Hard collinear gluon radiation and multiple scattering in a medium

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Hard collinear gluon radiation and multiple scattering in a medium

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The energy loss of hard jets produced in the Deep-Inelastic scattering (DIS) off a large nucleus is considered in the collinear limit. In particular, the single gluon emission cross section due to multiple scattering in the medium is calculated. Calculations are carried out in the higher-twist scheme, which is extended to include contributions from multiple transverse scatterings on both the produced quark and the radiated gluon. The leading length enhanced parts of these power suppressed contributions are resummed. Various interferences between such diagrams lead to the Landau-Pomeranchuk-Migdal (LPM) effect. We resum the corrections from an arbitrary number of scatterings and isolate the leading contributions which are suppressed by one extra power of the hard scale Q^2. All powers of the emitted gluon forward momentum fraction y are retained. We compare our results with the previous calculation of single scattering per emission in the higher-twist scheme as well as with multiple scattering resummations in other schemes. It is found that the leading $(1/Q^2)$ contribution to the double differential gluon production cross section, in this approach, is equivalent to that obtained from the single scattering calculation once the transverse momentum of the final quark is integrated out. We comment on the generalization of this formalism to Monte-Carlo routines.

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I. INTRODUCTION

The modification of hard jets in dense media [1–3], and the use of this modification to measure the partonic structure of both confined or deconfined matter is now a considerably mature science [4]. While the modification of full jet structure is only now being measured, the modification of the yield of leading hadrons, (a reduction or suppression of the yield) has been measured by the HERMES experiment at DESY for Deep-Inelastic Scattering (DIS) on a large nucleus [5] and by both the STAR and PHENIX detectors [6] at the Relativistic Heavy Ion Collider (RHIC) for jets in heavy-ion collisions. On the theory side, there now exist at least four sophisticated jet modification schemes which describe the single particle yields within a pQCD based phenomenology. It has now become customary to refer to these as the Gyulassy-Levai-Vitev (GLV) scheme [7], the Arnold-Moore-Yaffe scheme (AMY) [8], the Armesto-Salgado-Wiedemann scheme [9] and the higher-twist (HT) approach [10–13]. Of these, the GLV, ASW and HT approaches have also been applied to the case of DIS on a large nucleus [14–16] using almost an identical formalism to that used in the case of hot matter. All these calculations obtained a much smaller value of the leading transport coefficient in cold nuclear matter than for heavy-ion collisions. This represents one of the leading pieces of evidence for the formation of a new type of matter in high energy heavy-ion collisions.

The transport coefficient in question, $\hat{q}$, defined as the transverse momentum squared gained by a hard parton per unit length, has been identified as the leading effect of the medium on a hard parton [2, 12] for the case of light flavor suppression. In the case of DIS on a nucleus, the $\hat{q}$, assumed to scale with the nucleon density, is a constant, as a function of time, over most of the space through which the parton traverses (ignoring edge effects in the Woods-Saxon distribution). The case of heavy-ion collisions requires more sophistication: $\hat{q}$ is usually assumed to scale with an intrinsic quantity such as the entropy density $s \propto T^3$ (T is the local temperature). In a dynamically expanding medium, T depends on both the location in space and time. Hence a complete calculation requires an underlying realistic model of the medium. Three of the formalisms (ASW, AMY and HT) have been extended to calculations where the evolution of the medium is treated with a 3-D relativistic fluid dynamical simulation [17].

Current experiments have, however, progressed far beyond a measurement of light flavor single particle inclusive suppression and now include measurements of heavy-flavor suppression and azimuthal anisotropy [18, 19] as well as triggered distributions of associated particles. Theory calculations have also progressed, especially in the case of two particle correlations. While the GLV [14, 20], ASW [21] and AMY [22] approaches have attempted to describe the away side associated yield using a leading order partonic cross section, the HT has incorporated a Next-to-Leading Order (NLO) hard cross section [23]. Along with this, the HT approach has also remained the sole formalism to incorporate multi-hadron fragmentation functions in an effort to explain the near side associated yield [24].

In spite of these developments, the HT approach still suffers from the disadvantage that, unlike the ASW and the AMY schemes, it is a single scattering scheme. In its original form [10], the HT approach only contained one scattering and one emission in the medium. In subsequent efforts [25], the HT approach was extended to include multiple emissions by iterating the single-scattering-single-emission kernel by means of a Dokshitzer-Gribov-Lipatov-Altarelli-Parisi

(DGLAP) [26] evolution equation. While this does include the case where a hard parton or its radiated gluons scatter multiple times, these scatterings are included incoherently, i.e., there is at most one scattering within the formation time of a single produced gluon. The applicability of this assumption can be quickly estimated: imagine a typical high energy DIS event (say at the HERMES experiment [27]) with a $Q^2 \sim 10\text{GeV}^2$ and a forward energy $\nu \sim 30\text{GeV}$. The produced quark is off-shell by at most Q^2 and this depreciates to a non-perturbatively small value in a series of iterative steps. Consider a step where the drop is $\Delta Q \sim 2\text{GeV}$ with a momentum fraction of 1/2, the formation length of this radiation is,

$$\tau_f \sim \frac{\nu}{2(\Delta Q)^2} \sim 1\text{fm}. \tag{1}$$

Assuming that there is at most one scattering per nucleon, given that double scattering in a nucleon involves twist-4 matrix elements, and there is approximately one nucleon per fm of length, this particular emission will encounter one scattering during the formation of the radiated gluon. However, for smaller drops in Q^2, smaller initial values of Q^2 and higher energies ν, this simple approximation breaks down. The limit of at most one scattering per radiation cannot hold in the case of jets at RHIC either, as the density of scatterers is much larger than in cold nuclear matter. Such considerations require the extension of the HT scheme to include multiple coherent scatterings per emitted gluon. It is the object of this paper to carry out this extension.

This paper completes the efforts begun in Refs. [12, 13] by computing the single gluon emission cross section from multiple scattering of a hard quark in a dense medium. The prior efforts included the simpler problems of evaluating the all twist contribution to the propagation of a hard quark produced in DIS on a large nucleus without gluon radiation and also the emission of electromagnetic radiation from this hard quark. Similar to the case of photon radiation at all twist in Ref. [13], we commence by computing and studying the triple differential distribution to split the struck hard quark into a quark and a gluon separated by a transverse momentum $\vec{l}_{q\perp} + l_\perp$. The gluon carries a large transverse momentum $l_\perp$ and a fraction y of the original quark light-cone momentum, and the outgoing quark carries a similar large transverse momentum $l_{q,\perp}$ and a fraction $1-y$ of the original quark momentum. In the absence of scattering $l_{q,\perp} = -l_\perp$. Physically meaningful expressions will be derived in this notation. We then compute the integral over all these variables to compute the additive contribution of scattering induced gluon emission to the hadronic tensor. In order to aid in the resummation of the multiple scattering calculation, we introduce a slight shift in notation by setting $l_{q,\perp} \to l_{q,\perp} - l_\perp$. As a result, beyond Eq. (79) $l_{q_\perp}$ will represent the small net transverse momentum of the quark-gluon pair. The effect of multiple scattering on the $l_{q,\perp}$ distribution is then resummed to all orders while the effect on the distribution of the much larger $l_\perp$ is considered term-by-term in a power series expansion in inverse powers of $l_\perp^2$.

Such a separation of distributions is sensible in the HT approach as we explicitly consider the case where $l_\perp^2$, the squared transverse momentum of the radiated gluon, limited by the scale of the hard interaction Q^2, is considered to be much larger than $k_\perp^2$, the squared transverse momentum of the exchanged or scattering gluons. This choice of scale separation places this calculation in a different part of kinematic space than that explored by GLV, ASW or AMY, where no such scale separation is assumed. In Sect. VI we will discuss comparisons with ASW and AMY: these are meant merely as diagrammatic comparisons and will not imply a comparison in the region of kinematic space explored by these other formalisms.

In a real experiment, the size of the electromagnetic fine structure constant α_{EM} ensures that multiple hard photon bremsstrahlung is suppressed; however, given the size of α_s at the typical scales involved, the produced quark will radiate gluons multiple times. In this effort, only the differential single gluon emission spectrum will be calculated. The inclusion of multiple emissions, which involves an iteration of the single gluon emission kernel, followed by the eventual numerical calculation and comparison with experimental data, will be presented in a separate effort.

The paper is organized as follows: Sec. II will introduce the basic notation by evaluating the leading twist contribution to the differential collinear gluon radiation spectrum from a hard quark produced in DIS on a large nucleus. Sec. III will identify an inclusive class of diagrams that yield leading contributions to the all-twist gluon emission spectrum. The collinear approximation will be invoked, integrations over the irrelevant momenta will be carried out followed by simplifications in the numerator structure. Sec. IV will carry out the various pole integrations and identify the source of the Landau-Pomeranchuk-Migdal effect. Sec. V will include discussions of the color factors and length enhancements of different classes of all-twist diagrams. Sec. VI will evaluate the all-twist contributions in two different limits and compare with the calculations in the ASW and AMY formalisms. In Sec. VII we present our concluding discussions.

II. LEADING TWIST AND COLLINEAR GLUON RADIATION

In this section we calculate the next-to-leading order correction to semi-inclusive DIS on a large nucleus with a quark in the final state. By next-to-leading order, we simply mean including one interaction term in the amplitude and complex conjugate, which converts a single quark in to a quark and a gluon. As in the previous efforts [12, 13] we will assume and refrain from an attempt to discuss factorization [28, 29]. We re-derive this straightforward result to familiarize the reader with our conventions and the choice of gauge.

Consider the semi-inclusive process of DIS off a nucleus in the Breit frame where one quark with a transverse momentum $l_{q_\perp}$ and a bremsstrahlung gluon with transverse momentum $l_\perp$ are produced (see Fig. 1),

$$\mathcal{L}(L_1) + A(p) \longrightarrow \mathcal{L}(L_2) + q(l_{q_\perp}) + G(l_\perp) + X. \quad (2)$$

In the above equation, L_1 and L_2 represent the momentum of the incoming and outgoing leptons. The incoming nucleus of atomic mass A is endowed with a momentum Ap. In the final state, all high momentum hadrons $(h_1, h_2, ...)$ with momenta $p_1, p_2, \ldots$ are detected and their momenta summed to obtain the quark jet momentum (l_q) and gluon jet momentum (l) and X denotes that the process is semi-inclusive.

Our choice of light-cone component notation for four vectors $(p \equiv [p^+, p^-, \vec{p}_\perp])$ is somewhat different from the regular notation, i.e.,

$$p^+ = \frac{p^0 + p^3}{2}; \;\; p^- = p^0 - p^3. \quad (3)$$

In spite of the asymmetric definition, the formal structure of all dot products will be identical to the standard notation. The kinematics is defined in the Breit frame as sketched in Fig. 1. In such a frame, the incoming virtual photon γ^* and the nucleus have momentum four vectors q, P_A given as,

$$q = L_2 - L_1 \equiv \left[\frac{-Q^2}{2q^-}, q^-, 0, 0\right], \qquad P_A \equiv A[p^+, 0, 0, 0].$$

In this frame, the Bjorken momentum fraction of the incoming quark with respect to its parent nucleon which has the average momentum $[p^+, 0, 0, 0]$ is given as $x_B = Q^2/2p^+q^-$. After the scattering with the virtual photon, the final state quark has a four momentum $\sim [0, q^-, 0, 0]$ i.e., a large momentum in the negative z or negative light cone direction. This outgoing quark will radiate a gluon. The radiated gluon has a transverse momentum of $l_\perp$ and carries a fraction y of the forward momentum q^- of the quark originating in the hard scattering, *i.e.*,

$$y = \frac{l^-}{q^-}. \quad (4)$$

The double differential cross section of the semi inclusive process with a final state quark (which will eventually become a jet) with transverse momentum $l_{q_\perp}$ and a final state gluon with transverse momentum $l_\perp \gg \Lambda_{QCD}$ may be expressed as

$$\frac{E_{L_2} d\sigma}{d^3L_2 d^2l_{q_\perp} d^2l_\perp dy} = \frac{\alpha_{em}^2}{2\pi s Q^4} L_{\mu\nu} \frac{dW^{\mu\nu}}{d^2l_{q_\perp} d^2l_\perp dy}, \quad (5)$$

where $s = (p + L_1)^2$ is the total invariant mass of the lepton nucleon system. In the single photon exchange

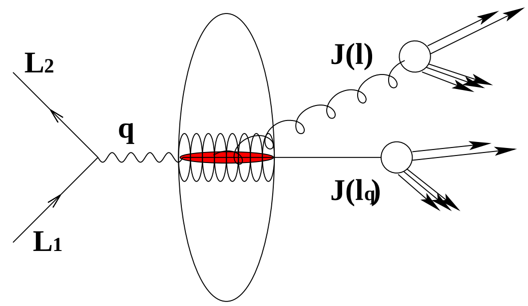

FIG. 1: The Lorentz frame chosen for the process where a nucleon in a large nucleus is struck by a hard space-like photon leading to the production of an outgoing parton and a radiated gluon.

approximation, the leptonic part of the cross section is described by the particular form of the leptonic tensor denoted as $L^{\mu\nu}$ given as,

$$L_{\mu\nu} = \frac{1}{2}\mathbf{Tr}[\not{L}_1 \gamma_\mu \not{L}_2 \gamma_\nu]. \quad (6)$$

In the notation used in this paper, $|A; p\rangle$ represents the spin averaged initial state of an incoming nucleus with A nucleons with a momentum p per nucleon. The general final hadronic or partonic state is defined as $|X\rangle$. As a result, the semi-inclusive hadronic tensor in the nuclear state $|A\rangle$ may be defined as

$$\begin{aligned} W^{A^{\mu\nu}} &= \sum_X (2\pi)^4 \delta^4(q + P_A - p_X) \\ &\times \langle A; p | J^\mu(0) | X \rangle \langle X | J^\nu(0) | A; p \rangle \\ &= 2\text{Im}\left[\int d^4 y_0 e^{iq\cdot y_0} \langle A; p | J^\mu(y_0) J^\nu(0) | A; p \rangle\right] \quad (7) \end{aligned}$$

In the equation above, the sum $(\sum_X)$ runs over all possible hadronic states and $J^\mu = Q_q \bar{\psi}_q \gamma^\mu \psi_q$ is the hadronic electromagnetic current, where, Q_q is the charge of a quark of flavor q in units of the positron charge e. It is understood that the quark operators are written in the interaction picture and two factors of the electromagnetic coupling constant α_{em} have already been extracted and included in Eq. (5). The leptonic tensor will not be discussed further. The focus in the remaining shall lie exclusively on the hadronic tensor. This tensor will be expanded order by order in a partonic basis with one on-shell gluon in the final state and leading twist and maximally length enhanced higher twist contributions will be isolated.

The leading twist contribution is obtained by expanding the products of currents to next-to-leading order in α_s to account for the radiated gluon. This contribution may be expressed diagrammatically as in Fig. 1. This represents the process where a hard quark, produced from one nucleon in a deep-inelastic scattering on a nucleus, proceeds to radiate a hard gluon and then exits the nucleus without further interaction. Diagrams where the hard gluon is emitted from the incoming quarks are suppressed in our choice of light cone gauge. In the following, we analyze this contribution in some detail. Indeed, there is no new information presented in the current section and the discussion of such contributions is now well established [26]. The approximations made in this section will form the basis for the analysis of single inclusive gluon radiation at all twist. The semi-inclusive hadronic tensor may be expressed as,

$$
\begin{aligned}
W_0^{A\mu\nu} &= AC_p^A W_0^{\mu\nu} \quad (8) \\
&= AC_p^A \int d^4y_0 \langle p|\bar{\psi}(y_0)\gamma^\mu \widehat{\mathcal{O}^{00}} \gamma^\nu \psi(0)|p\rangle \\
&= C_p^A \int d^4y_0 \mathbf{Tr}[\frac{\gamma^-}{2}\gamma^\mu \frac{\gamma^+}{2}\gamma^\nu] F(y_0)\mathcal{O}^{00}(y_0).
\end{aligned}
$$

In the above equation, C_p^A expresses the probability to find a nucleon state with momentum p inside a nucleus with A nucleons. In the collinear limit, the incoming parton is assumed to be endowed with very high forward momentum ($p_0{}^+ = x_0 p^+, p_0^- \to 0$) with negligible transverse momentum $p_{0\perp} \ll p_0^+$. Within the kinematics chosen, the incoming virtual photon also has no transverse momentum. As a result, the produced final state parton also has a vanishingly small transverse momentum (*i.e.*, with a distribution $\delta^2(\vec{p}_\perp)$). In this limit, the leading spin projection of the pieces which represent the initial state and final state may be taken. The factors,

$$
\gamma^+ = \frac{\gamma^0 + \gamma^3}{2} \;\; ; \quad \gamma^- = \gamma^0 - \gamma^3, \quad (9)
$$

are used to obtain the spin projections along the leading momenta of the outgoing state and the incoming state. The coefficients of these projections are the two functions,

$$
F(y_0) = A\langle p|\bar{\psi}(y_0)\frac{\gamma^+}{2}\psi(0)|p\rangle \quad (10)
$$

and (in a notation where the superscript on the operator $\mathcal{O}^{00}$ implies that the quark undergoes no scattering in the initial or final state)

$$
\begin{aligned}
\mathcal{O}^{00} &= \mathbf{Tr}\left[\frac{\gamma^-}{2}\widehat{\mathcal{O}^{00}}\right] \quad (11) \\
&= \int \frac{d^4l}{(2\pi)^4} d^4z d^4z' \frac{d^4l_q}{(2\pi)^4}\frac{d^4p_0}{(2\pi)^4}\frac{d^4p_0'}{(2\pi)^4} \\
&\times \mathbf{Tr}\left[\frac{\gamma^-}{2}\frac{-i(\not{p}_0 + \not{q})}{(p_0+q)^2 - i\epsilon} i\gamma^\alpha \not{l}_q 2\pi\delta(l_q^2)\right. \\
&\times \left. G_{\alpha\beta}(l) 2\pi\delta(l^2)(-i\gamma^\beta)\frac{i(\not{p}_0' + \not{q})}{(p_0'+q)^2 + i\epsilon}\right] \\
&\times e^{iq\cdot y_0} e^{-i(p_0+q)\cdot(y_0-z)} e^{-il\cdot(z-z')} e^{-il_q\cdot(z-z')} \\
&\times e^{-i(p_0'+q)\cdot z'} g^2.
\end{aligned}
$$

Integrating over z and z' yields the two four-dimensional δ-functions: $\delta^4(p_0 + q - l - l_q)$ and $\delta^4(p_0 - p_0')$.

The reader will have noted that we have ignored various projections such as those which arise from the ($\perp$)-components of the γ matrices, *e.g.*,

$$
C_P^A \mathbf{Tr}\left[\frac{\gamma^{\perp_i}}{2}\gamma^\mu \frac{\gamma^{\perp_j}}{2}\gamma^\nu\right] F_{\perp_i}(y_0)\mathcal{O}^{00}_{\perp_j}. \quad (12)
$$

This approximation may be justified in the high energy, collinear limit $l_\perp^2/y << Q^2$ where such contributions are suppressed compared to those of Eq. (8). As pointed out, in this effort, we will often adjudicate the importance of different terms using scaling arguments inspired by SCET. The kinematic regime explored in this article is $l_\perp \sim \lambda Q$ and both $y, (1-y) \lesssim 1$. In this region, $l_\perp^2/y \sim \lambda^2 Q^2 \ll Q^2$. Another interesting regime is the soft y limit where $y \sim \lambda$; even in this region one may ignore the non-transverse projections of the virtual photon. In so doing, the focus of the remainder of this article has been limited to projections where the incoming virtual photon is transverse.

The on-shell δ-function over l is used to set $l^+ = l_\perp^2/2l^-$. The other on-shell δ-function, instills the condition,

$$
\begin{aligned}
\delta(l_q^2) &= \delta[(p_0 + q - l)^2] \\
&\simeq \delta[-Q^2 + 2p_0^+(q^- - l^-) - 2q^+l^- - 2q^-l^+] \\
&= \frac{1}{2p^+q^-}\delta\left[x_0(1-y) - x_B(1-y) - \frac{l_\perp^2}{2p^+q^-y}\right] \\
&= \frac{\delta[x_0 - x_B - x_L]}{2p^+q^-(1-y)}, \quad (13)
\end{aligned}
$$

where, the collinear condition that $p_{0_\perp} \to 0$ has been used to simplify the final equation. The new *momentum fraction* x_L has been introduced:

$$
x_L = \frac{l_\perp^2}{2p^+q^-y(1-y)} = \frac{1}{p^+\tau_f} \sim \lambda^2, \quad (14)
$$

where y has already been defined as the momentum fraction of the radiated photon (l^-/q^-) and $\tau_f \sim 1/(\lambda^2 Q)$ is the formation time of the radiated gluon.

The factor $G_{\alpha\beta}(l)$ in Eq. (11) represents the radiated photon's spin sum. In this effort, the light cone gauge ($A^- = 0$) will be assumed, *i.e.*,

$$
G_{\alpha\beta}(l) \;=\; -g^{\mu\nu} + \frac{l^\mu n^\nu + l^\nu n^\mu}{n\cdot l}, \quad (15)
$$

where we have introduced the light cone vector $n \equiv [1,0,0,0]$ which yields $l\cdot n = l^-$. Note that with this choice of gauge, the largest component of the vector potentials from the initial states may still be regarded as the (+)-components.

Substituting the above simplifications in Eq. (11), leads to the simplified form for the final state projection:

$$\begin{aligned}
\mathcal{O}^{00} &= \int \frac{d^4p_0}{(2\pi)^4}\frac{dl^- d^2l_\perp}{(2\pi)^3 2l^-} g^2 e^{-ip_0\cdot y_0} \\
&\times \mathbf{Tr}\left[\frac{\gamma^-}{2}\frac{\gamma^+ q^-}{2p^+q^-(x_0 - x_B - x_{D0} - i\epsilon)}\right. \\
&\times \left\{\gamma_\perp^\alpha \gamma^-([x_0 - x_B]p^+ - l^+)\gamma_\perp^\beta(-g_{\perp\alpha\beta})\right. \\
&- \frac{\gamma_\perp\cdot l}{l^-}\gamma_\perp\cdot l\gamma^- - \gamma^-\gamma_\perp\cdot l\frac{\gamma_\perp\cdot l}{l^-} \\
&+ \left.\gamma^-\gamma^+(q^- - l^-)\gamma^-\frac{2l^+}{l^-}\right\} \\
&\times \left.\frac{\gamma^+ q^-}{2p^+q^-(x_0 - x_B - x_{D0} + i\epsilon)}\right] \\
&\times 2\pi\frac{\delta[x_0 - x_B - x_L]}{2p^+q^-(1-y)}. \qquad (16)
\end{aligned}$$

The δ-function may be used to carry out the $p_0^+ = x_0p^+$ integral; the absence of p_0^- and $p_{0_\perp}$ from the integrands allows for these integrals to be carried out and constrain the locations $y_0^+, y_{0_\perp}$ to the origin. Further simplifications may be carried out by noting that $\gamma^\pm$ anticommutes with $\gamma_\perp$, while $\{\gamma^+,\gamma^-\} = 2\mathbf{1}$ (*i.e.*, twice the unit matrix in spinor space) and $\{\gamma^\pm,\gamma^\pm\} = 0$. Replacing $l^- = q^-y$, in Eq. (16), one obtains,

$$\begin{aligned}
\mathcal{O}^{00} &= \delta(y_0^+)\delta^2(y_{0_\perp})\int\frac{dyd^2l_\perp}{(2\pi)^3 2y}e^{-i(x_B+x_L)p^+y_0^-}p^+ \\
&\times \frac{g^2 4(q^-)^2}{(2p^+q^-)^2 4p^+q^-(1-y)x_L^2} \\
&\times \left[2(x_Lp^+ - l^+) + \frac{2l_\perp^2}{l^-} + 2(1-y)\frac{2l^+}{y}\right] \\
&= \delta(y_0^+)\delta^2(y_{0_\perp})\frac{\alpha_s}{2\pi}\int\frac{dydl_\perp^2}{l_\perp^2}\frac{2-2y+y^2}{y}. \qquad (17)
\end{aligned}$$

Reintroduction of the final state projection $\mathcal{O}^{00}$ in Eq. (8), produces the well known and physically clear formula for the differential semi-inclusive hadronic tensor with single photon emission in the final state,

$$\begin{aligned}
\frac{dW_0^A\mu\nu}{dydl_\perp^2} &= C_p^A 2\pi\sum_q Q_q^2 f_q^A(x_B+x_L)(-g_\perp^{\mu\nu}) \\
&\times \frac{\alpha_s}{2\pi}\frac{1}{l_\perp^2}P_{q\to q\gamma}(y). \qquad (18)
\end{aligned}$$

In the equation above, $f_q^A(x_B+x_L)$ represent the parton distribution function of a quark with flavor q and electric charge Q_q in units of the electron charge e, in a nucleus with momentum Ap where the parton carries a fraction (x_B+x_L) of the momentum p,*i.e.*,

$$\begin{aligned}
f_q^A(x_B+x_L) &= A\int\frac{dy_0^-}{2\pi}e^{-i(x_B+x_L)p^+y_0^-} \\
&\times \frac{1}{2}\langle p|\bar\psi(y_0^-)\gamma^+\psi(0)|p\rangle. \qquad (19)
\end{aligned}$$

In Eq. (18), the factor $P_{q\to q\gamma}(y) = (2-2y+y^2)/y$ is the quark-to-photon splitting function; it represents the probability that a quark will radiate a photon which will carry away a fraction y of its forward momentum. The projection $g_\perp^{\mu\nu} = g^{\mu\nu} - g^{\mu-}g^{\nu+} - g^{\mu+}g^{\nu-}$.

In DIS experiments, one has an experimental handle on the Bjorken variable x_B. In this article we focus on the region where $x_B \lesssim 1$. As a result, given that $l_\perp \sim \lambda Q$, we may approximate $f(x_B+x_L) \simeq f(x_B)$. As the parton produced immediately after the hard scattering has a vanishingly small transverse momentum, the transverse momentum of the final produced quark is simply the negative of the radiated photon's transverse momentum, *i.e.*, $\vec{l}_{q_\perp} = -\vec{l}_\perp$. As a result, the differential hadronic tensor for the transverse momentum distribution of the final quark is given as

$$\begin{aligned}
\frac{dW_0^{\mu\nu}}{dydl_\perp^2 d^2l_{q_\perp}} &\simeq C_p^A 2\pi\sum_q Q_q^2 f_q^A(x_B)(-g_\perp^{\mu\nu}) \\
&\times \frac{\alpha_s}{2\pi}\frac{1}{l_\perp^2}P_{q\to q\gamma}(y)\delta^2(\vec{l}_\perp + \vec{l}_{q_\perp}). \qquad (20)
\end{aligned}$$

III. GLUON RADIATION FROM MULTIPLE SCATTERING

In this section, we start by taking the part factorized from the hard cross section denoted as $\mathcal{O}$ and expand it as a power series in the hard scale Q^2. The leading term was denoted as $\mathcal{O}^{00}$ in the preceding section and only included the process of vacuum radiation without any scattering. Here we consider the effect of multiple scattering on both the quark and the radiated gluon. As in prior attempts in this direction, we will not discuss the issue of factorization [28–31]. The focus will be on deriving the physical leading effect of multiple scattering on the process of single gluon radiation.

We start with the part denoted as $\mathcal{O}^{N,N}_{q,p;n,m}$ i.e., the term with N scatterings on either side and with the gluon radiated from location q,p on the quark amplitude and complex conjugate and scattering a total of $n \le N-q$ and $m \le N-p$ times. This is represented in Fig. 2. The Feynman integral for this diagram can be written down in position space, with the quark propagator in the amplitude (left hand side of the cut) from y'_i to y'_{i+1} written as,

$$S(y_{i+1}-y_i) = \int\frac{d^4{q'}_{i+1}}{(2\pi)^4}\frac{i\,{\not q'}_{i+1}e^{-i{q'}_{i+1}\cdot(y'_{i+1}-y'_i)}}{{q'}^2_{i+1}+i\epsilon}. \quad (21)$$

The quark propagators in the complex conjugate (right hand side of the cut) are identical with the signs of the $i\epsilon$ and the i in the numerator reversed. We then shift the momenta as $q'_{i+1} = q + \sum_{j=0}^i p'_i$ where p'_0 is the momentum of the original incoming quark which is struck with the virtual photon and the other p'_i's are the momenta of the gluons off which the final out going quark scatters. Thus we replace the variable being integrated as $dq'_{i+1} \to dp'_i$. Similar replacements are made in the complex conjugate. In the the gluon propagators, the replacements are made starting from the cut end, e.g., the $i^{\rm th}$ propagator momentum in the complex conjugate $Q_i = l - \sum_{j=i}^{m} k_i$ where l is the momentum of the gluon propagator that is cut and the k_i's are the momenta being brought in by the "soft" gluons off which the hard gluon scatters. this replaces $dQ_i \to dk_i$ and similarly in the amplitude.

Having made all these shifts in momentum we can write down the full expression for the Feynman integral corresponding to the diagram of Fig. 2. We quote the full expression first and then will describe the various terms within. The full expression for the final state outgoing quark and gluon which scatter a total of N times in the amplitude and complex conjugate, with the gluon being produced at q in the amplitude scattering n times and at p in the complex conjugate scattering m times, is given as,

$$
\begin{aligned}
\mathcal{O}^{NN}_{q,p;n,m} = & \prod_{i=1}^{N-m}\prod_{k=1}^{N-n} d^4y_i d^4y'_k \frac{d^4p_{i-1}}{(2\pi)^4}\frac{d^4p'_{k-1}}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{d^4l_q}{(2\pi)^4}\prod_{i=1}^{m}\prod_{j=1}^{n} d^4\zeta_i d^4\zeta'_j \frac{d^4k_i}{(2\pi)^4}\frac{d^4k'_j}{(2\pi)^4} \\
& \times \mathbf{Tr}\left[\frac{1}{2}\prod_{i=0}^{p}\left\{\gamma^- \frac{-i\left\{(\sum_{j=0}^{i} \not p_j) + \not q\right\}}{[(\sum_{j=0}^{i} p_j)+q]^2 - i\epsilon}\right\}\gamma_{\alpha_0}\prod_{i=p}^{N-m-1}\left\{\frac{-i\left\{(\sum_{j=0}^{i}\not p_j) + \not q - \not l + \sum_{j=1}^{m} \not k_j\right\}}{[(\sum_{j=0}^{i} p_j) + q - l + (\sum_{j=1}^{m} k_j)]^2 - i\epsilon}\gamma^-\right\}\right. \\
& \times 2\pi\delta(l^2)2\pi\delta(l_q^2) G^{\alpha_0\beta'_0}_{a_0b'_0}(m,n)\ \not l_q \\
& \times \left.\prod_{k=N-n-1}^{q}\left\{\gamma^- \frac{i\left\{(\sum_{l=0}^{k}\not p'_l) + \not q - \not l + \sum_{l=1}^{n} \not k'_l\right\}}{[(\sum_{l=0}^{k} p'_l) + q - l + (\sum_{l=1}^{n} k'_l)]^2 + i\epsilon}\right\}\gamma_{\beta_0}\prod_{k=q}^{1}\left\{\frac{i\left\{(\sum_{l=0}^{k}\not p'_l) + \not q\right\}}{[(\sum_{l=0}^{k} p'_l) + q]^2 + i\epsilon}\gamma^-\right\}\frac{\not p'_0 + \not q}{(p'_0+q)^2 + i\epsilon}\right] \\
& \times \prod_{i=0}^{N-m-1} e^{-ip_i\cdot y_i}\prod_{j=1}^{m} e^{-ik_j\cdot\zeta_j}\prod_{i=0}^{N-n-1} e^{ip'_i\cdot y'_i}\prod_{j=1}^{n} e^{ik'_j\cdot\zeta'_j} \\
& \times e^{-iy_{(N-m)}\cdot\left\{l_q - \left(\sum_{i=0}^{N-m-1} p_i\right) - q + l - \sum_{i=1}^{m} k_i\right\}} e^{iy'_{(N-n)}\cdot\left\{l_q - \left(\sum_{j=0}^{N-n-1} p'_j\right) - q + l - \sum_{j=1}^{n} k'_j\right\}} \\
& \times \left\langle A\left|\prod_{i=1}^{N-m} -igt^{a_i}A^+_{a_i}(y_i)\prod_{k=N-n}^{1} igt^{b_k}A^+_{b_k}(y'_k)\prod_{j=1}^{m} gf^{b_ja_jc_j}A^+_{c_j}(\zeta_j)\prod_{l=1}^{n} gf^{c'_la'_lb'_l}A^+_{c'_l}(\zeta'_l)\right|A\right\rangle .
\end{aligned}
\tag{22}
$$

The gluon line $G^{\alpha_0\beta'_0}_{a_0b'_0}$ (this refers to combination of gluon propagators and scattering vertices extending from the amplitude side to the complex conjugate) is given as,

$$
G^{\alpha_0\beta'_0}_{a_0b'_0}(m,n) = \prod_{i=1}^{m}\left[\frac{-i\bar{G}^{\alpha_{i-1}\beta_i}_{a_{i-1}b_i}}{\left[l - \sum_{j=i}^{m} k_j\right]^2 - i\epsilon}(-g_\perp)_{\beta_i\alpha_i}2l^-\right](-g_\perp)_{\alpha_m\beta'_n}\delta^{a^m b'^n}\prod_{k=n}^{1}\left[2l^-(-g_\perp)_{\beta'_i\alpha'_i}\frac{i\bar{G}^{\alpha'_i\beta'_{i-1}}_{a_ib_{i-1}}}{[l - \sum_{l=k}^{n} k'_l]^2 + i\epsilon}\right]. \tag{23}
$$

The Feynman diagram corresponding to this equation is Fig. 2. The first line of Eq. (22) contains a series of integrals over coordinates and momenta. The coordinates y_i, where $1 < i < N-m$, represent the locations where the quark scatters in the amplitude; the coordinates y'_k, where $1 < k < N-n$, represent the scattering of the quark in the complex conjugate. The momenta p_{i-1}, are the incoming momenta into the quark line with p_0 being the momentum of the incoming quark which is struck by the virtual photon. The remaining momenta from p_1 to p_{N-m-1} are momenta of the gluons off which the quark scatters. The same is true for the p'_k momenta in the complex conjugate. The coordinates ζ_i (ζ'_j), where $1 < i < m$ ($1 < j < n$) represent the locations where the emitted gluon scatters in the amplitude (complex conjugate). The momenta carried by each of these insertions is characterized by k_i (k'_j) where $1 < i < m$ ($1 < j < n$) in the amplitude (complex conjugate). The two remaining momenta l and l_q represent the final on-shell momenta of the exiting quark and gluon as in the previous section.

The second line of Eq. (22) represents the quark propagators on the amplitude side. The third line contains the cut propagators and the full set of gluon propagators from α_0 to β'_0 in one expression $G^{\alpha_0\beta'_0}_{a_0b'_0}$ which is decom-

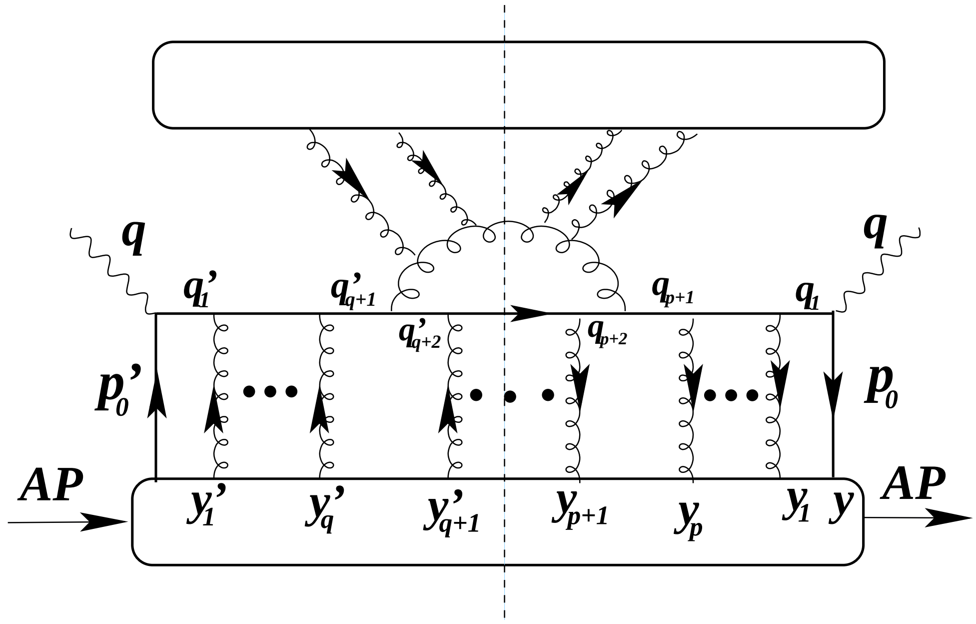

FIG. 2: DIS on a nucleus with a produced quark and an emitted gluon scattering N times.

posed in Eq. (23). The Greek superscripts represent the vector indices where the gluon is first emitted from the quark line and the Latin subscripts represent the color at this emission. The fourth line contains the quark propagators on the complex conjugate side of the cut. The fifth line contains the phase factors for all but the cut quark propagator. This last phase factor is separately expressed in the sixth line. The seventh line contains the expectation in the nucleus for the $N+N$ gluon field operators in the amplitude and complex conjugate. Here $|A\rangle$ represents the nuclear state [32].

Eq. (23), contains all the gluon propagators for the emitted gluon which undergoes m scatterings in the amplitude and n scatterings in the complex conjugate, where, $\bar{G}^{\alpha_{i-1}\beta_i}_{a_{i-1}b_i}$ represents the numerators of the gluon propagators after the emission both on the amplitude and complex conjugate side. In the rest of this section we will discuss various simplifications that arise due to our choice of kinematics for the radiated gluon: $l_\perp \sim \lambda Q$ and $l^- = yq^- \sim Q$ where Q is the hard scale in the DIS process.

Eqs. (22,23) are not completely general and have a set of approximations already included. While calculations are carried out in the negative light-cone gauge $A^-_{a_i} = 0$, there remain three other components of the gluon field in covariant products such as $\gamma_{\mu_i} A^{\mu_i}_{a_i}$ which arise with each quark-gluon vertex. As shown in a recent study [33], as well as in Appendix A, in the case of the Breit frame, the dominant component of the vector potential (in $A^- = 0$ gauge), for gluons radiated from partons with collinear momenta in the (+)-direction, is the (+)-component which scales as $A^+ \sim \lambda^2 Q$ compared to the ($\perp$)-component which scales as $A_\perp \sim \lambda^3 Q$.

The numerator of the terms in the second and fourth line of Eq. (22) have the general form $\ldots \gamma_{\mu_i} q_i^{\mu_i} \gamma_{\nu_i} A^{\nu_i}_{a_i} t_{a_i} \ldots$. While this includes all four components of the quark momentum and the gauge field, the ($-$)-component of the quark momenta is $q_i^- = q^- + \sum_j p_j^- \simeq q^- \sim Q$ and dominates over the ($\perp$) and (+)-components which scale as λQ and $\lambda^2 Q$ respectively. This is also true of the gauge field as mentioned above. As a result, the leading term in each product of quark propagator and gluon insertion is $\ldots \gamma^+ q^- \gamma^- A^+_{a_i} t^{a_i} \ldots$ if the quark propagator is situated before the radiation of the outgoing gluon and is $\ldots \gamma^+ (q^- - l^-) \gamma^- A^+_{a_i} t^{a_i} \ldots$ for quark propagators which appear after the radiated gluon. This is identical to the case of photon radiation from a multiply scattering quark produced in DIS [13].

The major difference in this article from Ref. [13], besides the scaling relations used, is the scattering encountered by the radiated gluon. This is somewhat non-trivial in $A^- = 0$ gauge. We begin by elucidating the various propagator and three gluon vertex structures. We will only consider three gluon vertices as four gluon vertices indicate a scattering of two gluon fields at the same location. As pointed out in Ref. [12], such contributions are not length enhanced and are thus suppressed. From the effective theory perspective, such contributions are not contained in the leading Lagrangian [33].

The gluon propagator in $A^- = 0$ gauge for a gluon with four-momentum l contains the constant unit vector $n^\mu = (1,0,0,0)$ and is given as (suppressing color indices)

$$G^{\mu\nu}(l) = \frac{i\bar{G}^{\mu\nu}}{l^2 + i\epsilon} = i\frac{-g^{\mu\nu} + \frac{l^\mu n^\nu + l^\nu n^\mu}{l\cdot n}}{l^2 + i\epsilon} \quad (24)$$

All components where either of the Lorentz indices is ($-$), are vanishing, i.e.,

$$G^{-+} = G^{+-} = i\frac{-1+1}{l^2} = 0,$$
$$G^{--} = G^{\perp -} = 0. \quad (25)$$

The leading components (where the numerator has the largest power of λ) are those where both Lorentz indices are transverse, i.e.,

$$(G_\perp)_{\mu\nu} \equiv G_{\perp_\mu \perp_\nu} = i\frac{-g_{\perp_\mu \perp_\nu}}{l^2+i\epsilon} = i\frac{-g_{\perp\mu\nu}}{l^2+i\epsilon}. \quad (26)$$

The meaning of the notation $l_{\perp_\mu} \equiv (l_\perp)_\mu$ is one where the index, which may be referred to as μ or $\perp_\mu$ can only assume values $\mu = \perp_\mu = 1,2$, and $(l_\perp)^\mu = l^{\perp_\mu} = -l_{\perp_\mu} = -(l_\perp)_\mu$ and $(g_\perp)_{\mu\nu} \equiv g_{\perp_\mu\perp_\nu} = g^{\perp_\mu\perp_\nu} \equiv (g_\perp)^{\mu\nu} = -\delta^{\perp_i\perp_j}$. The $\perp\perp$ components are followed by the $(+\perp)$ and the $(++)$ components, where

$$G^{+\perp_i} = i\frac{\bar{G}^{+\perp_i}}{l^2+i\epsilon} = i\frac{l^{\perp_i}}{(l^-)[l^2+i\epsilon]},$$
$$G^{++} = i\frac{\bar{G}^{++}}{l^2+i\epsilon} = i\frac{2l^+}{l^-[l^2+i\epsilon]}. \quad (27)$$

The three gluon vertex contracted with the gauge field may also be simplified as,

$$\Gamma^{bac}_{\beta\alpha\gamma}(l,k)A^\gamma_c(k) = gf^{bac} \quad (28)$$
$$\times \; [g_{\beta\alpha}(k-2l)\cdot A_c + A_{c\alpha}(l-2k)_\beta + A_{c\beta}(l+k)_\alpha]$$
$$\simeq \; gf^{bac}g_{\beta\alpha}(-2l^-)A^+_c + A_{c\alpha}l^- g^+_\beta + A_{c\beta}l^- g^+_\alpha.$$

When the three gluon vertex is used in combination with gluon propagators on either side, the dominant term in terms of counting powers of λ from the contraction of Lorentz indices is the first term on the third line of Eq. (28), i.e.,

$$\ldots \bar{G}^{\alpha_{i-1}\beta_i}_{a_{i-1}b_i}\Gamma^{b_i a_i c_i}_{\beta_i\alpha_i\gamma_i}A^{\gamma_i}_{c_i}\bar{G}^{\alpha_i\beta_{i+1}}_{a_i b_{i+1}}\ldots \quad (29)$$
$$= \; \ldots\left(-g_\perp{}^{\alpha_{i-1}\beta_i}\right)gf^{a_{i-1}b_{i+1}c_i}A^+_{c_i}2l^-$$
$$\times \; \left(-g_{\perp\beta_i\alpha_i}\right)\left(-g_\perp{}^{\alpha_i\beta_{i+1}}\right)\ldots.$$

The approximation of replacing the numerators of all the gluon propagators with the transverse projector $g_{\perp\mu\nu}$ is consistent for all except the first and the last propagators which connect with the quark line. Here the Lorentz index that contracts with the γ-matrix may assume both a $+$ or a $\perp$ component yielding the same power counting for the complete expression. As a result, in Eq. (23), the numerators of the first and last gluon propagators may contain either a $(\bar{G}^{+\perp})^{\mu\nu}$ or a $g_\perp{}^{\mu\nu}$. The numerators of all other gluon propagators contain solely the sum over transverse polarizations to obtain the leading power counting.

As a result of the above simplifications, the A^μ fields in Eq. (22) have all been replaced by A^+. The contracting γ^μ matrices are replaced by simply γ^-. The momentum dependent part of the three gluon vertices have been replaced by simply $2l^-$ as indicated by Eq. (29). There remain the simplifications on the numerators of the quark propagator, which essentially amount to replacing all the numerators with γ^+q^- expect for the propagators that surround the gluon emission vertex. The contraction with either a $(\bar{G}^{+\perp})^{\mu\nu}$ or a $g^{\mu\nu}_\perp$ leads to three terms from the numerator of the three propagators connecting to the gluon emission vertex,

$$V^\nu \simeq \gamma^+q^-\gamma_\mu\frac{n^\mu\left(l_\perp - \sum_{j=1}^m k^j_\perp\right)^\nu}{l^-}\gamma^+(q^- - l^-) \quad (30)$$
$$+ \; \gamma^\alpha_\perp\left(\sum_{i=0}^p p^i_\perp\right)_\alpha \gamma_{\perp\mu}(-g^{\mu\nu}_\perp)\gamma^+(q^- - l^-)$$
$$+ \; \gamma^+q^-\gamma_{\perp\mu}(-g^{\mu\nu}_\perp)\gamma^\alpha_\perp\left(\sum_{i=0}^p p^i_\perp + \sum_{j=1}^m k^j_\perp - l_\perp\right)_\alpha.$$

A similar structure arises on the complex conjugate side as well. Invoking all these simplifications leads to a considerable simplification of the numerator structure of Eq. (22).

Following the stated rules of power counting, the denominator of the various propagators in Eqs. (22,23) may be simplified and expressed in terms of the usual momentum fractions [10, 13]. For the first set of quark denominators, up to the gluon emission vertex, we introduce the fractions,

$$x^i_D = \frac{|p^i_\perp|^2 + 2\sum_{j=0}^{i-1}p^i_\perp\cdot p^j_\perp}{2p^+q^-} \sim \lambda^2. \quad (31)$$

While written without vector attributes, it should be understood that $p^i_\perp$ for all i are two dimensional vectors transverse to the direction of propagation of the jet. Using this, we may write the i^{th} denominator as,

$$D_{i\leq p} = 2p^+q^-\left[\left(\sum_{j=0}^i x_i - x^i_D\right) - x_B\right], \quad (32)$$

where $x_i = p^+_i/p^+ \sim \lambda^2$ and is thus included in the denominator. Ignored are terms which depend on p^-_i which introduce $\lambda^2 Q$ corrections to $q^- \sim Q$. For the denominators beyond the emitted photon, besides the fractions x^i_D which are defined similar to Eq. 31, we introduce the momentum fractions which connect the final outgoing transverse momentum of the radiated gluon with the incoming transverse momentum which strikes the hard quark,

$$x^i_L = \frac{|l_\perp|^2 - 2yl_\perp\cdot\sum_{k=0}^i p^k_\perp}{2p^+q^-y(1-y)}. \quad (33)$$

Another set of transverse momentum fractions arise from the combination of the incoming transverse momenta striking the gluon line and those striking the quark line,

$$(x^m_T)^i = \frac{2p^i_\perp\cdot\sum_{j=1}^m k^j_\perp}{2p^+q^-}. \quad (34)$$

Finally there are the momentum fractions that connect the transverse momentum of the final radiated gluon with the soft transverse momenta striking it,

$$(z^m_L)_i = \frac{2l_\perp\cdot k^i_\perp - |k^i_\perp|^2 - k^i_\perp\cdot\sum_{j=i+1}^m k^j_\perp}{2p^+q^-}. \quad (35)$$

The forward momentum fractions of these incoming gluons denoted as,

$$z_i = \frac{k_i^+}{p^+}. \tag{36}$$

With these fractions, the denominator of the $i^{\rm th}$ quark propagator after the gluon emission can be expressed as,

$$\begin{aligned} D_{i>p} &= 2p^+q^-(1-y)\left[\left(\sum_{j=0}^{i} x_j - \frac{x_D^j + (x_T^m)^j}{1-y}\right)\right. \\ &- \left. x_B - x_L^i + \left(\sum_{k=1}^{m} z_k + \frac{(z_L^m)_k}{1-y}\right)\right]. \end{aligned} \tag{37}$$

The denominator of the $i^{\rm th}$ gluon propagator has a much simpler expression in terms of these momentum fractions,

$$D_i^g = -2p^+q^-y\left[\sum_{j=i}^{m} z_j - \frac{(z_L^m)_j}{y}\right]. \tag{38}$$

To Substitute the above simplifications, requires that we separate the trace over spinor indices contracted with the vector indices in the numerators of the gluon propagators in a numerator factor $\mathcal{N}$, and the remaining denominators of all propagators in a denominator factor $\mathcal{D}$, which are coupled by the set of integrals over position and momentum, i.e.,

$$\mathcal{O}^{NN}_{q,p;n,m} = \int \mathfrak{D}(y)\mathfrak{D}(p)\mathcal{N}\cdot\mathcal{D}, \tag{39}$$

where, $\mathfrak{D}(y)$ represents all position integrals on the amplitude and complex conjugate side of the matrix element and $\mathfrak{D}(p)$ represents all the integrals over momenta.

The numerator factor, containing all the γ matrices from the quark line and all metric tensors from the gluon line, keeping only the leading terms in λ, simplifies as,

$$\begin{aligned} \mathcal{N} &= -\mathbf{Tr}\left[\frac{1}{2}(\gamma^-\gamma^+q^-)^p\gamma^-\left\{\gamma^+q^-\gamma_\mu\frac{n^\mu\left(l_\perp - \sum_{j=1}^m k_\perp^j\right)^\nu}{l^-}\gamma^+(q^--l^-) + \gamma_\perp^\alpha\left(\sum_{i=0}^p p_\perp^i\right)_\alpha \gamma_{\perp\mu}(-g_\perp^{\mu\nu})\gamma^+(q^--l^-)\right.\right. \\ &+ \left.\gamma^+q^-\gamma_{\perp\mu}(-g_\perp^{\mu\nu})\gamma_\perp^\alpha\left(\sum_{i=0}^p p_\perp^i + \sum_{j=1}^m k_\perp^j - l_\perp\right)_\alpha\right\}\{\gamma^-\gamma^+(q^--l^-)\}^{N-m-p-1}\gamma^-\gamma^+(q^--l^-) \\ &\times \{\gamma^-\gamma^+(q^--l^-)\}^{N-n-q-1}\gamma^-\left\{\gamma^+(q^--l^-)\gamma_\rho\frac{n^\rho\left(l_\perp - \sum_{j=1}^n k'^j_\perp\right)_\nu}{l^-}\gamma^+q^-\right. \\ &+ \left.\left.\gamma^+(q^--l^-)\gamma_\perp^\rho(-g_\perp)_{\rho\nu}\gamma_\perp^\beta\left(\sum_{i=0}^q p'^i_\perp\right)_\beta + \gamma_\perp^\beta\left(\sum_{i=0}^q p'^i_\perp + \sum_{j=1}^n k'^j_\perp - l_\perp\right)_\beta \gamma_\perp^\rho(-g_\perp)_{\rho\nu}\gamma^+q^-\right\}(\gamma^-\gamma^+q^-)^q\right] \end{aligned} \tag{40}$$

$$\begin{aligned} &= -\frac{1}{2}\mathbf{Tr}\left[(2q^-)^p\gamma^-\left\{\gamma^+q^-\gamma_\mu\frac{n^\mu\left(l_\perp - \sum_{j=1}^m k_\perp^j\right)^\nu}{l^-}\gamma^+(q^--l^-) + \gamma_\perp^\alpha\left(\sum_{i=0}^p p_\perp^i\right)_\alpha \gamma_{\perp\mu}(-g_\perp^{\mu\nu})\gamma^+(q^--l^-)\right.\right. \\ &+ \left.\gamma^+q^-\gamma_{\perp\mu}(-g_\perp^{\mu\nu})\gamma_\perp^\alpha\left(\sum_{i=0}^p p_\perp^i + \sum_{j=1}^m k_\perp^j - l_\perp\right)_\alpha\right\}\{2(q^--l^-)\}^{N-m-p-1}2(q^--l^-) \\ &\times \{2(q^--l^-)\}^{N-n-q-1}\gamma^-\left\{\gamma^+(q^--l^-)\gamma_\rho\frac{n^\rho\left(l_\perp - \sum_{j=1}^n k'^j_\perp\right)_\nu}{l^-}\gamma^+q^-\right. \\ &+ \left.\left.\gamma^+(q^--l^-)\gamma_\perp^\rho(-g_\perp)_{\rho\nu}\gamma_\perp^\beta\left(\sum_{i=0}^q p'^i_\perp\right)_\beta + \gamma_\perp^\beta\left(\sum_{i=0}^q p'^i_\perp + \sum_{j=1}^n k'^j_\perp - l_\perp\right)_\beta \gamma_\perp^\rho(-g_\perp)_{\rho\nu}\gamma^+q^-\right\}(2q^-)^q\right] \end{aligned} \tag{41}$$

The entire set of transverse components of the metric tensor $(-g_\perp{}^{\mu\nu})$ that arise from the propagators and vertices of the radiated gluon line lead to the contraction between the ν component of the gluon emission vertex in the amplitude with the same component in the complex conjugate. The overall minus sign is due to the fact that there is one more propagator in the gluon line than vertex i.e., the cut propagator.

The equation above for $\mathcal{N}$ does not contain any $(+)$ or $(-)$ components of the incoming momenta: p_i, p'_i, k_i, k'_i. The only non-transverse components are the large $(-)$ components of the outgoing quark (q^-) and radiated gluon (l^-). This allows for the integrations over all $(+)$ and $(-)$ components of the incoming momenta to be carried out solely by considering the denominators of the various propagators. These set of integrations are carried out in the next section.

The remaining expression for the numerator of $\mathcal{O}^{NN}_{q,p;n,m}$ can be further simplified to obtain a more suggestive form. Isolating the coefficient C_L of $(\vec{l}_\perp - \sum_{i=1}^m \vec{k}^i_\perp)\cdot(\vec{l}_\perp - \sum_{j=1}^n \vec{k'}^j_\perp)$ yields four terms, which may be summed to obtain,

$$C_L = \frac{(2q^-)^{2N-m-n+1}}{y}(1-y)^{2N-m-n-p-q-1}\left[\frac{4}{y}(1-y)^2 + 2(1-y) + 2(1-y) + 2y\right] \quad (42)$$
$$= \frac{(2q^-)^{2N-m-n+1}}{y}(1-y)^{2N-m-n-p-q-1}\left[\frac{2}{y}\left(2-2y+y^2\right)\right] = \frac{(2q^-)^{2N-m-n+1}}{y}(1-y)^{2N-m-n-p-q-1}2P(y).$$

The coefficients of $y\sum_{i=1}^p \vec{p}^i_\perp \cdot \left(\vec{l}_\perp - \sum_{j=1}^n \vec{k'}^j_\perp\right)$, $\left(\vec{l}_\perp - \sum_{i=1}^m \vec{k}^i_\perp\right)\cdot y\sum_{j=1}^q \vec{p'}^j_\perp$ and $y^2\sum_{i=1}^p \vec{p}^i_\perp \cdot \sum_{j=1}^q \vec{p'}^j_\perp$ are similar and may be included with the contributions containing C_L to obtain the general numerator as

$$\mathcal{N} = \frac{(2q^-)^{2N-m-n+1}}{y}(1-y)^{2N-m-n-p-q-1}2P(y)\left(\vec{l}_\perp - \sum_{i=1}^m \vec{k}^i_\perp - y\sum_{i=1}^p \vec{p}^i_\perp\right)\cdot\left(\vec{l}_\perp - \sum_{j=1}^n \vec{k'}^j_\perp - y\sum_{j=1}^q \vec{p'}^j_\perp\right). \quad (43)$$

Ignoring the factors of $y, (1-y)$ and q^- on the right hand side, the remaining expression for the numerator has a very simple interpretation. While $P(y)$ is simply the regular vacuum splitting function, the two terms, $\left(\vec{l}_\perp - \sum_{i=1}^m \vec{k}^i_\perp - y\sum_{i=1}^p \vec{p}^i_\perp\right)$ and $\left(\vec{l}_\perp - \sum_{j=1}^n \vec{k'}^j_\perp - y\sum_{j=1}^q \vec{p'}^j_\perp\right)$ are the transverse momentum of the radiated gluon immediately after the radiative vertex in the amplitude and complex conjugate respectively. In the next section we will simplify the denominator by carrying out a series of integrations over the light-cone components of the incoming gluon momenta. The numerator factor in Eq. (43) does not contain these components and thus plays no role in these integrations.

IV. THE POLES STRUCTURE FOR SINGLE GLUON EMISSION FROM N SCATTERING

In this section, the integrals over the light cone components of the incoming gluon momenta will be carried out. As mentioned in the preceding section, for this part of the integration, we only need to consider the denominators of all the propagators. The phase factors for the last scatterings on the quark at locations $y_{(N-m)}$ may be simplified, by the introduction of an $(N-m)^{\text{th}}$ momentum, using the four δ-function as

$$1 = \int d^4p_{(N-m)}\delta^4\left(l_q - \left(\sum_{i=0}^{N-m-1} p_i\right) - q + l - \sum_{i=1}^m k_i - p_{(N-m)}\right). \quad (44)$$

Using the $\pm$ components of this four dimensional δ-function and the definitions of the various momentum fractions, the argument of the $\delta(l_q^2)$ function may be simplified as,

$$\int dl_q^+ dl_q^- \delta(l_q^2)\delta^2\left(l_q^\pm - \sum_{i=0}^{N-n} p_i^\pm - q^\pm + l^\pm - \sum_{i=1}^m k_i^\pm\right) = \int dl_q^+ dl_q^- \frac{\delta\left(l_q^+ - \frac{{l_q}_\perp^2}{2l_q^-}\right)}{2l_q^-}\delta^2\left(l_q^\pm - \sum_{i=0}^{N-n} p_i^\pm - q^\pm + l^\pm - \sum_{i=1}^m k_i^\pm\right)$$
$$\simeq \frac{1}{2p^+q^-(1-y)}\delta\left(\sum_{i=0}^{N-m} x_i - \frac{x_D^i + (x_T^m)^i}{(1-y)} - x_B - x_L^{(N-m)} + \sum_{j=1}^m z_j + \frac{(z_L^m)_j}{(1-y)}\right). \quad (45)$$

We also perform the integration over l^+ using the on-shell delta function,

$$\frac{d^4l}{(2\pi)^4}2\pi\delta(l^2) = \int \frac{dl^- d^2l_\perp}{(2\pi)^3}\frac{1}{l^-}. \quad (46)$$

We make the substitution, $l^- = q^- y$ and express all further expressions in terms of the radiated momentum fraction y. This simplifies the last two phase factors in the expression for $\mathcal{O}^{NN}_{q,p;n,m}$ at locations $y_{(N-m)}$ and $y'_{(N-n)}$. Including only the denominators, δ-functions (associated factors of 2π and i) and phase factors from Eq. 22, we obtain,

$$
\begin{aligned}
\mathfrak{D}^{NN}_{q,p;n,m} =& \prod_{i=1}^{N-m}\prod_{k=1}^{N-n} d^4y_i d^4y'_k \frac{dx_{i-1}d^3p_{i-1}}{(2\pi)^4}\frac{dx'_{k-1}d^3p'_{k-1}}{(2\pi)^4}\frac{dyd^2l_\perp}{(2\pi)^3 y}\frac{dx_{(N-m)}d^3p_{(N-m)}}{(2\pi)^4}\prod_{i=1}^{m}\prod_{j=1}^{n} d^4\zeta_i d^4\zeta'_j \frac{dz_i d^3k_i}{(2\pi)^4}\frac{dz'_j d^3k'_j}{(2\pi)^4} \\
\times\;& d^2l_{q\perp}\delta^2\left(l_{q\perp} + l_\perp - \sum_{i=0}^{(N-m)} p^i_\perp - \sum_{j=1}^{m} k^j_\perp\right)\prod_{i=0}^{p}\left\{\frac{-i}{2q^-\left[\left(\sum_{j=0}^{i} x_i - x^i_D\right) - x_B - i\epsilon\right]}\right\} \\
\times\;& \prod_{i=p}^{N-m-1}\left\{\frac{-i}{2q^-(1-y)\left[\left(\sum_{j=0}^{i} x_j - \frac{x^j_D + (x^m_T)^j}{(1-y)}\right) - x_B - x^i_L + \left(\sum_{j=1}^{m} z_j + \frac{(z^m_L)_j}{1-y}\right) - i\epsilon\right]}\right\} \\
\times\;& \frac{1}{(p^+)^2}\frac{2\pi}{2q^-(1-y)}\delta\left(\sum_{i=0}^{N-m} x_i - \frac{x^i_D + (x^m_T)^i}{(1-y)} - x_B - x_L^{(N-m)} + \sum_{j=1}^{m} z_j + \frac{(z^m_L)_j}{1-y}\right) \\
\times\;& \prod_{k=N-n-1}^{q}\left\{\frac{i}{2q^-(1-y)\left[\left(\sum_{l=0}^{k} x'_l - \frac{x'^{\,l}_D - (x'^m_T)^l}{1-y}\right) - x_B - x'^k_L + \left(\sum_{l=1}^{n} z'_l + \frac{z'^{\,l}_L}{1-y}\right) + i\epsilon\right]}\right\} \\
\times\;& \prod_{k=q}^{0}\left\{\frac{i}{2q^-\left[\left(\sum_{l=0}^{k} x'_l - x'^l_D\right) - x_B + i\epsilon\right]}\right\} \\
\times\;& \prod_{i=1}^{m}\frac{-i}{-2q^- y\left[\sum_{k=i}^{m} z_k - \frac{(z^m_L)_k}{y} + i\epsilon\right]}\prod_{j=1}^{n}\frac{i}{-2q^- y\left[\sum_{l=j}^{n} z_l - \frac{(z^n_L)_l}{y} - i\epsilon\right]} \\
\times\;& \prod_{i=0}^{N-m} e^{-ip_i\cdot(y_i - y'_{(N-n)})}\prod_{j=1}^{m} e^{-ik_j\cdot(\zeta_j - y'_{(N-n)})}\prod_{i=0}^{N-n-1} e^{ip'_i\cdot(y'_i - y'_{(N-n)})}\prod_{j=1}^{n} e^{ik'_j\cdot(\zeta'_j - y'_{(N-n)})}.
\end{aligned}
\tag{47}
$$

The extra factor of $(1/2p^+)^2$ is due to the fact that there are two extra propagators than integrals over (+)-components of momentum. The gluon propagators have an over all negative sign compared to the quark propagators. This is due to convention of calculating the momentum in the i^{th} gluon propagator in the amplitude as,

$$
\begin{aligned}
& \frac{i}{\left[l - \sum_{j=i}^{q} k^j\right]^2 + i\epsilon} \\
\simeq\; & \frac{i}{\left[-2l^- \sum_{j=i}^{n} k^+_j + l_\perp\cdot\sum_{j=i}^{n} k^j_\perp - \left|\sum_{j=i}^{n} k^j_\perp\right|^2 + i\epsilon\right]} \\
=\; & \frac{i}{-2p^+q^- y\left[\sum_{j=i}^{n} z_j - \frac{(z^m_L)_j}{y} - i\epsilon\right]}.
\end{aligned}
\tag{48}
$$

The reader will note that besides the phase factors, the (−)-components of the incoming gluon momenta have been neglected from the integrand. As a result, we can integrate over the (−)-components of all the incoming gluon momenta isolating the entire process on the negative light cone, e.g.,

$$
\int \frac{dp^-_i}{2\pi} e^{-ip^-_i(y^+_i - y'^+_{(N-m)})} = \delta(y^+_i - y'^+_{(N-m)}). \tag{49}
$$

Like wise integrating over the set of momenta p'^-_i, k^-_i and k'^-_i yields the sets of δ-functions $[\delta(y'^+_i - y'^+_{(N-m)}), \delta(\zeta^+_i - y'^+_{(N-m)}), \delta(\zeta'^+_i - y'^+_{(N-m)})\ ]$ that constrains every vertex to the y'^+_{N-m} surface. The δ-function that arises from the integration over p'^-_0, equates, y'^+_{N-m} and thus all other (+)-locations to the origin ($y'^-_0 = 0$). This removes all negative light-cone components of momentum from the derivation. It should be pointed out that, in the higher twist formalism, these components are responsible for elastic energy loss [34, 35]. By integrating out these components we specialize to the case of only radiative energy loss. In reality, as is obvious from the formulation of the Feynman integral of Eq. (22), elastic and radiative loss are coupled, with each contributing to the magnitude of the other. We leave the coupled calculation of elastic and radiative loss to a future effort.

The integrations over the momentum fractions x_i, x'_i, z_i and z'_i requires a certain amount of care as regards the order in which these have to be performed in order to have the simplest pole structure. We focus only on the remaining light-cone components of the phase factor,

$$\Gamma^+ = \prod_{i=0}^{N-m} e^{-ix_i p^+ (y_i^- - y'^-_{(N-n)})} \prod_{j=1}^{m} e^{-iz_j p^+ (\zeta_j^- - y'^-_{(N-n)})} \times \prod_{i=0}^{N-n-1} e^{ix'_i p^+ (y'^-_i - y'^-_{(N-n)})} \prod_{j=1}^{n} e^{iz'_j p^+ (\zeta'^+_j - y'^+_{(N-n)})}. \quad (50)$$

The remaining delta function over the (+)-components of the momentum may be used to set the last momentum fraction $x_{(N-m)} (\equiv p^+_{(N-m)}/p^+)$. Similar to the case of photon production and multiple scattering in Refs. [13] and [12], we obtain,

$$x_{(N-m)} = x_B + x_L^{(N-m)} - \left(\sum_{k=1}^{m} z_k + \frac{(z_L^m)_k}{1-y} \right) + \sum_{i=0}^{N-m} \frac{x_D^i + x_T^i}{1-y} - \sum_{i=0}^{N-m-1} x_i. \quad (51)$$

This converts the light-cone phase factor to,

$$\Gamma^+ = \prod_{i=0}^{N-m} e^{-ix_i p^+ (y_i^- - y^-_{(N-m)})} \prod_{j=1}^{m} e^{-iz_j p^+ (\zeta_j^- - y^-_{(N-m)})} \prod_{i=0}^{N-n-1} e^{ix'_i p^+ (y'^-_i - y'^-_{(N-n)})} \prod_{j=1}^{n} e^{iz'_j p^+ (\zeta'^-_j - y'^-_{(N-n)})}$$
$$\times \exp\left[-ip^+ \left(y^-_{(N-m)} - y'^-_{(N-n)} \right) \left\{ x_B + x_L^{(N-m)} + \sum_{i=0}^{N-m} \frac{x_D^i + x_T^i}{1-y} - \sum_{k=1}^{m} \frac{(z_L^m)_k}{1-y} \right\} \right]. \quad (52)$$

The remaining integrations over momentum fractions involve contour integrations around the various poles of the $2N+2$ propagators. The simplest means to carry this out is to start by integrating the penultimate momentum fractions on either side of the cut quark line, $x_{(N-m-1)}$ and $x'_{(N-n-1)}$ and proceed towards the lower momentum fractions. In this way, the fraction being integrated over appears in at most one propagator. The integration over $x_{(N-m-1)}$ requires the contour to be closed at $+i\infty$ and, as a result, the positions $y^-_{(N-m)}$ and $y^-_{(N-m-1)}$ need to be ordered with $y^-_{(N-m)} > y^-_{(N-m-1)}$. The isolated residue of this integration yields,

$$\int \frac{dx_{(N-m-1)}}{2\pi} \frac{e^{-ix_{(N-m-1)} \cdot p^+ \left(y^-_{(N-m-1)} - y^-_{(N-n)} \right)}}{\left(\sum_{j=0}^{N-m-1} x_j - \frac{x_D^j + (x_T^m)^j}{(1-y)} \right) - x_B - x_L^{(N-m-1)} + \left(\sum_{j=1}^{m} z_j + \frac{(z_L^m)_j}{1-y} \right) - i\epsilon}$$
$$= i\theta\left(y^-_{(N-m)} - y^-_{(N-m-1)} \right) \exp\left[-ip^+ \left(y^-_{(N-m-1)} - y^-_{(N-n)} \right) \right.$$
$$\left. \times \left\{ \left(\sum_{j=0}^{N-m-1} \frac{x_D^j + (x_T^m)^j}{(1-y)} \right) + x_B + x_L^{(N-m-1)} - \left(\sum_{j=1}^{m} z_j + \frac{(z_L^m)_j}{1-y} \right) - \sum_{j=0}^{N-m-2} x_j \right\} \right] \quad (53)$$

Similarly, the integration over $x'_{(N-n-1)}$ requires the contour to be closed at $-i\infty$ and thus orders $y'^-_{(N-n)} > y'^-_{(N-n-1)}$.

Integrating over single poles such as the above may be continued until x_{p+1} in the complex conjugate (*c.c.*) and x'_{q+1} in the amplitude. At this point, there are two denominators in the *c.c.* both of which contain factors of x_p and z_1, and two denominators in the amplitude both of which contain factors of x'_q and z'_1. Equation 47, after all the contour integrations up to x_p and x'_q have been performed may be expressed as (with integral signs implied),

$$\begin{aligned}
\mathfrak{D}^{NN}_{q,p;n,m} = & \prod_{i=1}^{N-m}\prod_{k=1}^{N-n} d^3y_i d^3y'_k \frac{d^2p_\perp^{i-1}}{(2\pi)^2}\frac{d^2{p'}_\perp^{k-1}}{(2\pi)^2}\frac{dy\, d^2l_\perp}{(2\pi)^3 y}\frac{d^2p_\perp^{(N-m)}}{(2\pi)^2}\prod_{i=1}^{m}\prod_{j=1}^{n} d^3\zeta_i d^3\zeta'_j \frac{dz_i d^2k_\perp^i}{(2\pi)^3}\frac{d{z'}_j d^2{k'}_\perp^k}{(2\pi)^3} \\
\times & \; d^2l_{q\perp}\delta^2\left(l_{q\perp}+l_\perp-\sum_{i=0}^{(N-m)}p_\perp^i-\sum_{j=1}^{m}k_\perp^j\right)\left\{\prod_{i=0}^{p}\frac{dx_i}{2\pi}\frac{-i}{2q^-\left[\left(\sum_{j=0}^{i}x_i-x_D^i\right)-x_B-i\epsilon\right]}\right\} \\
\times & \left\{\frac{-i\prod_{i=p+1}^{N-m-1}\theta(y^-_{i+1}-y^-_i)}{2q^-(1-y)\left[\left(\sum_{j=0}^{p}x_j-\frac{x_D^j+(x_T^m)^j}{(1-y)}\right)-x_B-x_L^p+\left(\sum_{j=1}^{m}z_j+\frac{(z_L^m)_j}{1-y}\right)-i\epsilon\right]}\right\} \\
\times & \frac{1}{(p^+)^2}\left(\frac{1}{2q^-(1-y)}\right)^{N-m-p-1}\frac{1}{2q^-(1-y)}\left(\frac{1}{2q^-(1-y)}\right)^{N-n-q-1} \\
\times & \left\{\prod_{k=q}^{0}\frac{dx_k}{2\pi}\frac{i}{2q^-\left[\left(\sum_{l=0}^{k}x'_l-{x'}_D^l\right)-x_B+i\epsilon\right]}\right\} \\
\times & \left\{\frac{i\prod_{j=q+1}^{N-n-1}\theta(y^-_{j+1}-y^-_j)}{2q^-(1-y)\left[\left(\sum_{l=0}^{q}x'_l-\frac{{x'}_D^{\,l}-({x'}_T^m)^l}{1-y}\right)-x_B-{x'}_L^q+\left(\sum_{l=1}^{n}z'_l+\frac{{z'}_L^l}{1-y}\right)+i\epsilon\right]}\right\} \\
\times & \prod_{i=1}^{m}\frac{-i}{-2q^-y\left[\sum_{k=i}^{m}z_k-\frac{(z_L^m)_k}{y}+i\epsilon\right]}\prod_{j=1}^{n}\frac{i}{-2q^-y\left[\sum_{l=j}^{n}z_l-\frac{(z_L^n)_l}{y}-i\epsilon\right]} \\
\times & \prod_{i=0}^{N-m-1}e^{ip_\perp^i\cdot(y_\perp^i-{y'}_\perp^{(N-n)})}\prod_{j=1}^{m}e^{ik_\perp^j\cdot(\zeta_\perp^j-{y'}_\perp^{(N-n)})}\prod_{i=0}^{N-n-1}e^{-i{p'}_\perp^i\cdot({y'}_\perp^i-{y'}_\perp^{(N-n)})}\prod_{j=1}^{n}e^{-i{k'}_\perp^j\cdot({\zeta'}_\perp^j-{y'}_\perp^{(N-n)})} \\
\times & \prod_{i=p+1}^{N-m}e^{-ip^+y_i^-\left(\frac{x_D^i+(x_T^m)^i-\delta x_L^i}{1-y}\right)}\prod_{j=q+1}^{N-n}e^{ip^+{y'}_j^-\left(\frac{{x'}_D^j+({x'}_T^m)^j-\delta {x'}_L^j}{1-y}\right)}\prod_{i=0}^{p}e^{-ip^+x_i\left(y_i^--y^-_{p+1}\right)}\prod_{j=0}^{q}e^{ip^+{x'}_j\left({y'}_j^--{y'}^-_{q+1}\right)} \\
\times & \prod_{i=1}^{m}e^{-ip^+z_i\left(\zeta_i^--y^-_{p+1}\right)}\prod_{j=1}^{n}e^{ip^+{z'}_j\left({\zeta'}_j^--{y'}^-_{q+1}\right)}\times e^{-ip^+y^-_{(p+1)}\left[\left(\sum_{k=0}^{p}\frac{x_D^k+(x_T^m)^k}{(1-y)}\right)+x_B+x_L^p-\left(\sum_{k=1}^{m}\frac{(z_L^m)_j}{1-y}\right)\right]} \\
\times & \; e^{ip^+{y'}^-_{(q+1)}\left[\left(\sum_{l=0}^{q}\frac{{x'}_D^l+({x'}_T^m)^l}{(1-y)}\right)+x_B+{x'}_L^p-\left(\sum_{l=1}^{n}\frac{({z'}_L^n)_j}{1-y}\right)\right]}.
\end{aligned} \tag{54}$$

The $p_\perp, y_\perp$ are two dimensional vectors with a Euclidean metric. The new momentum fractions, δx_L^i and $\delta {x'}_L^j$ are defined as in Refs. [13],

$$\begin{aligned}
x_L^i - x_L^{i-1} &= \frac{-2yl_\perp\cdot\sum_{k=0}^{i}p_\perp^k+2yl_\perp\cdot\sum_{k=0}^{i-1}p_\perp^k}{2p^+q^-y(1-y)} \\
&= \frac{-2l_\perp\cdot p_\perp^i}{2p^+q^-(1-y)} = -\frac{\delta x_L^i}{1-y}.
\end{aligned} \tag{55}$$

The momentum fraction on the amplitude side, $\delta {x'}_L^j$, has an analogous definition and like δx_L^i depends only on the transverse momenta of the gluons which strike the propagating quark:

$$\begin{aligned}
{x'}_L^j - {x'}_L^{j-1} &= \frac{-2yl_\perp\cdot\sum_{l=0}^{j}{p'}_\perp^l+2yl_\perp\cdot\sum_{l=0}^{j-1}{p'}_\perp^l}{2p^+q^-y(1-y)} \\
&= \frac{-2l_\perp\cdot {p'}_\perp^j}{2p^+q^-(1-y)} = -\frac{\delta {x'}_L^j}{1-y}.
\end{aligned} \tag{56}$$

The first set of non-trivial integrations are those over x_p and x'_q, which involve two poles. It should be pointed out that this is not a true double pole and are indeed two separate single poles: one may simply take the sum of the residues at the two poles. Isolating the integration over x_p, we obtain,

$$
\begin{aligned}
&\int \frac{dx_p}{2\pi} \frac{-ie^{-ix_p p^+\left(y_p^- - y_{(p+1)}^-\right)}}{\left[\sum_{i=0}^{p} x_i - x_D^i - x_B - i\epsilon\right]} \frac{-i}{\left[\left\{\sum_{i=0}^{p} x_i - \frac{x_D^i + (x_T^m)^i}{1-y}\right\} - x_B - x_L^p + \left\{\sum_{i=1}^{m} z_i + \frac{(z_L^m)_i}{1-y}\right\} - i\epsilon\right]} \\
&= -i\theta\left(y_{(p+1)}^- - y_p^-\right) e^{ip^+ \sum_{i=0}^{p-1} x_i \left(y_p^- - y_{(p+1)}^-\right)} e^{-ip^+\left(x_B + \left\{\sum_{i=0}^{p} x_D^i\right\}\right)\left(y_p^- - y_{(p+1)}^-\right)} \qquad (57) \\
&\times \left[\frac{e^{-ip^+\left(x_L^p + \left\{\sum_{i=0}^{p} \frac{y x_D^i + (x_T^m)^i}{1-y}\right\} - \left\{\sum_{i=1}^{m} z_i + \frac{(z_L^m)_i}{1-y}\right\}\right)\left(y_p^- - y_{(p+1)}^-\right)}}{x_L^p + \left\{\sum_{i=0}^{p} \frac{y x_D^i + (x_T^m)^i}{1-y}\right\} - \left\{\sum_{i=1}^{m} z_i + \frac{(z_L^m)_i}{1-y}\right\} - i\epsilon} - \frac{1}{x_L^p + \left\{\sum_{i=0}^{p} \frac{y x_D^i + (x_T^m)^i}{1-y}\right\} - \left\{\sum_{i=1}^{m} z_i + \frac{(z_L^m)_i}{1-y}\right\} + i\epsilon}\right].
\end{aligned}
$$

There is a completely analogous expression for the integral over x_q' in the amplitude which involves the integral over two poles. The remaining integrations over the x_i for $i < p$ and over x_j' with $j < q$ are rather trivial and consist of simple contour integrations as performed in Refs. [12, 35]. When the set of propagators and associated phase factors in the equation above is combined with the propagators and phase factors in the gluon propagators there exist two poles for the momentum fraction z_1 in the complex conjugate and z_1' in the amplitude. Isolating the integrand of the z_1 integration, we obtain,

$$
\begin{aligned}
&\int \frac{dz_1}{2\pi} \frac{i}{\sum_{i=1}^{m} z_i - \frac{(z_L^m)_i}{y} + i\epsilon} \left[\frac{ie^{-ip^+\left(x_L^p + \left\{\sum_{i=0}^{p} \frac{y x_D^i + (x_T^m)^i}{1-y}\right\} - \left\{\sum_{i=1}^{m} z_i + \frac{(z_L^m)_i}{1-y}\right\}\right)\left(y_p^- - y_{(p+1)}^-\right)} e^{-ip^+ z_1\left(\zeta_1^- - y_{p+1}^-\right)}}{\left\{\sum_{i=1}^{m} z_i + \frac{(z_L^m)_i}{1-y}\right\} - x_L^p - \left\{\sum_{i=0}^{p} \frac{y x_D^i + (x_T^m)^i}{1-y}\right\} + i\epsilon}\right. \\
&\left. - \frac{ie^{-ip^+ z_1\left(\zeta_1^- - y_{p+1}^-\right)}}{\left\{\sum_{i=1}^{m} z_i + \frac{(z_L^m)_i}{1-y}\right\} - x_L^p - \left\{\sum_{i=0}^{p} \frac{y x_D^i + (x_T^m)^i}{1-y}\right\} - i\epsilon}\right] \\
&= \theta(\zeta_1^- - y_p^-) e^{-ip^+\left(x_L^p + \left\{\sum_{i=0}^{p} \frac{y x_D^i + (x_T^m)^i}{1-y}\right\} - \left\{\sum_{i=2}^{m} z_i + \sum_{i=1}^{m} \frac{(z_L^m)_i}{1-y}\right\}\right)\left(y_p^- - y_{(p+1)}^-\right)} \\
&\times \left\{\frac{ie^{-ip^+\left(-\sum_{i=2}^{m} z_i + \sum_{i=1}^{m} \frac{(z_L^m)_i}{y}\right)\left(\zeta_1^- - y_p^-\right)} - ie^{-ip^+\left(-\sum_{i=2}^{m} z_i - \sum_{i=1}^{m} \frac{(z_L^m)_i}{1-y} + x_L^p + \left\{\sum_{i=0}^{p} \frac{y x_D^i + (x_T^m)^i}{1-y}\right\}\right)\left(\zeta_1^- - y_p^-\right)}}{\sum_{i=1}^{m} \frac{(z_L^m)_i}{y(1-y)} - x_L^p - \left\{\sum_{i=0}^{p} \frac{y x_D^i + (x_T^m)^i}{1-y}\right\}}\right\} \\
&- \left\{\frac{\theta(\zeta_1^- - y_{p+1}^-) ie^{-ip^+\left(-\sum_{i=2}^{m} z_i + \sum_{i=1}^{m} \frac{(z_L^m)_i}{y}\right)\left(\zeta_1^- - y_{p+1}^-\right)} + \theta(y_{p+1}^- - \zeta_1^-) ie^{-ip^+\left(-\sum_{i=2}^{m} z_i - \sum_{i=1}^{m} \frac{(z_L^m)_i}{1-y} + x_L^p + \left\{\sum_{i=0}^{p} \frac{y x_D^i + (x_T^m)^i}{1-y}\right\}\right)\left(\zeta_1^- - y_{p+1}^-\right)}}{\sum_{i=1}^{m} \frac{(z_L^m)_i}{y(1-y)} - x_L^p - \left\{\sum_{i=0}^{p} \frac{y x_D^i + (x_T^m)^i}{1-y}\right\}}\right\} \\
&= \frac{e^{ip^+\left(\sum_{i=2}^{m} z_i - \sum_{i=1}^{m} \frac{(z_L^m)_i}{y}\right)\left(\zeta_1^- - y_{p+1}^-\right)}}{(-x_L^{pm})} \left[\theta(\zeta_1 - y_p^-)\left\{ie^{-ip^+ x_L^{pm}(y_p^- - y_{p+1}^-)} - ie^{-ip^+ x_L^{pm}(\zeta_1^- - y_{p+1}^-)}\right\}\right. \\
&\left. - i\theta(\zeta_1^- - y_{p+1}^-) - i\theta(y_{p+1}^- - \zeta_1^-) e^{-ip^+ x_L^{pm}(\zeta_1^- - y_{p+1}^-)}\right]. \qquad (58)
\end{aligned}
$$

Where the aptly named momentum fraction x_L^{pm} is expressed as

$$
x_L^{pm} = x_L^p + \sum_{i=0}^{p} \frac{y x_D^i + (x_T^m)_i}{1-y} - \sum_{i=1}^{m} \frac{(z_L^m)_i}{y(1-y)} = \frac{\left|l_\perp - \sum_{i=1}^{m} k_\perp^i - y \sum_{j=0}^{p} p_\perp^j\right|^2}{2p^+ q^- y(1-y)}. \qquad (59)
$$

Similarly, the integral on the amplitude side is given as

$$\frac{e^{-ip^+\left(\sum\limits_{k=2}^{n} z'_k - \frac{(z'^n_L)_k}{y}\right)(\zeta'^-_1 - y'^-_{q+1})}}{x'^{qn}_L}\left[\theta(\zeta'_1 - y'^-_q)\left\{ie^{ip^+x'^{qn}_L(y'^-_q - y'^-_{q+1})} - ie^{ip^+x'^{qn}_L(\zeta'^-_1 - y'^-_{q+1})}\right\}\right.$$
$$\left. -\ i\theta(\zeta'^-_1 - y'^-_{q+1}) - i\theta(y'^-_{q+1} - \zeta'^-_1)e^{ip^+x'^{qn}_L(\zeta'^-_1 - y'^-_{q+1})}\right], \tag{60}$$

where the momentum fraction x'^{qn}_L on the amplitude side is given as,

$$x'^{qn}_L \ = \ \frac{\left|l_\perp - \sum\limits_{l=1}^{n} k'^l_\perp - y\sum\limits_{k=0}^{q} p'^k_\perp\right|^2}{2p^+q^-y(1-y)}. \tag{61}$$

The remaining integrations over the variables z_2 to z_m, z'_s to z'_n, x_{p-1} to x_0 and x'_{q-1} to x'_0 can now be performed to complete all contour integrations. The resulting expression for $\mathfrak{D}^{NN}_{q,p;n,m}$ is now completely ordered in the negative light-cone direction, i.e., all the y^-, y'^-, ζ^-, ζ'^- locations are strongly ordered:

$$\mathfrak{D}^{NN}_{q,p;n,m} = \prod_{i=1}^{N-m}\prod_{k=1}^{N-n} d^3y_i d^3y'_k \frac{d^2p^{i-1}_\perp}{(2\pi)^2}\frac{d^2p'^{\,k-1}_\perp}{(2\pi)^2}\frac{dyd^2l_\perp}{(2\pi)^3y}\frac{d^2p^{(N-m)}_\perp}{(2\pi)^2}\prod_{i=1}^{m}\prod_{j=1}^{n} d^3\zeta_i d^3\zeta'_j \frac{d^2k^i_\perp}{(2\pi)^3}\frac{d^2k'^k_\perp}{(2\pi)^3}d^2l_{q\perp}$$
$$\times\ \delta^2\left(l_{q\perp} + l_\perp - \sum_{i=0}^{(N-m)} p^i_\perp - \sum_{j=1}^{m} k^j_\perp\right)\left(\frac{1}{2q^-}\right)^{p+1}\frac{1}{2q^-(1-y)}\frac{1}{x^{pm}_L}\frac{1}{2p^+}\left(\frac{1}{2q^-(1-y)}\right)^{N-m-p-1}\frac{1}{2q^-(1-y)}$$
$$\times\ \left(\frac{1}{2q^-(1-y)}\right)^{N-n-q-1}\frac{1}{2p^+}\frac{1}{x'^{qn}_L}\frac{1}{2q^-(1-y)}\left(\frac{1}{2q^-}\right)^{q+1}\left(\frac{1}{2q^-}\right)^{n}\left(\frac{1}{2q^-}\right)^{m}$$
$$\times\ \left[\theta(\zeta^-_1 - y^-_p)\left\{e^{-ip^+x^{pm}_L y^-_p} - e^{-ip^+x^{pm}_L\zeta^-_1}\right\} - \theta(\zeta^-_1 - y^-_{p+1})e^{-ip^+x^{pm}_L y^-_{p+1}} - \theta(y^-_{p+1} - \zeta^-_1)e^{-ip^+x^{pm}_L\zeta^-_1}\right]$$
$$\times\ \left[\theta(\zeta'^-_1 - y'^-_q)\left\{e^{ip^+x'^{qn}_L y'^-_q} - e^{ip^+x'^{qn}_L\zeta'^-_1}\right\} - \theta(\zeta'^-_1 - y'^-_{q+1})e^{ip^+x'^{qn}_L y'^-_{q+1}} - \theta(y'^-_{q+1} - \zeta'^-_1)e^{ip^+x'^{qn}_L\zeta'^-_1}\right]$$
$$\times\ e^{-ix_Bp^+y^-_0 + p^0_\perp\cdot y^0_\perp}\prod_{i=1}^{p}\theta(y^-_i - y^-_{i-1})e^{-ix^i_Dp^+y^-_i + ip^i_\perp\cdot y^i_\perp}\prod_{i=p+1}^{N-m}\theta(y^-_i - y^-_{i-1})e^{-ip^+y^-_i\left(\frac{x^i_D + (x^m_T)^i - \delta x^i_L}{1-y}\right) + ip^i_\perp\cdot y^i_\perp}$$
$$\times\ \theta(\zeta^-_m > \zeta^-_{m-1} > \ldots > \zeta^-_1)\prod_{i=1}^{m} e^{-ip^+\zeta^-_i\frac{(z^m_L)_i}{y} + ik^i_\perp\cdot\zeta^i_\perp}\theta(\zeta'^-_n > \zeta'^-_{n-1} > \ldots > \zeta'^-_1)\prod_{j=1}^{n} e^{ip^+\zeta'^-_j\frac{(z'^n_L)_j}{y} - ik'^j_\perp\cdot\zeta'^j_\perp}$$
$$\times\ \prod_{j=1}^{q}\theta(y'^-_j - y'^-_{j-1})e^{ix'^j_Dp^+y'^-_j - ip'^j_\perp\cdot y'^j_\perp}\prod_{j=q+1}^{N-n}\theta(y'^-_j - y'^-_{j-1})e^{ip^+y'^-_j\left(\frac{x'^j_D + (x'^n_T)^j - \delta x'^j_L}{1-y}\right) - ip'^j_\perp\cdot y'^j_\perp}. \tag{62}$$

In the equation above, the exponentials containing the dot products of transverse positions and momenta have changed. We have defined a new transverse momentum,

$$\vec{p'}^{N-n}_\perp \ = \ \sum_{i=0}^{N-m}\vec{p}^i_\perp + \sum_{j=1}^{m}\vec{k}^j_\perp - \sum_{i=0}^{N-n-1}\vec{p'}^i_\perp - \sum_{j=1}^{n}\vec{k'}^i_\perp. \tag{63}$$

Since there is no integration over $y'^0_\perp$, we may shift the $p'^0_\perp$ integration as $d^2p'^0_\perp \to d^2p'^{N-n}_\perp$. This allows the nesting of the various interactions, where transverse momenta from the amplitude and complex conjugate may be combined in pairs. This has to be done in a way so as to obtain the maximal enhancement in length and color and will be carried out in the next section.

V. COLOR FACTORS AND LENGTH ENHANCEMENT OF MATRIX ELEMENTS

With the completion of the contour integrals over both light cone components of the incoming momenta and the simplification of the numerator Dirac structure, there remains the sum over all interactions and the simplification of the color factors. Unlike the case of photon radiation, the color factor in this case is very diagram dependent and depends strongly on where the radiated gluons attach to the quark line as well as the number of cross connections between the scattering of the gluon and that of the final quark. The determination of the color factor has to be carried out in tandem with the estimation of the length enhancement achieved by each diagram. As a result, in contrast to the case of photon radiation the first step is to decompose the expectation of the $2N$ gluon operators in the nucleon state. This is followed by an evaluation of its color factor and length enhancement. Only after this can the sum over emission points of the radiated gluon be carried out.

A. Color factor and length enhancement for the fully symmetric diagram

Given the different color factors for different diagrams we need an organization scheme. We begin with the completely symmetric case where $m = n < N$ and $p = q < N$. In this case we may nest the various interactions as indicated by the diagram in Fig. 3. Each blob represents one nucleon off which the quark or gluon may scatter. As in the case of transverse broadening [12] and photon production [13], we will ignore the expectation of more than two operators per nucleon state. We will not explicitly consider the expectation of two successive gluon operators on the same side of the cut line in the same nucleon state. These do not induce any change in the transverse momentum distribution and constitute unitarity corrections. They are included in the final result by a normalization of the resulting transverse momentum distribution [12].

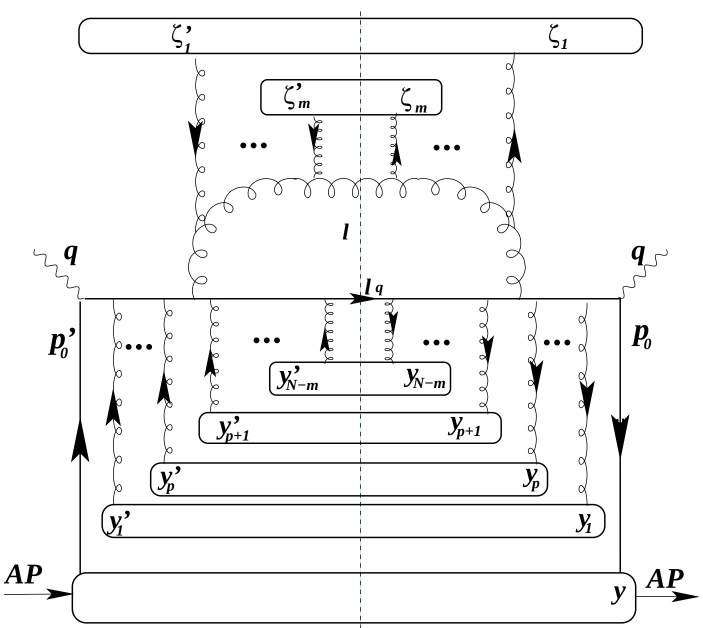

FIG. 3: DIS on a nucleus with a produced quark and an emitted gluon scattering N times symmetrically in the amplitude and complex conjugate.

The decomposition of the nuclear state as indicated by Fig. 3, may be carried out as in Refs. [12, 13]: The nucleus is approximated as a weakly interacting homogeneous gas of nucleons. Such an approximation is only sensible at very high energy, where, due to time dilation, the nucleons appear to travel in straight lines almost independent of each other over the interval of the interaction of the hard probe. In a sense, all forms of correlators between nucleons (spin, momentum, etc.) are assumed to be rather suppressed. As a result, the expectation of the $2N$ gluon and 2 quark operators in the nuclear state may be decomposed as

$$
\begin{aligned}
&\langle A;p|\bar{\psi}(y^-,y_\perp)\gamma^+\psi(0)\prod_{i=1}^{2N}A^+_{a_i}(y_i)|A;p\rangle \\
&= AC^A_{p_1}\langle p_1|\bar{\psi}(y^-,y_\perp)\gamma^+\psi(0)\prod_{i=1}^{2N}A^+_{a_i}(y_i)|p_1\rangle \\
&+ C^A_{p_1,p_2}\langle p_1|\bar{\psi}(y^-,y_\perp)\gamma^+\psi(0)|p_1\rangle \\
&\times \langle p_2|\prod_{i=1}^{2N}A^+_{a_i}(y_i)|p_2\rangle + \ldots, \qquad (64)
\end{aligned}
$$

where, the factor $C^A_{p_1}$ represents the probability to find a nucleon in the vicinity of the location $\vec{y}$, which is a number of order unity (it is the probability that one of A nucleons distributed in a volume of size cA will be found within a nucleon size sphere centered at $\vec{y}$). The remaining coefficients $C^A_{p_1,\ldots}$ represent the weak position correlations between different nucleons. The overall factor of A arises from the determination of the origin (the location 0 in the equation above) in the nucleus, which may be situated on any of the A nucleons. Solely for the current discussion, we reintroduce the quark operators $\bar{\psi}(y^-,y_\perp)$ and $\psi(0)$ in the above equation.

It is clear from the above decomposition that the largest contribution arises from the term where the expectation of each partonic operator is evaluated in separate nucleon states as the $\vec{y}_i$ integrations may be carried out over the nuclear volume. As a nucleon is a color singlet, any combination of quark or gluon field strength insertions in a nucleon state must itself be restricted to a color singlet combination. As a result, the expectation of single partonic operators in nucleon states is vanishing. The first (and hence largest) non-zero contribution emanates from the terms where the quark operators in the singlet color combination are evaluated in a nucleon state and the $2N$ gluons are divided into pairs of singlet combinations, with each singlet pair evaluated in a separate nucleon state. This reduces the discussion to $N + 1$ nucleons, one in which the original quark is struck by the virtual photon and the remaining N off which the struck quark scatters on its way out of the nucleus. As has been demonstrated in Ref. [36], under the assumption that there are no correlations between the nucleons, the combinatorial factor reduces to,

$$C^A_{p_1,\ldots,p_{N+1}} = AC^A_{p_1}\left(\frac{\rho}{2p^+}\right)^N, \tag{65}$$

where, $C^A_{p_1}$ varies with the nuclear density profile ρ and is equal to unity within the nuclear radius for a hard sphere distribution.

Further simplifications arise in the evaluation of gluon pairs in a singlet combination in the nucleon states by carrying out the $y_\perp$ integrations. The basic object under consideration is (ignoring the longitudinal positions and color indices on the vector potentials)

$$\begin{aligned}
&\int d^2y^i_\perp d^2y'^j_\perp \langle p|A^+(\vec{y}^i_\perp)A^+(\vec{y'}^j_\perp)|p\rangle \\
&\times\; e^{-ix^ip^+y^-_i} e^{ip^i_\perp\cdot y^i_\perp} e^{ix'^jp^+y'^-_j} e^{-ip'^j_\perp\cdot y'^j_\perp} \\
&=\; (2\pi)^2\delta^2(\vec{p}^i_\perp - \vec{p'}^j_\perp)\int d^2y_\perp e^{-ix^ip^+(y^-_i - y'^-_j)} \\
&\times\; e^{ip_\perp\cdot y_\perp}\langle p|A^+(\vec{y}_\perp/2)A^+(-\vec{y}_\perp/2)|p\rangle,
\end{aligned} \tag{66}$$

where, $y_\perp$ is the transverse gap between the two gluon insertions and $p_\perp = (p^i_\perp + p^j_\perp)/2$. The fractions x_i and x'_i represent any of the possible types of momentum fractions x_D, x_T, x_L and any combination thereof which appear in the expression for $\mathcal{O}^{N,N}_{p,p;m,m}$. The physics of the above equation is essentially the transverse translation symmetry of the two gluon correlator in a very large nucleus.

One will note that the two dimensional delta function over the transverse momenta has removed an integration over the transverse area of the nucleus thus reducing the overall A enhancement that may be obtained. This is then used to equate the transverse momenta emanating from the two gluon insertions in the amplitude and complex conjugate amplitude. This also simplifies the longitudinal phase factors which now depends solely on the longitudinal positions of the two gluon insertions. The assumption of short distance color correlation length (an assumption valid both in the deconfined plasma as well as in color confined cold nuclear matter) leads to the constraint that the longitudinal positions of the gluon operators from the amplitude and complex conjugate, whose expectation values are sought in the same nucleon state, cannot be more further apart than the color correlation length, i.e.,

$$\begin{aligned}
&\int dy^- dy'^- \langle p|A^+(y^-,\vec{y}_\perp/2)A^+(y'^-,-\vec{y}_\perp/2)|p\rangle \\
\simeq &\int dy^- y^-_c \langle p|A^+(y^-,\vec{y}_\perp/2)A^+(y^-,-\vec{y}_\perp/2)|p\rangle
\end{aligned} \tag{67}$$

In other words the two locations y^- and y'^- have been restricted to be in close proximity of each other and each single nucleon expectation of two gluon operators may introduce at most one factor of length enhancement from the remaining y^- integration.

The requirement that the two gluon operators be in the color singlet state, may be enforced by averaging over the colors of the two gluon operators,

$$\langle p|A^+_a A^+_b|p\rangle = \frac{\delta^{ab}}{(N_c^2-1)}\langle p|A^+_a A^+_a|p\rangle. \tag{68}$$

A similar constraint on the quark antiquark expectation, yields,

$$\langle p|\bar{\psi}_i\gamma^+\psi_j|p\rangle = \frac{\delta_{ij}}{N_c}\langle p|\bar{\psi}\gamma^+\psi|p\rangle. \tag{69}$$

The expectation of the quark operators as well as the associated phase factor $\exp\left(-ix_Bp^+y^-_0 + ip^0_\perp\cdot y^0_\perp\right)$ and combinatorial factors $AC^A_{p_1}$ may now be extracted and combined to form the nuclear structure function of the struck quark. Given these simplifications, the full expression for the multiple scattering of the struck quark off N gluons in the symmetric pattern of Fig. 3, $\mathcal{O}^{NN}_{p,p;m,m}$, may be written as,

$$\mathcal{O}^{NN}_{p,p;m,m} = \prod_{i=1}^{N-n} dy_i^- \frac{d^2p^i_\perp}{(2\pi)^2} \frac{dy d^2 l_\perp}{(2\pi)^3 y} \prod_{i=1}^{m} d\zeta_i^- \frac{d^2k^i_\perp}{(2\pi)^2} d^2 l_{q\perp} \frac{\delta^2\left(l_{q\perp} + l_\perp - \sum_{i=0}^{(N-m)} p^i_\perp - \sum_{j=1}^{m} k^j_\perp\right) P(y)}{\left(\vec{l}_\perp - \sum_{i=1}^{m} \vec{k}^i_\perp - y \sum_{i=0}^{p} \vec{p}^i_\perp\right)^2}$$
$$\times \left[\theta(\zeta_1^- - y_p^-)\left\{2 - e^{-ip^+ x_L^{pm}(\zeta_1^- - y_p^-)} - e^{ip^+ x_L^{pm}(\zeta_1^- - y_p^-)}\right\} - \theta(\zeta_1^- - y_{p+1}^-)\left\{2e^{-ip^+ x_L^{pm}(y_{p+1}^- - y_p^-)}\right.\right.$$
$$\left.\left. - \ 2e^{-ip^+ x_L^{pm}(y_{p+1}^- - \zeta^-)}\right\} - \theta(y_{p+1}^- - \zeta_1^-)\theta(\zeta_1^- - y_p^-)\left\{2e^{-ip^+ x_L^{pm}(\zeta_1^- - y_p^-)} - 2\right\}\right]$$
$$\times \left(\frac{\rho}{2p^+}\right)^N \prod_{i=1}^{p} \left\{\theta(y_i^- - y_{i-1}^-) e^{ip^i_\perp \cdot y^i_\perp} \frac{\delta^{a_i a_i'} \langle p|A^{o_i} A^{o_i}|p\rangle}{N_c^2 - 1}\right\} \prod_{i=p+1}^{N-m} \left\{\theta(y_i^- - y_{i-1}^-) e^{ip^i_\perp \cdot y^i_\perp}\right.$$
$$\times \left.\frac{\delta^{a_i a_i'} \langle p|A^{o_i} A^{o_i}|p\rangle}{N_c^2 - 1}\right\} \theta(\zeta_m^- > \zeta_{m-1}^- > \ldots > \zeta_1^-) \prod_{i=1}^{m} e^{ik^i_\perp \cdot \zeta^i_\perp} \frac{\delta^{c_i c_i'} \langle p|A^{o_i} A^{o_i}|p\rangle}{N_c^2 - 1}$$
$$\times \frac{1}{N_c} \mathbf{Tr}\left[\prod_{i=1}^{p} t^{a_i} t^{b_0} \prod_{i=p+1}^{N-m} t^{a_i} \prod_{i=N-m}^{p} t^{a_i'} t^{b_0'} \prod_{i=p}^{1} t^{a_i'}\right] \prod_{i=1}^{m} (f^{b_{i-1} c_i b_i}) \delta^{b_n b_n'} \prod_{i=m}^{1} (-f^{b_i' c_i' b_{i-1}'}), \quad (70)$$

where, $\langle p|A^{o_i} A^{o_i}|p\rangle$ is short hand for

$$\int d\delta y_i e^{-ip^+ x^i_G \delta y_i^-} \quad (71)$$
$$\times \ \langle p|A^{o_i}(y_i^- + \delta y_i^-/2, y^i_\perp/2) A^{o_i}(y_i^- - \delta y_i^-/2, -y^i_\perp/2)|p\rangle.$$

The overall color factor for the Feynman rule above may be evaluated by calculating the trace over the color matrices and dividing by the factors obtained from averaging over colors, which yields,

$$C^{N,N}_{p,p;m,m} = \frac{(C_F)^{(N-m+1)} C_A^m}{N_c (N_c^2 - 1)^N} = \frac{\left(\frac{(N_c^2-1)}{2N_c}\right)^{(N-m+1)} N_c^m}{N_c (N_c^2 - 1)^N} = \frac{N_c^2 - 1}{2N_c} \left(\frac{1}{2N_c}\right)^{N-m} \left(\frac{N_c}{N_c^2 - 1}\right)^m . \quad (72)$$

Under the assumption that the parton propagates through a long extended medium, matrix elements are enhanced by the length traversed through the medium. Without any constraints except those of Eq. (67) which connect the scattering locations on the amplitude with those on the complex conjugate, the maximal length enhancement factor is

$$\left(\int_0^L dy^-\right)^N = L^N. \quad (73)$$

One can isolate the leading length enhanced contribution for a given configuration by analyzing the various θ-function constraints that limit the length integrations. For the diagram of Fig. 3 corresponding to Eq. (70), we note that all the scattering points on the quark and the gluon are ordered in the $(-)$-light-cone coordinate i.e., $\zeta_m^- > \ldots > \zeta_1^-$, $y_{N-m}^- > \ldots > y_0^-$. In addition the gluon scattering points for the first term in square brackets are forced to be ahead of y_p^- on the quark, i.e., $\zeta_1^- > y_p^-$. For the second term in square brackets $\zeta_1^- > y_{p+1}^-$, while for the third term $y_{p+1}^- > \zeta_1^- > y_p^-$.

The largest length factor for each of the terms may be estimated by performing all integrations over the entire length with a unit integrand. For a single *stream* of constrained integrals, this procedure yields the well known result,

$$\int_0^L dy_0 \int_{y_0}^L dy_1 \ldots \int_{y_{N+1}}^L dy_N = \frac{\left(\int_0^L dy\right)^N}{N!}. \quad (74)$$

For the case of two separate streams joined at a point y_p as in the case the first term in the square brackets of Eq. (70), this procedure yields the length enhancement factor,

$$L^{N,N}_{p,p;m,m;1} = \frac{(N-p)! \left(\int_0^L dy\right)^N}{(N-m-p)!(m)!N!}. \quad (75)$$

To make it easier to analyze these factors we make the replacement, $m_q = N - m - p$ (m_q is the number of times the quark scatters after radiating the gluon), which yields,

$$L^{N,N}_{p,p;m,m;1} = \frac{(m + m_q)! \left(\int_0^L dy\right)^N}{m_q! m! N!}. \quad (76)$$

Similarly, the length enhancement factor for the second term in square brackets is given as,

$$L^{N,N}_{p,p;m,m;2} = \frac{(N-p-1)!\left(\int_0^L dy\right)^N}{(N-m-p-1)!(m)!N!} = \frac{(m+m_q-1)!\left(\int_0^L dy\right)^N}{(m_q-1)!(m)!N!}. \quad (77)$$

Finally the length enhancement factor for the third factor with the square brackets of Eq. (70) may be obtained by noting that ζ_1^- is locked between y_p^- and y_{p+1}^-, thus this is similar to the the first length enhancement factor but with the replacements $m \to m-1$, $p \to p+1$, i.e.,

$$L^{N,N}_{p,p;m,m;3} = \frac{(N-p-1)!\left(\int_0^L dy\right)^N}{(N-m-p)!(m-1)!N!} = \frac{(m+m_q-1)!\left(\int_0^L dy\right)^N}{(m_q)!(m-1)!N!}. \quad (78)$$

Inspection of these terms suggests that the largest contributions arise from the case where $m+m_q \to N$, i.e., most of the scattering happens after the gluon has been radiated. We can now complete the analysis of this term by expanding the transverse dependence of the hard part. In this case, the hard part consists of everything except the matrix elements and the associated phase factors which contain the transverse momentum controlled by the matrix elements. The first step is to institute the condition that $l_\perp \gg k^i_\perp$. This simplifies the momentum fractions x_L^{pm} which appear in the phase factors to $x_L = l_\perp^2/(2p^+q^-y(1-y))$. One can also expand the sum of transverse momenta in the denominator as,

$$\frac{1}{\left(\vec{l}_\perp - \sum_{i=1}^n \vec{k}^i_\perp - y\sum_{i=1}^p \vec{p}^i_\perp\right)} = \frac{1}{l_\perp^2} + \frac{2\vec{l}_\perp\cdot\left(\sum_{i=1}^n \vec{k}^i_\perp + y\sum_{i=1}^p \vec{p}^i_\perp\right)}{l_\perp^4} - \frac{\left(\sum_{i=1}^n \vec{k}^i_\perp + y\sum_{i=1}^p \vec{p}^i_\perp\right)^2}{l_\perp^4} - \dots \quad (79)$$

Since contributions are dominated by smaller values of p, as a first approximation, one may neglect the contribution with large p and focus on the $p=0$ limit. Note that in the case of multiple emissions leading to the change of a fragmentation function, the momentum fractions carried out by the gluon is restricted to a small fraction of unity. This consideration adds additional weight to the neglect of these terms, especially for the case of leading hadron suppression calculations. We may further shift the $l_{q\perp}$ integration as $l_{q\perp} = l_{q\perp} - l_\perp$, to represent the difference in the transverse momentum of the radiated gluon and the final quark. Incorporating the above mentioned simplifications, expanding the transverse momentum dependent term above and the transverse momentum dependent δ-function yields,

$$\prod_{i,j=1}^{m,n}\frac{1}{2}\frac{\partial^2}{\partial^2 k^i_\perp}\frac{\partial^2}{\partial^2 p^j_\perp}\left[\left\{\frac{1}{l_\perp^2} + \frac{2\vec{l}_\perp\cdot\left(\sum\limits_{l=1}^m \vec{k}^l_\perp\right) - \left(\sum\limits_{l=1}^m \vec{k}^l_\perp\right)^2}{l_\perp^4} + \frac{\left(2\vec{l}_\perp\cdot\sum\limits_{l=1}^m \vec{k}^l_\perp\right)^2}{l_\perp^6}\right\}\right. \quad (80)$$
$$\left.\times\,\delta^2\left(\vec{l}_{q\perp} - \sum_{l=1}^m \vec{k}^l_\perp - \sum_{l=1}^n \vec{p}^l_\perp\right)\right]_{\vec{k}^i_\perp=\vec{p}^j_\perp=0}|\vec{k}^i_\perp|^2|\vec{p}^j_\perp|^2$$
$$= \left(\frac{1}{2}\right)^N\left[\frac{\left(\nabla^2_{l_{q\perp}}\right)^N\delta^2\left(\vec{l}_{q\perp}\right)}{l_\perp^2} - \frac{2m\vec{l}_\perp\cdot\nabla_{l_{q\perp}}\left(\nabla^2_{l_{q\perp}}\right)^{N-1}\delta^2\left(\vec{l}_{q\perp}\right)}{l_\perp^4} + \frac{4m\left(\nabla^2_{l_{q\perp}}\right)^{N-1}\delta^2\left(\vec{l}_{q\perp}\right)}{l_\perp^4}\right]\prod_{i,j=1}^{m,m_q}|\vec{k}^i_\perp|^2|\vec{p}^j_\perp|^2.$$

The factors of incoming transverse momentum, both on the gluon line ($|k^i_\perp|^2$) as well as those on the quark line ($|p^j_\perp|^2$) may be combined with the position dependent matrix elements to yield the Lorentz force correlator obtained in Ref. [12] (suppressing color indices and longitudinal momentum and position arguments):

$$\int d^2y^i_\perp|\vec{k}^i_\perp|^2 e^{i\vec{k}^i_\perp\cdot\vec{y^i}_\perp}\langle p|A^+(y_\perp/2)A^+(-y_\perp/2)|p\rangle \quad (81)$$
$$= \int d^2y^i_\perp e^{i\vec{k}^i_\perp\cdot\vec{y^i}_\perp}\frac{1}{2}\langle p|F^{+,\alpha_i}(y_\perp/2)F^+_{\alpha_i,}(-y_\perp/2)|p\rangle.$$

In the equation above, F^{+,α_i} represent the gluon field strength operators. It should be noted that α_i only runs over the transverse components. In the full expression for Eq. (3), the incoming transverse momenta $k^i_\perp$, $p^j_\perp$ appear only in the exponentials and may be integrated out to confine the transverse locations to the origin. Hence we define the longitudinal position dependent coefficient,

$$\bar{D}(y_i^-) = 2\pi^2\alpha_s\int\frac{d\delta y_i^-}{2\pi}\int\frac{d^2k_\perp}{(2\pi)^2}d^2y^i_\perp e^{-i\frac{|\vec{k}^i_\perp|^2}{2q^-}y_i^- + i\vec{k}^i_\perp\cdot\vec{y^i}_\perp}$$
$$\times\langle p|F^{+\mu}(y_i^- + \delta y_i^-, y^i_\perp/2)F^+_\mu(y_i^-, -y^i_\perp/2)|p\rangle \quad (82)$$
$$\simeq 2\pi^2\alpha_s\int\frac{d\delta y_i^-}{2\pi}\langle p|F^{+\mu}(y_i^- + \delta y_i^-)F^+_\mu(\delta y_i^-)|p\rangle.$$

The last line results when we ignore the longitudinal momentum fraction $|\vec{k}^i_\perp|^2/2q^-$. This is a rather drastic approximation, as the length y_i^- may be long enough that this factor may not be negligible. Ignoring this term should be taken as a short hand for the diffusion coefficient $\bar{D}(y_i^-)$. While, in the remaining part of the paper, we will express $\bar{D}$ without the longitudinal momentum fraction, it should always be understood to be present. The full definition of D should always be understood to be given by the first two lines of Eq. (82).

This last simplification allows one to express the cross section for N symmetric scattering with no scattering prior to emission (i.e., restricting to only the first θ-function of Eq. (70) with $p \to 0$) as,

$$\begin{aligned}
\mathcal{O}^{N,N}_{0,0;m,m} &= \int \frac{dy d^2 l_\perp}{(2\pi)^3} d^2 l_q \frac{4\pi\alpha_s C_F P(y)}{l_\perp^2 y} \left[\frac{\left(\nabla^2_{l_{q\perp}}\right)^N \delta^2\left(\vec{l}_{q\perp}\right)}{N!} \frac{N! C_A^m C_F^{N-m}}{m!(N-m)!} \left(\frac{\rho \int_0^{L^-} dy_i^- \bar{D}(y_i^-)}{2p^+(N_c^2-1)} \right)^{N-1} \right. \\
&+ \left\{ -\frac{2\vec{l}_\perp \cdot \nabla_{l_{q\perp}} \left(\nabla^2_{l_{q\perp}}\right)^{N-1} \delta^2\left(\vec{l}_{q\perp}\right)}{(N-1)! l_\perp^2} + \frac{4\left(\nabla^2_{l_{q\perp}}\right)^{N-1} \delta^2\left(\vec{l}_{q\perp}\right)}{(N-1)! l_\perp^2} \right\} \\
&\times \left. \frac{(N-1)!}{(m-1)!(N-m)!} C_A^m C_F^{N-m} \left(\frac{\rho \int_0^L dy^- \bar{D}(y^-)}{2p^+(N_c^2-1)} \right)^{N-1} \right] \\
&\times \rho \int d\zeta_1^- \frac{\bar{D}(\zeta_1^-)}{2p^+(N_c^2-1)} \left\{ 2 - 2\cos\left(p^+ x_L \zeta_1^-\right) \right\}
\end{aligned} \quad (83)$$

Summing over m for fixed N combines the gluon and quark color factors, as

$$\sum_{m=0}^{N} \frac{N!}{m!(N-m)!} C_A^m C_F^{N-m} = (C_A + C_F)^N. \quad (84)$$

As in the case of photon production we absorb the color, phase and density factors into the definition of two new diffusion coefficients:

$$\begin{aligned}
D &= \frac{(C_F + C_A)\rho\bar{D}}{(2p^+(N_c^2-1))}, \quad (85) \\
E(x_L)^\pm &= (C_F + C_A)\rho \int d\zeta_1^- \frac{\bar{D}(\zeta_1^-)}{2p^+(N_c^2-1)} e^{\pm i p^+ x_L \zeta_1^-}.
\end{aligned}$$

Summing over N allows us to resum the multiple derivatives of the two dimensional delta function as [37],

$$\begin{aligned}
&\sum_{N=0}^{\infty} \frac{\left(\int dy^- D(y^-)\right)^N \left(\nabla^2_{l_{q\perp}}\right)^N}{N!} \delta^2(l_{q\perp}) \\
&= \frac{e^{-\frac{{l_q}_\perp^2}{4\int dy^- D(y^-)}}}{4\pi \int dy^- D(y^-)}. \quad (86)
\end{aligned}$$

Using the above identity one may easily resum Eq. (83) over all scatterings. The final result is a very simple extension of the result of next-to-leading twist.

B. Color factor and length enhancement for more complicated diagrams

After demonstrating the outcome of a rather simple and leading case, the color and length factors of more complicated diagrams will be outlined in this subsection. The entire set of diagrams may be divided into six different types with different color and length enhancement factors. Any diagram may be constructed as a combination of these six types of diagrams. The first and simplest type of scattering are those which occur before the gluon is radiated, as shown in Fig. 4. All scatterings on the amplitude and complex conjugate are correlated in pairs and ordered in time. The general color factor for p scatterings is given as,

$$\begin{aligned}
&\mathbf{Tr}\left[\prod_{i=1}^{m} t^{a'_i} t^{b'_0} \hat{C} t^{a_0} \prod_{i=m}^{1} t^{a_i} \right] \frac{1}{N_c} \prod_{i=1}^{m} \frac{\delta^{a_i a'_i}}{N_c^2-1} \\
&= \frac{C_F^m}{N_c(N_c^2-1)^m} \mathbf{Tr}\left[t^{b'_0} \hat{C} t^{a_0} \right]. \quad (87)
\end{aligned}$$

As all the scatterings are correlated and ordered the diagram has the obvious length enhancement factor of $(y_p)^p/p!$ where we have assumed that the quark starts at the origin and the p^{th} scattering occurs at the location y_p. The $N_c \times N_c$ dimensional operator $\hat{C}$ represents further scattering encountered by the quark and gluon pair after emission. This particular type of scattering has no color entanglement effects with other types of scattering.

The next type of scattering diagrams are sketched in Fig. 5 and involve the interference of scattering that oc-

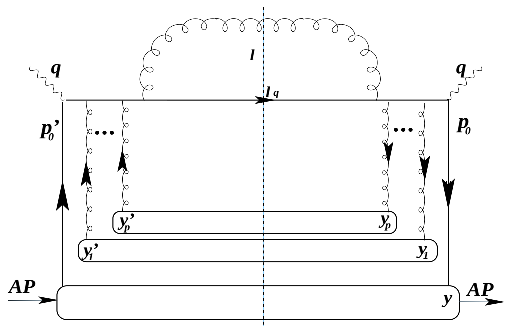

FIG. 4: Simplest example of scattering prior to emission.

curs before the emission of the gluon correlated with scattering on the quark produced after the emission. The color factor for this type of diagram is given as

$$\mathbf{Tr}\left[\prod_{i=1}^{m} t^{a'_i} t^{b'_0} \hat{C} \prod_{i=m}^{1} t^{a_i} t^{a_0}\right] \frac{1}{N_c} \prod_{i=1}^{m} \frac{\delta^{a_i a'_i}}{N_c^2 - 1}$$
$$= \frac{(C_F - C_A/2)^m}{N_c (N_c^2 - 1)^m} \mathbf{Tr}\left[t^{b'_0} \hat{C} t^{a_0}\right]. \qquad (88)$$

Very similar to this type of diagram is the case where

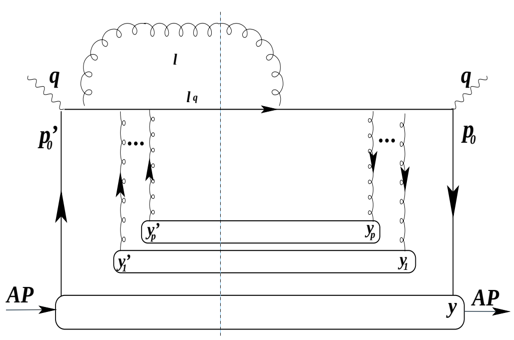

FIG. 5: An interference effect where scattering of the quark before the emission is correlated with scattering of the quark after the emission.

the scattering on the quark before emission are correlated with scattering on the gluon after emission. The color factor for this type of diagram, shown in Fig. 6 is given as,

$$\mathbf{Tr}\left[\prod_{i=1}^{m} (-i t^{a'_i}) t^{b'_0} \hat{C} t^{a_0}\right] \frac{1}{N_c} \prod_{i=1}^{m} f^{b_{i-1} a_i b_i} \delta^{b_0 a_0} \hat{C}_2^{b^m b'_0}$$
$$\times \prod_{i=1}^{m} \frac{\delta^{a_i a'_i}}{N_c^2 - 1} = \frac{(C_A/2)^m \, \mathbf{Tr}\left[t^{b_m} t^{b'_0} \hat{C}\right]}{N_c (N_c^2 - 1)^m} \hat{C}_2^{b^m b'_0}. \qquad (89)$$

Since these scatterings are also ordered and correlated, we find that the length enhancement factor in this case for m scattering is simply $(y_{m+p} - y_p)^m / m!$ where y_p is the location where the scattering begins and y_{p+m} is the location of the last scattering of this type. The color contributions from such diagrams are completely untangled with the case of Figs. 4 and 5 and also with later scattering on the quark and gluon after emission. The operators $\hat{C}$ and $\hat{C}_2^{b^m b'_0}$ indicate further scattering.

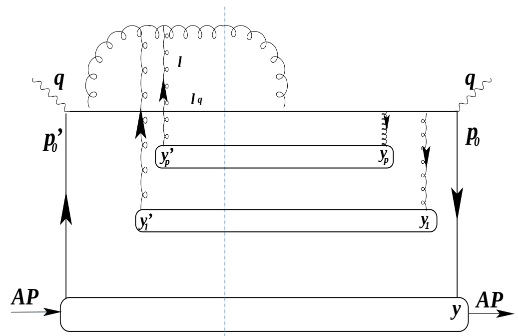

FIG. 6: An interference effect where the scattering of the quark before emission is correlated with the scattering of the gluon after emission.

The remaining three types of scattering are all in the final state. Two of these are rather simple and involve correlated ordered scattering of the final quark or the gluon which yields the obvious multiplicative color factors of $C_F^n/(N_c^2-1)^n$ or $C_A^n/(N_c^2-1)^n$ and have no interference with each other yielding separate length enhancement factors of $(y_f - y_{p+m})^n/n!$. As these are final state effects, y_f is the extent of the medium and y_{p+m} is the location of the last interference term between scattering in the initial and final states. As these color and length factors are rather trivial no diagram corresponding to these is presented.

The most complicated situation, which is ignored by most other formalisms is the interference between quark scattering in the final state and gluon scattering in the final state along with entangled diagrams where such scattering interchanges with ordered correlated scattering on the final quark and gluon. Analyzing the purely interfering case of the type of diagrams shown in Fig. 7 with n scatterings, the length enhancement factor may be immediately surmised to be of the form $(y_f - y_{p+m})^n/n!$. Note that these contributions and those of scattering purely on the final quark and the gluon may become entangled. However at the same order of coupling, the former will always yield a larger length enhancement factor in the limit of multiple scattering.

The color factor for diagrams of the type shown in Fig. 7, may be calculated as,

$$C_n = \mathbf{Tr}\left[t^{c_n} \prod_{i=1}^{n} (-i t^{a'_i}) t^{c^0}\right] \frac{1}{N_c} \prod_{j=0}^{n} f^{c_{j-1} a_j c_j} \frac{\delta^{a_j a'_j}}{N_c^2 - 1} \qquad (90)$$

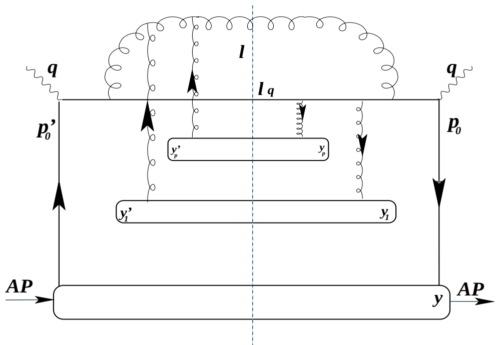

FIG. 7: An interference effect where the scattering of the the quark after emission is correlated with the scattering of the gluon after emission.

There is no general formula for terms of this type. For the case of $n = 1$, we obtain, $C_1 = \frac{C_F i C_A/2}{N_c^2-1}$; for $n = 2$ we obtain $C_2 = 0$. For $n = 3$ we obtain, $C_3 = \frac{iN_c}{N_c(N_c^2-1)^2}$. Further scatterings cannot be done analytically and numerical methods yield that these color factors are very suppressed compared to all other types of scattering at the same order of coupling. These will henceforth be ignored in the remaining analysis. Diagrams with interchanging attachments where the first scattering on a quark in the amplitude interferes with a gluon in the complex conjugate followed by a scattering on the gluon in the amplitude interfering with a scattering on the quark in the complex conjugate will yield similar color factors. There remain the case of entangled diagrams, which may be shown to be suppressed as well and will also be ignored.

VI. THE MULTIPLE SCATTERING RESUMMED SINGLE GLUON EMISSION KERNEL.

The analysis of the previous sections will now be incorporated to obtain the leading result for the all-twist resummed single gluon emission kernel. At this juncture we will also attempt to connect with other leading formalisms for jet energy loss. In particular we will discuss two limits of the calculation which deal with the results of Refs. [8] (referred to as the Arnold-Moore-Yaffe (AMY) formalism) and those of Refs. [9] (referred to as the Armesto-Salgado-Wiedemann (ASW) formalism). We would remind the reader once again that such comparisons are meant to be diagrammatic, i.e., similar diagrams are included, however the current calculation will always be carried out in the kinematic region where $l_\perp^2 \gg k_\perp^2$, which is different from the other formalisms.

We begin this final section with a discussion of two types of diagrams which have not yet been discussed. Virtual contributions for which the cut line only goes through one side of the radiated gluon without cutting the radiated gluon or the final quark. These virtual corrections are important for inclusive observables and tend to unitarize the single gluon emission kernel. In single hadron inclusive [25] (and also for parts of multi-hadron inclusive calculations [36]) these diagrams are incorporated by insisting the product of the final fragmentation function and the integral over the integrated splitting function be subtracted from the real contribution. In the case of calculations of the energy lost in a single gluon emission or the change in the partonic distributions, which are more exclusive i.e., depend on the momentum carried out by both quark and gluon, these diagrams play no role as one is more interested in solely the real contributions.

The other type of diagrams are those which have an unequal number of scattering on either side of the cut in either real of virtual diagrams. Such diagrams unitarize the effect of multiple scatterings. As shown in Ref. [12], summing over all types of multiple scattering and integrating over the transverse momentum of a given line should yield a factor of order unity i.e., no length enhancement. This effect is introduced into the calculation by seeking normalized solutions of the diffusion equation that results from the multiple scattering expansion.

A. The limit of only gluon scattering and leading twist

The is essentially the limit of large N_c, where we assume that only the gluon scatters. As demonstrated by the color factors derived in the previous section, the difference between a quark and a gluon scattering in the medium induces a factor of $C_A/C_F = N_c^2/[2(N_c^2-1)]$ per scattering, which is enhanced n times for n scattering. Thus in the limit of multiple scattering, the naive expectation is that such diagrams dominate over those where the quark also scatters. This is not entirely accurate as for a given order the length enhancement factor for the case of equal scattering on the final quark and the gluon is length enhanced over the case of scattering solely on the quark or the gluon. The actual situation is intermediate between these two cases. In the small y limit, the scattering on the initial quark is suppressed by y. Also for a computation of the energy loss, the scattering on the final quark does not yield any change in the momentum dependent part of the expression for the energy loss. This is because, no factor of the transverse momentum imparted to the final quark appears in any of the terms of the integrand except for the transverse momentum δ-function. The scattering on the final outgoing quark however does introduce a non-trivial change in the phase factor which will be considered in the next subsection.

One may easily obtain the expressions corresponding to the limit of only gluon scattering by setting $m = N$ in Eq. (83). All factors of C_F which arise from the scattering of the quark disappear. The sole factor of C_F corresponds to the gluon emission vertex. As a result, the only type of transport coefficient that plays a role is the gluon transport coefficient $\hat{q}$, i.e., the diffusion coefficient of Eq. (85) simplifies to

$$D = \frac{C_A \rho \bar{D}}{2p^+(N_c^2 - 1)}. \tag{91}$$

The transverse broadening of a parton which has progressed a distance L^- , is given as

$$l_\perp^2 = 4\int_0^{L^-} d\zeta^- D(\zeta^-) = \int_0^{L^-} d\zeta^- \frac{\hat{q}}{2} = \int_0^L d\zeta \hat{q}. \tag{92}$$

The factor of 1/2 along with $\hat{q}$ is due to our notation of light cone variables where $\zeta^- = 2\zeta$ and the definition of $\hat{q}$ as $l_\perp^2/L$. Thus we obtain,

$$\hat{q} = 8D = \frac{8\pi^2\alpha_s C_A}{N_c^2 - 1}\rho x_T G(x_T) \tag{93}$$

The $N^{\rm th}$ order transverse momentum derivatives on the transverse momentum δ-function may now be resummed. To be able to resum the result, one has to extract the first length integration over ζ_1^- and order the subsequent scatterings. As a result, we obtain,

$$\sum_{N=1}^{\infty} \mathcal{O}^{N,N}_{0,0;m,m} = \sum_{N=1}^{\infty} \int \frac{dy d^2l_\perp d^2 l_{q_\perp}}{2\pi^2}\frac{\alpha_s C_F P(y)}{l_\perp^2 y}\int_0^{L^-} d\zeta^- D(\zeta^-)\left\{2 - 2\cos\left(p^+ x_L \zeta^-\right)\right\} \tag{94}$$
$$\times \left[\frac{\left(\nabla^2_{l_{q_\perp}}\right)^N \delta^2\left(\vec{l}_{q_\perp}\right)}{(N-1)!} + N\left\{-\frac{2\vec{l}_\perp\cdot\nabla_{l_{q_\perp}}\left(\nabla^2_{l_{q_\perp}}\right)^{N-1}\delta^2\left(\vec{l}_{q_\perp}\right)}{(N-1)!l_\perp^2} + \frac{4\left(\nabla^2_{l_{q_\perp}}\right)^{N-1}\delta^2\left(\vec{l}_{q_\perp}\right)}{(N-1)!l_\perp^2}\right\}\right]\left(\int_{\zeta^-}^{L^-} dy^- D(y^-)\right)^{N-1}.$$

In order to sum over N, and indeed even in the derivation of the expression above for a fixed N, we have ignored terms which are suppressed by powers of N in the large N limit. While this allows us to resum the series above into a closed form, it has the disadvantage of not reducing to the known limit at $N = 1$ [10]. Summing over N we obtain the single gluon emission kernel in the ASW limit of multiple scattering as,

$$\sum_{N=1}^{\infty} \mathcal{O}^{N,N}_{0,0;m,m} = \int \frac{dy d^2l_\perp d^2 l_{q_\perp}}{2\pi^2}\frac{\alpha_s C_F P(y)}{l_\perp^2 y}\int_0^{L^-} d\zeta^- D(\zeta^-)\left\{2 - 2\cos\left(p^+ x_L \zeta^-\right)\right\}\left[\left(\frac{4 - 2\vec{l}_\perp\cdot\nabla_{l_{q_\perp}}}{l_\perp^2}\right)\frac{e^{-\frac{{l_{q_\perp}}^2}{4\int dy^- D(y^-)}}}{4\pi\int dy^- D(y^-)}\right.$$
$$\left. + \nabla^2_{l_{q_\perp}}\frac{e^{-\frac{{l_{q_\perp}}^2}{4\int dy^- D(y^-)}}}{4\pi\int dy^- D(y^-)} - \left(\int_{\zeta^-}^{L^-} dy^- D(y^-)\right)\left\{\frac{2\vec{l}_\perp\cdot\nabla_{l_{q_\perp}}\nabla^2_{l_{q_\perp}} - 4\nabla^2_{l_{q_\perp}}}{l_\perp^2}\right\}\frac{e^{-\frac{{l_{q_\perp}}^2}{4\int dy^- D(y^-)}}}{4\pi\int dy^- D(y^-)}\right]. \tag{95}$$

The above expression represents the leading contribution at all twist to the double differential distribution of a hard quark and a gluon produced from a hard initial quark. While we keep referring to this as the ASW limit, it should be pointed out that we are evaluating this in the range $l_\perp^2 \gg k_\perp^2$ to $l_\perp^2 \gtrsim k_\perp^2$, whereas in the original ASW papers, the evaluation is carried out toward the lower limit.

In the derivation of Eq. (95) from Eq. (94) we have summed over an arbitrary number of scatterings on the quark and gluon. To carry out this sum, we have used the property derived in Ref. [12], that,

$$\frac{\partial \phi(y^-, l_{q\perp})}{\partial y^-} = D(y^-)\frac{\partial^2\phi(y^-, l_{q\perp})}{\partial^2 l_{q\perp}}, \tag{96}$$

where $\phi(L^-, l_{q\perp}) = \sum_{N=0}^{\infty}\left[\int_0^{L^-} dy^- D(y^-)\nabla^2_{l_{q\perp}}\right]^N \delta^2(l_{q\perp})$.

Equation (96) is a diffusion equation which represents the transverse momentum distribution as a function of length travelled L^-. In the calculation so far, we have only included the central cut diagrams, i.e., diagrams with an equal number of scatterings on both sides of the cut. However, we can also include the diagrams with unequal number of scatterings on either side of the cut. While these diagrams produce a smaller transverse momentum for the outgoing quark gluon dipole, they contain equal factors of length enhancement and several of these terms are negative. The effect of including these diagrams (as worked out in Ref. [12]) is that the integral of the distribution ϕ in transverse momentum remain unitary for any length traversed, i.e.,

$$\frac{\partial \int d^2 l_{q\perp}\phi(L^-, l_{q\perp})}{\partial L^-} = 0. \tag{97}$$

The solution to both these conditions, is given by the

Gaussian function,

$$\phi(y^-, l_{q\perp}) = \frac{\exp\left[-\frac{l_{q\perp}^2}{4 \int_0^{y^-} dx^- D(x^-)} \right]}{4\pi \int_0^{y^-} dx^- D(x^-)}. \quad (98)$$

Guided by our assumption that $l_\perp$ is large, we have expanded the complete result of Eq. (70) as a series in $k_\perp^2/l_\perp^2$ keeping terms up to $1/l_\perp^4$. Expanding further will lead to more terms. However, the Eq. (95) has a special property when applied to inclusive observables such as energy loss. To derive the single gluon emission kernel from the double differential distribution above, we integrate over $\vec{l}_{q\perp}$ and over the angle of gluon emission. One will immediately note the identity:

$$\int d^2 l_{q\perp} \frac{e^{-\frac{l_{q\perp}^2}{4\int dy^- D(y^-)}}}{4\pi \int dy^- D(y^-)} = 1 \quad (99)$$

as well as the vanishing contributions,

$$\int d^2 l_{q\perp} \nabla_{l_{q\perp}}^2 \frac{e^{-\frac{l_{q\perp}^2}{4\int dy^- D(y^-)}}}{4\pi \int dy^- D(y^-)} = 0, \quad (100)$$

$$\& \quad \int d^2 l_{q\perp} l_\perp \cdot \nabla_{l_{q\perp}} \frac{e^{-\frac{l_{q\perp}^2}{4\int dy^- D(y^-)}}}{4\pi \int dy^- D(y^-)} = 0.$$

The second identity is true only in the case of unpolarized observables. Integrating over $l_{q\perp}$, we obtain the simple and well know result for single scattering induced emission [10],

$$\sum_{N=1}^{\infty} \mathcal{O}_{0,0;m,m}^{N,N} = \int \frac{dy dl_\perp^2}{2\pi} \frac{\alpha_s C_F P(y)}{l_\perp^2 y} \quad (101)$$
$$\times \int_0^{L^-} d\zeta^- 4D(\zeta^-) \left[2 - 2\cos\left(p^+ x_L \zeta^-\right)\right].$$

While the above result may come as a surprise, the primary reason for the cancellation of the large number of terms is the unitarity of the transverse momentum distribution due to multiple scattering. The sum over diagrams, where the scatterings are not nested as in Fig. 3, restores unitarity to the distribution and thus enforces the choice of a normalized Gaussian solution to the diffusion equation [Eq. (86)]. In obtaining the above result we ignored higher transverse momentum derivatives which contained larger powers of $l_\perp^2$ in the denominator. Including these will introduce corrections to the above simple form. This is tantamount to restricting the calculation to the next-to-leading twist part of the all twist calculation. As would have been expected this yields the same result as the small y limit of the single scattering calculation [10].

B. The full calculation and comparison with the AMY and ASW schemes

In this subsection, we incorporate the effect of multiple scattering on both the quark and the gluon in the calculation of the single gluon emission kernel. In the preceding section, the various types of scattering were divided into six categories. Of these, two types which involve the interference between the scattering of the gluon and the quark after emission have very small color factors and do not possess larger length enhancements compared to other diagrams. These will be ignored in this effort. This leaves only four different types of diagrams that need to be considered: scattering on the quark before emission, interference between scattering of the quark or gluon after emission with scattering before emission and separate scattering of the produced quark and gluon. Diagrams which entangle these contributions tend to be suppressed in length factors. Thus these four contributions can be considered one after the other separately.

In order to incorporate all these effects. We divide the length of the medium traversed from y_0^- to L^- into four regions. The first part from y_0^- to y_Q^- within which only the initial quark scatters and the gluon emission has not occurred in either amplitude or complex conjugate (Fig.4). From $y_C > y_Q$ to y_E where there is interference between the scattering of the quark in the amplitude (complex conjugate) with scattering of the quark or gluon in the complex conjugate (amplitude) [Fig. 5 or Fig. 6]. These are referred to as the cross terms (hence indicated as y_C) and continue till the later gluon emission at y_E. We also keep track of the location ζ_C which is the first scattering of the gluon in the complex conjugate (amplitude). The last region is from $y_I > y_E$ to L^- which represents the part where the produced quark and gluon scatter independently. We also keep track of $\zeta_I \geq y_E$ which is the first scattering of the gluon in this regime. Note that $\zeta_C \leq \zeta_I \leq L^-$. The gluon emission kernel will depend on all these locations.

Unlike in the preceding subsection, where only one diffusion coefficient was required as there was only one type of scattering (that on the final state gluons), there are four different types of scattering in this subsection. As a result, four different types of diffusion coefficients will be required which differ in their overall color factor only. We denominate these as D_{GG} (or simply D_G) where both scatterings in the amplitude and complex-conjugate occur on the gluon,

$$D_{GG} = \frac{C_A \rho \bar{D}}{2p^+(N_c^2 - 1)}. \quad (102)$$

The standard transport coefficient in jet-quenching $\hat{q} = 8D_{GG}$. The case where one scattering is on the gluon and one on the quark is denoted as D_{GQ} or D_{QG} interchangeably, and given as

$$D_{GQ} = \frac{(C_A/2) \rho \bar{D}}{2p^+(N_c^2 - 1)}. \quad (103)$$

The case where both scatterings are on the quark and one occurs prior to gluon emission and one occurs after will be designated as $D_{Q\bar{Q}}$ and has the expression,

$$D_{Q\bar{Q}} = \frac{(C_F - C_A/2)\rho\bar{D}}{2p^+(N_c^2 - 1)}. \quad (104)$$

It may come as a surprise that this diffusion coefficient is actually negative. Multiple such insertions of interfering terms where the quark in the amplitude scatters before the emission and the quark in the complex conjugate scatters after the emission lead to a shrinking of the transverse momentum between the radiated gluon and the final outgoing quark. The final diffusion coefficient is the case where the the quark scatters in both amplitude and complex conjugate and both happen either before or after the gluon emission. The diffusion coefficient in this case is designated as D_{QQ}, or simply D_Q, and given as

$$D_{QQ} = \frac{(C_F)\rho\bar{D}}{2p^+(N_c^2 - 1)}. \quad (105)$$

The derivation of the full double differential distribution of the cross section as a function of the transverse momentum of the quark and the gluon is rather complicated and will be presented elsewhere. In this article, we will simply derive an expression for the energy loss distribution. From the analysis of the preceding sub-section on the ASW limit we note that in the differential distribution, all terms which involve extra derivatives of the delta function vanish in the integrated expression. Also terms which involve one integral to be kept separate from the rest due to a phase factor depending on this location will primarily be involved in the higher twist contributions. Thus terms with more than one phase factor are suppressed by larger powers of $l_\perp^2$. Thus to obtain the leading contribution only the terms with one phase factor need to be retained.

In the interest of simplicity we will not derive the extra contribution from diagrams where the quark scatters before the emission without interfering with any other type of scattering. These will be included separately at the end of the section. These diagrams were shown to have smaller length enhancement factors. They proportional to y^2 and are thus very suppressed in the small y limit. As a result, they play little role in the computation of the medium modified fragmentation function [11] where the range of the gluon momentum fraction is limited by $0 < y < 1 - z$. The fraction z is the momentum fraction of the detected hadron with respect to the initial hard quark's momentum and is thus also the minimum momentum fraction that must be carried by the final quark to be able to fragment and produce such a hadron. These diagrams also do not contain a vacuum like contribution in either amplitude or complex conjugate and arise as a purely medium dependent piece. As a result, the neglect of these terms focuses this part of the calculation to manifestly involve the vacuum contribution in either the amplitude or complex conjugate. As a result, this part of the calculation is somewhat orthogonal to the calculations in the formalism of Arnold, Moore and Yaffe (AMY) where none of the terms contain a vacuum contribution. This part of the calculation, however has a considerable overlap with the calculation of Armesto Salgado and Wiedemann (ASW) except that in the ASW case there is no Taylor expansion in the exchanged momentum i.e., there is no explicit assumption $l_\perp \gg k_\perp$.

Invoking the above mentioned simplifications, we write the expression for $\mathcal{O}^{r+s+m+n,q+m+n}_{0,q;s+m,m}$ where $q = r+s$. This represents the operator expression for the case where the quark emits a gluon at the origin in the amplitude and after q scatterings in the complex conjugate. The q scatterings interfere with r scatterings on the produced quark and s scatterings on the gluon in the amplitude. After these interfering terms, the produced quark and gluon scatter m and n times in both amplitude and complex conjugate. The full expression is written as,

$$
\begin{aligned}
\mathcal{O}^{q+m+n,q+m+n}_{0,q;s+m,m} =& \int \frac{dy d^2 l_\perp d^2 l_{q\perp}}{2\pi^2} \alpha_s C_F P(y) C_A^m C_F^n (C_F - C_A/2)^r (C_A/2)^s \delta^2\left(l_{q\perp} - \sum_{i=1}^{s} k_\perp^i - \sum_{j=1}^{r} p_\perp^j - \sum_{l=1}^{m} k_\perp^l - \sum_{k=1}^{n} p_\perp^k \right) \\
&\times \frac{l_\perp - \sum_{i=1}^{s} k_\perp^i - \sum_{l=1}^{m} k_\perp^l}{\left(l_\perp - \sum_{i=1}^{s} k_\perp^i - \sum_{l=1}^{m} k_\perp^l \right)^2} \cdot \frac{l_\perp - y\sum_{i=1}^{s} k_\perp^i - \sum_{l=1}^{m} k_\perp^l}{\left(l_\perp - y\sum_{i=1}^{s} k_\perp^i - \sum_{l=1}^{m} k_\perp^l \right)^2} \\
&\times \prod_{i=1}^{q} \int dy_i^- \frac{\int d^3\delta y_i \rho \langle p| A^+(y_i^- + \delta y_i^-, 0) A^+(y_i^-, -\delta y_\perp^i)|p\rangle}{2p^+(N_c^2-1)} e^{ik_\perp^i \delta y_\perp^i} \\
&\times \prod_{j=1}^{n} \int dy_j^- \frac{\int d^3\delta y_j \rho \langle p| A^+(y_j^- + \delta y_j^-, 0) A^+(y_j^-, -\delta y_\perp^j)|p\rangle}{2p^+(N_c^2-1)} e^{ik_\perp^j \delta y_\perp^j} \\
&\times \prod_{l=1}^{m} \int d\zeta_l^- \frac{\int d^3\delta \zeta_l \langle p| A^+(\zeta_l^- + \delta \zeta_l^-, 0) A^+(\zeta_l^-, -\delta \zeta_\perp^l)|p\rangle}{(N_c^2-1)} e^{ik_\perp^l \delta \zeta_\perp^l} \\
&\times \left[\theta(\zeta_I^- - y_E^-) \left\{ e^{-ip^+ x_L y_E^-} - e^{-ip^+ x_L \zeta_I^-} \right\} - \theta(\zeta_I^- - y_I^-) e^{-ip^+ x_L y_I^-} - \theta(y_I^- - \zeta_I^-) e^{-ip^+ x_L \zeta_I^-} \right] \\
&\times \left[\theta(\zeta_C^- - y_0^-) \left\{ e^{ip^+ x_L y_0^-} - e^{ip^+ x_L \zeta_C^-} \right\} - \theta(\zeta_C^- - y_C^-) e^{ip^+ x_L y_C^-} - \theta(y_C^- - \zeta_C^-) e^{ip^+ x_L \zeta_C^-} \right]. \qquad (106)
\end{aligned}
$$

As mentioned above, we will consider the case where the initial state scattering on the quark is ignored thus we set $y_0^- = 0$. The least power suppressed contribution in this limit is the term containing the $e^{ip^+ x_L y_0^-} = 1$ factor.

The second line in Eq. (106) may be simplified as,

$$
\frac{1}{l_\perp^2} - \frac{(1-y+y^2)\left(\sum_{i=1}^{s} k_\perp^i \right)^2}{l_\perp^4} - \frac{\left(\sum_{i=1}^{m} k_\perp^i \right)^2}{l_\perp^4} + 2(1+y^2) \frac{\left(l_\perp \cdot \sum_{i=1}^{s} k_\perp^i \right)^2}{l_\perp^6} + 4 \frac{\left(l_\perp \cdot \sum_{i=1}^{m} k_\perp^i \right)^2}{l_\perp^6}. \qquad (107)
$$

In the equation above, we have only retained only terms which are suppressed by at most $l_\perp^4$. This is all that is required for the case of terms containing the $e^{ip^+ x_L y_0^-} = 1$ factor. Terms containing two separate phases will require the products $(\sum_{i=1}^{s} k_\perp^i)^2 (\sum_{i=1}^{m} k_\perp^i)^2$ which appear in the above expansion suppressed by $l_\perp^6$. While it is not difficult to include these terms and these will be included in future numerical calculations, we refrain from including them here as we illustrate the leading behavior of the multiple scattering gluon radiation spectrum.

Expanding the δ-function and the (non-phase) momentum dependent part as a series in transverse momentum, we obtain,

$$
\begin{aligned}
\mathcal{O}^{q+m+n,q+m+n}_{0,q;s+m,m} =& \int \frac{dy d^2 l_\perp d^2 l_{q\perp}}{2\pi^2} \alpha_s C_F P(y) C_A^m C_F^n (C_F - C_A/2)^r (C_A/2)^s \left[\frac{\left(\nabla^2_{l_{q\perp}} \right)^{q+m+n}}{l_\perp^2} + \frac{4(sy+m)\left(\nabla^2_{l_{q\perp}} \right)^{q+m+n-1}}{l_\perp^4} \right] \\
&\times \delta^2(l_{q\perp}) \prod_{i=1}^{q} \int dy_i^- \frac{\rho \bar{D}(y_i^-)}{2p^+(N_c^2-1)} \prod_{j=1}^{n} \int dy_j^- \frac{\rho \bar{D}(y_j^-)}{2p^+(N_c^2-1)} \prod_{l=1}^{m} \int d\zeta_l^- \frac{\rho \bar{D}(\zeta_l^-)}{2p^+(N_c^2-1)} \qquad (108) \\
&\times \left[\theta(\zeta_I^- - y_E^-) \left\{ e^{-ip^+ x_L y_E^-} - e^{-ip^+ x_L \zeta_I^-} \right\} - \theta(\zeta_I^- - y_I^-) e^{-ip^+ x_L y_I^-} - \theta(y_I^- - \zeta_I^-) e^{-ip^+ x_L \zeta_I^-} \right]
\end{aligned}
$$

In the equation above, the q integrals run from $y_0 = 0$ to ζ_I or y_I which ever comes first. The q integrals are all ordered and r of them attach to a quark on both sides of the cut. For terms without the $exp[-ip^+ x_L y_E^-]$, this yields the length enhancement factor of,

$$
\begin{aligned}
&\prod_{i=1}^{q} \int dy_i^- \theta(y_i^- - y_{i-1}^-) \frac{(r+s)!}{r!s!} \\
&= \frac{\left(\int_{y_0}^{min[y_I,\zeta_I]} dy \right)^r}{r!} \frac{\left(\int_{y_0}^{min[y_I,\zeta_I]} dy \right)^s}{s!} \qquad (109)
\end{aligned}
$$

For the term containing the y_E^- phase factor, if y_E scatters on a quark on both sides, then we take the $r^{\rm th}$ integration out from the product. If y_E scatters on a gluon then we take the $s^{\rm th}$ integration out from the ordered product, yielding the factor,

$$\int_0^{min[y_I,\zeta_I]} dy_E^- \frac{\rho \bar{D}(y_E^-)}{2p^+(N_c^2-1)} e^{-ip^+x_L y_E^-} \tag{110}$$
$$\times \left[\frac{\left(\int_0^{y_E^-} dy\right)^{r-1}}{(r-1)!} \frac{\left(\int_0^{y_E^-} dy\right)^{s}}{s!} + \frac{\left(\int_0^{y_E^-} dy\right)^{r}}{(r)!} \frac{\left(\int_0^{y_E^-} dy\right)^{s-1}}{(s-1)!} \right].$$

As a result, including all the length dependent terms, associated with the y_E^- phase factor, we get,

$$\int_0^{L^-} dy_E^- \frac{\rho \bar{D}(y_E^-)}{2p^+(N_c^2-1)} e^{-ip^+x_L y_E^-}$$
$$\times \left[\frac{\left(\int_0^{y_E^-} dy\right)^{r-1}}{(r-1)!} \frac{\left(\int_0^{y_E^-} dy\right)^{s}}{s!} + \frac{\left(\int_0^{y_E^-} dy\right)^{r}}{(r)!} \frac{\left(\int_0^{y_E^-} dy\right)^{s-1}}{(s-1)!} \right]$$
$$\times \frac{\left(\int_{y_E^-}^{L^-} d\zeta\right)^{m}}{m!} \frac{\left(\int_{y_E^-}^{L^-} d\zeta\right)^{n}}{n!}. \tag{111}$$

When combined with the transverse momentum derivatives one notes that there is no contribution from the terms containing $r-1$ (as these will always contain an extra $\nabla^2_{l_{q\perp}}$ which is not being summed) i.e., where y_E^- represents the scattering on a quark. Summing over r, s, m, n, and integrating over $l_{q\perp}$, the only surviving term is,

$$\mathcal{O}(1) = \int \frac{dy d^2 l_\perp}{2\pi^2} \alpha_s C_F P(y)$$
$$\times \int_0^{L^-} dy_E^- 4y D_{GQ}(y_E^-) e^{-ip^+x_L y_E^-}. \tag{112}$$

Note that this term is explicitly y dependent and thus subleading as $y \to 0$.

Similarly for terms with the $e^{-ip^+x_L\zeta_I^-}$ phase factor, the length dependent terms, after summation over all insertions and integration over $l_{q\perp}$, yields,

$$\mathcal{O}(2) = -\int \frac{dy d^2 l_\perp}{2\pi^2} \alpha_s C_F P(y)$$
$$\times\ 2 \int_0^{L^-} d\zeta^- 4 D_G(\zeta^-) e^{-ip^+x_L\zeta^-}. \tag{113}$$

The two equations above, represent terms which are not squares of amplitudes but rather interference terms. Thus we also need to include the complex conjugates. The result of summing the two equations above with their complex conjugates may be easily obtained by converting $e^{-ip^+x_L y_E^-}$ to $2\cos(p^+x_L y_E^-)$ in Eq.(112) and $e^{-ip^+x_L\zeta^-}$ to $2\cos(p^+x_L\zeta^-)$ in Eq (113).

In the results above, we have only obtained terms which contain a cosine of the usual phase argument $(x_L p^+ \zeta^-)$ but not the terms proportional to unity. The combination of these yields the LPM effect. To obtain the terms which don't contain the argument $x_L p^+ \zeta^-$, we have to consider terms with at least two of the sums restricted, i.e., terms where $r = 0, s = 0$ or terms with $m = 0, n = 0$. Considering these latter contributions changes the overall phase factor, e.g., when $r = 0, s = 0$ one will have to replace y_C^- with y_I^- and ζ_C^- with ζ_I^- in Eq. (106). While this limit provides the zeroth order contributions to the various diffusion terms that go with the $\cos(p^+x_L\zeta^-)$, it also introduces new contributions, i.e., the terms which don't depend on the cosine, which have to be included separately. This we carry out in the following.

The reader will note that all the sums over the number of scatterings start from a minimum of 1, e.g., in the first line of Eq. (106). This requires that all types of scatterings considered have to have taken place at least once. We also need to include the cases where a particular type of scattering did not take place. The simplest situation is the case where $m = n = r = s = 0$. In this case there will not be any of the phase factors or in-medium matrix elements and an inspection of Eq. (106) will reveal that this yields simply the pure vacuum contribution of Eq. (17). The second case is where we set r, s the cross scattering to 0. Beyond all other trivial changes, this modifies the phase factor to yield,

$$\Gamma = \left[\theta(\zeta_I^-)\left\{1 - e^{-ip^+x_L\zeta_I^-}\right\} \right. \tag{114}$$
$$\left. -\ \theta(\zeta_I^- - y_I^-) e^{-ip^+x_L y_I^-} - \theta(y_I^- - \zeta_I^-) e^{-ip^+x_L\zeta_I^-} \right]$$
$$\times \left[\theta(\zeta_I^-)\left\{1 - e^{ip^+x_L\zeta_I^-}\right\} \right.$$
$$\left. -\ \theta(\zeta_I^- - y_I^-) e^{ip^+x_L y_I^-} - \theta(y_I^- - \zeta_I^-) e^{ip^+x_L\zeta_I^-} \right].$$

We ignore the contributions that contain y_I^- as there is no corresponding $p_\perp$ that appears in the amplitude, thus these will vanish when integrated over $l_{q\perp}$. Also all terms which contain the phase factor $\exp(\pm ip^+x_L\zeta_I^-)$ give leading contributions when $y_I^- > \zeta_I^-$. This allows the integrals over quark scatterings to be lumped together and included in the broadening of the two dimensional δ-function over $l_{q\perp}$. Thus the leading contributions (i.e., the least suppressed in Q^2) which contain at most one location ζ_I are given as,

$$\left(1 - e^{-ip^+x_L\zeta_I^-} - \theta(y_I^- - \zeta_I^-) e^{-ip^+x_L\zeta_I^-}\right)$$
$$\times \left(1 - e^{ip^+x_L\zeta_I^-} - \theta(y_I^- - \zeta_I^-) e^{ip^+x_L\zeta_I^-}\right)$$
$$= 2 + 3\theta(y_I^- - \zeta_I^-) - 2\cos(p^+x_L\zeta_I^-)$$
$$-\ 2\theta(y_I^- - \zeta_I^-)\cos(p^+x_L\zeta_I^-). \tag{115}$$

Along with these one will have to consider the terms which depend on the location y_I^-, i.e.,

$$\theta(\zeta_I^- - y_I^-)\left[-2\cos(p^+x_L y_I^-) + 1\right]. \tag{116}$$

There is one extra factor of $\theta(y_I^- - \zeta_I^-)$ without a cosine in Eq. (115) and one less factor if $\theta(\zeta_I^- - y_I^-)$ (without a cosine) in Eq. (116). Since these terms have no extra dependence on any position they are essentially equal, i.e., these are symmetric across the line $y_I^- = \zeta_i^-$. As a result, the term containing the extra factor or $\theta(y_I^- - \zeta_I^-)$ can be converted to a term containing an $\theta(y_I^- - \zeta_I^-)$ and included with Eq. (116). Such a conversion is required in the $l_\perp \gg k_\perp^i$ limit, as any term which cannot be cast in the final form,

$$\int dl_\perp^2 \frac{2 - 2\cos(x_L p^+ \zeta^-)}{l_\perp^4}, \tag{117}$$

up to multiplicative factors, diverges as $1/l_\perp^4$ at small $l_\perp$. Such a divergence calls into question the approximation of Taylor expanding in the softer momenta $k_\perp^i$. Such a term also dominates over the vacuum radiation term at small $l_\perp$ and thus will invalidate the use of an in-medium DGLAP evolution equation to incorporate the effect of multiple radiations. All terms, at this order of approximation, are finite at $l_\perp \to 0$ due to the destructive interference of the LPM effect.

We now consolidate terms which contain one more integral over light-cone position ζ_I^- than transverse derivative on the δ function, in Eq. (108). These are of two types, those which remain as $n \to 0$ and those which vanish in this limit. Terms of the first type may be expressed as,

$$\int_0^{L^-} d\zeta_I 4D_G \left[2 - 2\cos(p^+ x_L \zeta_I^-)\right] \tag{118}$$
$$\times \sum_{m=1,n=0}^{\infty} \frac{\left(\int_{\zeta_I^-}^{L^-} d\zeta^- D_G \nabla^2_{l_{q_\perp}}\right)^{m-1} \left(\int_0^{L^-} dy^- D_Q \nabla^2_{l_{q_\perp}}\right)^n \delta^2(l_{q_\perp})}{(m-1)!n!}.$$

Summing over m,n reduces the second line of the above equation to the broadened Gaussian,

$$\frac{\exp\left[-\frac{|l_{q_\perp}|^2}{4\int_0^{L^-} dy^- D_Q + 4\int_{\zeta_I^-}^{L^-} d\zeta^- D_G}\right]}{4\pi\left(\int_0^{L^-} dy^- D_Q + \int_{\zeta_I^-}^{L^-} d\zeta^- D_G\right)}. \tag{119}$$

The terms in Eq. (108) which only exist for $n \geq 1$ and contain one more integral over the light-cone location ζ_I^- than transverse derivative on the δ function can be expressed as,

$$\int_0^{L^-} d\zeta_I 4D_G \left[2 - 2\cos(p^+ x_L \zeta_I^-)\right] \tag{120}$$
$$\times \sum_{m=1,n=1}^{\infty} \frac{\left(\int_{\zeta_I^-}^{L^-} d\zeta^- D_G \nabla^2_{l_{q_\perp}}\right)^{m-1} \left(\int_{\zeta_I^-}^{L^-} dy^- D_Q \nabla^2_{l_{q_\perp}}\right)^n \delta^2(l_{q_\perp})}{(m-1)!n!}.$$

To sum over m,n one of the n integrals have to be extracted from the sum. Denoting this location as y_L^-, the result of the sum over m,n for the second line of the above equation yields,

$$\frac{\int dy_L^- D_Q \nabla^2_{l_{q_\perp}} \exp\left[-\frac{|l_{q_\perp}|^2}{4\int_{\zeta_I^-}^{y_L^-} dy^- D_Q + 4\int_{\zeta_I^-}^{L^-} d\zeta^- D_G}\right]}{4\pi\left(\int_{\zeta_I^-}^{y_L^-} dy^- D_Q + \int_{\zeta_I^-}^{L^-} d\zeta^- D_G\right)}. \tag{121}$$

Integrating the above term over $l_{q_\perp}$ yields zero due to Eq. (100).

Combining these contributions, yields the leading medium dependent contribution to the gluon emission spectrum from multiple scattering in the small y limit as,

$$\frac{dN_1}{dy dl_\perp^2} = \frac{\alpha_s C_F}{2\pi l_\perp^4} P(y) \times \int_0^{L^-} d\zeta^- 4D_G(\zeta^-) \left[2 - 2\cos\left(p^+ x_L \zeta^-\right)\right]. \tag{122}$$

Note that this is identical to the contribution in the previous subsection where we insisted that only the gluon scatters.

The next-to-leading contribution in the small y limit may be derived similarly from the contributions in Eq. (112), including in addition the contributions which do not contain an exponential or a cosine factor, as

$$\frac{dN_2}{dy dl_\perp^2} = -\frac{\alpha_s C_F}{2\pi l_\perp^4} y P(y) \times \int_0^{L^-} d\zeta^- 4D_{GQ}(\zeta^-) \left[2 - 2\cos\left(p^+ x_L \zeta^-\right)\right]. \tag{123}$$

We now add the contributions where the initial quark scatters independently, i.e., where none of these scatterings interfere with scatterings on the radiated gluon or on the final quark. These diagrams contribute terms suppressed by one extra power of $l_\perp^2$ only in the limit where $r, s = 0$. These may be written down in analogy with Eq. (122) by replacing $k_\perp^i \to y p_\perp^i$ and D_G with D_Q. As a result, this y suppressed contribution is given as,

$$\frac{dN_3}{dy dl_\perp^2} = \frac{\alpha_s C_F}{2\pi l_\perp^4} y^2 P(y) \times \int_0^{L^-} d\zeta^- 4D_Q(\zeta^-) \left[2 - 2\cos\left(p^+ x_L \zeta^-\right)\right]. \tag{124}$$

While the expressions above may be used in an energy loss calculation to determine the medium modified fragmentation function, it cannot be use for the Monte-Carlo simulation of jet propagation in medium. That will require the full double differential expressions in $l_\perp$ and $l_{q_\perp}$. While these are not presented here, they may be straightforwardly derived from Eq. (106) and will be discussed elsewhere.

Though not discussed in this article, the calculation of the medium modified fragmentation function or in the

Monte-Carlo simulation of a hard jet, one requires probability decreasing contributions which counter the effect of the probability increasing expressions such as Eq. (122). In the case of the medium modified fragmentation functions, these are essentially introduced as unitarity (or virtual) corrections. In the case of jet simulations, one will traditionally derive a Sudakov factor [38] from the real splitting correction.

VII. DISCUSSIONS, CONCLUSIONS AND OUTLOOK

In this article, we have extended the higher twist formalism by deriving the leading length-enhanced contribution to the single gluon splitting kernel at all-twist. Physically this includes the process of gluon emission from a hard quark which undergoes multiple scattering within the formation time of the produced collinear radiation.

We considered the propagation of a hard quark produced in DIS on a large nucleus. The hard virtual photon strikes a hard incoming quark in a nucleon which then proceeds to burrow through the nucleus in the direction of the hard photon. The hard quark is off-shell at the point of production and in the absence of the remaining nucleons would proceed to radiate multiple gluons prior to hadronization. We focused on one such hard emission $l_\perp \gg \Lambda_{QCD}$ which occurred inside the nucleus. The presence of the medium modifies the cross section to radiate the hard gluon due to multiple scattering of the propagating quark and the radiated gluon.

The main focus in the current effort has been on the calculation of the triple differential cross section to produced a single gluon with transverse momentum $l_\perp$, momentum fraction y and a final outgoing quark with transverse momentum $l_{q_\perp}$ (and momentum fraction $1-y$). The series of diagrams which contain the leading length enhanced corrections to this process were identified. These diagrams were then evaluated in the collinear and hard approximation $q^- \ggg l_\perp \gg k_\perp$ where the first inequality suggests that the jet energy is logarithmically larger than the transverse momentum of the radiation, which by the second inequality is power enhanced compared to the transverse momentum imparted from the medium.

Incorporating the various simplifications that result from the limits mentioned above, we evaluated the leading terms in the triple differential cross section. Besides the effect of multiple scattering on vacuum like splitting diagrams, we also obtained new contributions which represent the physical effects of a hard incoming gluon hitting a near on shell quark and the forcing it off shell and inducing a radiation. This interferes with the vacuum like emission as well as with other processes where on-shell quark splits, radiating a gluon and either the out-going quark or gluon goes off-shell. The off-shell parton then quickly loses its excess virtuality by encountering another collision with an incoming hard gluon.

After a discussion of length enhancements and color factors of different types of diagrams, we focused on diagrams which yield the leading power corrections to the single gluon cross section. Among these, the leading corrections in the gluon momentum fraction y with the largest color factors arose from processes where the final produced quark or gluon scattered independently or where there was interference between the scattering on the quark prior to emission and that on the quark or gluon after emission.

Following this, the double differential cross section to radiate a hard gluon was evaluated. This was evaluated in two separate cases: the case of only gluon scattering and the case which included all types of scattering. The leading corrections in a power expansion involving the hard scale were then isolated. In these diagrams, the insistence that the transverse momentum of the radiated gluon is large in comparison with the transverse momentum of the incoming gluons from the medium make this calculation somewhat orthogonal to that of AMY. Integrating the transverse momentum of the final produced quark reduced these contributions to those obtained from a single scattering analysis. This implies that the leading effect of multiple scattering is to broaden the transverse momentum distributions of the produced quark and gluon leaving the energy loss kernel unchanged. This justifies a similar assumption make in Ref. [39], where it was assumed that the broadening of a radiated gluon could be considered independently from the emission cross section.

In this paper we have restricted the calculation to only the analytical component. The numerical evaluation of the medium modified fragmentation function followed by the phenomenological implementation in DIS on a large nucleus and in jets in high energy heavy ion collisions will be carried out in a separate calculation. These will include the effect of multiple such stimulated emissions.

Along with the various terms outlined in the preceding section, we will also include a series of other terms which were not explicitly evaluated in this effort. Principle among these are the contributions suppressed by Q^4 (or rather $l_\perp^4$) and greater. The neglect of such contributions is justified at large Q^2. For a $Q^2 \sim 3-5\text{GeV}^2$, as is the case for the HERMES experiments, and for large enough nuclei, such power corrections may also have to be included.

The DGLAP equation in vacuum, is a coupled equation which connects the evolution of the quark fragmentation function with that of the gluon. This will also be the case for energy loss in a medium. Thus a complete in-medium evolution will also require a calculation of the in-medium splitting function for a gluon to two gluons or a quark anti-quark pair. The derivation of such a splitting function will proceed in the same manner as in this article and will be presented in a future effort.

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IX. APPENDIX A: λ SCALING OF THE A^+ FIELD

We use the linear response formula to ascertain the power counting of the A^+ field. Suppressing the color superscripts we obtain,

$$A^\mu(x) = \int d^4y \mathcal{D}^{\mu\nu}(x-y) J_\nu(y). \quad (125)$$

In the equation above, $\mathcal{D}$ is the gluon propagator and at leading order in the covariant gauge is given as,

$$\mathcal{D}^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-ig^{\mu\nu} e^{-ik\cdot(x-y)}}{k^2+i\epsilon}. \quad (126)$$

In Eq. (125), $J^\nu(y) = \bar{\psi}(y)\gamma^\nu\psi(y)$ is the current of partons in the target which generates the gluon field. The fermionic operator may be decomposed as,

$$\psi(y) = \int \frac{dp^+ d^2p_\perp}{(2\pi)^3 \sqrt{p^+ + \frac{p_\perp^2}{2p^+}}} \sum_s u^s(p) a_p^s e^{-ip\cdot y} + . \quad (127)$$

The scaling of the fermionic operator depends on the range of momentum which are selected from the in-state by the annihilation operator. Note that this influences both the scaling of the annihilation operator a_p as well as the bispinor $u(p)$. The power counting of the annihilation operator may be surmised from the standard anti-commutation relation,

$$\{a_p^r, a_{p'}^{s\,\dagger}\} = (2\pi)^3 \delta^3(\vec{p} - \vec{p}')\delta^{rs}. \quad (128)$$

and the power counting of the bispinor from the normalization condition,

$$\sum_s u_p^s \bar{u}_p^s = \not{p} = \gamma^- p^+ + \gamma^+ p^- - \gamma_\perp \cdot p_\perp. \quad (129)$$

Substituting the equation for the current in Eq. 126, and integrating out y, we obtain,

$$A^+ \simeq \int \frac{d^3p d^3q}{(2\pi)^6 \sqrt{p^+}\sqrt{q^+}} \frac{-ie^{-i(p-q)\cdot x}}{(p-q)^2} a_q^\dagger a_p \bar{u}(q)\gamma^+ u(q) \quad (130)$$

If the incoming and out going momenta p and q scale as collinear momenta in the (+)-direction, i.e., $p \sim Q(1, \lambda^2, \lambda)$, then we get, $\delta^3(\vec{p} - \vec{p}') \sim [\lambda^2 Q^3]^{-1}$, as one of the momenta will involve the large (+)-component and the remaining are the small transverse components. Thus the annihilation (and creation) operator scales as $\lambda^{-1}Q^{-3/2}$. Also in the spin sum $\not{p} \sim Q$ and thus $u(p) \sim u(q) \sim Q^{1/2}$; one can check that the γ^+ projects out the large components in u and $\bar{u}$ in the expression $\bar{u}(q)\gamma^+ u(p)$. We also institute the Glauber condition that $p^+ - q^+ \sim \lambda^2 Q$, $p^- - q^- \sim \lambda^2 Q$ and $p_\perp - q_\perp \sim \lambda Q$.

Using these scaling relations we correctly find that the bispinor scales as $\lambda Q^{3/2}$. However, to obtain the correct scaling of the gauge field A^+ one needs to institute the approximation that $q^+ = p^+ + k^+$ where $k^+ \sim \lambda^2 Q$. This condition is introduced by insisting that the (+) momentum of the incoming and outgoing state, which control the scaling of $a_q^\dagger$ and a_p, are separated by $k^+ \sim \lambda^2 Q$. This is used to shift the $dq^+ \to dk^+$ and as a result we obtain the scaling of the A^+ field as $\lambda^2 Q$. Following a similar derivation, with the replacement $\gamma^+ \to \gamma^\perp$ we obtain the scaling of the $A^\perp \sim \lambda^3 Q$.

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