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Two-loop corrections to partition function of Pohlmeyer-reduced theory for $AdS_5 \times S^5$ superstring

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Abstract

Pohlmeyer reduction of $AdS_5 \times S^5$ superstring, involving solution of Virasoro conditions in terms of coset current variables, leads to a set of equations of motion following from an action containing a bosonic $Sp(2,2) \times Sp(4)/[SU(2)]^4$ gauged WZW term, an integrable potential and a fermionic part coupling bosons from the two factors. The original superstring and the reduced model are in direct correspondence at the classical level but their relation at the quantum level remains an open question. As was found earlier, the one-loop partition functions of the two theories computed on the respective classical backgrounds match; here we explore the fate of this relation at the two-loop level. We consider the example of the reduced theory solution corresponding to the long folded spinning string in AdS. The logarithm of the $AdS_5 \times S^5$ superstring partition function computed on the spinning string background is known to be proportional to the universal scaling function which depends on the string tension $\frac{\sqrt{\lambda}}{2\pi}$. Its "quantum" part is $f(\lambda) = a_1 + \frac{1}{\sqrt{\lambda}} a_2 + ...$, where the one-loop term is $a_1 = -3\ln 2$ and the two-loop term is the negative of the Catalan's constant, $a_2 = -\mathrm{K}$. We find that the counterpart of $f(\lambda)$ in the reduced theory is $\mathrm{f}(k) = \mathrm{a}_1 + \frac{2}{k}\mathrm{a}_2 + ...$, where k is the coupling of the reduced theory. Here the one-loop coefficient is the same as in string theory, $\mathrm{a}_1 = a_1$, while the two-loop one is $\mathrm{a}_2 = a_2 - \frac{1}{4}(a_1)^2$. Remarkably, the first Catalan's constant term here matches the string theory result if we identify the two couplings as $k = 2\sqrt{\lambda}$. Nevertheless, the presence of the additional $(a_1)^2 \sim (\ln 2)^2$ term suggests that the relation between the two quantum partition functions (if any) is not a simple equality. Similar results are found in the case of the $AdS_3 \times S^3$ string theory where $a_1 = -2\ln 2$ and $a_2 = 0$, while in the corresponding reduced theory $\mathrm{a}_1 = a_1$ and $\mathrm{a}_2 = a_2 - \frac{1}{4}(a_1)^2$.

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Contents

1 Introduction and summary

Pohlmeyer reduction applied to classical $AdS_5 \times S^5$ superstring theory leads [1, 2] to a generalised sine-Gordon type model described by a particular bosonic G/H gauged WZW theory with an integrable potential coupled to fermions. Since the Virasoro conditions are solved in the process of the Pohlmeyer reduction, the reduced theory may be viewed as an analog of a "light-cone gauge-fixed" version of the original string theory. While the conformal-gauge $AdS_5 \times S^5$ string theory

(ST) and the associated Pohlmeyer-reduced theory (PRT) are closely related at the classical level, PRT has important simplifying features being 2d Lorentz invariant and quadratic in fermions which have standard 2d kinetic terms. When expanded near the respective vacua, the two theories are described by the equivalent sets of 8+8 boson+fermion physical 2d fields. This raises a hope that PRT may be useful in an attempt to solve the $AdS_5 \times S^5$ ST from first principles.

The relation between the $AdS_5 \times S^5$ ST and PRT is established at the classical level and involves a transformation from coset currents to new fields in a way that solves the conformal gauge conditions algebraically (both theories originate from the same set of first-order equations for the currents). The classical solutions are thus in correspondence (though the values of the two actions on the associated solutions are, in general, different). There are also close similarities between the associated tree-level S-matrices [3] and the soliton spectra of the theories [4].

The relation between the $AdS_5 \times S^5$ ST and PRT at the quantum level is a priori unclear. Nevertheless, given their classical connection, and the integrability and UV finiteness of both theories [5], one may conjecture that the two quantum theories should also be closely related. The precise form of such a relation remains to be understood. An indication of a quantum relation is the equality of the one-loop partition functions of the two theories computed by expanding near "dual" solutions [6, 7]

$$Z^{(1)}_{\rm ST} = Z^{(1)}_{\rm PRT} \, . \tag{1.1}$$

While one may be tempted to view this one-loop relation as a consequence of the classical equivalence of the two theories (suggesting that determinants of small fluctuation operators found by perturbing the classical solutions should match), it is still a non-trivial test[1] of the correspondence between the underlying physical degrees of freedom of the two theories.

The aim of the present paper is to explore possible relations between the two quantum partition functions at the two-loop level. Since the two-loop computations in a non-trivial background are, in general, very complicated here we will consider the simplest string solution – the infinite spin (scaling) limit of the folded spinning (S, J) string in $AdS_3 \times S^1$ subspace of $AdS_5 \times S^5$ [8, 9] and the associated solution of the reduced theory. As a further simplification we will eventually consider the limit $J \to 0$ when the logarithm of the string theory worldsheet partition function is simply proportional to a function of the coupling constant (i.e. to the universal scaling function of string tension on the string theory side).

Our conclusion (in both $AdS_5 \times S^5$ and $AdS_3 \times S^3$ cases that we consider below) appears to be as follows: while the non-trivial parts of the two two-loop partition functions (coming from the most complicated two-loop integrals) appear to be direct correspondence, the reduced theory partition function contains an extra two-loop term proportional to the square of the one-loop coefficient. Thus if the two quantum partition functions are indeed related, this relation may be effectively non-linear. It is possible also that the matching of two partition functions may be restored by modifying the PRT action by a certain one-loop counterterm that may be required to maintain its quantum integrability, i.e. to preserve certain hidden (super)symmetries. These ideas remain to be explored.[2]

---

[1] One may, in principle, construct a pair of classically equivalent theories that have different one-loop partition functions.

[2] At the moment we do not have a natural suggestion for a local counterterm that would restore the two-loop equivalence.

1.1 Quantum partition function in string theory

Let us first review the known structure of the two-loop string partition in the long spinning string background [9, 10, 11, 12, 13]. The (S,J) spinning string background in the large spin limit has the following form in terms of the $AdS_5\times S^5$ embedding coordinates (below $S=\sqrt{\lambda}\mathcal{S},\ J=\sqrt{\lambda}\mathcal{J}$ where $\frac{\sqrt{\lambda}}{2\pi}$ is string tension)

$$\begin{aligned} Y_0+iY_5=\cosh(\ell\sigma)\,e^{i\kappa\tau}\,, \qquad Y_1+iY_2=\sinh(\ell\sigma)\,e^{i\kappa\tau}\,, \qquad Y_{3,4}=0\,, \\ X_1+iX_2=e^{i\mu\tau}\,, \qquad X_{3,4,5,6}=0\,, \end{aligned} \tag{1.2}$$

where the parameters $\kappa\gg 1$, $\ell\gg 1$ and μ are related by[3]

$$\kappa^2=\ell^2+\mu^2\ , \qquad \ell=\frac{1}{\pi}\ln\mathcal{S}\gg 1\ , \qquad \mathcal{J}=\mu. \tag{1.3}$$

We will eventually be interested in the limit $\mu\to 0$ when $\kappa\to\ell$ is the only scale in the problem so that one may introduce the rescaled worldsheet coordinates $\sigma'=\kappa\sigma,\ \tau'=\kappa\tau$ which in the $\kappa=\ell\to\infty$ limit span the whole 2-plane. We shall define $V_2=\int d\tau' d\sigma'=\kappa^2\bar{V}_2$ as the resulting infinite volume factor. The logarithm of the resulting quantum partition function is given by[4]

$$\Gamma_{\rm ST}=-\ln Z_{\rm ST}=\frac{1}{2\pi}\ f(\lambda)\ V_2\ , \tag{1.4}$$

$$f(\lambda)=a_1+\frac{a_2}{\sqrt{\lambda}}\ +\ O(\frac{1}{(\sqrt{\lambda})^2})\ , \tag{1.5}$$

$$a_1=-3\ln 2\ , \qquad a_2=a_{2B}+a_{2F}={\rm K}-2{\rm K}=-{\rm K}\ . \tag{1.6}$$

Here a_1 is the one-loop and a_2 is the two-loop contribution (K is the Catalan's constant). In a_2 we indicated separately the part coming from purely bosonic graphs and graphs involving fermions. The spectrum of the string fluctuation modes [8] includes (in the $\mu=0$ limit and after rescaling of masses by κ): one AdS_3 mode with $m^2=4$, two AdS_5 modes "transverse" to AdS_3 with $m^2=2$, five S^5 modes with $m^2=0$ and eight fermionic modes with $m^2=1$. Contributions proportional to K originate from two-loop "sunset" graphs with three propagators that are expressed in terms of the following momentum integrals

$$I[m_i^2,m_j^2,m_k^2]\equiv\int\frac{d^2q_id^2q_jd^2q_k}{(2\pi)^4}\frac{\delta^{(2)}(q_i+q_j+q_k)}{(q_i^2+m_i^2)(q_j^2+m_j^2)(q_k^2+m_k^2)}\ , \tag{1.7}$$

$$I[4,2,2]=\frac{1}{(4\pi)^2}{\rm K}\ , \qquad I[2,1,1]=\frac{2}{(4\pi)^2}{\rm K}\ . \tag{1.8}$$

Here both the bosonic $I[4,2,2]$ and the fermionic $I[2,1,1]$ contributions involve the "transverse" AdS_5 modes with $m^2=2$. Since such modes are absent in the case of the $AdS_3\times S^3$ superstring theory one expects to find there no Catalan constant contribution. Indeed, as we will show in Appendix B, in this case

$$AdS_3\times S^3: \qquad \mathrm{a}_1=-2\ln 2\ , \qquad \mathrm{a}_2=0\ . \tag{1.9}$$

[3]Note that our notation here differs from [11, 12]: $\mu_{\rm here}=\nu_{\rm there}$, $\ell_{\rm here}=\mu_{\rm there}$ and $(\mathcal{J}/\ln\mathcal{S})_{\rm here}=\ell_{\rm there}$

[4]Once one includes the contribution of the classical action, the full scaling function (or "cusp anomaly") is given by $\hat{f}=\sqrt{\lambda}+f$.

1.2 Quantum partition function in reduced theory

Let us now summarize the results of the corresponding two-loop computation in reduced theory described in detail in the main part of this paper.

The Green-Schwarz $AdS_5 \times S^5$ superstring theory is based on the F/G supercoset where $F = PSU(2,2|4)$ and[5] $G = [Sp(2,2) \times Sp(4)]$ with the action having the following symbolic form [14, 15]

$$I_{\rm ST} = \frac{\sqrt{\lambda}}{4\pi} \int {\rm STr}\big[J^{(2)} \wedge *J^{(2)} + J^{(1)} \wedge J^{(3)}\big] \ , \tag{1.10}$$

where $J^{(2)}$ and $J^{(1),(3)}$ are the bosonic coset and the fermionic components of the $PSU(2,2|4)$ current. The associated Pohlmeyer reduced theory is given by a G/H gauged WZW model (with $G = Sp(2,2) \times Sp(4)$ and $H = [SU(2)]^4$) deformed with an integrable potential and coupled to two-dimensional fermions (originating from projections of the fermionic currents $J^{(1),(3)}$). Its action is, symbolically, [1]

$$I_{\rm PRT} = \frac{k}{8\pi} \int d^2\sigma \quad \Big[L_{gWZW}(g,A) + {\rm STr}\big(\mu^2 g^{-1}TgT + \Psi D(A)\Psi + \mu \Psi g^{-1}\Psi g\big)\Big] \ . \tag{1.11}$$

Here $g \in G$ is related to the bosonic coset current, A_a are components of the H gauge field, T is a constant matrix chosen in the reduction procedure whose commutant ($[H,T] = 0$) defines the subgroup H of G and μ is an arbitrary mass scale parameter. μ^2 may be interpreted as a gauge-prescribed value of the classical AdS_5 or S^5 stress tensor in the original string theory (i.e. μ is playing the role of p^+ in the corresponding string light-cone gauge): the simplest "vacuum" configuration corresponds to the BMN geodesic with μ proportional to the angular momentum in S^5.

The coupling constant k of the reduced theory is undetermined by the classical reduction procedure. If the quantum string theory and the quantum reduced theory are to be related at all, k should be related to the string tension or $\sqrt{\lambda}$. Observing that the μ-dependent terms in the reduced theory Lagrangian (1.11) are exactly equal to the superstring Lagrangian in (1.10) (with the components of the coset current replaced by its reduced theory values $J_+^{(2)} = \mu T, \ J_-^{(2)} = \mu g^{-1}Tg$, etc.) one may conjecture that[6]

$$k = 2\sqrt{\lambda} \ . \tag{1.12}$$

While the matching of the one-loop partition functions (1.1) is not sensitive to the values of the two coupling constants (as they do not enter the determinants of the quadratic fluctuation operators) the comparison of higher-loop quantum corrections crucially depends on a relation like (1.12).

Our aim below will be to compute the two-loop correction to the partition function of the PRT defined by (1.11) expanded near a solution which is a counterpart of the long spinning string solution (1.2). In the reduced theory the parameter μ of the solution in (1.2) becomes identified with the μ in the PRT action (1.11). While we shall keep μ non-zero at the intermediate stages, to be able to obtain the explicit two-loop result we will take the $\mu \to 0$ limit in the final expression, i.e. we will do the two-loop quantum PRT computation for the counterpart of the long spinning string with $J = 0$. In that case the logarithm of the quantum partition function in the reduced

[5] The canonical notation for $Sp(2,2) \subset SU(2,2)$ is $USp(2,2)$.

[6] Note that it is not clear if k is to be quantized in (1.11) as the reduced theory is defined in Minkowski 2-space and is massive.

theory has a similar form as in string theory (cf. (1.4),(1.5))

$$\Gamma_{\rm PRT} = -\ln Z_{\rm PRT} = \frac{1}{2\pi}\,{\rm f}(k)\ V_2\ , \qquad (1.13)$$

$${\rm f}(k) = {\rm a}_1 + \frac{2{\rm a}_2}{k}\ +\ O(\frac{1}{k^2})\ . \qquad (1.14)$$

Explicit results for the coefficients $\rm a_n$ that we found are (cf. (1.6))

$${\rm a}_1 = -3\ln 2\ , \qquad {\rm a}_2 = \bar{\rm a}_2 + \tilde{\rm a}_2\ , \qquad \bar{\rm a}_2 = -{\rm K}\ , \qquad \tilde{\rm a}_2 = -\frac{1}{4}({\rm a}_1)^2 = -\frac{9}{4}(\ln 2)^2\ . \qquad (1.15)$$

The value of the one-loop coefficient $\rm a_1$ matches the string theory one in (1.6), in agreement with (1.1). The Catalan constant term in $\bar{\rm a}_2$ has exactly the same coefficient as in the string partition function in (1.6) provided we assume the identification of couplings in (1.12).Moreover, the pattern of the bosonic and fermionic contributions (i.e. $+{\rm K} - 2{\rm K} = -{\rm K}$) turns out to be exactly the same as in the string theory expression in (1.6).

While the mass spectra of the quadratic fluctuation Lagrangians are equivalent, the interaction vertices could, in principle, generate additional nontrivial contributions in PRT, e.g. proportional to $I[4,4,4]$ in (1.7), which are not related to the Catalan's constant [10]. However, all such extra non-trivial integrals happen not to appear in PRT. We view this and the matching of the Catalan's constant as a strong indication that the two quantum theories are indeed closely connected.

At the same time, there is an additional $\tilde{\rm a}_2 \sim (\ln 2)^2$ term in the reduced theory two-loop coefficient $\rm a_2$ which is absent in the string theory two-loop coefficient a_2. To be precise, we did not manage to compute the value of the coefficient of $(\ln 2)^2$ term directly – we inferred it following a close analogy with the $AdS_3 \times S^3$ case where an alternative approach is available. The computational procedure we used in the $AdS_5 \times S^5$ PRT (called "first approach" below) led, in fact, to an IR divergent result $\tilde{\rm a}_2 = -\frac{5}{4}(\ln 2)^2 - \ln 2\,\ln m_0$, where $m_0 \to 0$ is an IR cutoff. We believe this should be an artifact of our approach in $AdS_5 \times S^5$ case in which the unphysical (non-coset) massless excitations were not explicitly decoupled.[7]

To test the expectation that an alternative computational procedure that does not involve unphysical propagating degrees of freedom should lead to an IR finite result for $\tilde{\rm a}_2$ we have repeated the same two-loop computation in a very similar but simpler setting of the reduced theory for the $AdS_3 \times S^3$ superstring [16]. While same approach as used in $AdS_5 \times S^5$ case here led again to an IR divergent coefficient, $\tilde{\rm a}_2^{(1)} = -\frac{2}{3}(\ln 2)^2 - \frac{4}{3}\ln 2\,\ln m_0$, the "second approach" based on integrating out the 2d gauge fields and gauge-fixing g led to a consistent finite two-loop result

$${\rm AdS_3 \times S^3}: \qquad {\rm f}(k_3) = {\rm a}_1 + \frac{2{\rm a}_2}{k_3}\ +\ O(\frac{1}{k_3^2})\ , \qquad (1.16)$$

$${\rm a}_1 = -2\ln 2\ , \qquad {\rm a}_2 = -\frac{1}{4}({\rm a}_1)^2 = -(\ln 2)^2\ . \qquad (1.17)$$

The coupling constants in the $AdS_5 \times S^5$ and $AdS_3 \times S^3$ are related by

$$k = k_5 = 2k_3\ . \qquad (1.18)$$

[7] This approach involves imposing the H gauge on the fluctuation of the 2d gauge field component, e.g. A_+, and treating the remaining bosonic degrees of freedom, i.e. g and A_- (some of which are unphysical and massless) on an equal footing. An additional subtlety may be related to a particular way of taking the $\mu \to 0$ limit.

Once again, the one-loop coefficient here is the same as in (1.6) and the absence of the more complicated contributions like the Catalan's constant is also consistent with the vanishing of the string theory two-loop coefficient in (1.9).

It remains to be understood if the apparent disagreement of the two-loop coefficients a_2 and a_2 in string and reduced theories by precisely the square of the one-loop coefficient is still hinting at some relation between the two universal scaling functions.

The rest of the paper is organized as follows. In section 2 we shall first review the structure of the reduced theory and explain the approach to perturbative calculations in it based on a field redefinition using Polyakov-Wiegmann identity and gauge fixing imposed on A. We shall present the fluctuation Lagrangian and list the basic types of two-loop diagrams we are going to compute below.

In section 3 we consider the $AdS_3 \times S^3$ reduced theory. We start with presenting the two-loop computation using the first approach explained in section 2 and then consider an alternative second approach based on integrating out 2d gauge fields and gauge-fixing imposed on g. We shall compare the results of the two approaches and suggest a resolution of the IR divergence problem of the first approach that should restore the equivalence between the two approaches. The resulting finite two-loop coefficient is given in (1.17).

In section 4 we present the analogous computation in the $AdS_5 \times S^5$ reduced theory. We first discuss the one-loop approximation where the result for the partition function matches the string theory result. We then consider the two-loop computation based on the first approach. Using a direct analogy with the $AdS_3 \times S^3$ case we propose a modification of the two-loop result that makes it IR finite. The final expression for the two-loop coefficient is given by the same Catalan's constant term as found in string theory plus an additional term proportional to the square of the one-loop coefficient.

Appendix A summarizes our supermatrix notation. In Appendix B we present the computation of the two-loop universal scaling function coefficient a_2 in the $AdS_3 \times S^3$ superstring theory, concluding that it vanishes, i.e., in contrast to the $AdS_5 \times S^5$ string case, it does not contain the Catalan's constant term. In Appendix C we include some details of the one-loop computation in section 4.1. In Appendix D we summarize the computation of the two-loop partition function of the reduced $AdS_5 \times S^5$ theory in the vacuum case and show that it vanishes.

2 Perturbative expansion of the Pohlmeyer reduced theory near a classical background

In this section we shall briefly review the action of reduced theory for string theory in $AdS_5 \times S^5$ and $AdS_3 \times S^3$ and then consider its perturbative expansion around a classical configuration.

2.1 Reduced theory action

The Green-Schwarz action in $AdS_n \times S^n$ spacetime can be formulated as a sigma-model action on the supercoset F/G with $F = PSU(1,1|2) \times PSU(1,1|2)$ and $G = SU(1,1) \times SU(2)$ for $n = 3$, and $F = PSU(2,2|4)$ and $G = Sp(2,2) \times Sp(4)$ for $n = 5$. The corresponding reduced theory is a G/H gauged Wess-Zumino-Witten model with an integrable potential and two-dimensional fermionic fields

$$
\begin{aligned}
I_{\rm PRT} = \frac{k}{8\pi}\int d^2\sigma\Big[\mathcal{L}_{\rm gWZW} + \ {\rm STr}\big(\mu^2\, g^{-1}TgT \\
+ \ \Psi_{_L}TD_+\Psi_{_L} + \Psi_{_R}TD_-\Psi_{_R} + \ \mu\, g^{-1}\Psi_{_L}g\Psi_{_R}\big)\Big]\ ,
\end{aligned}
\tag{2.1}
$$

where $\mathcal{L}_{\rm gWZW}$ is the Lagrangian of the symmetrically gauged WZW model,

$$
\begin{aligned}
\int d^2\sigma\ \mathcal{L}_{\rm gWZW} = {\rm STr}\Big[\frac{1}{2}\int d^2\sigma g^{-1}\partial_+ g g^{-1}\partial_- g - \frac{1}{6}\int d^3\sigma g^{-1}dg\, g^{-1}dg\, g^{-1}dg \\
+ \int d^2\sigma\left(A_+\,\partial_- g g^{-1} - A_-\, g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+A_-\right)\Big]\,.
\end{aligned}
\tag{2.2}
$$

Here $g\in G$ and $A_\pm$ take values in the algebra of H which is $[U(1)]^2$ in the $AdS_3\times S^3$ case and $[SU(2)]^4$ in the $AdS_5\times S^5$ case. We use the notation $\partial_\pm = \partial_\tau \pm \partial_\sigma,\ \ D=\partial+[A,\ ]$.

In general, the normalization of the $AdS_n\times S^n$ reduced theory action (2.1) depends on an index of the corresponding matrix representation, with k differing by 2 in $n=3$ and $n=5$ cases (see [3]). We will formally use the same normalization in both cases; when comparing the $AdS_3\times S^3$ and $AdS_5\times S^5$ results we should set $k=k_5=2k_3$ as in (1.18).

The constant matrix T is chosen as [1]

$$
\begin{aligned}
n=3 \quad &: \qquad T=\tfrac{i}{2}{\rm diag}(1,-1,1,-1)\,, \\
n=5 \quad &: \qquad T=\tfrac{i}{2}{\rm diag}(1,1,-1,-1,1,1,-1,-1)\,.
\end{aligned}
\tag{2.3}
$$

Since $[H,T]=0$, the full action is invariant under the H gauge transformations,

$$
\begin{aligned}
g\to h^{-1}g\,h\,, \qquad & A_\pm \to h^{-1}A_\pm h + h^{-1}\partial_\pm h\ , \\
\Psi_{_{R,L}} \to h^{-1}\Psi_{_{R,L}}h\,, \qquad & h\in H\ .
\end{aligned}
\tag{2.4}
$$

2.2 Gauge fixing and parameterization based on the Polyakov-Wiegmann identity

In general, a bosonic string solution corresponds to a bosonic solution of the reduced theory given by some non-trivial background (g_0, A_{0+}, A_{0-}). To compute the quantum partition of the reduced theory on such classical background one needs to fix an H gauge. It is natural to identify the physical degrees of freedom as corresponding to the coset part of fluctuations of g, $\delta g\in {\rm alg}(G/H)$, while the fluctuations of g along $\mathfrak{h}$ =alg(H) and A_+, A_- are "unphysical"; dim$\mathfrak{h}$ of the latter should be gauge-fixed and the rest integrated out. Then there are several possible choices:

(i) impose H-gauge on the fluctuations of the gauge field, e.g., on A_+;

(ii) impose H-gauge on the fluctuations of g;

(iii) impose some "mixed" gauge on the fluctuations of g and $A_\pm$.

In each of these cases one may either integrate out the remaining unphysical fluctuations from the very beginning or treat them on the same footing with the physical fluctuations of g in the loop expansion.

For example, in computing the perturbative S-matrix near the trivial vacuum $g_0=\mathbf{I},\ A_{0\pm}=0$ in [3] the gauge $A_+=0$ was imposed. Then the constraint following from integration over A_- was solved explicitly by eliminating from the action the unphysical part of g in terms of the physical one, ending up with a local action for the 8+8 physical massive bosonic+fermionic fluctuations only. In the case of the reduced theory for the $AdS_3 \times S^3$ string one may impose a gauge on g, then integrate out A_+, A_- ending up with a non-linear action for the 4+4 physical fluctuations. A "mixed" gauge fixing was used in [6, 7] in discussing the one-loop partition function for fluctuations near a solution corresponding to a string moving in the $AdS_3 \times S^3$ part of $AdS_5 \times S^5$. This gauge led to the decoupling of the unphysical fluctuations from the physical ones at the level of the quadratic fluctuation action.

Directly extending each of these approaches to the two-loop level in the reduced theory for $AdS_5 \times S^5$ string appears to be rather cumbersome. Imposing $A_+ = 0$ gauge and then trying to solve the A_--constraint produces a complicated non-local quartic fluctuation action. Integrating out A_+, A_- first and gauge-fixing g also leads to very involved fluctuation action. It is also unclear how to find a "mixed" gauge which would ensure the decoupling of the unphysical fluctuations beyond the quadratic fluctuation level.

In discussing $AdS_5 \times S^5$ case in this paper we shall follow a different approach which may be viewed as a version of (i). It is based on gauge-fixing A_+ combined with a particular field redefinition of g while formally keeping the remaining "massless" unphysical fluctuations on the same footing with the "massive" physical ones in the two-loop computation. We shall first change the variables from A_+, A_- to $U, \tilde{U}$ as

$$A_+ = U\partial_+ U^{-1} , \qquad A_- = \tilde{U}\partial_- \tilde{U}^{-1} , \qquad U,\ \tilde{U} \in H , \tag{2.5}$$

and then use the Polyakov-Wiegmann identity to rewrite the PRT Lagrangian in (2.1) as

$$\begin{aligned} &\mathcal{L}_{\rm PRT} = \mathcal{L}(\tilde{g}) - \mathcal{L}_{\rm WZW}(U^{-1}\tilde{U}) \ , \\ &\mathcal{L}(\tilde{g}) = \mathcal{L}_{\rm WZW}(\tilde{g}) + {\rm STr}\Big[\mu^2 \tilde{g}^{-1} T \tilde{g} T + \tilde{\Psi}_{_L} T \partial_+ \tilde{\Psi}_{_L} + \tilde{\Psi}_{_R} T \partial_- \tilde{\Psi}_{_R} + \mu \tilde{g}^{-1} \tilde{\Psi}_{_L} \tilde{g} \tilde{\Psi}_{_R}\Big] , \end{aligned} \tag{2.6}$$

where

$$\tilde{g} = U^{-1} g\, \tilde{U} , \qquad \tilde{\Psi}_{_L} = U^{-1} \Psi_{_L} U , \qquad \tilde{\Psi}_{_R} = \tilde{U}^{-1} \Psi_{_R} \tilde{U} . \tag{2.7}$$

Such a form of the PRT action was used previously in [5] to demonstrate the UV finiteness of the reduced model. An advantage of this parametrization is that the unphysical degrees of freedom contained in $A_\pm$ are isolated in the "ghost-like" $\mathcal{L}_{\rm WZW}(U^{-1}\tilde{U})$ term but one is still to deal with the unphysical $\mathfrak{h}$-part of the fluctuations of $\tilde{g}$. The action corresponding to (2.6) remains H gauge-invariant under $U' = h^{-1}U,\ \tilde{U}' = h^{-1}\tilde{U}$. This requires an H gauge fixing; one natural option is to fix (the fluctuation of) U to be trivial. For example, if U has a trivial classical background, after gauge-fixing $U = 1$ the resulting action will be equivalent to the one found in (2.2) in the gauge $A_+ = 0$: setting $A_- = \tilde{U}\partial_-\tilde{U}^{-1}$ one then gets $\mathcal{L} = \mathcal{L}_{\rm WZW}(g) - \tilde{U}\partial_-\tilde{U}^{-1} g^{-1}\partial_+ g + ...$, and finally $\tilde{U}$-part can be decoupled by a redefinition of g.

At the level of the classical equations following from (2.2) one can always choose the on-shell gauge $A_+ = A_- = 0$ [1]; in this case only g (and thus also $\tilde{g}$) will have a non-trivial background, i.e. the contribution of the path integral over $\tilde{U}$ will be trivial. There will still be "unphysical" degrees of freedom contained in the fluctuations of $\tilde{g}$: in the case of $G = Sp(2,2) \times Sp(4)$ we will have $10 + 10$ bosonic fluctuations with $6 + 6$ corresponding to $H = [SU(2)]^4$ part and $4 + 4$ being the "physical" coset ones. The "unphysical" degrees of freedom should of course effectively decouple (and cancel against other "ghost" contributions and contribution of determinant of the change of variables (2.5)) in the final expression for the quantum partition function but this decoupling may not be manifest.

Figure 1: Bosonic two-loop diagrams. In (a) momentum conservation implies $q_i + q_j + q_k = 0$.

2.3 Structure of two-loop quantum corrections

Let us now consider the expansion of the Lagrangian $\mathcal{L}(\tilde{g})$ in (2.6) near a classical bosonic solution $\tilde{g}_0$. In what follows we shall omit tilde on g. Introducing the fluctuations of g taking values in the algebra $\mathfrak{g}$ of G

$$g = g_0 e^{\eta} = g_0\big(1 + \eta + \frac{1}{2}\eta^2 + \ldots\big) , \qquad \eta \in \mathfrak{g} , \tag{2.8}$$

we find for the quadratic, cubic and quartic terms in the fluctuation Lagrangian

$$\begin{aligned}\mathcal{L}^{(2)} &= \mathrm{STr}\Big[\frac{1}{2}\mathcal{D}_+\eta\partial_-\eta - \frac{\mu^2}{2}\left[\eta, g_0^{-1}Tg_0\right][\eta, T] \\ &\quad + \Psi_R T\partial_-\Psi_R + \Psi_L T\partial_+\Psi_L + \mu g_0^{-1}\Psi_L g_0\Psi_R\Big] ,\end{aligned} \tag{2.9}$$

$$\begin{aligned}\mathcal{L}^{(3)} &= \mathrm{STr}\Big[-\frac{1}{6}\left[\eta, \mathcal{D}_+\eta\right]\partial_-\eta - \frac{\mu^2}{6}\left[\eta, g_0^{-1}Tg_0\right][\eta,\, [\eta, T]] \\ &\quad + \mu\left(g_0^{-1}\Psi_L g_0\eta\Psi_R - \eta g_0^{-1}\Psi_L g_0\Psi_R\right)\Big] ,\end{aligned} \tag{2.10}$$

$$\begin{aligned}\mathcal{L}^{(4)} &= \mathrm{STr}\Big[\frac{1}{24}\left[\eta,\, [\eta, \mathcal{D}_+\eta]\right]\partial_-\eta + \frac{\mu^2}{24}\left[\eta,\, \left[\eta, g_0^{-1}Tg_0\right]\right][\eta,\, [\eta, T]] \\ &\quad + \mu\big(\frac{1}{2}g_0^{-1}\Psi_L g_0\eta^2\Psi_R + \frac{1}{2}\eta^2 g_0^{-1}\Psi_L g_0\Psi_R - \eta g_0^{-1}\Psi_L g_0\eta\Psi_R\big)\Big] ,\end{aligned} \tag{2.11}$$

where $\mathcal{D}_+ = \partial_+ + \left[g_0^{-1}\partial_+ g_0\,,\ \right]$.

Under the G/H coset decomposition of the algebra $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ (induced by T, which selects $H \subset G$ such that $[H, T] = 0$, see Appendix A) we have

$$\eta = \eta^{\parallel} + \eta^{\perp} , \qquad \eta^{\parallel} \in \mathfrak{m}, \qquad \eta^{\perp} \in \mathfrak{h} , \qquad [\eta^{\perp}, T] = 0 . \tag{2.12}$$

Here $\eta^{\parallel}$ describes the "physical" fluctuations.

Our aim will be to compute the two-loop corrections to the partition function of this theory in the case of a special background corresponding to the infinite spin limit of the folded string (1.2). This is a homogeneous background for which the coefficients in the fluctuation Lagrangian will be constant; at the end we will take the limit $\mu \to 0$; then there will be only one common scale κ that can be absorbed into the infinite volume factor V_2 appearing in the logarithm of the partition function or the "quantum effective action" (1.13).

The diagrams contributing to two-loop partition functions are shown in Figures 1, 2, 3. Let us describe the general form of the two-loop corrections on the example of the bosonic diagrams in Figure 1. In computing the two-loop partition function we shall formally rotate to euclidean

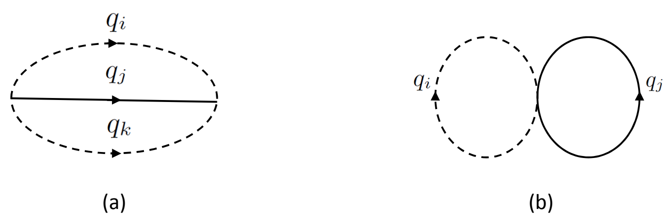

Figure 2: Diagrams with bosonic (solid lines) and fermionic (dashed lines) propagators.

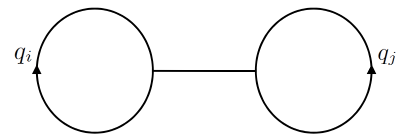

Figure 3: Tadpole diagrams. The intermediate line is bosonic while loops can be either bosonic or fermionic.

worldsheet time and thus consider the euclidean signature propagators. Writing the bosonic part of the fluctuation Lagrangian in (2.9), (2.9), (2.11) as

$$\mathcal{L}_B^{(2)} + \mathcal{L}_B^{(3)} + \mathcal{L}_B^{(4)} = \frac{1}{2}\Phi_I \triangle_{IJ} \Phi_J + \frac{1}{3!} V_{IJK} \Phi_I \Phi_J \Phi_K + \frac{1}{4!} V_{IJKL} \Phi_I \Phi_J \Phi_K \Phi_L + \ldots \ , \tag{2.13}$$

where Φ_I stand for fluctuation field components and assuming that all coefficients are constant one finds that the one-loop contribution to the logarithm of the quantum partition function $\Gamma = -\ln Z$ is given by $\frac{1}{2}\mathrm{Tr} \ln \triangle$ while 1PI part of the two-loop term in Γ is given by the sum of the "sunset" and "double-bubble" graph contributions:

$$\Gamma^{(2)}_{\text{sunset}} = -\frac{1}{12}\frac{8\pi}{k} V_2 \int \frac{d^2 q_i d^2 q_j}{(2\pi)^2} V_{IJK} V_{I'J'K'} \triangle^{-1}_{II'} \triangle^{-1}_{JJ'} \triangle^{-1}_{KK'} \,, \tag{2.14}$$

$$\Gamma^{(2)}_{\text{double-bubble}} = \frac{1}{8}\frac{8\pi}{k} V_2 \int \frac{d^2 q_i d^2 q_j}{(2\pi)^2} V_{IJKL} \triangle^{-1}_{IJ} \triangle^{-1}_{KL} \,. \tag{2.15}$$

Here $-\frac{1}{12}$ and $\frac{1}{8}$ are combinatorial factors, and $\frac{8\pi}{k}$ comes from the overall factor in front of the action (2.1). Contributions of graphs with fermionic propagators have similar structure (with overall minus sign due to fermionic loop).

In general, individual diagram contributions are gauge-dependent, so some graphs may or may not appear depending on the particular gauge fixing. In the corresponding two-loop computation in string theory non-1PI ("tadpole") diagrams did not contribute in the conformal gauge [10, 11], but were non-vanishing in the light-cone gauge [13]. In the present case of the reduced theory we will also have non-trivial contributions coming from the tadpole graphs in Figure 3. Moreover, while in lightcone string theory only fermion loops contribute to non-1PI graphs, in the reduced theory both bosonic and fermionic loops yield nontrivial contributions. The intermediate bosonic line connecting the two loops there has zero momentum, and the zero-momentum limit may be subtle. Since several physical components of the propagators (see sections 3 and 4) will vanish in that limit, we will set the momentum of the intermediate line to zero only after doing the integration in the two loops.

Let us now comment on the structure of relevant momentum integrals. As in the string theory computation [10, 11], we shall assume that all power-like UV divergent terms can be regularized away using an analytic regularization scheme.[8] The logarithmic UV divergences should cancel out [5] and we shall verify this below. Our aim will be to compute the finite part of Γ. Some of the two-loop integrals are expressed in terms of products of simple one-loop integrals,

$$I[m^2] = \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2+m^2} \, . \tag{2.16}$$

It is useful to rewrite it as

$$I[m^2] = I[1] - \frac{1}{4\pi} \ln m^2 \, , \tag{2.17}$$

isolating the UV divergent part $I[1]$. Double-bubble and tadpole diagram contributions will be given in terms of sums of products $I[m_i^2]I[m_j^2]$. The sunset diagram contributions are expressed in terms of the following integral

$$I[m_i^2, m_j^2, m_k^2] = \int \frac{d^2q_i \, d^2q_j \, d^2q_k}{(2\pi)^4} \frac{\delta^{(2)}(q_i+q_j+q_k)}{(q_i^2+m_i^2)(q_j^2+m_j^2)(q_k^2+m_k^2)} \, . \tag{2.18}$$

This integral (already mentioned above in (1.7),(1.8)) is UV finite and also IR finite for nonzero m_i, m_j and m_k.

3 Reduced theory for $AdS_3 \times S^3$ string

We shall first compute the two-loop corrections in the reduced theory corresponding to the $AdS_3 \times S^3$ superstring. The aim will be to compare with the superstring result found in Appendix B. In this case $G/H = (SU(1,1) \times SU(2))/[U(1)]^2$ and the action is given by (2.1) with $k = 2k_3$. In section 3.1 we shall discuss the computation of two-loop corrections in this theory using the first approach based on (2.6) as described in section 2.2. In section 3.2 we shall consider an alternative approach where one imposes a gauge fixing on g and integrates out $A_\pm$. As was shown in [16], the resulting model is the sum of the complex sinh-Gordon and complex sine-Gordon models coupled to fermions. Here only the physical degrees of freedom are present and the two-loop computation is straightforward.

Let us start with presenting the classical background $(g_0, A_{0\pm})$ in the reduced theory that corresponds to the long (S, J) string solution (1.2). g is a direct product of the "A" (i.e. AdS_3) and "S" (i.e. S^3) parts corresponding to $SU(1,1)$ and $SU(2)$, with embedding into $SU(2)$ being trivial. If we choose the basis in $\mathfrak{su}(1,1)$ as $R_1 = \sigma_1$, $R_2 = i\sigma_3$ and $R_3 = \sigma_2$ (σ_i are the Pauli matrices) then the G/H coset can be parametrized by the Euler angles (ϕ, χ), i.e. assuming vector $H = U(1)$ gauge fixing on g we have $g = \exp\left(\frac{1}{2}\chi R_2\right)\exp\left(\phi R_1\right)\exp\left(\frac{1}{2}\chi R_2\right)$. Then, a classical background corresponding to a string solution with trivial (BMN vacuum) S^3 part can be written as

$$g = \begin{pmatrix} g_A & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \qquad g_A = \begin{pmatrix} e^{i\chi}\cosh\phi & \sinh\phi \\ \sinh\phi & e^{-i\chi}\cosh\phi \end{pmatrix}, \tag{3.1}$$

$$A_\pm = \begin{pmatrix} A_{\pm A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad A_{+A} = -\frac{i}{2}\partial_+\chi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_{-A} = \frac{i}{2}\cosh^2\!\phi \, \partial_-\chi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

[8] Power divergent terms should cancel out provided all measure factors are properly accounted for.

where the values of $A_\pm$ are determined by solving the classical equations following from (2.1). The fields ϕ and χ appear in the complex sinh-Gordon theory corresponding to string in $AdS_3 \times S^1$ and are related to the global AdS_3 string coordinates by

$$\partial_+ Y^P \partial_- Y_P = -\mu^2 \cosh 2\phi\,, \qquad \epsilon_{QRSP} Y^Q \partial_+ Y^R \partial_- Y^S \partial_\pm^2 Y^P = 4\mu^3 \cosh^2\phi\, \partial_\pm \chi\,. \tag{3.2}$$

The classical reduced theory background corresponding to the string solution in (1.2) is

$$\phi_0 = \ln \frac{\kappa + \sqrt{\kappa^2 - \mu^2}}{\mu}\,, \qquad \chi_0 = \frac{\mu^2 - \kappa^2}{\mu}\sigma\,. \tag{3.3}$$

Equivalently, the classical PRT background we are interested in can be represented as

$$g_0 = \begin{pmatrix} g_{0A} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \qquad g_{0A} = \begin{pmatrix} \frac{\kappa}{\mu} v_\sigma^* & \frac{\ell}{\mu} \\ \frac{\ell}{\mu} & \frac{\kappa}{\mu} v_\sigma \end{pmatrix}, \tag{3.4}$$

$$A_{0\pm} = \begin{pmatrix} A_{0\pm A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad A_{0\pm A} = \frac{i\kappa^2}{2\mu} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.5}$$

$$v_\sigma \equiv e^{\frac{i\ell^2\sigma}{\mu}}\,, \qquad v_\tau \equiv e^{\frac{i\kappa^2\tau}{\mu}}\,, \qquad w \equiv v_\tau v_\sigma\,. \tag{3.6}$$

Here we introduced also the functions v_τ and w that will be often used below. The value of the classical reduced theory action on this background is

$$\Gamma^{(0)} = I_{\rm PRT} = \frac{k_3}{4\pi} V_2 \frac{\mu^4 - \kappa^4}{\mu^2}\,. \tag{3.7}$$

3.1 Approach based on PW identity and gauge-fixing A

The above classical solution written in terms of the variables $U, \tilde{U}$ and $\tilde{g}$ in (2.5),(2.7) becomes

$$U_0 = \begin{pmatrix} u & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \qquad \tilde{U}_0 = \begin{pmatrix} \tilde{u} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \qquad u = \tilde{u} = \begin{pmatrix} v_\tau^{*1/2} & 0 \\ 0 & v_\tau^{1/2} \end{pmatrix}, \tag{3.8}$$

$$\tilde{g}_0 = U_0^{-1} g_0 \tilde{U}_0 = \begin{pmatrix} \tilde{g}_{0A} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \qquad \tilde{g}_{0A} = \begin{pmatrix} \frac{\kappa}{\mu} v_\sigma^* & \frac{\ell}{\mu} v_\tau \\ \frac{\ell}{\mu} v_\tau^* & \frac{\kappa}{\mu} v_\sigma \end{pmatrix}. \tag{3.9}$$

We shall now discuss the computation of quantum corrections on this background using the approach described in sections 2.2 and 2.3, i.e. using the fluctuation Lagrangian in (2.9), (2.9), (2.11).

3.1.1 One-loop contribution

Using the parametrization described in Appendix A, let us introduce the following bosonic fluctuation fields in (2.12):

$$\eta^\parallel = \begin{pmatrix} \eta_A^\parallel & 0 \\ 0 & \eta_S^\parallel \end{pmatrix}, \qquad \eta^\perp = \begin{pmatrix} \eta_A^\perp & 0 \\ 0 & \eta_S^\perp \end{pmatrix}, \tag{3.10}$$

$$\eta_A^\parallel = \begin{pmatrix} 0 & w(a_1 + ia_2) \\ w^*(a_1 - ia_2) & 0 \end{pmatrix}, \qquad \eta_S^\parallel = \begin{pmatrix} 0 & b_1 + ib_2 \\ b_1 - ib_2 & 0 \end{pmatrix}, \tag{3.11}$$

$$\eta_A^\perp = \begin{pmatrix} ic & 0 \\ 0 & -ic \end{pmatrix}, \qquad \eta_S^\perp = \begin{pmatrix} id & 0 \\ 0 & -id \end{pmatrix}. \tag{3.12}$$

The fields a_1, a_2, b_1, b_2 represent the physical (coset) fluctuations while c and d are "unphysical" ones. The factors $w = v_\tau v_\sigma$ are introduced so that to make the coefficients in the resulting fluctuation Lagrangian constant. Then the "A" and "S" parts of the quadratic fluctuation Lagrangian (2.9) are found to be (the action is $I = \frac{k_3}{4\pi}\int d^2\sigma\, \mathcal{L}$)

$$\mathcal{L}^{(2)} = \mathcal{L}^{(2)}_{\rm A} + \mathcal{L}^{(2)}_{\rm S}\,, \tag{3.13}$$

$$\mathcal{L}^{(2)}_{\rm A} = \sum_{i=1,2} \partial_+ a_i \partial_- a_i + 2\left(\mu \partial_+ a_2 + M_2 \partial_- a_2\right) a_1 - \partial_+ c\, \partial_- c - 4 M_1 a_1 \partial_- c\,, \tag{3.14}$$

$$M_1 = \frac{\kappa\sqrt{\kappa^2-\mu^2}}{\mu^2}\,, \qquad M_2 = \frac{2\kappa^2-\mu^2}{\mu}\,, \tag{3.15}$$

$$\mathcal{L}^{(2)}_{\rm S} = \sum_{i=1,2}\left(\partial_+ b_i \partial_- b_i - \mu^2 b_i^2\right) + \partial_+ d \partial_- d\,. \tag{3.16}$$

The spectrum of massive physical fluctuations is exactly the same as in the corresponding string theory [9]: (3.14) describes two physical fluctuations with frequencies $\sqrt{n^2+2\kappa^2 \pm 2\sqrt{\kappa^4+n^2\mu^2}}$, (where n is spatial momentum number on worldsheet circle) while (3.16) describes two physical fields whose characteristic frequencies are $\sqrt{n^2+\mu^2}$. The one-loop partition function following from (3.14) and (3.16)

$$\Big(\det\left(\partial_+\partial_-\right)^2 \det\left(\partial_+\partial_- + \mu^2\right)^2 \det\left[\partial_+^2\partial_-^2 + 2\partial_+\partial_-\left(2\kappa^2-\mu^2\right) + \left(\partial_+^2+\partial_-^2\right)\mu^2\right]\Big)^{-1/2}, \tag{3.17}$$

differs from the bosonic part of the string theory result only by the unphysical massless field contribution $[\det(\partial_+\partial_-)]^{-1}$. The latter is canceled out once we account for (i) the Jacobian of the transformation (2.5), and (ii) the contribution of the $U^{-1}\tilde{U}$ dependent WZW term in (2.6). The latter gives only massless contribution since according to (3.8) $U^{-1}\tilde{U}$ has trivial background and we may gauge-fix the fluctuation of U to be zero (which corresponds to the $\delta A_+ = 0$ gauge).

The fermionic fluctuations can be parametrized as follows:

$$\Psi_{_R} = \begin{pmatrix} 0 & \mathfrak{X}_R \\ \mathfrak{Y}_R & 0 \end{pmatrix}, \qquad \Psi_{_L} = \begin{pmatrix} 0 & \mathfrak{X}_L \\ \mathfrak{Y}_L & 0 \end{pmatrix}, \tag{3.18}$$

$$\mathfrak{X}_R = \begin{pmatrix} 0 & (\alpha_1+i\alpha_2)t_{_{1+}} \\ (\alpha_3+i\alpha_4)t^*_{_{2+}} & 0 \end{pmatrix}, \quad \mathfrak{Y}_R = \begin{pmatrix} 0 & (-i\alpha_3-\alpha_4)t_{_{2+}} \\ (i\alpha_1+\alpha_2)t^*_{_{1+}} & 0 \end{pmatrix}$$

$$\mathfrak{X}_L = \begin{pmatrix} 0 & (\beta_1+i\beta_2)t_{_{1-}} \\ (\beta_3+i\beta_4)t^*_{_{2-}} & 0 \end{pmatrix}, \quad \mathfrak{Y}_L = \begin{pmatrix} 0 & (-i\beta_3-\beta_4)t_{_{2-}} \\ (i\beta_1+\beta_2)t^*_{_{1-}} & 0 \end{pmatrix}$$

$$t_{_{1\pm}} \equiv e^{i\frac{\ell^2(\tau\pm\sigma)}{2\mu}}\,, \qquad t_{_{2\pm}} \equiv e^{i\frac{\left(\kappa^2+\mu^2\right)\tau\pm\ell^2\sigma}{2\mu}}\,, \tag{3.19}$$

where the component fields α_k, β_k are real Grassmann. The rescaling factors $t_{_{1\pm}}, t_{_{2\pm}}$ are introduced to make the coefficients in the resulting fermionic part of the quadratic fluctuation Lagrangian (2.9) constant:

$$\mathcal{L}^{(2)}_{\rm F} = \sum_{i=1}^{4}\left(\alpha_i \partial_- \alpha_i + \beta_i \partial_+ \beta_i\right) + 2\mu\left(\alpha_3\alpha_4 + \beta_3\beta_4\right) + 2\kappa\left(\alpha_1\beta_2 - \alpha_2\beta_1 - \alpha_3\beta_4 + \alpha_4\beta_3\right)\,. \tag{3.20}$$

The resulting fermionic characteristic frequencies are: $2\times\sqrt{n^2+\kappa^2}$, $\sqrt{n^2+\kappa^2}+\mu$, $\sqrt{n^2+\kappa^2}-\mu$. These are equivalent to the string theory fluctuation spectrum with $\pm\mu$ shifts being due to an overall τ-dependent rotation of the fluctuations (these cancel out in the sum over frequencies or in the resulting functional determinant). Indeed, the fermionic contribution to the one-loop partition function following from (3.20) is

$$\Big[\det\big(\partial_+\partial_- + \kappa^2\big)\Big]^2 \ \det\Big[\partial_+^2\partial_-^2 + 2\kappa^2\partial_+\partial_- + \mu^2\big(\partial_+^2 + \partial_-^2\big) + \big(\kappa^2 - \mu^2\big)^2\Big] \ . \tag{3.21}$$

Here the determinant of the 4-th order operator can be factorized as follows:

$$\begin{aligned}
&\det\Big(\big[\partial_+\partial_- + i\mu\,(\partial_+ + \partial_-) - \mu^2 + \kappa^2\big]\big[\partial_+\partial_- - i\mu\,(\partial_+ + \partial_-) - \mu^2 + \kappa^2\big]\Big) \\
&= \det\big[e^{-i\mu\tau}\big(\partial_+\partial_- + \kappa^2\big)e^{i\mu\tau}\big] \ \det\big[e^{i\mu\tau}\big(\partial_+\partial_- + \kappa^2\big)e^{-i\mu\tau}\big] \ .
\end{aligned} \tag{3.22}$$

and thus the fermionic one-loop contribution is equivalent to the string theory one: $[\det(\partial_+\partial_- + \kappa^2)]^4$. Combining the bosonic and fermionic contributions we find in the limit $\mu \to 0$ the following expression for the one-loop correction to the effective action ($V_2 = \kappa^2\bar{V}_2$)

$$\begin{aligned}
\Gamma^{(1)} &= \frac{1}{2}\bar{V}_2 \int \frac{d^2q}{(2\pi)^2}\big[\ln(q^2 + 4\kappa^2) + 3\ln q^2 - 4\ln(q^2 + \kappa^2)\big] \\
&= 2\kappa^2\bar{V}_2\big(I[4] - I[1]\big) = \frac{1}{2\pi}(-2\ln 2)V_2 \ ,
\end{aligned} \tag{3.23}$$

where the first equality is proved by differentiating over κ^2. Thus the resulting one-loop coefficient in (1.14) is given by the same value (1.6) as in the $AdS_3 \times S^3$ string theory: $\mathrm{a}_1 = a_1 = -2\ln 2$.

Let us note that in the above approach based on (2.6) the limit $\mu \to 0$, though regular at the level of the quantum partition function, appears to be singular at the level of the fluctuation Lagrangian (M_1, M_2 in (3.15) blow up). This may be attributed to a special nature of the field redefinition/gauge choice we used. Indeed, in the "mixed" gauge approach used in [7] (where unphysical fluctuations were explicitly decoupled from physical ones) it was found that this limit is well-defined in the fluctuation Lagrangian. However, this "decoupling" gauge does not appear to have a useful extension beyond the quadratic fluctuation level.

3.1.2 Two-loop contribution

As we have seen above, we have $2+2$ physical and $1+1$ unphysical bosonic fluctuations which are coupled together. One possibility would be to integrate out the unphysical fluctuation first, getting a (non-local) effective Lagrangian for the physical fluctuations. Here we shall treat all fluctuations on an equal footing; an alternative approach will be discussed in the next subsection.

Labeling the bosonic fluctuations as

$$\Phi_I = \{\Phi_{A\,i}, \Phi_{S\,j}\} \ , \qquad \Phi_{A\,i} = \{a_1, a_2, c\} \, , \qquad \Phi_{S\,i} = \{b_1, b_2, d\} \, , \tag{3.24}$$

we find the following (euclidean) bosonic propagator in the "A" and "S" sectors:

$$\begin{aligned}
\triangle_A^{-1}(q) &= \tfrac{1}{D_2}\begin{pmatrix}
-\frac{q^2}{2} & -\frac{\hat\kappa^2 q_- + iq^1\hat\mu^2}{\hat\mu} & -\frac{\hat\kappa q_-\sqrt{\hat\kappa^2 - \hat\mu^2}}{\hat\mu} \\
\frac{\hat\kappa^2 q_- + iq^1\hat\mu^2}{\hat\mu} & \frac{q_+\left(q_+^2\hat\mu^2 - 4\hat\kappa^2\left(\hat\kappa^2 - \hat\mu^2\right)\right)}{2q_-\hat\mu^2} & \frac{2\hat\kappa\sqrt{\hat\kappa^2 - \mu^2}\left(\hat\kappa^2 q_- + iq_1\hat\mu^2\right)}{q_+\hat\mu^2} \\
\frac{\hat\kappa q_-\sqrt{\hat\kappa^2 - \hat\mu^2}}{\hat\mu} & \frac{2\hat\kappa\sqrt{\hat\kappa^2 - \hat\mu^2}\left(\hat\kappa^2 q_- + iq_1\hat\mu^2\right)}{q_+\hat\mu^2} & \frac{q^4\hat\mu^2 - 4\left(q^1\hat\mu^2 - i\hat\kappa^2 q_-\right)^2}{2q^2\hat\mu^2}
\end{pmatrix} , \\
\triangle_S^{-1}(q) &= \mathrm{diag}\left(-\tfrac{1}{2(q^2 + \hat\mu^2)}, -\tfrac{1}{2(q^2 + \hat\mu^2)}, -\tfrac{1}{2q^2}\right) \ , \\
q_\pm &= q_0 \pm iq_1 \ , \qquad q^2 = q_0^2 + q_1^2 \, , \qquad D_2 = q^4 + 4\hat\kappa^2 q^2 - 4\hat\mu^2 {q_1}^2 \, ,
\end{aligned} \tag{3.25}$$

where we rescaled 2d momentum q by ℓ and

$$\hat{\kappa} = \frac{\kappa}{\ell} , \qquad \hat{\mu} = \frac{\mu}{\ell} \tag{3.26}$$

are the parameters that are fixed in the limit $\ell \to \infty$. The fermionic propagator following from (3.20) has similar structure. The two-loop graphs to be computed were described in section 2.3. In the limit $\mu \to 0$ (i.e. $\hat{\mu} \to 0$, $\hat{\kappa} \to 1$) we are interested in the physical mass spectrum includes one bosonic mode with $m^2 = 4$ and 4 fermionic modes with $m^2 = 1$ plus 3 massless bosonic modes. The $\mu \to 0$ limit can be taken once we simplify the integrands of the two-loop integrals. The general structure of the two-loop integrals appearing in the contributions of the sunset diagrams in Figure 1(a) and Figure 2(a) is

$$\mathcal{I}_{m_i^2, m_j^2, m_k^2} = \int \frac{d^2 q_i \, d^2 q_j \, d^2 q_k}{(2\pi)^4} \frac{\mathcal{F}(q_i, q_j, q_k)}{q_i^{n_i} q_j^{n_j} q_k^{n_k} (q_i^2 + m_i^2)(q_j^2 + m_j^2)(q_k^2 + m_k^2)} . \tag{3.27}$$

while the double-bubble diagrams in Figure 1(b) and Figure 2(b) lead to

$$\mathcal{I}_{m_i^2, m_j^2} = \int \frac{d^2 q_i \, d^2 q_j}{(2\pi)^4} \frac{\mathcal{F}(q_i, q_j)}{q_i^{n_i} q_j^{n_j} (q_i^2 + m_i^2)(q_j^2 + m_j^2)} . \tag{3.28}$$

Here m_i^2, m_j^2, m_k^2 can take values 0,1 or 4 and $\mathcal{F}$ are some polynomial functions of momenta. The absence of modes with $m^2 = 2$ in the $AdS_3 \times S^3$ case suggests the absence of contributions proportional to the Catalan's constant. Using an analytic regularization scheme and tensor manipulations described in [13] all the integrals can be expressed in terms of products of two $I[m^2]$ factors in (2.16). In addition to the 1PI diagrams there are non-1PI diagrams in Figure 3. Since the bosonic propagator in (3.25) vanishes for $q = 0$, one is to define them by first keeping the momentum of the intermediate bosonic line non-zero and setting it to zero only after doing the one-loop integrals.

The resulting expression for the two-loop effective action can be written as

$$\Gamma^{(2)} = \frac{8\pi}{k_3} V_2 \sum_n J_n , \tag{3.29}$$

where J_n are contributions of different types of diagrams:

$$\begin{aligned}
J_{\text{boson sunset}} &= -\tfrac{1}{24}(6I[4]I[0] + 6I[4]I[4]) , \\
J_{\text{boson double-bubble}} &= \tfrac{1}{16}(-4I[4]I[0] - 4I[4]I[4]) , \\
J_{\text{fermion-boson sunset}} &= \tfrac{1}{8}(12I[1]I[0] + 4I[4]I[1] - 8I[1]I[1]) , \\
J_{\text{fermion-boson double-bubble}} &= -\tfrac{1}{8}(8I[1]I[0] - 8I[4]I[1]) , \\
J_{\text{boson-boson tadpole}} &= -\tfrac{1}{16}(-\tfrac{8}{3}I[4]I[0] - \tfrac{16}{3}I[4]I[4]) , \\
J_{\text{boson-fermion tadpole}} &= \tfrac{1}{16}(-\tfrac{8}{3}I[1]I[0] - \tfrac{40}{3}I[4]I[1]) , \\
J_{\text{fermion-fermion tadpole}} &= -\tfrac{1}{16}(-8I[1]I[1]) .
\end{aligned} \tag{3.30}$$

The sums of the 1PI and non-1PI (tadpole) contributions are given by

$$J_{\text{1PI}} = -\frac{1}{2}\big(I[4] - I[1]\big)\big(I[4] + I[0] - 2I[1]\big) , \tag{3.31}$$

$$J_{\text{tadpole}} = \frac{1}{6}\big(I[4] - I[1]\big)\big(2I[4] + I[0] - 3I[1]\big) . \tag{3.32}$$

These are separately UV finite but IR divergent due to the presence of $I[0]$. Notice also that both expressions are proportional to the coefficient $I[4] - I[1]$ appearing in the one-loop result (3.23).

The total coefficient is then (using (2.17))

$$\sum_n J_n = -\frac{1}{6}\big(I[4] - I[1]\big)\big(I[4] + 2I[0] - 3I[1]\big)$$
$$= -\frac{1}{24\pi^2}\,(\ln 2)^2 - \frac{1}{12\pi^2}\ln 2 \ln m_0 \ . \tag{3.33}$$

This expression is still IR divergent: we introduced an IR cutoff $m_0 \to 0$ to define $I[0]$.

While the presence of the $(\ln 2)^2$ contribution (absent in the corresponding string theory result found in Appendix B) is an unambiguous result, the appearance of the IR divergence should be an artifact of our computational procedure. It may be related, in particular, to mixing between massless unphysical and physical modes and/or to a possible ambiguity in how the limit $\mu \to 0$ was taken. To support this expectation, in the next subsection we shall repeat the above two-loop computation using a different approach: by first integrating out $A_\pm$ and gauge-fixing g so that to explicitly eliminate all unphysical (non-coset) degrees of freedom from the fluctuation Lagrangian. The resulting two-loop correction will be found to be IR finite.

3.2 Approach based on integrating out gauge fields and gauge-fixing g

In the case of the reduced theory for the $AdS_3 \times S^3$ string it is straightforward to integrate out $A_\pm$ and gauge-fix g to get an action for the physical degrees of freedom only, which may then be used for computing the two-loop correction. The resulting action is the sum of the complex sinh-Gordon and the complex sine-Gordon models coupled to two-dimensional fermions [16]. Depending on whether one starts with the axial-gauged or vector-gauged WZW model one gets the "tanh-tan" (t-t) model or the "coth-cot" (c-c) model (these names refer to functions in the kinetic terms of the 2+2 coset bosonic degrees of freedom). The two models are related by the 2d duality and lead to equivalent results for the partition function when expanded near the respective classical solutions corresponding to the long folded string (i.e. near (3.3) or its 2d dual analog). It is useful to consider both models in parallel as this provides an extra check on our computation. The Lagrangians of the two models are (the action is normalized as $I = \frac{k}{8\pi}\int d^2\sigma\ \mathcal{L} = \frac{k_3}{4\pi}\int d^2\sigma\ \mathcal{L}$)

$$\begin{aligned}
\mathcal{L}_{\rm t-t} = {} & \partial_+\varphi\partial_-\varphi + \tan^2\varphi\ \partial_+\theta\partial_-\theta + \partial_+\phi\partial_-\phi + \tanh^2\phi\ \partial_+\chi\partial_-\chi + \tfrac{\mu^2}{2}\left(\cos 2\varphi - \cosh 2\phi\right) \\
& +\alpha\partial_-\alpha + \beta\partial_-\beta + \gamma\partial_-\gamma + \zeta\partial_-\zeta + \lambda\partial_+\lambda + \xi\partial_+\xi + \rho\partial_+\rho + \sigma\partial_+\sigma \\
& + \tan^2\varphi\left[\partial_+\theta\left(\lambda\xi - \rho\sigma\right) - \partial_-\theta\left(\alpha\beta - \gamma\zeta\right)\right] - \tanh^2\phi\left[\partial_+\chi\left(\lambda\xi - \rho\sigma\right) - \partial_-\chi\left(\alpha\beta - \gamma\zeta\right)\right] \\
& - \left(\alpha\beta - \gamma\zeta\right)\left(\lambda\xi - \rho\sigma\right)\left(\tfrac{1}{\cos^2\varphi} - \tfrac{1}{\cosh^2\phi}\right) - 2\mu\Big(\cosh\phi\cos\varphi\left(\lambda\gamma + \xi\zeta - \rho\alpha - \sigma\beta\right) \\
& + \sinh\phi\sin\varphi\big[\cos(\chi+\theta)\left(-\rho\zeta + \sigma\gamma + \lambda\beta - \xi\alpha\right) \\
& - \sin(\chi+\theta)\left(\lambda\alpha + \xi\beta + \rho\gamma + \sigma\zeta\right)\big]\Big)\ ,
\end{aligned} \tag{3.34}$$

$$
\begin{aligned}
\mathcal{L}_{\rm c-c} &= \partial_+\varphi\partial_-\varphi + \cot^2\varphi\ \partial_+\theta\partial_-\theta + \partial_+\phi\partial_-\phi + \coth^2\phi\ \partial_+\chi\partial_-\chi + \tfrac{\mu^2}{2}\left(\cos 2\varphi - \cosh 2\phi\right) \\
&+\alpha\partial_-\alpha + \beta\partial_-\beta + \gamma\partial_-\gamma + \zeta\partial_-\zeta + \lambda\partial_+\lambda + \xi\partial_+\xi + \rho\partial_+\rho + \sigma\partial_+\sigma \\
&- \cot^2\varphi\left[\partial_+\theta\left(\lambda\xi - \rho\sigma\right) - \partial_-\theta\left(\alpha\beta - \gamma\zeta\right)\right] + \coth^2\phi\left[\partial_+\chi\left(\lambda\xi - \rho\sigma\right) - \partial_-\chi\left(\alpha\beta - \gamma\zeta\right)\right] \\
&- \left(\alpha\beta - \gamma\zeta\right)\left(\lambda\xi - \rho\sigma\right)\left(\tfrac{1}{\sin^2\varphi} + \tfrac{1}{\sinh^2\phi}\right) - 2\mu\Big(\sinh\phi\sin\varphi\left(\lambda\gamma + \xi\zeta - \rho\alpha - \sigma\beta\right) \\
&+ \cosh\phi\cos\varphi\Big[\cos(\chi+\theta)\left(\rho\zeta - \sigma\gamma - \lambda\beta + \xi\alpha\right) \\
&- \sin(\chi+\theta)\left(\lambda\alpha + \xi\beta + \rho\gamma + \sigma\zeta\right)\Big]\Big)\,.
\end{aligned}
\tag{3.35}
$$

Here ϕ, θ correspond to bosonic degrees of freedom related to AdS_3, φ, χ correspond to S^3 part and α, β, γ, ζ, λ, ξ, ρ, σ are real fermionic fields.

For technical reasons (to make the expansion near the vacuum point regular) it is useful to generalize the reduced theory solution (3.3) by introducing also a similar non-trivial background in the "S^3" part of the model. Namely, we may start with the reduced theory background corresponding to the following generalization of the long spinning string in (1.2):

$$
\begin{array}{ccc}
Y_0 + iY_5 = \cosh(\ell\sigma)\, e^{i\kappa\tau}\,, & Y_1 + iY_2 = \sinh(\ell\sigma)\, e^{i\kappa\tau}\,, & Y_{3,4} = 0\ , \\
X_1 + iX_2 = \frac{1}{\sqrt{2}}e^{i\omega\tau + i{\rm n}\sigma}\,, & X_3 + iX_4 = \frac{1}{\sqrt{2}}e^{i\omega\tau - i{\rm n}\sigma}\,, & X_{5,6} = 0\ , \\
\kappa^2 = \ell^2 + \mu^2\,, & \mu^2 = {\rm n}^2 + \omega^2\,. &
\end{array}
\tag{3.36}
$$

This solution represents a superposition of a string with large spin in AdS_3 and a circular string with two large equal spins in S^3. The corresponding classical solutions in tanh-tan and coth-cot models (related again by 2d duality) are [7]

$$
{\rm t-t:}\qquad
\begin{array}{ll}
\phi_0 = \ln\frac{\kappa + \sqrt{\kappa^2 - \mu^2}}{\mu}\,, & \chi_0 = \frac{\kappa^2}{\mu}\tau\,, \\
\varphi_0 = \frac{1}{2}{\rm arccos}\left(\frac{2\omega^2}{\mu^2} - 1\right)\,, & \theta_0 = \frac{\omega^2}{\mu}\tau\,,
\end{array}
\tag{3.37}
$$

$$
{\rm c-c:}\qquad
\begin{array}{ll}
\phi_0 = \ln\frac{\kappa + \sqrt{\kappa^2 - \mu^2}}{\mu}\,, & \chi_0 = \frac{\mu^2 - \kappa^2}{\mu}\sigma\,, \\
\varphi_0 = \frac{1}{2}{\rm arccos}\left(\frac{2\omega^2}{\mu^2} - 1\right)\,, & \theta_0 = \frac{\omega^2 - \mu^2}{\mu}\sigma\,.
\end{array}
\tag{3.38}
$$

Below we shall consider the one-loop and two-loop corrections in the models (3.34) and (3.35) expanded near these solutions. We will eventually be interested in the limit

$$
\mu \to 0, \qquad \omega \to 0, \qquad \kappa \to \ell\ \gg 1\ , \qquad {\rm n} \to 0\ . \tag{3.39}
$$

3.2.1 One-loop contribution

Expanding the bosonic parts of the reduced theory Lagrangians (3.34) and (3.35) near the classical solutions (3.37) and (3.38),

$$
\phi = \phi_0 + \delta\phi\,, \qquad \chi = \chi_0 + \delta\chi\,, \qquad \varphi = \varphi_0 + \delta\phi\,, \qquad \theta = \theta_0 + \delta\theta\,, \tag{3.40}
$$

leads to the following quadratic fluctuation Lagrangians

$$
\begin{aligned}
\mathcal{L}^{(2B)}_{\rm t-t} &= \partial_-\delta\phi\partial_+\delta\phi + \partial_-\delta\varphi\partial_+\delta\varphi - 4\left(\kappa^2 - \mu^2\right)\delta\phi^2 - 4\left(\omega^2 - \mu^2\right)\delta\varphi^2 + \frac{\kappa^2 - \mu^2}{\kappa^2}\partial_-\delta\chi\partial_+\delta\chi \\
&+ \frac{2\mu\sqrt{\kappa^2 - \mu^2}}{\kappa}\delta\phi\left(\partial_-\delta\chi + \partial_+\delta\chi\right) + \frac{\mu^2 - \omega^2}{\omega^2}\partial_-\delta\theta\partial_+\delta\theta + \frac{2\mu\sqrt{\mu^2 - \omega^2}}{\omega}\delta\varphi\left(\partial_-\delta\theta + \partial_+\delta\theta\right)\,,
\end{aligned}
\tag{3.41}
$$

$$\mathcal{L}^{(2B)}_{\rm c-c} = \partial_-\delta\phi\partial_+\delta\phi + \partial_-\delta\varphi\partial_+\delta\varphi - 4\kappa^2\delta\phi^2 - 4\omega^2\delta\varphi^2 + \frac{\kappa^2}{\kappa^2-\mu^2}\partial_-\delta\chi\partial_+\delta\chi$$
$$+\frac{2\kappa\mu}{\sqrt{\kappa^2-\mu^2}}\delta\phi\left(\partial_-\delta\chi - \partial_+\delta\chi\right) + \frac{\omega^2}{\mu^2-\omega^2}\partial_-\delta\theta\partial_+\delta\theta + \frac{2\mu\omega}{\sqrt{\mu^2-\omega^2}}\delta\varphi\left(\partial_-\delta\theta - \partial_+\delta\theta\right) . \quad (3.42)$$

The resulting bosonic factors in the one-loop partition functions are equal and are also the same as in the original string theory[9]

$$Z^{(1B)}_{\rm t-t} = Z^{(1B)}_{\rm c-c} = \Big(\det\big[\partial_+^2\partial_-^2 + 2(2\kappa^2-\mu^2)\partial_+\partial_- + \mu^2(\partial_+^2+\partial_-^2)\big]\Big)^{-1/2}$$
$$\times\Big(\big[(\partial_+^2\partial_-^2 + 2(2\omega^2-\mu^2)\partial_+\partial_- + \mu^2(\partial_+^2+\partial_-^2)\big]\Big)^{-1/2} , \quad (3.43)$$

To simplify the fermionic Lagrangians in (3.34), (3.35) (making the coefficients in them constant) it is useful to rotate the fermionic fluctuations in the following way

$$\alpha+i\beta \to (\alpha+i\beta)e^{\mathcal{B}} , \quad \gamma+i\zeta \to (\gamma+i\zeta)e^{\mathcal{B}^*} , \quad \lambda+i\xi \to (\lambda+i\xi)e^{\mathcal{B}^*} , \quad \rho+i\sigma \to (\rho+i\sigma)e^{\mathcal{B}} ,$$
$$\mathcal{B}_{\rm t-t} = i\frac{\kappa^2+\omega^2}{2\mu}\tau , \qquad \mathcal{B}_{\rm c-c} = i\frac{\kappa^2-\omega^2}{2\mu}\sigma . \quad (3.44)$$

The corresponding fermionic one-loop determinants are then found to be

$$Z^{(1F)}_{\rm t-t} = \Big[\det\big[\partial_+^2\partial_-^2 + \mu^2\left(\partial_+^2+\partial_-^2\right) + 2\left(\kappa^2-\mu^2+\omega^2\right)\partial_+\partial_- + \left(\kappa^2-\omega^2\right)^2\big]\Big]^2$$
$$Z^{(1F)}_{\rm c-c} = \Big[\det\left(\partial_+\partial_- + \kappa^2-\mu^2+\omega^2\right)\Big]^4 . \quad (3.45)$$

Despite looking different, these two expressions can be shown to be equivalent.

In the limit $\mu,\omega\to 0$ we recover the expression found in the approach of section 3.1 equal also to the string theory result:

$$Z^{(1)} = Z^{(1B)}Z^{(1F)} = \big[\det(\partial_+\partial_- + 4\kappa^2)\big]^{-1/2}\big[\det(\partial_+\partial_-)\big]^{-3/2}\big[\det(\partial_+\partial_- + \kappa^2)\big]^4 . \quad (3.46)$$

3.2.2 Two-loop contribution

Since in the present case the Lagrangians (3.34), (3.35) found after integrating out gauge fields in (2.1) contain quartic fermionic terms, in addition to diagrams in Figure 1 and Figure 2 we will also have to compute the fermionic double-bubble diagram in Figure 4.[10] Applying the redefinitions in (3.44) one finds that the coefficients in the cubic and quartic fermionic terms in the fluctuation Lagrangian are constant. Hence the computation of two-loop corrections is similar to that in the first approach in section 3.1

As we are interested in the result in the $\mu,\omega\to 0$ limit, one should be careful to keep track of possible ambiguities in taking this limit that may be present in the individual diagrams by introducing the parameter

$$r \equiv \frac{\omega}{\mu} . \quad (3.47)$$

[9]The corresponding characteristic frequencies of the 4 bosonic fluctuations are $\sqrt{n^2+2\kappa^2\pm 2\sqrt{\kappa^4+n^2\mu^2}}$ and $\sqrt{n^2+2\omega^2\pm 2\sqrt{n^2\mu^2+\omega^4}}$. These match the string-theory expressions [9, 17].

[10]One may wonder also if we should account for a local one-loop counterterm [3] originating from integrating out $A_\pm$. This counterterm leads, however, only to power-divergent two-loop corrections which (along with similar contributions from other diagrams) are to be regularized away.

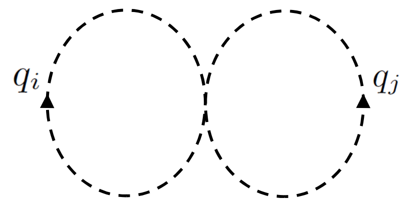

Figure 4: Fermionic double-bubble diagram.

The results for the contributions of individual diagrams to $\Gamma^{(2)}$ in (3.29) are found to be (cf. (3.30))

$$\begin{aligned}
\mathrm{t-t}: \quad & J_{\text{boson double-bubble}} = \tfrac{1}{16}\left(-8I[4]I[4]\right) , \\
& J_{\text{fermion-boson sunset}} = \tfrac{1}{8}\left(8I[1]I[0] + 4I[4]I[1] - 2\tfrac{1+2r^2}{r^2}I[1]I[1]\right) , \\
& J_{\text{fermion-boson double-bubble}} = -\tfrac{1}{8}\left(8I[1]I[0] - 4I[4]I[1]\right) , \\
& J_{\text{fermion-fermion double-bubble}} = \tfrac{1}{16}\left(4\tfrac{1-2r^2}{r^2}I[1]I[1]\right) , \\
& J_{\text{tadpole}} = -\tfrac{1}{16}\left(-8I[1]I[1]\right) ,
\end{aligned} \tag{3.48}$$

$$\begin{aligned}
\mathrm{c-c}: \quad & J_{\text{boson double-bubble}} = \tfrac{1}{16}\left(-8I[4]I[4]\right) , \\
& J_{\text{fermion-boson sunset}} = \tfrac{1}{8}\left(8I[1]I[0] + 4I[4]I[1] - \tfrac{6-4r^2}{1-r^2}I[1]I[1]\right) , \\
& J_{\text{fermion-boson double-bubble}} = -\tfrac{1}{8}\left(8I[1]I[0] - 4I[4]I[1]\right) , \\
& J_{\text{fermion-fermion double-bubble}} = \tfrac{1}{16}\left(-4\tfrac{1-2r^2}{1-r^2}I[1]I[1]\right) , \\
& J_{\text{tadpole}} = -\tfrac{1}{16}\left(-8I[1]I[1]\right) .
\end{aligned} \tag{3.49}$$

Summing up the 1PI contributions we find that IR-divergent and r-dependent terms cancel out and we get the same result in the two models:

$$J_{\text{1PI}} = -\frac{1}{2}I[4]I[4] + I[4]I[1] - I[1]I[1] , \tag{3.50}$$

$$J_{\text{tadpole}} = \frac{1}{2}I[1]I[1] , \tag{3.51}$$

so that the total is (using (2.17))

$$\Gamma^{(2)} = \frac{8\pi}{k_3}V_2\sum_n J_n , \qquad \sum_n J_n = -\frac{1}{2}\big(I[4] - I[1]\big)^2 = -\frac{1}{8\pi^2}\left(\ln 2\right)^2 . \tag{3.52}$$

Combining everything together we find that the effective action for the $AdS_3 \times S^3$ model is (cf. (1.17))

$$\Gamma^{(2)} = \frac{1}{\pi k_3}\,\mathrm{a}_2\, V_2 , \qquad \mathrm{a}_2 = -\frac{1}{4}(\mathrm{a}_1)^2 = -(\ln 2)^2 . \tag{3.53}$$

Let us now compare these results with those (3.31),(3.32),(3.33) found in the first approach in section 3.1. We observe that 1PI contributions in (3.31) and (3.50) contain the same $I[4]I[4]$ and $I[1]I[1]$ terms;[11] also, the fermion-fermion $I[1]I[1]$ tadpole terms are the same. Given that the

[11]Note that in contrast to the result (3.30) in the first approach in section 3.1 here we have no bosonic sunset contribution but the bosonic double-bubble contribution gives the same $I[4]I[4]$ term as the sum of the bosonic sunset and double-bubble contributions in (3.30). This should not be too surprising as the contributions of individual diagrams may be different in different gauges.

final result should be UV and IR finite, then the expression in (3.52) is a natural outcome. This suggests that it is the tadpole contribution (3.32) in the first approach that is to be blamed for the IR problem found there: it should not actually contain the $I[4]I[4]$ term if the two approaches are to agree. Then if instead of (3.31) one would take

$$\mathcal{J}'_{\rm tadpole} = \frac{1}{6}\big(I[4]-I[1]\big)\big(3I[0]-3I[1]\big) = \frac{1}{2}\big(I[4]-I[1]\big)\big(I[0]-I[1]\big)\ , \tag{3.54}$$

then the sum of (3.54) with the 1PI contribution (3.31) in the first approach would exactly match the result (3.52) of the second approach.[12]

It is also interesting to note that the final two-loop result in (3.52) is proportional to the square of the coefficient in the one-loop contribution in (3.23). These observations will guide us in interpreting and fixing the two-loop result in the case of the reduced theory for $AdS_5\times S^5$ string where we will only have the expression found using the first approach.

4 Reduced theory for $AdS_5 \times S^5$ string

Let us now carry out the similar computation in the reduced $AdS_5\times S^5$ theory. Here there are more fields and following the second approach based on integrating out gauge fields first appears to be difficult. For that reason here we will follow the first approach described in section 2.2. We will unambiguously determine the coefficient of the Catalan's constant term and match it with the string theory result. As in the $AdS_3\times S^3$ case discussed in section 3.1, in this approach there will be a non-canceling IR divergence which should be an artifact of mixing of physical and unphysical modes in this approach. A close analogy with the $AdS_3\times S^3$ case will motivate a modification of the tadpole contribution that will lead to IR finite two-loop $(\ln 2)^2$ term.

The reduced theory solution corresponding to the long (S,J) folded string (1.2) here has a similar structure to (3.4),(3.6). Following [6, 7], here we shall choose it in the τ-dependent form (cf. (3.4))[13]

$$g_0 = \begin{pmatrix} g_A & \mathbf{0} \\ \mathbf{0} & \mathbf{1}\end{pmatrix}, \qquad g_A = \begin{pmatrix} 0 & \frac{\kappa}{\mu}v_\tau^* & -\frac{\ell}{\mu}v_\tau^* & 0 \\ -\frac{\kappa}{\mu}v_\tau & 0 & 0 & \frac{\ell}{\mu}v_\tau \\ \frac{\ell}{\mu}v_\tau^* & 0 & 0 & -\frac{\kappa}{\mu}v_\tau^* \\ 0 & -\frac{\ell}{\mu}v_\tau & \frac{\kappa}{\mu}v_\tau & 0\end{pmatrix}, \qquad v_\tau = e^{i\frac{\kappa^2\tau}{\mu}} \tag{4.1}$$

[12] The replacement of (3.32) by (3.54) is formally achieved by replacing $I[4]$ in the second factor in (3.32) by $I[0]$. This may be related to a subtlety in how the two massive AdS_3 modes (which are mixed for $\mu\neq 0$) are treated in the tadpole contributions in the limit when $\mu\to 0$: in that limit one of them has $m^2=4$ and the other one becomes massless.

[13] In general, a choice of the reduced theory solution corresponding to a given string theory solution is not unique as one may apply an on-shell $H\times H$ gauge transformation. For example, one may start with a σ-dependent solution,

$$g'_A = \begin{pmatrix} \frac{\kappa}{\mu}v_\sigma^* & 0 & 0 & \frac{\ell}{\mu} \\ 0 & \frac{\kappa}{\mu}v_\sigma & \frac{\ell}{\mu} & 0 \\ 0 & \frac{\ell}{\mu} & \frac{\kappa}{\mu}v_\sigma^* & 0 \\ \frac{\ell}{\mu} & 0 & 0 & \frac{\kappa}{\mu}v_\sigma\end{pmatrix}, \qquad A'_{+A} = A'_{-A} = \frac{i\kappa^2}{2\mu}\Sigma\,.$$

One may expect that the result for the quantum partition function for the two solutions should be the same. In fact, we have checked that the individual diagram contributions in the two cases are indeed the same.

$$A_{\pm 0} = \begin{pmatrix} A_{\pm A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad A_{+A} = \frac{i(\ell^2+\kappa^2)}{2\mu}\,\Sigma\ , \quad A_{-A} = \frac{i\mu}{2}\Sigma\ , \quad \Sigma = \mathrm{diag}(1,-1,1,-1)\ . \tag{4.2}$$

The corresponding solutions for the fields $\tilde{g},\ U,\tilde{U}$ in (2.6),(2.5),(2.7) are (cf. (3.8),(3.9))

$$U_0 = \begin{pmatrix} u & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \qquad \tilde{U}_0 = \begin{pmatrix} \tilde{u} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \tag{4.3}$$

$$u = \tilde{u} = \begin{pmatrix} w^{*1/2} & 0 & 0 & 0 \\ 0 & w^{1/2} & 0 & 0 \\ 0 & 0 & w^{*1/2} & 0 \\ 0 & 0 & 0 & w^{1/2} \end{pmatrix}, \qquad w \equiv v_\tau v_\sigma = e^{i\frac{\kappa^2\tau+\ell^2\sigma}{\mu}}\ , \tag{4.4}$$

$$\tilde{g}_0 = U_0^{-1} g_0 \tilde{U}_0 = \begin{pmatrix} \tilde{g}_A & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{g}_A = \begin{pmatrix} 0 & \frac{\kappa}{\mu}v_\sigma & -\frac{\ell}{\mu}v_\tau^* & 0 \\ -\frac{\kappa}{\mu}v_\sigma^* & 0 & 0 & \frac{\ell}{\mu}v_\tau \\ \frac{\ell}{\mu}v_\tau^* & 0 & 0 & -\frac{\kappa}{\mu}v_\sigma \\ 0 & -\frac{\ell}{\mu}v_\tau & \frac{\kappa}{\mu}v_\sigma^* & 0 \end{pmatrix}. \tag{4.5}$$

Since $U_0^{-1}\tilde{U}_0 = 1$, the two-loop contribution of the WZW term $I[U^{-1}\tilde{U}]$ in (2.6) will be trivial. The value of the classical reduced theory action on this background is ($k = k_5$)

$$\Gamma^{(0)} = I_{\rm PRT} = \frac{k}{4\pi} V_2\, \frac{(\kappa^2-\mu^2)^2}{\mu^2}\ . \tag{4.6}$$

4.1 One-loop contribution

The analysis of quadratic fluctuation Lagrangian is similar to the one in section 3.1.1 though more involved as now the fluctuation fields have more components. The resulting one-loop partition function will match again the corresponding string theory result. To ensure that the coefficients in the fluctuation Lagrangian (2.9) are constant we parametrize the fluctuation fields as follows:

$$
\eta^{\|}=\left(\begin{array}{cc}\eta_A^{\|} & 0\\ 0 & \eta_S^{\|}\end{array}\right)\,,\qquad \eta_A^{\|}=\left(\begin{array}{cccc}0 & 0 & a_1+ia_2 & (a_3+ia_4)w\\ 0 & 0 & (a_3-ia_4)w^* & -a_1+ia_2\\ a_1-ia_2 & (a_3+ia_4)w & 0 & 0\\ (a_3-ia_4)w^* & -a_1-ia_2 & 0 & 0\end{array}\right)\,,
$$

$$
\eta_S^{\|}=\left(\begin{array}{cccc}0 & 0 & b_1+ib_2 & b_3+ib_4\\ 0 & 0 & -b_3+ib_4 & b_1-ib_2\\ -b_1+ib_2 & b_3+ib_4 & 0 & 0\\ -b_3+ib_4 & -b_1-ib_2 & 0 & 0\end{array}\right)\,,
$$

$$
\eta^{\perp}=\left(\begin{array}{cc}\eta_A^{\perp} & 0\\ 0 & \eta_S^{\perp}\end{array}\right)\,,\qquad \eta_A^{\perp}=\left(\begin{array}{cccc}ic_1 & (c_2+ic_3)w & 0 & 0\\ (-c_2+ic_3)w^* & -ic_1 & 0 & 0\\ 0 & 0 & ic_4 & (c_5+ic_6)w\\ 0 & 0 & (-c_5+ic_6)w^* & -ic_4\end{array}\right)\,,
$$

$$
\eta_S^{\perp}=\left(\begin{array}{cccc}id_1 & d_2+id_3 & 0 & 0\\ -d_2+id_3 & -id_1 & 0 & 0\\ 0 & 0 & id_4 & d_5+id_6\\ 0 & 0 & -d_5+id_6 & -id_4\end{array}\right)\,. \tag{4.7}
$$

Here $\eta^{\|}$ represent 4+4 physical (coset) fluctuations and $\eta^{\perp}$ represent 6+6 unphysical fields. The "S^5" part of the fluctuation Lagrangian is simply

$$
\mathcal{L}_{\rm S}^{(2)}=2\sum_{i=1}^{4}\left(\partial_+b_i\partial_-b_i-\mu^2b_i^2\right)+\sum_{j=1}^{6}\partial_+d_j\partial_-d_j\,, \tag{4.8}
$$

describing 4 mass μ degrees of freedom as in string theory. The "AdS_5" part splits into the subsectors of mixed fluctuations: one subsector contains a_1, a_2 and the off-diagonal part of $\eta_A^{\perp}$ — c_2, c_3,c_5 and c_6; the other contains a_3, a_4 and the diagonal part of $\eta_A^{\perp}$ — c_1 and c_4. Explicitly,

$$
\begin{aligned}
\mathcal{L}_{\rm A}^{(2)}&=\mathcal{L}_1^{(2)}+\mathcal{L}_2^{(2)}\,, &(4.9)\\
\mathcal{L}_1^{(2)}&=2\sum_{i=1,2}\left[\partial_+a_i\partial_-a_i-\left(2\kappa^2-\mu^2\right)a_i^2\right]-4M_1\left(\mu c_2+\partial_-c_3+\mu c_5+\partial_-c_6\right)a_1\\
&\quad+4M_1\left(\partial_-c_2-\mu c_3-\partial_-c_5+\mu c_6\right)a_2-\sum_{j=2,3,5,6}\left[\partial_+c_j\partial_-c_j+\left(2\kappa^2-\mu^2\right)c_j^2\right]\\
&\quad-2\left(\mu\partial_+c_3+M_2\partial_-c_3\right)c_2-2\left(\mu\partial_+c_6+M_2\partial_-c_6\right)c_5\,,\\
\mathcal{L}_2^{(2)}&=2\sum_{i=3,4}\partial_+a_i\partial_-a_i+4\left(\mu\partial_+a_4+M_2\partial_-a_4\right)a_3-\sum_{j=1,4}\partial_+c_j\partial_-c_j+4M_1(\partial_-c_1+\partial_-c_4)a_3\,,
\end{aligned}
$$

where the constants $M_1=\frac{\kappa\sqrt{\kappa^2-\mu^2}}{\mu^2}$, $M_2=\frac{2\kappa^2-\mu^2}{\mu}$ are the same as in (3.15). The bosonic contribution to the one-loop partition function is then

$$
Z^{(1B)}=Z_{\rm A}^{(1B)}Z_{\rm S}^{(1B)}\,,\qquad Z_{\rm S}^{(1B)}=\Big([\det\left(\partial_+\partial_-+\mu^2\right)]^4\ [\det\left(\partial_+\partial_-\right)]^6\Big)^{-1/2}\,, \tag{4.10}
$$

$$
\begin{aligned}
Z_{\rm A}^{(1B)}=&\Big([\det\left(\partial_+\partial_-+2\kappa^2-\mu^2\right)]^2\ [\det(\partial_+\partial_-)]^2\ [\det\left(\partial_-^2+\mu^2\right)]^2\ [\det\left(\partial_+^2+\mu^2\right)]^2\\
&\times\det\left[\partial_+^2\partial_-^2+2\left(2\kappa^2-\mu^2\right)\partial_+\partial_-+\mu^2(\partial_+^2+\partial_-^2)\right]\Big)^{-1/2}\,.
\end{aligned} \tag{4.11}
$$

Observing that

$$\det(\partial_\pm^2+\mu^2)=\det(\partial_\pm+i\mu)\ \det(\partial_\pm-i\mu)=\det(e^{-i\mu\tau}\partial_\pm e^{i\mu\tau})\det(e^{i\mu\tau}\partial_\pm e^{-i\mu\tau})\,, \qquad (4.12)$$

and accounting for the massless determinants coming from the Jacobian of transformation (2.5) and the quantum fluctuations of U in the WZW term in (2.6) one concludes that the bosonic part of the one-loop partition function is the same as in the corresponding $AdS_5\times S^5$ string theory.

The parametrization of the fermionic fluctuations

$$\Psi_{_R}=\begin{pmatrix}0 & \mathfrak{X}_R\\ \mathfrak{Y}_R & 0\end{pmatrix}, \qquad \Psi_{_L}=\begin{pmatrix}0 & \mathfrak{X}_L\\ \mathfrak{Y}_L & 0\end{pmatrix},$$

$$\mathfrak{X}_R=\begin{pmatrix}0 & 0 & (\alpha_1+i\alpha_2)t_+ & (\alpha_3+i\alpha_4)t_+\\ 0 & 0 & (-\alpha_3+i\alpha_4)t_+^* & (\alpha_1-i\alpha_2)t_+^*\\ (\alpha_5+i\alpha_6)t_+ & (\alpha_7-i\alpha_8)t_+ & 0 & 0\\ (\alpha_7+i\alpha_8)t_+^* & (-\alpha_5+i\alpha_6)t_+^* & 0 & 0\end{pmatrix}, \qquad t_\pm\equiv e^{i\frac{\kappa^2\tau\pm\ell^2\sigma}{2\mu}}$$

$$\mathfrak{Y}_R=\begin{pmatrix}0 & 0 & (-\alpha_6-i\alpha_5)t_+^* & (-\alpha_8-i\alpha_7)t_+\\ 0 & 0 & (\alpha_8-i\alpha_7)t_+^* & (-\alpha_6+i\alpha_5)t_+\\ (\alpha_2+i\alpha_1)t_+^* & (\alpha_4-i\alpha_3)t_+ & 0 & 0\\ (\alpha_4+i\alpha_3)t_+^* & (-\alpha_2+i\alpha_1)t_+ & 0 & 0\end{pmatrix},$$

$$\mathfrak{X}_L=\begin{pmatrix}0 & 0 & (\beta_1+i\beta_2)t_-^* & (\beta_3+i\beta_4)t_-^*\\ 0 & 0 & (\beta_3-i\beta_4)t_- & (-\beta_1+i\beta_2)t_-\\ (\beta_5+i\beta_6)t_-^* & (-\beta_7+i\beta_8)t_-^* & 0 & 0\\ (\beta_7+i\beta_8)t_- & (\beta_5-i\beta_6)t_- & 0 & 0\end{pmatrix},$$

$$\mathfrak{Y}_L=\begin{pmatrix}0 & 0 & (-\beta_6-i\beta_5)t_- & (-\beta_8-i\beta_7)t_-^*\\ 0 & 0 & (-\beta_8+i\beta_7)t_- & (\beta_6-i\beta_5)t_-^*\\ (\beta_2+i\beta_1)t_-^* & (-\beta_4+i\beta_3)t_- & 0 & 0\\ (\beta_4+i\beta_3)t_-^* & (\beta_2-i\beta_1)t_- & 0 & 0\end{pmatrix} \qquad (4.13)$$

leads to the fluctuation Lagrangian with constant coefficients

$$\mathcal{L}_{\rm F}=2\Big[\sum_{i=1}^{8}(\alpha_i\partial_-\alpha_i+\beta_i\partial_+\beta_i)-\mu\,(\alpha_1\alpha_2+\alpha_3\alpha_4+\alpha_5\alpha_6-\alpha_7\alpha_8-\beta_1\beta_2-\beta_3\beta_4-\beta_5\beta_6+\beta_7\beta_8)$$
$$+2\kappa\,(\alpha_1\beta_4+\alpha_2\beta_3-\alpha_3\beta_2-\alpha_4\beta_1-\alpha_5\beta_8+\alpha_6\beta_7+\alpha_7\beta_6-\alpha_8\beta_5)\Big]\,. \qquad (4.14)$$

It describes 8 fermionic degrees of freedom with characteristic frequencies equivalent (up to overall shifts)[14] to $\sqrt{n^2+\kappa^2}$. The fermionic contribution to the one-loop partition function following from (4.14) is

$$Z^{(1F)}=\det\Big[\partial_+^2\partial_-^2+2\partial_+\partial_-\kappa^2+\frac{1}{4}\mu^2\left(\partial_+^2+\partial_-^2\right)+\frac{1}{16}\left(4\kappa^2-\mu^2\right)^2\Big]\,. \qquad (4.15)$$

By the same argument as in (3.21),(3.22) this determinant can be shown to be equivalent to[15] $[\det(\partial_+\partial_-+\kappa^2)]^2$.

[14] These shifts reflect particular redefinitions of the fermionic fields we have chosen.

[15] The relation between determinants implies rotation of the basis. This extra rotation leads, however, to non-constant coefficients in the cubic and quartic terms in the fluctuation Lagrangian and for this reason keep using the rotation by $t_\pm$ as defined above.

The final expression for the one-loop partition function is thus the same as in string theory [9]. In the $\mu \to 0$ limit we get, as in (3.23), the familiar result [8] (see (1.15))

$$
\begin{aligned}
\Gamma^{(1)} &= \frac{1}{2}\bar{V}_2 \int \frac{d^2q}{(2\pi)^2}\big[\ln(q^2+4\kappa^2)+2\ln(q^2+2\kappa^2)+5\ln q^2-8\ln(q^2+\kappa^2)\big] \\
&= 2\kappa^2\bar{V}_2\big(I[4]+I[2]-2I[1]\big) = \frac{1}{2\pi}\mathrm{a}_1 V_2\ , \qquad \mathrm{a}_1=-3\ln 2\ .
\end{aligned} \tag{4.16}
$$

4.2 Two-loop contribution

As in the $AdS_3 \times S^3$ case in the first approach discussed in section 3.1 the two-loop computation uses the fluctuation Lagrangian in (2.9), (2.9), (2.11) expanded near the above classical background (4.5). Here one treats $4+4$ physical and $6+6$ unphysical bosonic fluctuations on an equal footing. Let us first consider the purely bosonic contributions given by diagrams in Figure 1. As discussed in Appendix C, it is useful to make $O(2)$ transformations of the unphysical fluctuations (see (C.3) and (C.9)) to obtain four decoupled subsectors so that the propagator takes a block-diagonal form. If we label the bosonic fluctuations as

$$
\Phi_I=\{\Phi_A,\Phi_S\},\quad \Phi_A=\{a_1,c_2,c_3,a_2,c_5,c_6,a_3,a_4,c_1,c_4\}\ ,\qquad \Phi_S=\{b_1,\dots,b_4,d_1,\dots,d_6\} \tag{4.17}
$$

then the euclidean-signature propagator in the A-sector becomes

$$
\triangle_A^{-1}(q)=\begin{pmatrix} \mathcal{M}_1(q) & 0 & 0 & 0 \\ 0 & \mathcal{M}_1(q) & 0 & 0 \\ 0 & 0 & \mathcal{M}_2(q) & 0 \\ 0 & 0 & 0 & \frac{1}{2q^2} \end{pmatrix}, \tag{4.18}
$$

where $\mathcal{M}_1(q)$ and $\mathcal{M}_2(q)$ are 3×3 matrices (we again rescale q by ℓ, i.e. $\hat\kappa=\frac{\kappa}{\ell},\ \hat\mu=\frac{\mu}{\ell}$)

$$
\mathcal{M}_1(q)=\tfrac{1}{D_1}\begin{pmatrix}
-\frac{4\hat\kappa^4-4\hat\kappa^2\hat\mu^2+\hat\mu^2\left(q_+^2+\hat\mu^2\right)}{4\hat\mu^2\left(q_-^2+\hat\mu^2\right)} & \frac{\hat\kappa\sqrt{\hat\kappa^2-\hat\mu^2}\left(2\hat\kappa^2-\hat\mu^2\right)}{\sqrt{2}\hat\mu^2\left(q_+^2+\hat\mu^2\right)} & -\frac{\hat\kappa\sqrt{\hat\kappa^2-\hat\mu^2}q_+}{\sqrt{2}\hat\mu\left(q_+^2+\hat\mu^2\right)} \\
\frac{\hat\kappa\sqrt{\hat\kappa^2-\hat\mu^2}\left(2-\hat\mu^2\right)}{\sqrt{2}\hat\mu^2\left(q_+^2+\hat\mu^2\right)} & -\frac{4\hat\kappa^2\left(\hat\kappa^2-\hat\mu^2\right)\left(q_-^2+\hat\mu^2\right)+\hat\mu^2\left(q^4-\hat\mu^4\right)}{2\hat\mu^2\left(q^4+2\hat\mu^2q^2-4\hat\mu^2q_0^2+\hat\mu^4\right)} & \frac{\hat\kappa^2q_+\left(q_-^2+\hat\mu^2\right)+iq^1\hat\mu^2\left(q^2-\hat\mu^2\right)}{\hat\mu\left(q^4+2\hat\mu^2q^2-4\hat\mu^2q_0^2+\hat\mu^4\right)} \\
\frac{\hat\kappa\sqrt{\hat\kappa^2-\hat\mu^2}q_+}{\sqrt{2}\hat\mu\left(q_+^2+\hat\mu^2\right)} & -\frac{\hat\kappa^2q_+\left(q_-^2+\hat\mu^2\right)+iq^1\hat\mu^2\left(q^2-\hat\mu^2\right)}{\hat\mu\left(q^4+2\hat\mu^2q^2-4\hat\mu^2q_0^2+\hat\mu^4\right)} & \frac{q^4-\hat\mu^4}{2\left(q^4+2\hat\mu^2q^2-4\hat\mu^2q_0^2+\hat\mu^4\right)}
\end{pmatrix},
$$

$$
\mathcal{M}_2(q)=\tfrac{1}{D_2}\begin{pmatrix}
-\frac{q^2}{4} & -\frac{\hat\kappa^2q_-+iq^1\hat\mu^2}{2\hat\mu} & \frac{\hat\kappa q_-\sqrt{\hat\kappa^2-\hat\mu^2}}{\sqrt{2}\hat\mu} \\
\frac{\hat\kappa^2q_-+iq^1\hat\mu^2}{2\hat\mu} & \frac{q_+\left(q_+^2\hat\mu^2-4\hat\kappa^2\left(\hat\kappa^2-\hat\mu^2\right)\right)}{4q_-\hat\mu^2} & -\frac{\sqrt{2}\hat\kappa\sqrt{\hat\kappa^2-\mu^2}\left(\hat\kappa^2q_-+iq_1\hat\mu^2\right)}{q_+\hat\mu^2} \\
-\frac{\hat\kappa q_-\sqrt{\hat\kappa^2-\hat\mu^2}}{\sqrt{2}\hat\mu} & -\frac{\sqrt{2}\hat\kappa\sqrt{\hat\kappa^2-\hat\mu^2}\left(\hat\kappa^2q_-+iq_1\hat\mu^2\right)}{q_+\hat\mu^2} & \frac{q^4\hat\mu^2-4\left(q^1\hat\mu^2-i\hat\kappa^2q_-\right)^2}{2q^2\hat\mu^2}
\end{pmatrix},
$$

$$
q_\pm=q_0+iq_1\ ,\qquad q^2=q_0^2+q_1^2\ ,\qquad D_1=q^2+2\hat\kappa^2-\hat\mu^2\ ,\qquad D_2=q^4+4\hat\kappa^2q^2-4\hat\mu^2{q_1}^2\ . \tag{4.19}
$$

The propagator in the S-sector is ($\mathbf{I}_n=\mathrm{diag}(1,...,1)$)

$$
\triangle_S^{-1}(q)=\mathrm{diag}\left(-\frac{1}{4\left(q^2+\hat\mu^2\right)}\mathbf{I}_4,-\frac{1}{2q^2}\mathbf{I}_6\right)\ . \tag{4.20}
$$

As the two-loop computation for finite μ appears to be quite complicated, we consider only the limit $\mu \to 0$. This limit (i.e. $\hat{\mu} \to 0,\ \hat{\kappa} \to 1$) can be smoothly taken once we simplify the integrands of the two-loop integrals. Below we will summarize the results for the contributions of different types of two-loop diagrams found after taking this limit.

4.2.1 Bosonic 1PI contributions

Plugging the bosonic fluctuation fields (4.7) into $\mathcal{L}^{(3)}$ in (2.9) one finds the vertices for the sunset diagrams in Figure 1(a). Sunset diagrams will be expressed in terms of the integrals (3.28) where $\mathcal{F}(q_i, q_j, q_k)$ is a polynomial function of momenta. In the A-sector, the vertices contained in the fluctuation Lagrangian are of the three types. The first type includes the vertices $V_{A\,ijk}$ with $(i,j,k) = (\{7,8,9\},\{1,2,3\},\{1,2,3\})$ or $(i,j,k) = (\{7,8,9\},\{4,5,6\},\{4,5,6\})$. In the $\mu \to 0$ limit we have (see (4.19)) $D_1 \to q^2+2$ and $D_2 \to (q^2+4)q^2$, so that using these vertices we obtain the integral $\mathcal{I}_{422}$ containing $I[4,2,2]$ and thus the Catalan's constant in (1.8). The vertices $V_{A\,ijk}$ with $(i,j,k) = (\{7,8,9\},\{7,8,9\},\{7,8,9\})$ lead to the integral $\mathcal{I}_{444}$. A nontrivial finite part of this integral may contain $I[4,4,4]$ but as in the corresponding string-theory computation such term does not actually appear. The third type of vertices is $V_{A\,ijk}$ with $(i,j,k) = (10,\{1,2,3\},\{4,5,6\})$ yielding the integral $\mathcal{I}_{022}$.

Explicitly, one finds for the resulting contribution to the two-loop effective action (cf. (3.29); here $k = k_5$):

$$\Gamma^{(2)} = \frac{8\pi}{k} V_2 \sum_n J_n \ , \tag{4.21}$$

$$J_{\text{boson sunset}} = \mathcal{I}_{422} + \mathcal{I}_{444} + \mathcal{I}_{022} \ , \qquad \mathcal{I}_{422} = 2I[4,2,2] - \frac{1}{2}I[2]I[0] - \frac{3}{2}I[4]I[2] \, ,$$
$$\mathcal{I}_{444} = -\frac{1}{4}I[4]I[0] - \frac{1}{4}I[4]I[4] \, , \qquad \mathcal{I}_{022} = -\frac{1}{2}I[2]I[2] \, , \tag{4.22}$$

where we used the notation in (2.16). The contribution from the S-sector is trivial as the corresponding part of the reduced theory solution is the vacuum one.[16]

The diagrams in Figure 1(b) lead to momentum integrals of the type (3.28). In the A-sector one finds:

$$J_{\text{boson double-bubble}} = \mathcal{I}_{22} + \mathcal{I}_{44} \ , \qquad \mathcal{I}_{22} = -\tfrac{1}{2}I[2]I[2] \, , \qquad \mathcal{I}_{44} = -\tfrac{1}{4}I[4]I[0] - \tfrac{1}{4}I[4]I[4] \, . \tag{4.23}$$

The S-sector again does not lead to a non-trivial contribution.

Summing up (4.22) and (4.23) we find the total contribution of two-loop bosonic 1PI diagrams

$$J_{\text{boson 1PI}} = 2I[4,2,2] - \frac{1}{2}I[2]I[0] - \frac{1}{2}I[4]I[0] - I[2]I[2] - \frac{3}{2}I[4]I[2] - \frac{1}{2}I[4]I[4] \, , \tag{4.24}$$

which contains the Catalan's constant term

$$I[4,2,2] = \frac{1}{16\pi^2}\mathrm{K} \, . \tag{4.25}$$

[16]Explicitly, one finds

$$\int \frac{d^2q_i\, d^2q_j}{(2\pi)^4} \frac{q_i^+ q_j^- - q_i^- q_j^+}{q_i^2 q_j^2 (q_i+q_j)^2} = 0$$

due to obvious symmetry of the momentum-space integral under interchange $i \leftrightarrow j$.

4.2.2 Fermionic 1PI contributions

Let us now consider the 1PI diagrams with the fermionic propagators in Figure 2. As the fermionic fluctuations have mass $\hat{\kappa} = 1$ the integrals arising from the fermionic sunset diagram are of the type $\mathcal{I}_{m^2,1,1}$, where m is a mass of the bosonic fluctuation (i.e. $m^2 = 4, 2, 0$). Explicitly, we find that the nonvanishing contributions are

$$\begin{aligned} J_{\text{fermion-boson sunset}} &= \mathcal{I}_{211} + \mathcal{I}_{411} + \mathcal{I}_{011} \ , \qquad \mathcal{I}_{211} = -2I[2,1,1] + I[1]I[1] - 2I[2]I[1] \ , \\ \mathcal{I}_{411} &= I[4]I[1] - \frac{1}{2}I[1]I[1] \, , \qquad \mathcal{I}_{011} = 9I[1][0] - \frac{9}{2}I[1]I[1] \, . \end{aligned} \tag{4.26}$$

The fermionic double-bubble diagrams are expressed in terms of the integrals $\mathcal{I}_{m^2,1}$, where the m is again the bosonic fluctuation mass:

$$\begin{aligned} J_{\text{fermion-boson double-bubble}} &= \mathcal{I}_{41} + \mathcal{I}_{21} + \mathcal{I}_{01} \ , \\ \mathcal{I}_{01} = -9I[1]I[0] \, , \quad & \mathcal{I}_{21} = 6I[2]I[1] \, , \quad \mathcal{I}_{41} = I[1]I[0] + 2I[4]I[1] \, . \end{aligned} \tag{4.27}$$

Combining (4.26) and (4.27) we get

$$J_{\text{fermion 1PI}} = -2I[2,1,1] + I[1]I[0] - 4I[1]I[1] + 4I[2]I[1] + 3I[4]I[1] \, , \tag{4.28}$$

which again contains the Catalan's constant term since

$$I[2,1,1] = \frac{1}{8\pi^2}\mathrm{K} \, . \tag{4.29}$$

Combining the bosonic (4.24) and the fermionic (4.28) 1PI contributions together we find

$$J_{\text{1PI}} = -\frac{1}{8\pi^2}\mathrm{K} - \frac{1}{2}\big(I[4] + I[2] - 2I[1]\big)\big(I[4] + 2I[2] + I[0] - 4I[1]\big) \ . \tag{4.30}$$

We observe that as in the corresponding string theory computation [10] the fermionic Catalan's constant contribution is twice and opposite in sign to the bosonic one. Also, as in the $AdS_3 \times S^3$ reduced theory case (3.31), the second term in (4.30) is UV finite but IR divergent and is proportional to the same combination $I[4] + I[2] - 2I[1]$ which appears in the one-loop result (4.16).

4.2.3 Tadpole contributions and total result for the two-loop coefficient

The non-1PI diagrams relevant in the present case are shown in Figure 3.[17] We find for their contributions (including the $\frac{1}{8}$ combinatorial factor)

$$J_{\text{boson-boson tadpole}} = \frac{1}{6}\big(I[2]I[0] + I[4]I[0] + 3I[2]I[2] + 5I[4]I[2] + 2I[4]I[4]\big) \, , \tag{4.31}$$

$$J_{\text{boson-fermion tadpole}} = -\frac{1}{3}\big(I[1]I[0] + 6I[2]I[1] + 5I[4]I[1]\big) \, , \tag{4.32}$$

$$J_{\text{fermion-fermion tadpole}} = 2I[1]I[1] \, . \tag{4.33}$$

[17]To evaluate them we again use the prescription that momentum of the intermediate line is set to zero only at the end of the computation.

The total tadpole contribution is thus

$$J_{\rm tadpole} = \frac{1}{6}\big(I[4]+I[2]-2I[1]\big)\big(2I[4]+3I[2]+I[0]-6I[1]\big)\ . \tag{4.34}$$

Like the corresponding expression (3.32) found using first approach in the $AdS_3\times S^3$ case, this coefficient is UV finite but IR divergent and is proportional to the one-loop combination in (4.16).

Combining together (4.30) and (4.34) we find the following expression for the coefficient in the two-loop effective action (4.21) (cf. (1.15))

$$\sum_n J_n = J_{\rm 1PI} + J_{\rm tadpole} = \bar J + \tilde J\ , \qquad \bar J = -\frac{1}{8\pi^2}{\rm K}\ , \tag{4.35}$$

$$\tilde J = -\frac{1}{6}\big(I[4]+I[2]-2I[1]\big)\big(I[4]+3I[2]+2I[0]-6I[1]\big)\ . \tag{4.36}$$

The resulting two-loop coefficient thus contains, in addition to $\bar{\rm a}_2 = -{\rm K}$, also $\tilde{\rm a}_2 = 8\pi^2\tilde J = -\frac{5}{4}(\ln 2)^2 - \ln 2\,\ln m_0$ which is IR divergent.

The close similarity with the $AdS_3\times S^3$ case discussed in section 3 suggests that the problem with non-cancellation of IR divergences is due to a subtlety in how the tadpole contribution was computed. In particular, the analogy with the $AdS_3\times S^3$ case suggests that there should be no $I[4]I[4]$ term in the tadpole contribution in (4.34). Indeed, the results in the reduced $AdS_3\times S^3$ and reduced $AdS_5\times S^5$ theories are in direct agreement in what concerns $I[4]I[4]$ contributions coming from the 1PI graphs: this term enters (4.30) with the same coefficient $-\frac{1}{2}$ as in (3.31) or (3.50).

Accepting this natural suggestion, the tadpole term (4.34) should be replaced by[18]

$$\begin{aligned} J'_{\rm tadpole} &= \frac{1}{6}\big(I[4]+I[2]-2I[1]\big)\big(3I[2]+3I[0]-6I[1]\big) \\ &= \frac{1}{2}\big(I[4]+I[2]-2I[1]\big)\big(I[2]+I[0]-2I[1]\big)\ . \end{aligned} \tag{4.37}$$

Then the sum of (4.30) and (4.37) leads to $\tilde{\rm a}_2$ which is IR finite and, as in the $AdS_3\times S^3$ case in (3.52), is proportional to the square of the one-loop coefficient:

$$\tilde J = -\frac{1}{2}\big(I[4]+I[2]-2I[1]\big)^2\ , \qquad \text{i.e.} \qquad \tilde{\rm a}_2 = 8\pi^2\tilde J = -\frac{1}{4}({\rm a}_1)^2 = -\frac{9}{4}(\ln 2)^2\ . \tag{4.38}$$

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[18]Since the UV finiteness should be preserved this is effectively equivalent to replacing $I[4]$ in the second factor by $I[0]$, as in the $AdS_3\times S^3$ case.

A Matrix superalgebra

In this Appendix we will briefly summarize some notation used in this paper (for details see [1, 5, 6, 18]). The superalgebra $\mathfrak{su}(2,2|4)$ is spanned by 8×8 supermatrices $\mathfrak{f}$

$$\mathfrak{f} = \begin{pmatrix} \mathfrak{A} & \mathfrak{X} \\ \mathfrak{Y} & \mathfrak{D} \end{pmatrix}, \qquad \mathrm{STr}\,\mathfrak{f} \equiv \mathrm{tr}\,\mathfrak{A} - \mathrm{tr}\,\mathfrak{D} = 0\,, \tag{A.1}$$

$$\mathfrak{A}^\dagger = -\hat{\Sigma}\mathfrak{A}\hat{\Sigma}\,, \qquad \mathfrak{D}^\dagger = -\mathfrak{D}\,, \qquad \mathfrak{Y} = -\mathfrak{X}^\dagger\hat{\Sigma}\ , \qquad \hat{\Sigma} = \mathrm{diag}(1,1,-1,-1)\ , \tag{A.2}$$

where the 4×4 matrices $\mathfrak{A}$, $\mathfrak{D}$ are Grassmann even and $\mathfrak{X}$, $\mathfrak{Y}$ are Grassmann odd. $\mathfrak{A}$ belongs to $\mathfrak{u}(2,2)$ and $\mathfrak{D}$ belongs to $\mathfrak{u}(4)$. $\mathfrak{psu}(2,2|4)$ is the quotient of $\mathfrak{su}(2,2|4)$ over the remaining $\mathfrak{u}(1)$. $\mathfrak{psu}(2,2|4)$ admits a $\mathbb{Z}_4$ decomposition:

$$\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3\,, \qquad [\mathfrak{f}_r, \mathfrak{f}_s] \subset \mathfrak{f}_{r+s \bmod 4}\,, \tag{A.3}$$

$$\mathfrak{f}_0 = \frac{1}{2}\begin{pmatrix} \mathfrak{A} - K\mathfrak{A}^{\mathrm{t}}K & 0 \\ 0 & \mathfrak{D} - K\mathfrak{D}^{\mathrm{t}}K \end{pmatrix}, \qquad \mathfrak{f}_1 = \frac{1}{2}\begin{pmatrix} 0 & \mathfrak{X} - iK\mathfrak{Y}^{\mathrm{t}}K \\ \mathfrak{Y} + iK\mathfrak{X}^{\mathrm{t}}K & 0 \end{pmatrix},$$

$$\mathfrak{f}_2 = \frac{1}{2}\begin{pmatrix} \mathfrak{A} + K\mathfrak{A}^{\mathrm{t}}K & 0 \\ 0 & \mathfrak{D} + K\mathfrak{D}^{\mathrm{t}}K \end{pmatrix}, \qquad \mathfrak{f}_3 = \frac{1}{2}\begin{pmatrix} 0 & \mathfrak{X} + iK\mathfrak{Y}^{\mathrm{t}}K \\ \mathfrak{Y} - iK\mathfrak{X}^{\mathrm{t}}K & 0 \end{pmatrix},$$

$$K = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{A.4}$$

$\mathfrak{g} = \mathfrak{f}_0$ is the algebra of the group $G = Sp(2,2)\times Sp(4)$ and $\mathfrak{p} = \mathfrak{f}_2$ as the F/G coset part of the algebra $\mathfrak{f}$. The element T of the maximal Abelian subalgebra of $\mathfrak{p}$

$$T = \frac{i}{2}\,\mathrm{diag}\,(1,\,1,\,-1,\,-1,\,1,\,1,\,-1,\,-1) \tag{A.5}$$

defines a $\mathbb{Z}_2$ decomposition ($r = 0,1,2,3$)

$$\mathfrak{f} = \mathfrak{f}^{\parallel} \oplus \mathfrak{f}^{\perp}\,, \qquad [\mathfrak{f}^{\perp}, \mathfrak{f}^{\perp}] \subset \mathfrak{f}^{\perp}\,, \qquad [\mathfrak{f}^{\perp}, \mathfrak{f}^{\parallel}] \subset \mathfrak{f}^{\parallel}\,, \qquad [\mathfrak{f}^{\parallel}, \mathfrak{f}^{\parallel}] \subset \mathfrak{f}^{\perp}\,, \qquad \mathrm{STr}(\mathfrak{f}^{\parallel}\mathfrak{f}^{\perp}) = 0\,, \tag{A.6}$$

$$\mathfrak{f}^{\parallel}_r = -\,[T,[T,\mathfrak{f}_r]]\ , \qquad \mathfrak{f}^{\perp}_r = -\{T,\{T,\mathfrak{f}_r\}\}\,, \qquad \{\mathfrak{f}^{\parallel}_r, T\} = 0, \quad [\mathfrak{f}^{\perp}_r, T] = 0\ . \tag{A.7}$$

We set $\mathfrak{h} = \mathfrak{f}^{\perp}_0$, $\mathfrak{m} = \mathfrak{f}^{\parallel}_0$, so that $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$, $[\mathfrak{m},\mathfrak{h}] \subset \mathfrak{m}$. $\mathfrak{h}$ is the algebra of the subgroup H of G which commutes with T, $[\mathfrak{h}, T] = 0$. For the specific choice of the matrices Σ, K and T one can explicitly represent the general elements of $\mathfrak{m}$ and $\mathfrak{h}$ as follows (we use this when discussing the bosonic fluctuations in the reduced theory):

$$
\begin{aligned}
&\mathfrak{m} = \begin{pmatrix} \mathfrak{m}_{_A} & 0 \\ 0 & \mathfrak{m}_{_S} \end{pmatrix} , \qquad \mathfrak{m}_{_A} = \begin{pmatrix} 0 & 0 & a_1 + ia_2 & a_3 + ia_4 \\ 0 & 0 & a_3 - ia_4 & -a_1 + ia_2 \\ a_1 - ia_2 & a_3 + ia_4 & 0 & 0 \\ a_3 - ia_4 & -a_1 - ia_2 & 0 & 0 \end{pmatrix} , \\
&\mathfrak{m}_{_S} = \begin{pmatrix} 0 & 0 & b_1 + ib_2 & b_3 + ib_4 \\ 0 & 0 & -b_3 + ib_4 & b_1 - ib_2 \\ -b_1 + ib_2 & b_3 + ib_4 & 0 & 0 \\ -b_3 + ib_4 & -b_1 - ib_2 & 0 & 0 \end{pmatrix} , \\
&\mathfrak{h} = \begin{pmatrix} \mathfrak{h}_{_A} & 0 \\ 0 & \mathfrak{h}_{_S} \end{pmatrix} , \qquad \mathfrak{h}_{_A} = \begin{pmatrix} ic_1 & c_2 + ic_3 & 0 & 0 \\ -h_c + ic_3 & -ic_1 & 0 & 0 \\ 0 & 0 & ic_4 & c_5 + ic_6 \\ 0 & 0 & -c_5 + ic_6 & -ic_4 \end{pmatrix} , \\
&\mathfrak{h}_{_S} = \begin{pmatrix} id_1 & d_2 + id_3 & 0 & 0 \\ -d_2 + id_3 & -id_1 & 0 & 0 \\ 0 & 0 & id_4 & d_5 + id_6 \\ 0 & 0 & -d_5 + id_6 & -id_4 \end{pmatrix} .
\end{aligned}
\tag{A.8}
$$

Fermionic fields of the reduced theory take values in $\mathfrak{f}^{||}$, i.e. $\Psi_{_R} \in \mathfrak{f}^{||}_1$, $\Psi_{_L} \in \mathfrak{f}^{||}_3$:

$$
\begin{aligned}
&\mathfrak{f}^{||}_1 = \begin{pmatrix} 0 & \mathfrak{X}_1 \\ \mathfrak{Y}_1 & 0 \end{pmatrix} , \qquad \mathfrak{X}_1 = \begin{pmatrix} 0 & 0 & \alpha_1 + i\alpha_2 & \alpha_3 + i\alpha_4 \\ 0 & 0 & -\alpha_3 + i\alpha_4 & \alpha_1 - i\alpha_2 \\ \alpha_5 + i\alpha_6 & \alpha_7 - i\alpha_8 & 0 & 0 \\ \alpha_7 + i\alpha_8 & -\alpha_5 + i\alpha_6 & 0 & 0 \end{pmatrix} , \\
&\mathfrak{Y}_1 = \begin{pmatrix} 0 & 0 & -\alpha_6 - i\alpha_5 & -\alpha_8 - i\alpha_7 \\ 0 & 0 & \alpha_8 - i\alpha_7 & -\alpha_6 + i\alpha_5 \\ \alpha_2 + i\alpha_1 & \alpha_4 - i\alpha_3 & 0 & 0 \\ \alpha_4 + i\alpha_3 & -\alpha_2 + i\alpha_1 & 0 & 0 \end{pmatrix} , \\
&\mathfrak{f}^{||}_3 = \begin{pmatrix} 0 & \mathfrak{X}_3 \\ \mathfrak{Y}_3 & 0 \end{pmatrix} , \qquad \mathfrak{X}_3 = \begin{pmatrix} 0 & 0 & \beta_1 + i\beta_2 & \beta_3 + i\beta_4 \\ 0 & 0 & \beta_3 - i\beta_4 & -\beta_1 + i\beta_2 \\ \beta_5 + i\beta_6 & -\beta_7 + i\beta_8 & 0 & 0 \\ \beta_7 + i\beta_8 & \beta_5 - i\beta_6 & 0 & 0 \end{pmatrix} , \\
&\mathfrak{Y}_3 = \begin{pmatrix} 0 & 0 & -\beta_6 - i\beta_5 & -\beta_8 - i\beta_7 \\ 0 & 0 & -\beta_8 + i\beta_7 & \beta_6 - i\beta_5 \\ \beta_2 + i\beta_1 & -\beta_4 + i\beta_3 & 0 & 0 \\ \beta_4 + i\beta_3 & \beta_2 - i\beta_1 & 0 & 0 \end{pmatrix} .
\end{aligned}
\tag{A.9}
$$

The discussion of $\mathfrak{psu}(1,1|2)$ superalgebra relevant for the $AdS_3 \times S^3$ case is similar. Here the matrix $\mathfrak{f}$ in (A.1) is a 4×4 supermatrix with $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{X}$ and $\mathfrak{Y}$ being 2×2 matrices and

$$
\Sigma = K = \ \mathrm{diag}\,(1,\, -1) \ , \qquad T = \frac{i}{2}\ \mathrm{diag}\,(1,\, -1,\, 1,\, -1) \ , \tag{A.10}
$$

$$
\mathfrak{m} = \begin{pmatrix} \mathfrak{m}_{_A} & 0 \\ 0 & \mathfrak{m}_{_S} \end{pmatrix} , \quad \mathfrak{m}_{_A} = \begin{pmatrix} 0 & a_1 + ia_2 \\ a_1 - ia_2 & 0 \end{pmatrix} , \quad \mathfrak{m}_{_S} = \begin{pmatrix} 0 & b_1 + ib_2 \\ b_1 - ib_2 & 0 \end{pmatrix} ,
$$

$$\mathfrak{h} = \begin{pmatrix} \mathfrak{h}_{_A} & 0 \\ 0 & \mathfrak{h}_{_S} \end{pmatrix} , \qquad \mathfrak{h}_{_A} = \begin{pmatrix} ic & 0 \\ 0 & -ic \end{pmatrix} , \qquad \mathfrak{h}_{_S} = \begin{pmatrix} id & 0 \\ 0 & -id \end{pmatrix} . \tag{A.11}$$

The fermionic fields belong to

$$\begin{aligned} \mathfrak{f}^{\|}_1 &= \begin{pmatrix} 0 & \mathfrak{X}_1 \\ \mathfrak{Y}_1 & 0 \end{pmatrix}, \quad \mathfrak{X}_1 = \begin{pmatrix} 0 & \alpha_1 + i\alpha_2 \\ \alpha_3 + i\alpha_4 & 0 \end{pmatrix}, \quad \mathfrak{Y}_1 = \begin{pmatrix} 0 & -i\alpha_3 - \alpha_4 \\ i\alpha_1 + \alpha_2 & 0 \end{pmatrix} \\ \mathfrak{f}^{\|}_3 &= \begin{pmatrix} 0 & \mathfrak{X}_3 \\ \mathfrak{Y}_3 & 0 \end{pmatrix}, \quad \mathfrak{X}_3 = \begin{pmatrix} 0 & \beta_1 + i\beta_2 \\ \beta_3 + i\beta_4 & 0 \end{pmatrix}, \quad \mathfrak{Y}_3 = \begin{pmatrix} 0 & -i\beta_3 - \beta_4 \\ i\beta_1 + \beta_2 & 0 \end{pmatrix} \end{aligned} \tag{A.12}$$

B Two-loop computation in $AdS_3 \times S^3$ superstring theory

Here we shall discuss the computation of the two-loop correction to the energy of long folded string spinning in AdS_3 in critical $AdS_3 \times S^3 \times T^4$ superstring theory. In this case it is sufficient to consider just the $AdS_3 \times S^3$ supercoset theory as extra massless modes can be decoupled [19]. The calculation is very similar to the one in the $AdS_5 \times S^5$ case [10, 11]. The efficient approach is to map the infinite spin limit of the folded string solution to the Poincaré coordinates, where it is a critical point of the Euclidean action equivalent to the null-cusp solution. We shall follow the light-cone gauge approach developed in [13]. The strategy will be to compute the two-loop partition function on the corresponding classical background.

As was shown in [20, 21], the $AdS_3 \times S^3$ Green-Schwarz superstring action in the AdS light-cone gauge may be obtained by a simple truncation of the $AdS_5 \times S^5$ light-cone gauge action: one is to ignore the two boundary coordinates transverse to the light-cone, set to zero two of the six transverse coordinates and reduce the number of components of fermions from 4 to 2. Starting from the action in [20, 21], setting $p^+ = 1$ and Wick-rotating to the euclidean worldsheet signature by $\sigma \to i\sigma$ leads to (see, e.g., [13] for notation)

$$S_E = \frac{\sqrt{\lambda}}{4\pi} \int d\tau \int_0^\infty d\sigma \; L_E \ , \tag{B.1}$$

$$\begin{aligned} L_E = & \big[\dot z^M + \frac{\mathrm{i}}{z^2} z_N \eta_i (\rho^{MN})^i{}_j \eta^j\big]^2 + \ \frac{1}{z^4} z'^M z'^M + \mathrm{i}\big(\theta^i \dot\theta_i + \eta^i \dot\eta_i + \theta_i \dot\theta^i + \eta_i \dot\eta^i\big) \\ & - \ \frac{1}{z^2} (\eta^i \eta_i)^2 + \ 2\mathrm{i}\Big[\ \frac{1}{z^3} z^M \eta^i (\rho^M)_{ij} \theta'^j + \frac{1}{z^3} z^M \eta_i (\rho^\dagger_M)^{ij} \theta'_j\Big] \ . \end{aligned} \tag{B.2}$$

The form of the classical solution is the same as in the $AdS_5 \times S^5$ case:

$$z = \sqrt{\frac{\tau}{\sigma}} \ , \qquad x^+ = \tau \ , \qquad x^- = -\frac{1}{2\sigma} \ , \qquad x^+ x^- = -\frac{1}{2} z^2 \ , \tag{B.3}$$

where the $AdS_3 \times S^3$ metric is $ds^2 = \frac{1}{z^2}(dx^+ dx^- + dz_M dz_M)$, $M = (a, 4), \ a = 1, 2, 3$. The fluctuations around the classical solution are defined as

$$z = \sqrt{\frac{\tau}{\sigma}} \ \tilde z \ , \qquad \tilde z = e^{\tilde\phi} = 1 + \tilde\phi + \dots \ , \quad z^M = \sqrt{\frac{\tau}{\sigma}} \ \tilde z^M \ , \qquad \tilde z^M = e^{\tilde\phi} \tilde u^M \ , \tag{B.4}$$

$$\tilde u^a = \frac{y^a}{1 + \frac14 y^2} \ , \qquad \tilde u^4 = \frac{1 - \frac14 y^2}{1 + \frac14 y^2} \ , \qquad y^2 \equiv \sum_{a=1}^{3} (y^a)^2 \ , \tag{B.5}$$

$$\theta = \frac{1}{\sqrt{\sigma}} \tilde\theta \ , \qquad \eta = \frac{1}{\sqrt{\sigma}} \tilde\eta \ . \tag{B.6}$$

It is useful to do a further redefinition of the worldsheet coordinates $(\tau, \sigma) \to (t, s)$ (we will denote by (p_0, p_1) the corresponding two-dimensional momenta, i.e. $(p_0, p_1) = -\mathrm{i}(\partial_t, \partial_s)$)

$$t = \frac{1}{2} \ln \tau \ , \qquad s = \frac{1}{2} \ln \sigma \ , \qquad dtds = \frac{1}{4} \frac{d\tau d\sigma}{\tau\sigma} \ , \qquad \tau\partial_\tau = 2\partial_t \ , \qquad \sigma\partial_\sigma = 2\partial_s \ . \tag{B.7}$$

It leads then to the following euclidean action (B.1), (B.2):

$$S_E = \frac{\sqrt{\lambda}}{4\pi} \int dt \int_{-\infty}^{\infty} ds \ \mathcal{L} \ , \tag{B.8}$$

$$\begin{aligned} \mathcal{L} = & \left[\partial_t \tilde{z}^M + \tilde{z}^M + \frac{2\mathrm{i}}{\tilde{z}^2} \tilde{\eta}_i (\rho^{MN})^i{}_j \tilde{\eta}^j \tilde{z}_N\right]^2 + \frac{1}{\tilde{z}^4}\left(\partial_s \tilde{z}^M - \tilde{z}^M\right)^2 \\ & + 2\mathrm{i}(\tilde{\theta}^i \partial_t \tilde{\theta}_i + \tilde{\eta}^i \partial_t \tilde{\eta}_i + \tilde{\theta}_i \partial_t \tilde{\theta}^i + \tilde{\eta}_i \partial_t \tilde{\eta}^i) - \frac{4}{\tilde{z}^2} (\tilde{\eta}^2)^2 \\ & + 4\mathrm{i} \left[\frac{1}{\tilde{z}^3} \tilde{\eta}^i (\rho^M)_{ij} \tilde{z}^M (\partial_s \tilde{\theta}^j - \tilde{\theta}^j) + \frac{1}{\tilde{z}^3} \tilde{\eta}_i (\rho_M^\dagger)^{ij} \tilde{z}^M (\partial_s \tilde{\theta}_j - \tilde{\theta}_j)\right] . \end{aligned} \tag{B.9}$$

The normalization of the worldsheet coordinates was chosen so that the masses of the quadratic fluctuations reproduce the masses of the fluctuations around the closed folded string, i.e. the spectrum is given by one boson with $m^2 = 4$, three massless bosons and four fermions with $m^2 = 1$. This may be obtained from the spectrum of fluctuations in the $AdS_5 \times S^5$ case by truncating away two transverse AdS_5 fluctuations, two S^5 fluctuations and half of the fermions. With this normalization, the effective action is related to the cusp anomaly $\mathrm{f}(\lambda)$ as [13]

$$\Gamma = \frac{1}{2\pi} \hat{f}(\lambda) \ V_2 \ , \qquad V_2 = \int dt \int_{-\ell/2}^{\ell/2} ds \ , \qquad \ell = 2\pi\kappa = 2 \ln \mathcal{S} \ , \tag{B.10}$$

$$\hat{f} = \sqrt{\lambda} + f \ , \qquad f = \mathrm{a}_1 + \frac{1}{\sqrt{\lambda}} \mathrm{a}_2 + \mathcal{O}(\frac{1}{(\sqrt{\lambda})^2}) \, . \tag{B.11}$$

Evaluating the one-loop effective action implies that the one-loop coefficient in the cusp anomaly is given by

$$a_1 = -2 \ln 2 \, . \tag{B.12}$$

Before quoting the result for the two-loop correction it is instructive to discuss the expected differences compared to the known $AdS_5 \times S^5$ result which are due to the absence of the two bosonic fluctuations with $m^2 = 2$. The terms that are not given by products of factors of lower transcendentality arise solely from 3-propagator integrals (such as $I[2, 2, 4] = \frac{1}{16\pi^2}\mathrm{K}$ in AdS_5 case). From the bosonic diagrams we may expect $I[4, 4, 4]$, $I[4, 4, 0]$, $I[4, 0, 0]$ and $I[0, 0, 0]$ while the fermionic diagrams may contribute $I[1, 1, 4]$ and $I[1, 1, 0]$. The integral $I[4, 4, 4]$ is generated by the terms in the Lagrangian that depend only on the "radial" AdS_3 coordinate z. Such terms are the same as in the $AdS_5 \times S^5$ case; since there this contribution canceled out, it should not appear here either. A similar reasoning can be used to rule out all other 3-propagator integral contributions. Therefore, we should expect that the two-loop effective action should be given only by a sum of products $I[m_i^2]I[m_j^2]$ of one-loop integrals. Such products all canceled out in the $AdS_5 \times S^5$ case, and the same should happen here too.

Indeed, a direct calculation based on (B.8) shows that the relevant two-loop Feynman diagrams produce the following contributions

$$\Gamma^{(2)} = \frac{4\pi}{\sqrt{\lambda}} V_2 \sum_n J_n \ , \tag{B.13}$$

$$J_{\text{boson sunset}} = \frac{1}{2} I[4]I[4] \ , \qquad J_{\text{boson double-bubble}} = -\frac{1}{2} I[4]I[4] \ , \tag{B.14}$$

$$J_{\text{fermion-boson sunset}} = \frac{1}{4} \left(-3I[0]I[1] + 2I[1]I[1] + 4I[1]I[4]\right) \ , \tag{B.15}$$

$$J_{\text{fermion-boson double-bubble}} = -\frac{1}{4} \left(-3I[0]I[1] + 4I[1]I[4]\right) \ , \tag{B.16}$$

$$J_{\text{tadpole}} = -\frac{1}{2} I[1]I[1] \, . \tag{B.17}$$

As a result, the sum of all contributions is not only UV and IR finite but also vanishes,[19] $\sum_n J_n = 0$, i.e. in contrast to the $AdS_5 \times S^5$ superstring case where $a_2 = -\mathrm{K}$, in the $AdS_3 \times S^3$ case we find that

$$a_2 = 0 \, . \tag{B.18}$$

It would be interesting to reproduce this string theory result from the asymptotic Bethe ansatz conjectured in [19].

C Comments on one-loop computation in section 4

The quadratic fluctuation Lagrangian in section 4.1 looks different from the corresponding one in $AdS_5 \times S^5$ string theory but the two lead to equivalent sets of characteristic frequencies and the one-loop determinants. Here we shall comment on the structure of subsectors of the bosonic fluctuation Lagrangian (4.9). Let us start with $\mathcal{L}_1^{(2)}$ in (4.9) containing a_1 and a_2. Integrating out c_2,c_3,c_5 and c_6 gives[20]

$$\tilde{\mathcal{L}}_1^{(2)} = 2 \sum_{i=1,2} \left[\partial_+ a_i \partial_- a_i - \left(2\kappa^2 - \mu^2\right) a_i^2 + 4M_1^2 a_i \frac{\partial_+\partial_- + 2\kappa^2 - \mu^2}{\partial_+^2 + M_2^2} a_i \right] . \tag{C.1}$$

This looks different from the fluctuation Lagrangian found from the corresponding string action (and found also in the reduced theory by taking the $\mu \to 0$ limit in the "mixed" gauge where the physical and unphysical modes are decoupled [6])

$$\mathcal{L}_1 = 2 \sum_{i=1,2} \left[\partial_+ a_i \partial_- a_i - \left(2\kappa^2 - \mu^2\right) a_i^2 \right] , \tag{C.2}$$

but the two are closely related as one can factorise the operator $\partial_+\partial_- + 2\kappa^2 - \mu^2$ in (C.1).

The Lagrangians $\mathcal{L}_1^{(2)}$ in (4.9) and $\mathcal{L}_1$ are, in fact, related by a nonlocal transformation. To see this, it is useful to perform the following $O(2)$ rotations,

$$c_2 \to \tfrac{1}{\sqrt{2}} (c_2 - c_6) \ , \quad c_3 \to \tfrac{1}{\sqrt{2}} (c_3 + c_5) \ , \quad c_5 \to \tfrac{1}{\sqrt{2}} (c_2 + c_6) \ , \quad c_6 \to \tfrac{1}{\sqrt{2}} (c_3 - c_5) \ . \tag{C.3}$$

[19]Note that the bosonic contribution vanishes separately; the fermionic term vanish only after both the 1PI and non-1PI contributions are combined together.

[20]The resulting determinant of the operator $\mathcal{O} = \partial_+^2 + M_2^2$ is equivalent to the (square of) massless operator determinant.

Then $\mathcal{L}_1^{(2)}$ splits into smaller subsectors. One contains a_1, c_2 and c_3,

$$\begin{aligned}\mathcal{L}_{a_1}^{(2)} &= 2\left[\partial_+ a_1 \partial_- a_1 - \left(2\kappa^2-\mu^2\right) a_1^2\right] - 4M_1\left(\mu c_2 + \partial_- c_3\right) a_1 \\ &\quad - \sum_{j=2,3}\left[\partial_+ c_j \partial_- c_j + (2\kappa^2-\mu^2) c_j^2\right] - 2\left(\mu\partial_+ c_3 + M_2 \partial_- c_3\right) c_2 \,. \end{aligned} \tag{C.4}$$

Another contains a_2, c_5 and c_6 with a similar Lagrangian. To decouple a_1 from c_2, c_3 we may apply the nonlocal transformation

$$\begin{aligned} a_1 &\to a_1 - \frac{\sqrt{2}\kappa\sqrt{\kappa^2-\mu^2}}{\partial_+\partial_- + 2\kappa^2-\mu^2} c_2 - \frac{\sqrt{2}\kappa\sqrt{\kappa^2-\mu^2}\partial_-(\partial_-^2+\mu^2)}{\mu(\partial_+^2\partial_-^2-\mu^4)} c_3 \,, \\ c_2 &\to c_2 + \frac{\partial_+\mu^2\left(\partial_+\partial_- + 2\kappa^2-\mu^2\right) + \partial_-\left(\left(2\kappa^2-\mu^2\right)\partial_+\partial_- + \mu^4\right)}{\mu\left(\partial_+^2\partial_-^2-\mu^4\right)} c_3 \,, \end{aligned} \tag{C.5}$$

leading to

$$\begin{aligned}\mathcal{L}_{a_1}^{(2)} &= 2\left[\partial_+ a_1 \partial_- a_1 - \left(2\kappa^2-\mu^2\right) a_1^2\right] \\ &\quad + 2c_2 \frac{\partial_+^2\partial_-^2-\mu^4}{\partial_+\partial_- + 2\kappa^2-\mu^2} c_2 + c_3 \frac{\left(\partial_+\partial_- + 2\kappa^2-\mu^2\right)\left(\partial_-^2+\mu^2\right)\left(\partial_+^2+\mu^2\right)}{\partial_+^2\partial_-^2-\mu^4} c_3 \,. \end{aligned} \tag{C.6}$$

The physical part of this Lagrangian is the same as (C.2). The product of determinants resulting from integrating out c_2 and c_3 contains only trivial massless factors. The same is true in the a_2,c_5,c_6 sector.

Similar observations apply in the sectors containing a_3 and a_4 described by the Lagrangian $\mathcal{L}_2^{(2)}$ in (4.9). Integrating out c_3, c_4 directly leads to

$$\tilde{\mathcal{L}}_2^{(2)} = 2\sum_{i=3,4} \partial_+ a_i \partial_- a_i + 4\left(\mu\partial_+ a_4 + M_2\partial_- a_4\right) a_3 + 8M_1^2 a_3 \frac{\partial_-}{\partial_+} a_3 \,, \tag{C.7}$$

which looks different from the string theory counterpart

$$\mathcal{L}_2 = 2\sum_{i=3,4} \partial_+ a_i \partial_- a_i + 4\left(\kappa^2-\mu^2\right) a_3^2 + 4\mu\left(\partial_+ a_3 + \partial_- a_3\right) a_4 \,. \tag{C.8}$$

To find a transformation between $\mathcal{L}_2^{(2)}$ and (C.8) let us apply an $O(2)$ rotation

$$c_1 \to \tfrac{1}{\sqrt{2}}\left(c_1 + c_4\right) \,, \qquad c_4 \to \tfrac{1}{\sqrt{2}}\left(c_1 - c_4\right) \,, \tag{C.9}$$

and the following redefinition

$$c_1 \to c_1 + 2\sqrt{2}\tfrac{\kappa\sqrt{\kappa^2-\mu^2}}{\mu}\tfrac{1}{\partial_+} a_3 \,, \qquad a_4 \to -a_4 - 2\tfrac{\kappa^2-\mu^2}{\mu}\tfrac{1}{\partial_+} a_3 \,. \tag{C.10}$$

Then we get

$$\mathcal{L}_{a_3,a_4}^{(2)} = 2\sum_{i=3,4} \partial_+ a_i \partial_- a_i + 4\left(\kappa^2-\mu^2\right) a_3^2 + 4\mu\left(\partial_+ a_3 + \partial_- a_3\right) a_4 - \sum_{j=1,4} \partial_+ c_j \partial_- c_j \,, \tag{C.11}$$

where the physical part is the same as in (C.8).

D The two-loop computation in the vacuum case

In the main part of this paper we studied two-loop corrections near the folded string with large spin S in AdS_3 taking the limit $\mu \to 0$ in which the angular momentum in S^5 vanishes. Here we shall check that the two-loop correction vanishes in the opposite limit of the trivial reduced theory solution corresponding to the BMN vacuum, i.e. in the case when

$$\kappa \to \mu \ , \qquad \ell \to 0 \ . \tag{D.1}$$

In this case it is useful to define the "S" part of fluctuation fields with an additional rescaling by $\mathrm{w} = e^{i\mu\tau}$ as follows (cf. (4.7))

$$\begin{aligned}
\eta_S^{\parallel} &= \begin{pmatrix} 0 & 0 & b_1 + ib_2 & (b_3+ib_4)\mathrm{w} \\ 0 & 0 & (-b_3+ib_4)\mathrm{w}^* & b_1 - ib_2 \\ -b_1+ib_2 & (b_3+ib_4)\mathrm{w} & 0 & 0 \\ (-b_3+ib_4)\mathrm{w}^* & -b_1-ib_2 & 0 & 0 \end{pmatrix} , \\
\eta_S^{\perp} &= \begin{pmatrix} id_1 & (d_2+id_3)\mathrm{w} & 0 & 0 \\ (-d_2+id_3)\mathrm{w}^* & -id_1 & 0 & 0 \\ 0 & 0 & id_4 & (d_5+id_6)\mathrm{w} \\ 0 & 0 & (-d_5+id_6)\mathrm{w}^* & -id_4 \end{pmatrix} .
\end{aligned} \tag{D.2}$$

Also, the component fields of the fermionic fluctuations are to be defined as (cf. (4.13))

$$\mathfrak{X}_R = \begin{pmatrix} 0 & 0 & (\alpha_1+i\alpha_2)t_{1+} & (\alpha_3+i\alpha_4)t_{2+} \\ 0 & 0 & (-\alpha_3+i\alpha_4)t^*_{2+} & (\alpha_1-i\alpha_2)t^*_{1+} \\ (\alpha_5+i\alpha_6)t_{1+} & (\alpha_7-i\alpha_8)t_{2+} & 0 & 0 \\ (\alpha_7+i\alpha_8)t^*_{2+} & (-\alpha_5+i\alpha_6)t^*_{1+} & 0 & 0 \end{pmatrix} , \tag{D.3}$$

$$\mathfrak{Y}_R = \begin{pmatrix} 0 & 0 & (-\alpha_6-i\alpha_5)t^*_{1+} & (-\alpha_8-i\alpha_7)t_{2+} \\ 0 & 0 & (\alpha_8-i\alpha_7)t^*_{2+} & (-\alpha_6+i\alpha_5)t_{1+} \\ (\alpha_2+i\alpha_1)t^*_{1+} & (\alpha_4-i\alpha_3)t_{2+} & 0 & 0 \\ (\alpha_4+i\alpha_3)t^*_{2+} & (-\alpha_2+i\alpha_1)t_{1+} & 0 & 0 \end{pmatrix} , \tag{D.4}$$

$$\mathfrak{X}_L = \begin{pmatrix} 0 & 0 & (\beta_1+i\beta_2)t^*_{2-} & (\beta_3+i\beta_4)t^*_{1-} \\ 0 & 0 & (\beta_3-i\beta_4)t_{1-} & (-\beta_1+i\beta_2)t_{2-} \\ (\beta_5+i\beta_6)t^*_{2-} & (-\beta_7+i\beta_8)t^*_{1-} & 0 & 0 \\ (\beta_7+i\beta_8)t_{1-} & (\beta_5-i\beta_6)t_{2-} & 0 & 0 \end{pmatrix} , \tag{D.5}$$

$$\mathfrak{Y}_L = \begin{pmatrix} 0 & 0 & (-\beta_6-i\beta_5)t_{2-} & (-\beta_8-i\beta_7)t^*_{1-} \\ 0 & 0 & (-\beta_8+i\beta_7)t_{1-} & (\beta_6-i\beta_5)t^*_{2-} \\ (\beta_2+i\beta_1)t_{2-} & (-\beta_4+i\beta_3)t^*_{1-} & 0 & 0 \\ (\beta_4+i\beta_3)t_{1-} & (\beta_2-i\beta_1)t^*_{2-} & 0 & 0 \end{pmatrix} , \tag{D.6}$$

where

$$t_{1\pm} = e^{i\frac{\ell^2(\tau\pm\sigma)}{2\mu}} , \qquad t_{2\pm} = e^{i\frac{(\ell^2+2\mu^2)\tau\pm\ell^2\sigma}{2\mu}} . \tag{D.7}$$

Taking the limit (D.1) in the two-loop diagrams one finds cancellations between A and S sectors in each type of diagrams leading to the vanishing two-loop correction.

One can also check this cancellation directly, by expanding near the reduced theory counterpart of the BMN vacuum

$$g_0 = \mathbf{I}_{8\times 8}\,, \qquad A_\pm = 0\ . \tag{D.8}$$

In this case the τ,σ-dependent rescalings of fluctuations are not needed and 2d Lorentz invariance of the perturbation theory is manifest. One then finds for the individual diagram contributions to the coefficient in the two-loop effective action[21]

$$\begin{aligned}
&\text{bosonic sunset}: && J_A = -J_S = -\frac{3}{2}I[1]I[1]\,,\\
&\text{bosonic double}-\text{bubble}: && J_A = -\ J_S = -\frac{1}{2}I[1]I[1]\,,\\
&\text{fermionic sunset}: && J_A = -J_S = -6I[0]I[1] + 3[1]I[1]\,,\\
&\text{fermionic double}-\text{bubble}: && J_A = -J_S = -6I[0]I[1] - 4I[1]I[1]\,,\\
&\text{tadpole}: && J_A = -J_S = 0\,.
\end{aligned} \tag{D.9}$$

We conclude again that the sum of the A and S sector contributions vanishes.

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