r^7}(13 + 7\sin^2\theta), \quad a/r \to 0. \tag{4.4a}$$

The TE mode contributes

$$E^{\text{TE}} \sim \frac{\alpha_{zz}a^3}{4\pi r^7}\frac{7}{4}\cos^2\theta, \quad a/r \to 0. \tag{4.4b}$$

We see here the expected isotropic electric polarizability of a conducting sphere $\alpha_{\text{sp},E} = \mathbf{1}a^3$. We note that the TM result (4.4a) coincides with the result obtained from Eq. (2.1). The TE contribution is, in fact, the coupling between the electric polarizability of the atom and the magnetic polarizability of the sphere $\alpha_{\text{sp},M} = -\frac{a^3}{2}\mathbf{1}$ [19].

To see this, we first remind the reader of the CP interaction between isotropic atoms possessing both electric and magnetic polarizabilities [20],

$$U_{\text{CP}} = -\frac{23}{4\pi r^7}(\alpha_1^E\alpha_2^E + \alpha_1^M\alpha_2^M) + \frac{7}{4\pi r^7}(\alpha_1^E\alpha_2^M + \alpha_1^M\alpha_2^E). \tag{4.5}$$

When the atoms are not isotropic it is easy to deduce the generalization of this, using the methods described in Ref. [16], starting from the multiple-scattering coupling term between electric and magnetic dyadics,

$$E_{\text{em}} = -\frac{i}{2}\,\text{Tr}\ln\left(1 + \mathbf{\Phi}_0\mathbf{T}_1^E\cdot\mathbf{\Phi}_0\mathbf{T}_2^M\right) \approx -\frac{i}{2}\,\text{Tr}\,\mathbf{\Phi}_0\cdot\mathbf{V}_1^E\mathbf{\Phi}_0^e\cdot\mathbf{V}_2^M, \tag{4.6}$$

where the last form reflects weak coupling, and we are considering the interaction between one object having purely electric susceptibility and a second object having purely magnetic susceptibility, so

$$\mathbf{V}_1^E = 4\pi\boldsymbol{\alpha}_1^E\delta(\mathbf{r}-\mathbf{r_1}), \quad \mathbf{V}_2^M = 4\pi\boldsymbol{\alpha}_2^M\delta(\mathbf{r}-\mathbf{r_2}). \tag{4.7}$$

This formula is expressed in terms of the magnetic Green's dyadic,

$$\boldsymbol{\Phi}_0 = -\frac{\zeta^2}{4\pi R^3}\mathbf{R}\times(|\zeta|R+1)e^{-|\zeta|R}. \tag{4.8}$$

Then, an immediate calculation yields the electric-magnetic CP interaction

$$U_{\text{CP,EM}} = \frac{7}{8\pi R^7}\,\text{tr}(\hat{\mathbf{R}}\times\boldsymbol{\alpha}^E)(\hat{\mathbf{R}}\times\boldsymbol{\alpha}^M), \tag{4.9}$$

which indeed for isotropic polarizabilities gives the second term in Eq. (4.5). The result (4.4b) is now an immediate consequence for a conducting sphere interacting with an atom only polarizable in the z direction.

Evidently, no repulsion can occur in this CP limit where the conducting sphere is regarded as an anisotropically polarizable atom. In fact, numerical evaluation shows no repulsion occurs at any separation distance between the sphere and the atom.

V. ELECTROSTATIC FORCE BETWEEN A CONDUCTING ELLIPSOID AND A DIPOLE

In this section we return, for heuristic reasons, to the electrostatic situation of the interaction between a fixed dipole and a conducting body. Such have been given considerable attention lately [2, 13, 21]. Here we consider the interaction between a perfectly conducting ellipsoid polarized by a constant electric field and a fixed dipole. The polarization of the ellipsoid by the dipole is neglected at this stage. This is a much simpler calculation than the more interesting one of the interaction between a dipole and a ellipsoid, but we justify the inclusion of the details of the simpler calculation here because it allows us to approach the complexity of the full calculation. Elsewhere, we will present that calculation and the corresponding quantum Casimir-Polder calculation, building on the work of Ref. [22].

A. Ellipsoidal coordinates

Consider first a conducting uncharged solid ellipsoid with semiaxes $a > b > c$, centered at the origin $x = y = z = 0$. The semiaxis c lies along the z axis. The electrostatic potential ϕ in the external region can be described in terms of ellipsoidal coordinates ξ, η, ζ, corresponding to solutions for u of the cubic equation

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1. \tag{5.1}$$

The coordinate intervals are in general

$$\infty > \xi \geq -c^2, \quad -c^2 \geq \eta \geq -b^2, \quad -b^2 \geq \zeta \geq -a^2. \tag{5.2}$$

We will henceforth assume axial symmetry around the z axis. In that case, $b \to a$, $\zeta \to -a^2$, and the ellipsoidal coordinates ξ, η, ζ reduce to oblate spheroidal coordinates ξ and η restricted to the intervals

$$\infty > \xi \geq -c^2, \quad -c^2 \geq \eta \geq -a^2. \tag{5.3}$$

If $\rho = \sqrt{x^2+y^2}$ denotes the horizontal radius in the plane $z =$ constant, the cubic equation (5.1) reduces to the quadratic equation

$$u^2 - (\rho^2 - a^2 - c^2 + z^2)u - (\rho^2 - a^2)c^2 - z^2a^2 = 0 \tag{5.4}$$

for $u = (\xi, \eta)$. The solution for $u = \xi$ corresponds to the positive square root:

$$\xi = \frac{1}{2}(\rho^2 - a^2 - c^2 + z^2) + \frac{1}{2}\sqrt{(\rho^2 - a^2 + c^2)^2 + z^2(2\rho^2 + 2a^2 - 2c^2 + z^2)}. \tag{5.5}$$

At the surface of the ellipsoid, $\xi = 0$, whereas in the external region, $\xi > 0$. Note that in the xy plane ($z = 0$) the expression for ξ simplifies to $\xi = \rho^2 - a^2$, when $\rho > a$. The solution for $u = \eta$ corresponds to the same expression (5.5) but with the negative square root.

Surfaces of constant ξ and η are oblate spheroids and hyperboloids of revolution, the surfaces intersecting orthogonally. On the symmetry axis $\rho = 0$ one has $\xi = -c^2 + z^2$, $\eta = -a^2$. The relations between ξ, η and z, ρ are

$$z = \pm\sqrt{\frac{(\xi + c^2)(\eta + c^2)}{c^2 - a^2}}, \quad \rho = \sqrt{\frac{(\xi + a^2)(\eta + a^2)}{a^2 - c^2}}. \tag{5.6}$$

We will henceforth be concerned with the half-space $z \geq 0$ only.

B. Ellipsoid situated in a uniform electric field

Assume now that the ellipsoid is placed in a uniform electric field $\mathbf{E}_0$, directed along the z axis. We take the electrostatic potential ϕ to be zero on the ellipsoid surface. With quantities R_ξ and R_η defined as

$$R_\xi = (\xi + a^2)\sqrt{\xi + c^2}, \quad R_\eta = (\eta + a^2)\sqrt{\eta + c^2}, \tag{5.7}$$

the Laplace equation in the external region $\xi \geq 0$ can be written as

$$\nabla^2\phi \equiv \frac{4}{\xi - \eta}\left[\frac{R_\xi}{\xi + a^2}\frac{\partial}{\partial\xi}\left(R_\xi\frac{\partial\phi}{\partial\xi}\right) - \frac{R_\eta}{\eta + a^2}\frac{\partial}{\partial\eta}\left(R_\eta\frac{\partial\phi}{\partial\eta}\right)\right] = 0. \tag{5.8}$$

The potential due solely to $\mathbf{E}_0$ is

$$\phi_0 = -E_0 z, \tag{5.9}$$

and we write the full potential ϕ in the form

$$\phi = \phi_0[1 + F(\xi)], \tag{5.10}$$

so that $\phi_0 F$ denotes the modification due to the ellipsoid. The boundary condition at the surface is $F(0) = -1$.

Inserting Eq. (5.10) into Eq. (5.8) we find the following equation for F,

$$\frac{d^2F}{d\xi^2} + \frac{dF}{d\xi}\frac{d}{d\xi}\ln\left[R_\xi(\xi + c^2)\right] = 0. \tag{5.11}$$

The solution can be written as

$$\phi = \phi_0\left[1 - \frac{\int_\xi^\infty \frac{ds}{(s+c^2)R_s}}{\int_0^\infty \frac{ds}{(s+c^2)R_s}}\right]. \tag{5.12}$$

We can also express the solution in terms of the incomplete beta function, defined as

$$B_x(\alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt. \tag{5.13}$$

Some manipulation yields

$$\int_\xi^\infty \frac{ds}{(s + c^2)R_s} = \frac{1}{(a^2 - c^2)^{3/2}} B_{(a^2-c^2)/(\xi+a^2)}\left(\frac{3}{2}, -\frac{1}{2}\right), \tag{5.14}$$

and so we can write the final answer for the potential as

$$\phi = \phi_0\left[1 - \frac{B_{(a^2-c^2)/(\xi+a^2)}\left(\frac{3}{2}, -\frac{1}{2}\right)}{B_{1-c^2/a^2}\left(\frac{3}{2}, -\frac{1}{2}\right)}\right]. \tag{5.15}$$

For small values of x the following expansion may be useful,

$$B_x(\alpha,\beta) = \frac{x^\alpha}{\alpha}(1-x)^\beta \left[1 + \sum_{n=0}^{\infty} \frac{B(\alpha+1,n+1)}{B(\alpha+\beta,n+1)} x^{n+1}\right], \tag{5.16}$$

where $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the complete beta function. In our case, the limit $x \ll 1$ corresponds to the minor semiaxis c being only slightly less than the major semiaxis a.

In the following, we shall need the expression for the z component of the electric field, $E_z = -\partial\phi/\partial z$, at an arbitrary point (ρ, z) in the exterior region. It is here convenient first to differentiate the relation (5.4) $(u = \xi)$ with respect to z, keeping ρ constant, to obtain

$$\left(\frac{\partial \xi}{\partial z}\right)_\rho = \frac{2(\xi + a^2)}{\xi - \eta}\sqrt{\frac{(\xi + c^2)(\eta + c^2)}{c^2 - a^2}}. \tag{5.17}$$

With $x = (a^2 - c^2)/(\xi + a^2)$ we have

$$\frac{\partial B_x\left(\frac{3}{2},-\frac{1}{2}\right)}{\partial z} = \frac{\partial \xi}{\partial z}\frac{\partial x}{\partial \xi}\frac{\partial B_x\left(\frac{3}{2},-\frac{1}{2}\right)}{\partial x} = 2\frac{(a^2-c^2)}{(\xi+c^2)(\xi-\eta)}(-\eta - c^2)^{1/2}. \tag{5.18}$$

Then, from Eq. (5.15),

$$E_z = E_0\left[1 - \frac{B_{(a^2-c^2)/(\xi+a^2)}\left(\frac{3}{2},-\frac{1}{2}\right)}{B_{1-c^2/a^2}\left(\frac{3}{2},-\frac{1}{2}\right)} - \frac{2(a^2-c^2)^{1/2}(\xi+c^2)^{-1/2}(\eta+c^2)}{B_{1-c^2/a^2}\left(\frac{3}{2},-\frac{1}{2}\right)}\frac{1}{\xi-\eta}\right]. \tag{5.19}$$

For large values of z and arbitrary ρ the influence from the ellipsoid must evidently fade away, $E_z \to E_0$.

In the xy plane where $z = 0, \xi + a^2 = \rho^2$, $\eta + c^2 = 0$, we have

$$E_z(z=0) = E_0\left[1 - \frac{B_{(a^2-c^2)/\rho^2}\left(\frac{3}{2},-\frac{1}{2}\right)}{B_{1-c^2/a^2}\left(\frac{3}{2},-\frac{1}{2}\right)}\right]. \tag{5.20}$$

When $\rho = a$ (on the surface), $E_z(z=0) = 0$ as it should.

C. Force on a dipole

Assume now that a dipole $\mathbf{p} = p_z\hat{\mathbf{z}}$ is situated at rest in the position (ρ, z). The dipole is taken to be polarized in the z direction only. The value of z (≥ 0) is arbitrary, whereas the value of ρ is assumed constant. Thus, writing $\rho = a + L$, L is the constant horizontal distance between the dipole and the edge of the ellipsoid. The force F_z on the dipole is

$$F_z = \nabla_z(\mathbf{p}\cdot\mathbf{E}) = p_z\frac{\partial E_z}{\partial z}. \tag{5.21}$$

Note that we are ignoring the polarization of the ellipsoid by the field of the dipole; the ellipsoid acquires a dipole moment only because of the applied external field. We thus have to differentiate the expression (5.19) with respect to z. Performing the calculation along the same lines as above, we obtain

$$F_z = \frac{6p_z E_0}{B_{1-c^2/a^2}\left(\frac{3}{2},-\frac{1}{2}\right)}\frac{(a^2-c^2)\sqrt{-\eta-c^2}}{(\xi+c^2)(\xi-\eta)} \times\left[1-\frac{(\xi+a^2)(-\eta-c^2)}{(a^2-c^2)(\xi-\eta)}+\frac{2}{3}\frac{(\xi+c^2)(\eta+c^2)(\xi+\eta+2a^2)}{(a^2-c^2)(\xi-\eta)^2}\right]. \tag{5.22}$$

At $z=0$, the force vanishes as it should, since $\eta+c^2=0$ then.

Note that the force vanishes if $c/a\to 0$, that is, for a disk, because the integral representing the incomplete beta function diverges in the limit. (It is not to be interpreted as its analytic continuation.) This is not surprising, for in the limit of a disk, the electric field is just $\mathbf{E}_0$, the applied constant field. This is because inserting a perfectly conducting sheet perpendicular to the field line has no effect on the boundary conditions. See also the discussion in Chap. 4 of Ref. [23].

As a small check, we consider the limit of a sphere, $c^2\to a^2$. Then, according to Eq. (5.16), we have

$$B_{1-c^2/a^2}\left(\frac{3}{2},-\frac{1}{2}\right)\to\frac{2}{3}a^{-3}(a^2-c^2)^{3/2}, \tag{5.23}$$

and

$$\xi\approx\rho^2+z^2-c^2,\quad \eta=-c^2-\frac{\delta^2 z^2}{\rho^2+z^2}, \tag{5.24}$$

in terms of the ultimately vanishing quantity $\delta^2=a^2-c^2$. Then we immediately obtain

$$F_z=3p_zE_0\frac{a^3 z}{(\rho^2+z^2)^{7/2}}(3\rho^2-2z^2). \tag{5.25}$$

This result also follows immediately from the dipole-dipole interaction energy

$$U=-\frac{1}{r^5}(3\mathbf{r}\cdot\mathbf{p}_1\,\mathbf{r}\cdot\mathbf{p}_2-r^2\mathbf{p}_1\cdot\mathbf{p}_2), \tag{5.26}$$

when we take

$$\mathbf{p}_1=p_z\hat{\mathbf{z}},\quad \mathbf{p}_2=a^3E_0\hat{\mathbf{z}}. \tag{5.27}$$

The force on the sphere (5.25) is attractive at large distance, because the dipoles become essentially coaxial then, and repulsive at small distance, because the case of parallel dipoles in a plane is approached in that situation.

The same features hold for a general ellipsoid. For short distances, $z^2 \ll \rho^2 - a^2 + c^2$, we have

$$\xi = \rho^2 - a^2 + O(z^2), \quad \eta = -c^2 - \frac{z^2(a^2-c^2)}{\rho^2 - a^2 + c^2} + O(z^4), \tag{5.28}$$

and then the force is repulsive,

$$z \to 0: \quad F_z = \frac{6p_z E_0}{B_{1-c^2/a^2}(3/2,-1/2)} \frac{z(a^2-c^2)^{3/2}}{(\rho^2-a^2+c^2)^{5/2}}, \tag{5.29}$$

which reduces in the spherical case to

$$c \to a: \quad F_z = \frac{9p_z E_0 a^3 z}{\rho^5}, \tag{5.30}$$

which agrees with Eq. (5.25). And in the large distance limit, where $\xi \approx z^2$, $\eta \approx -a^2$, the force in general is attractive,

$$z \to \infty: \quad F_z = -\frac{4p_z E_0 (a^2-c^2)^{3/2}}{B_{1-c^2/a^2}(3/2,-1/2)} \frac{1}{z^4}, \tag{5.31}$$

which again has the expected limit,

$$c \to a: \quad F_z = -\frac{6p_z E_0 a^3}{z^4}. \tag{5.32}$$

VI. INTERACTION OF ANISOTROPIC ATOM WITH ANISOTROPIC DIELECTRIC

In view of the considerations of Sec. II, we might hope that repulsion could be achieved if an anisotropic atom were placed above an anisotropic dielectric medium. Consider such an atom, with polarizability only in the z direction, $\boldsymbol{\alpha} = \alpha\hat{\mathbf{z}}\hat{\mathbf{z}}$, a distance a above a dielectric with different permittivities in the z direction and the transverse directions,

$$\boldsymbol{\varepsilon} = \mathrm{diag}(\varepsilon_\perp, \varepsilon_\perp, \varepsilon_\parallel). \tag{6.1}$$

We will assume (see below) that $\varepsilon_\perp$, $\varepsilon_\parallel > 1$. The Casimir-Polder interaction is

$$E_{\rm CP} = -\alpha \int_{-\infty}^{\infty} d\zeta \left(\Gamma_{zz} - \Gamma^0_{zz}\right)(\mathbf{R},\mathbf{R}), \tag{6.2}$$

where the atom is located at $\mathbf{R} = (0,0,a)$. Here we have subtracted the free-space contribution. We can write the Green's dyadic in terms of a transverse Fourier transform,

$$\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}') = \int \frac{(d\mathbf{k}_\perp)}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')_\perp} \boldsymbol{\gamma}(z,z'), \tag{6.3}$$

where (assuming that $\mathbf{k}_\perp$ lies in the $+x$ direction)

$$\boldsymbol{\gamma}(z,z') = \begin{pmatrix} \frac{1}{\varepsilon_\perp}\frac{\partial}{\partial z}\frac{1}{\varepsilon'_\perp}\frac{\partial}{\partial z'}g^H & 0 & \frac{ik_\perp}{\varepsilon_\perp\varepsilon'_\parallel}\frac{\partial}{\partial z}g^H \\ 0 & -\zeta^2 g^E & 0 \\ -\frac{ik_\perp}{\varepsilon'_\perp\varepsilon_\parallel}\frac{\partial}{\partial z'}g^H & 0 & \frac{k_\perp^2}{\varepsilon_\parallel\varepsilon'_\parallel}g^H \end{pmatrix}. \tag{6.4}$$

We have followed Ref. [24] and used the notation $\varepsilon = \varepsilon(z)$, $\varepsilon' = \varepsilon(z')$. Here we have omitted δ-function terms that do not contribute in the point-splitting limit. The transverse electric and transverse magnetic Green's functions satisfy the differential equations

$$\left(-\frac{\partial^2}{\partial z^2} + k_\perp^2 - \omega^2\varepsilon_\perp\right) g^E(z,z') = \delta(z-z'), \tag{6.5a}$$

$$\left(-\frac{\partial}{\partial z}\frac{1}{\varepsilon_\perp}\frac{\partial}{\partial z} + \frac{k_\perp^2}{\varepsilon_\parallel} - \omega^2\right) g^H(z,z') = \delta(z-z'). \tag{6.5b}$$

It is rather straightforward to solve these equations and find the Casimir-Polder energy:

$$E_{\rm CP} = \frac{\alpha}{4\pi^2}\int_{-\infty}^{\infty} d\zeta \int (d\mathbf{k}_\perp)\frac{k_\perp^2}{2\kappa}\frac{\bar\kappa-\kappa}{\bar\kappa+\kappa}e^{-2\kappa a}, \tag{6.6}$$

where $\kappa^2 = k_\perp^2 - \omega^2$, $\bar\kappa = \sqrt{(k_\perp^2 - \omega^2\varepsilon_\parallel)/\varepsilon_\perp\varepsilon_\parallel}$. Checks of this result are the following:

$$\varepsilon_\perp \to \infty: \quad E_{\rm CP} \to -\frac{\alpha}{8\pi a^4}, \tag{6.7}$$

one-third of the usual Casimir-Polder interaction of an isotropic atom with a perfect conducting plate. This is what we would have for such an anisotropic atom above a isotropic conducting plate, because taking $\varepsilon_\perp \to \infty$ imposes the usual boundary condition that the tangential components of $\mathbf{E}$ vanish on the surface. In the other limit, we have no such simple correspondence,

$$\varepsilon_\parallel \to \infty: \quad E_{\rm CP} \to \frac{\alpha}{8\pi a^4}\left(1 + \frac{3}{2}\sqrt{\varepsilon_\perp} - 3\varepsilon_\perp + 3\sqrt{\varepsilon_\perp}(\varepsilon_\perp - 1)\ln\frac{\sqrt{\varepsilon_\perp}+1}{\sqrt{\varepsilon_\perp}}\right), \tag{6.8}$$

where the quantity in parentheses varies between $-1/2$ for $\varepsilon_\perp = 1$ and -1 as $\varepsilon_\perp \to \infty$.

We can check that in all cases, if we ignore dispersion, Eq. (6.6) yields an attractive result: $E_{\rm CP}$ scales like a^{-4} times a numerical integral which is always negative because $\bar\kappa^2 - \kappa^2 < 0$. Repulsion does not occur in this case because there is no breaking of translational invariance in the transverse direction.

In fact, the electromagnetic force density in an anisotropic nonmagnetic medium is (see Ref. [25], Eq. (1.2a))

$$\mathbf{f} = -\frac{1}{8\pi}E_i E_k \nabla\varepsilon_{ik}. \tag{6.9}$$

Assume that the single air-medium interface is flat, lying in the xy plane. Then the only nonvanishing component of the gradient $\nabla\varepsilon_{ik}$ is the vertical component $\partial_z\varepsilon_{ik}$. If the principal coordinate axes for ε_{ij} coincide with the x, y, z axes, then the surface force density $\int f_z\,dz$ (which is subsequently to be integrated across the surface $z = 0$), is directed upwards, because $\varepsilon_{\|,\perp} > 1$. The surface force acts in the direction of the optically thinner medium. Now, momentum conservation of the total system asserts that the force on a dipole above the surface acts in the downward direction. The dipole force has to be attractive.

That $\varepsilon > 1$ for an isotropic medium is a thermodynamical result. For an anisotropic medium, oriented such that the coordinate axes fall together with the crystallographic axes, one must analogously have $\varepsilon_{\|,\perp} > 1$. See, for instance, Sec. 14 in Ref. [26].

Note the contrast with the force on a dipole outside a dielectric wedge, studied in Ref. [13]. In the latter case, the normal surface force on the inclined (lower) surface necessarily has a vertical (z) component that is downward directed. Momentum conservation for the total system thus no longer forbids the force on the dipole to be repulsive.

VII. CONCLUSIONS

Earlier, we observed that Casimir-Polder repulsion along a direction perpendicular to the symmetry axis of a semi-infinite planar conductor or a conducting wedge and an anisotropically polarizable atom could be achieved in the region close to the conductor [13]. Here we have shown that anisotropically polarizable atoms can also repel in this sense, provided they are sufficiently anisotropic, and have perpendicular principal axes. We further show that such an atom may be repelled by a conducting cylinder, provided, at closest approach, it is sufficiently far away from the cylinder, whereas no such phenomenon occurs for a sphere and an anisotropic atom. We further discuss a new example of classical repulsion by considering a polarized ellipsoid interacting with a dipole. On the other hand, a system of an anisotropically polarizable atom interacting via fluctuation forces with an anisotropic dielectric half-space does not exhibit repulsion. Apparently, spatial anisotropy is also required for repulsion between electric bodies.

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[1] I. D. Dzyaloshinskii, E. M. Lifshitz, and L. P. Pitaevskii, Usp. Fiz. Nauk **73**, 381 (1961) [Engl. transl.: Sov. Phys. Usp. **4**, 153 (1961)].

[2] M. Levin, A. P. McCauley, A. W. Rodriguez, M. T. Homer Reid, and S. G. Johnson, Phys. Rev. Lett. **105**, 090403 (2010) [arXiv:1003.3487].

[3] J. Munday, F. Capasso, and V. A. Persegian, Nature **457**, 170 (2009).

[4] A. W. Rodriguez, A. P. McCauley, D. Woolf, F. Capasso, J. D. Joannopoulos, and S. G. Johnson, Phys. Rev. Lett. **104**, 160402 (2010) [arXiv:0912.2243].

[5] T. H. Boyer, Phys. Rev. **174**, 1764 (1968).

[6] C. M. Bender and K. A. Milton, Phys. Rev. D **50**, 6547 (1994) [arXiv:hep-th/9406048].

[7] K. A. Milton, Phys. Rev. D **55**, 4940 (1997) [arXiv:hep-th/9611078].

[8] E. K. Abalo, K. A. Milton and L. Kaplan, Phys. Rev. D **82**, 125007 (2010) [arXiv:1008.4778 [hep-th]].

[9] E. K. Abalo, K. A. Milton and L. Kaplan, in preparation, and arXiv:1112.0079.

[10] T. H. Boyer, Phys. Rev. A **9**, 2078 (1974).

[11] A. P. McCauley, Rongkuo Zhao, M. T. Homer Reid, A. W. Rodriguez, J. Zhou, F. S. S. Rosa, J. D. Joannopoulos, D. A. R. Dalvit, C. M. Soukoulis, and S. G. Johnson, Phys. Rev. B **82**, 165108 (2010) [arXiv:1006.5489].

[12] M. F. Maghrebi, Phys. Rev. D **83**, 045004 (2011) [arXiv:1012.1060].

[13] K. A. Milton, E. K. Abalo, P. Parashar, N. Pourtolami, I. Brevik and S. Å. Ellingsen, Phys. Rev. A **83**, 062507 (2011) [arXiv:1103.4386 [hep-th]].

[14] D. P. Craig and E. A. Power, Chem. Phys. Lett. **3**, 195 (1969).

[15] D. P. Craig and E. A. Power, Int. J. Quantum Chem. **3**, 903 (1969).

[16] K. A. Milton, P. Parashar and J. Wagner, in *The Casimir Effect and Cosmology*, ed. S. D. Odintsov, E. Elizalde, and O. B. Gorbunova, in honor of Iver Brevik (Tomsk State Pedagogical University) pp. 107-116 (2009) [arXiv:0811.0128].

[17] K. V. Shajesh and M. Schaden, arXiv:1112.1348.

[18] V. B. Bezerra, E. R. B. de Mello, G. L. Klimchitskaya, V. M. Mostepanenko and A. A. Saharian, Eur. Phys. J. C **71**, 1614 (2011) [arXiv:1103.3922 [hep-th]].

[19] J. Schwinger, L. L. DeRaad, Jr., K. A. Milton, and W.-y. Tsai, *Classical Electrodynamics* (Perseus/Westview, New York, 1998).

[20] G. Feinberg and J. Sucher, J. Chem. Phys. **48**, 3333 (1968).

[21] M. Levin and S. Johnson, Am. J. Phys. **79**, 843 (2011) .

[22] N. Graham, A. Shpunt, T. Emig, S. J. Rahi, R. L. Jaffe, and M. Kardar, Phys. Rev. D **83**, 125007 (2011) [arXiv:1103.5942 [quant-ph]].

[23] K. A. Milton and J. Schwinger, *Electromagnetic Radiation: Variational Methods, Waveguides and Accelerators* (Springer, Berlin, 2006).

[24] J. Schwinger, L. L. DeRaad, Jr., and K. A. Milton, Ann. Phys. (N.Y.) **115**, 1–23 (1978).

[25] I. Brevik, Physics Reports **52**, 133 (1979).

[26] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, (Pergamon, Oxford, 1984).