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Flows to Schrödinger Geometries

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Abstract

We construct RG flow solutions interpolating AdS and Schrödinger geometries in Abelian Higgs models obtained from consistent reductions of type IIB supergravity and M-theory. We find that $z = 2$ Schrödinger geometries can be realized at the minima of scalar potentials of these models, where a scalar charged under $U(1)$ gauge symmetry obtains a nonzero vacuum expectation value. The RG flows are induced by an operator deformation of the dual CFT. The flows are captured by fake superpotentials of the theories.

1 Introduction

Since the discovery of the AdS/CFT correspondence [1, 2, 3], there have been a lot of attempts to apply the correspondence to describe strongly-coupled quantum field theories in terms of classical gravity. In its early era, gravity duals of relativistic theories were basically considered. Now it also becomes important to consider applications to strongly-coupled non-relativistic systems. Gravity duals for non-relativistic conformal field theories were constructed in the Einstein gravity coupled to massive vector fields which break the relativistic conformal group to non-relativistic one [4, 5]. Soon after that they were embedded into string/M theories using so-called Null Melvin Twist solution generating technique [6, 7, 8]. These solutions are named the Schrödinger geometry after its symmetry group. Lifshitz geometry is also important for considering gravity dual of non-relativistic field theories, and is also obtained as a solution in the massive gauge theory [9, 10], which can be obtained from the Abelian Higgs model [11]. The embedding to the string/M theories are worked out in [12, 13, 14, 15, 16, 17]. But the situation is more subtle when it comes to embedding the Lifshitz geometries to string theory than to do the Schrödinger geometries so.

In this paper, we will consider a realization of the Schrödinger geometry in the Abelian Higgs models given by [18, 19], which are consistent truncations of type IIB SUGRA and M-theory.[1] These models have W-shaped scalar potentials with two extrema at the origin and the non-zero point of a charged scalar field. The Abelian Higgs model is massive gauge theory around the non-zero point, where the expectation value of the scalar field gives rise to the mass of the gauge field. Therefore, the Schrödinger geometry is expected to be realized there, while the asymptotically AdS space is obtained at the origin of the scalar potentials. It is, however, still non-trivial to embed the non-relativistic geometries to these models since the parameters in scalar potentials cannot be arbitrary varied as opposed to effectively constructed bottom-up models. We will see that it is possible to find the Schrödinger solutions in both cases, while it is not for the Lifshitz solutions.

In particular, since the AdS and the Schrödinger geometries are realized in the maximum and the minimum of the potential, respectively, there must be an interpolating solution between them. We will construct such a domain-wall solution that is dual to the RG flow to a non-relativistic CFT in the IR from a relativistic CFT in the UV.[2] To trigger the RG flow, we will consider an addition of the deformation term $J\mathcal{O}$ to the UV theory, where J

[1]More general reductions including them are considered in [20, 21].

[2]In a different context, flow solutions from AdS_4 spacetime to the five-dimensional Schrödinger geometry have been constructed in M-theory in [22], whose IR geometry was obtained in [23]. See also [24, 25] for some related works on geometric realization of Schrödinger symmetry.

is an external field coupled to the scalar operator $\mathcal{O}$ charged under a $U(1)$ symmetry. With this deformation, the $U(1)$ symmetry is supposed to be explicitly broken, and it is expected to flow to the other IR fixed point, where the IR geometry is the Schrödinger geometry in our case.

This paper is organized as follows. In Section 2, we will review how to embed the Schrödinger geometry into the Abelian Higgs model, and then we consider the string/M-theory construction of the Schrödinger geometries. In Section 3, we will construct the interpolating solutions. After discussing the asymptotics of the fields, we obtain domain-wall solutions numerically. We will discuss the puzzling aspects of our solutions and possible future directions in Section 4.

2 Schrödinger geometries from Abelian Higgs model

Gravity duals for non-relativistic conformal field theories can be constructed in the Einstein gravity coupled to massive vector fields [4, 5]. It is also pointed out that such massive gauge theories can be realized in the broken phase of Abelian Higgs models [5]. Let us summarize the construction briefly.

The metric of a gravity background dual to the boundary theories with Schrödinger symmetry is given by [4, 5]

$$ds^2 = \ell^2 \left[-\sigma^2 r^{2z} (dx^+)^2 + \frac{dr^2}{r^2} + r^2(-dx^+ dx^- + d\vec{x}^2) \right] , \tag{1}$$

where z is the dynamical exponent, ℓ the radius, and σ a parameter of the geometry. This arises from the Einstein gravity coupled to a massive gauge field,

$$S = \frac{1}{16\pi G_N} \int d^{d+3}x \sqrt{-g} \left(R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} A_\mu A^\mu \right) . \tag{2}$$

The solution (1) exists provided that the cosmological constant and the mass of the vector field are chosen as

$$\Lambda = -\frac{(d+1)(d+2)}{2\ell^2} , \qquad m^2 = \frac{z(z+d)}{\ell^2} , \tag{3}$$

and that the gauge field takes the form of

$$A_+ = 2\sigma\ell\sqrt{\frac{z^2-1}{z^2+8}}\, r^z \ . \tag{4}$$

The massive gauge theory can be obtained in the broken phase of the Abelian Higgs model

$$S = \frac{1}{16\pi G_N}\int d^{d+3}x\sqrt{-g}\left(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |(\partial_\mu - iqA_\mu)\psi|^2 - V(\psi)\right) \ , \tag{5}$$

in which the scalar field ψ obtains a nonzero vacuum expectation value in the minima of the scalar potential $V(\psi)$. The form of ψ is restricted as $\psi = \psi_0 e^{i\theta}$, where $\psi_0 \in \mathbb{R}$ corresponds to the vacuum expectation value of the scalar field. The phase θ can be absorbed by the gauge shift, and the resulting theory is the massive gauge theory

$$S = \frac{1}{16\pi G_N}\int d^{d+3}x\sqrt{-g}\left(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - q^2\psi_0^2 A_\mu A^\mu - V(\psi_0^2)\right) \ . \tag{6}$$

It follows that the Schrödinger geometry can be obtained in the Abelian Higgs model (5) with the following parameters [5],

$$q^2\psi_0^2 = \frac{z(z+d)}{2\ell^2} \ , \qquad V(\psi_0^2) = -\frac{(d+1)(d+2)}{\ell^2} \ . \tag{7}$$

We will consider the Schrödinger geometries in the models obtained from consistent truncations of ten-dimensional type IIB SUGRA and eleven-dimensional M-theory to five- and four-dimensions, respectively.[3] In both cases, we will obtain the solutions with the dynamical exponent $z = 2$.

[3]More general solutions are constructed by [26], where the internal spaces are taken as Sasaki-Einstein manifolds, while they are squashed in our case.

2.1 In IIB string theory

Let us start from the five-dimensional case of type IIB SUGRA. The action contains a metric, a $U(1)$ gauge field and a scalar [18],[4]

$$S = \frac{1}{16\pi G_N}\int d^5x\sqrt{-g}\Big[R - \frac{1}{4}F^2 + \frac{1}{12\sqrt{3}}\epsilon^{\lambda\mu\nu\sigma\rho}F_{\lambda\mu}F_{\nu\sigma}A_\rho \\ - \frac{1}{2}(\partial_\mu\eta)^2 - \frac{\sinh^2\eta}{2}(\partial_\mu\theta - \frac{\sqrt{3}}{L}A_\mu)^2 + \frac{3}{L^2}\cosh^2\frac{\eta}{2}(5-\cosh\eta)\Big] \ . \qquad (8)$$

This action is derived from a consistent truncation of ten-dimensional IIB SUGRA action by decomposing the metric as

$$ds^2 = \cosh\frac{\eta}{2}ds_5^2 + \frac{L^2}{\cosh\frac{\eta}{2}}\left[ds_V^2 + \cosh^2\frac{\eta}{2}(\zeta^A)^2\right] \ , \qquad (9)$$

where V is a four-dimensional Kähler-Einstein manifold with $R_{\mu\nu} = 6g_{\mu\nu}$, and $\zeta^A = \zeta + \frac{A}{\sqrt{3}L}$ and $\zeta = d\psi + \sigma$ such that $d\zeta = 2\omega$ with ω being a Kähler form [18]. The phase θ can be absorbed by the gauge shift. The Chern-Simons term can be ignored when the gauge field is assumed as $A \sim dx^+$. The potential minima for the scalar field η are at

$$\eta = \pm\log(2+\sqrt{3}) \ . \qquad (10)$$

The positive sign in (10) is chosen without loss of generality. The action reduces to the massive gauge theory around the minima, where the radius, the cosmological constant and the mass of the gauge field are read off as

$$\ell^2 = \frac{8}{9}L^2 \ , \qquad \Lambda = -\frac{27}{4L^2} \ , \qquad m^2 = \frac{9}{L^2} \ . \qquad (11)$$

Comparing them with (3), the $z = 2$ Schrödinger geometry is obtained in this model.[5] This solution is not supersymmetric because the $U(1)$ fibration of the internal manifold is stretched and there is no Killing spinor on it [27] as is mentioned in [18] for the AdS solution.

[4]Here we rescaled the gauge field $A_{ours} = \frac{2L}{\sqrt{3}}A_{theirs}$ to normalize the coefficient of the field strength.

[5]There exists the other solution of (3) with $z = -4$. We do not adopt it here.

2.2 In M-theory

Similarly we can use a four-dimensional theory obtained from a consistent truncation of M-theory on a seven-dimensional Sasaki-Einstein manifold [19, 28],

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\Big[R - \frac{1}{4}F^2 - \frac{1}{2}(\partial_\mu\eta)^2 \\ - \frac{\sinh^2\eta}{2}(\partial_\mu\theta - \frac{1}{L}A_\mu)^2 + \frac{1}{L^2}\cosh^2\frac{\eta}{2}(7-\cosh\eta)\Big] \ . \tag{12}$$

Here we need a constraint for the gauge field $F\wedge F = 0$, but this is always satisfied for the Schrödinger background. The potential minima are at

$$\eta = \pm\log(3+2\sqrt{2}) \ , \tag{13}$$

and we find the massive vector theory with the parameters

$$\ell^2 = \frac{3}{4}L^2 \ , \qquad \Lambda = -\frac{4}{L^2} \ , \qquad m^2 = \frac{8}{L^2} \ . \tag{14}$$

Then we obtain $z = 2$ by solving (3) and (14).[6] This solution is also non-supersymmetric for a similar reason to type IIB case [29, 30].

2.3 Comment on Lifshitz solutions

The massive vector theory also allows us to construct Lifshitz solutions of the form [9, 10]

$$ds^2 = -\left(\frac{r}{\ell}\right)^{2z}dt^2 + \ell^2\frac{dr^2}{r^2} + \left(\frac{r}{\ell}\right)^2 d\vec{x}^2_{d+1} \ , \tag{15}$$

by tuning the parameters such that

$$\Lambda = -\frac{z^2+dz+(d+1)^2}{2\ell^2} \ , \qquad m^2 = \frac{2z}{\ell^2} \ . \tag{16}$$

However, there are no physical solutions in four- and five-dimensional theories considered above. It would be interesting to look for a consistent truncation which affords parameters consistent with the relations (16).

[6] $z = -3$ is the other solution which satisfies (3).

3 Domain wall solutions

The actions given by (8) and (12) in five and four dimensions, respectively, have two vacua: one is supersymmetric at $\eta = 0$, and the other is non-supersymmetric at $\eta = \eta_* \neq 0$. As already studied in [21, 31], domain wall solutions interpolating two AdS vacua can be constructed. Here we expect a domain wall solution interpolating the AdS and the Schrödinger geometries.

We assume the following ansatz for the domain wall solution,

$$
\begin{aligned}
ds^2 =& L^2 \left(d\rho^2 + e^{2A(\rho)}((dx^i)^2 - dx^+ dx^-) - B(\rho) e^{2zA(\rho)} (dx^+)^2 \right) \ , \\
A_+ =& L a(\rho) e^{zA(\rho)} \ , \\
\eta =& \eta(\rho) \ ,
\end{aligned}
\tag{17}
$$

where $i = 1, \ldots, d$, and $d = 2$ and $d = 1$ for type IIB and M-theory, respectively. This ansatz is chosen such that the Schrödinger geometry in the IR will become manifest. We focus only on the case of zero temperature, and an operator deformation will be assumed in the UV. The equations of motion are given by

$$
\begin{aligned}
\eta'' + (d+2)A'\eta' - \partial_\eta V_{pot} &= 0 \ , \\
2(d+1)A'' + (d+1)(d+2)A'^2 + \frac{1}{2}\eta'^2 + V_{pot} &= 0 \ , \\
(d+1)(d+2)A'^2 - \frac{1}{2}\eta'^2 + V_{pot} &= 0 \ ,
\end{aligned}
\tag{18}
$$

where V_{pot} is a scalar potential for η,

$$
\begin{aligned}
B'' + (4z+d-2)A'B' - \Big[2(d+2-z)A'' - 2(2z^2 + (d-2)z - \frac{d^2+5d+2}{2})A'^2 + \frac{1}{2}\eta'^2 + V_{pot} \Big] B \\
= \frac{d(d+1)}{2} a^2 \sinh^2 \eta + (zaA' + a')^2 \ ,
\end{aligned}
\tag{19}
$$

and the Maxwell equation

$$
a'' + (2z+d)A'a' + \left[zA'' + z(z+d)A'^2 - \frac{d(d+1)}{2} \sinh^2 \eta \right] a = 0 \ . \tag{20}
$$

The second-order differential equations in (18) can be written as first-order equations [32]. One of the equations (18) is not independent, and the two independent ones can be obtained as the Euler-Lagrange equations for the following functional

$$E = \int_{-\infty}^{\infty} d\rho\, e^{(d+2)A(\rho)} \left[(\partial_\rho \eta)^2 - 2(d+1)(d+2)(\partial_\rho A)^2 + 2V_{pot} \right] \ . \tag{21}$$

When the potential for the scalar takes the form of

$$V_{pot} = \frac{1}{2}(\partial_\eta \mathcal{W})^2 - \frac{d+2}{4(d+1)}\mathcal{W}^2 \ , \tag{22}$$

the functional can be rewritten as

$$\begin{aligned} E = \int_{-\infty}^{\infty} d\rho\, e^{4A(\rho)} &\left[(\eta' \mp \partial_\eta \mathcal{W})^2 - 2(d+1)(d+2)(A' \pm \frac{1}{2(d+1)}\mathcal{W})^2 \right] \\ &\pm 2e^{(d+2)A(\rho)}\mathcal{W}\Big|_{-\infty}^{\infty} \ . \end{aligned} \tag{23}$$

Then it gives two first-order equations of motion,

$$A' = \mp \frac{1}{2(d+1)}\mathcal{W} \ , \qquad \eta' = \pm \partial_\eta \mathcal{W} \ . \tag{24}$$

Here the $\mathcal{W}$ is called as a "fake superpotential."

Let us find out the asymptotic behavior of the fields which solve the equations of motion around $\rho = \pm\infty$. We first consider the solution in the UV ($\rho \to \infty$). The scalar field η and metric A can be expanded as

$$\begin{aligned} \eta &= \eta_{(0)} e^{-\Delta_- \rho/L} + \eta_{(1)} e^{-\Delta_+ \rho/L} + \cdots \ , \\ A &= \frac{\rho}{L} - A_{(0)} e^{-2\Delta_- \rho/L} + \cdots \ , \end{aligned} \tag{25}$$

where $\Delta_\pm$ are the dimensions of η given by the solutions of $\Delta(\Delta - d - 2) = -m_\eta^2$, where m_η is the (negative) mass of η. The dot-parts include further exponentially suppressed contributions. The presence of $\eta_{(0)}$ corresponds to a deformation of the CFT in the UV by a term $\eta_{(0)}\mathcal{O}$ in the Lagrangian of the boundary theory, and the RG flow is caused by this operator deformation. The equations of motion (24) relates $A_{(0)}$ with $\eta_{(0)}$ as

$$A_{(0)} = \frac{\eta_{(0)}^2}{8(d+1)} \ . \tag{26}$$

For the Maxwell equation (20), we obtain the behavior of the gauge field

$$a = a_{(1)}\, e^{-z\rho/L} + a_{(2)} e^{-(z+d)\rho/L} + \cdots \ , \tag{27}$$

which is nothing but

$$A_+ = L(a_{(1)} + a_{(2)} e^{-d\rho/L} + \cdots) \ . \tag{28}$$

Finally, the function B takes a form of

$$\begin{aligned} B &= B_{(0)} e^{-m_-\rho/L} + B_{(1)} e^{-m_+\rho/L} + \cdots \ , \\ m_\pm &= \frac{1}{2}\left[4z + d - 2 \pm \sqrt{(4z+d-2)^2 - 8(2z^2 + (d-2)z - \frac{d^2+5d+2}{2}) + 4V_{UV}L^2}\right] \ , \end{aligned} \tag{29}$$

where V_{UV} is the value of the potential at $\rho = \infty$, and here we have used the fact that $\eta' = 0$ there.

It is straightforward also to find the asymptotic behavior of the fields in the IR ($\rho \to -\infty$). We impose the following boundary condition:

$$\begin{aligned} \eta &= \eta_* + \widetilde{\eta}_{(0)} e^{\widetilde{\Delta}_-\rho/\ell} + \widetilde{\eta}_{(1)} e^{\widetilde{\Delta}_+\rho/\ell} + \cdots \ , \\ A &= \frac{\rho}{\ell} - \widetilde{A}_{(0)} e^{2\widetilde{\Delta}_-\rho/\ell} + \cdots \ , \\ a &= a_0 + \cdots \ , \\ B &= a_0^2 + \cdots \ , \end{aligned} \tag{30}$$

where η_* is the vacuum expectation value of the scalar field at the minimum of the potential, and $\widetilde{\Delta}_\pm$ is the dimension of η in the IR. The fake superpotential $\mathcal{W}$ defined by (22) will be convenient to read off $\widetilde{\Delta}_\pm$. To derive the asymptotic forms of a and B in (30), we solved the equations of motion of IIB SUGRA (8) and M-theory (12) around the potential minimum. The constant a_0 appearing in the leading terms of a and B is responsible for the emergence of Schrödinger geometry in the IR. We will numerically solve the equations of motion to find domain wall solutions with non-zero a_0.

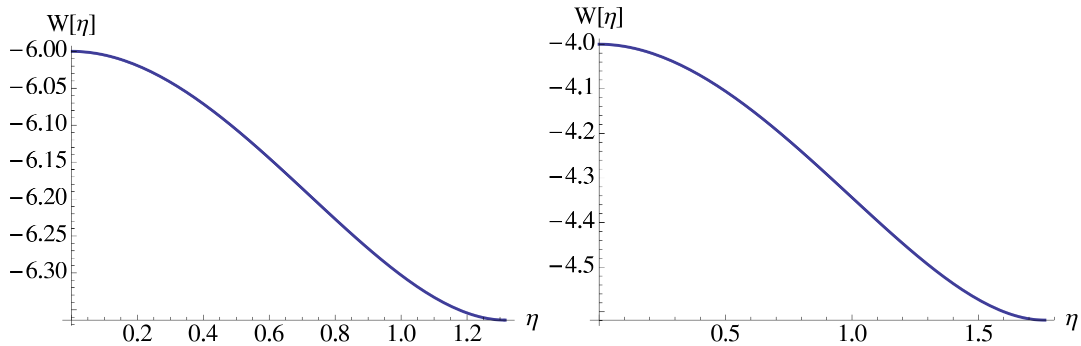

Figure 1: The fake superpotential interpolating supersymmetric ($\eta = 0$) and non-supersymmetric ($\eta = \eta_*$) vacua in IIB SUGRA (left) and M-theory (right).

3.1 Fake superpotential and asymptotic solution

For concreteness, we compute coefficients appearing in the exponents of the asymptotic expansion for our IIB SUGRA and M-theory cases. We shall set the AdS radius $L = 1$ for simplicity hereafter. In the computation, it is worthwhile to notice that the fake superpotential is useful to capture the RG flows and to determine the asymptotics.

We first consider the IIB SUGRA case, where $d = 2$ and the domain-wall solution we focus on has $z = 2$. We can solve (22) to obtain a fake superpotential shown in Fig. 1, which corresponds to the RG flow induced by the scalar source $\eta_{(0)}$ in the UV and flowing to the vacuum at $\eta = \eta_*$ in the IR. The superpotential can be expanded around $\eta = 0$ as $\mathcal{W} = -6 - \eta^2/2 + \cdots$, and we see that the leading behavior of η around $\rho = \infty$ corresponds to $\Delta_- = 1$. Since $V_{UV} = -12$ at the UV critical point $\eta = 0$, we obtain $m_- = 2$ from (29). At the other critical point in the IR, $\eta_* = \log(2 + \sqrt{3})$, the superpotential in Fig. 1 can be expanded as

$$\mathcal{W} = -\frac{9}{\sqrt{2}} + \frac{3(\sqrt{6} - \sqrt{2})}{4}(\eta - \eta_*)^2 + \cdots \ , \tag{31}$$

and then we obtain $\widetilde{\Delta}_- = 2\sqrt{3} - 2$.

We can repeat this analysis also in the M-theory case, where $d = 1$, $z = 2$, $V_{UV} = -6$. In the UV, we have $\Delta_- = 1$, and (29) gives $m_- = 2$. Around the IR critical point, $\eta_* = \log(3 + 2\sqrt{2})$, the superpotential can be expanded as

$$\mathcal{W} = -\frac{8}{\sqrt{3}} + \frac{(\sqrt{11} - \sqrt{3})}{2}(\eta - \eta_*)^2 + \cdots \ , \tag{32}$$

and then we obtain $\widetilde{\Delta}_- = (\sqrt{33} - 3)/2$.

As the asymptotic behavior in the UV is obtained, it is convenient to rewrite the metric (17) so that the AdS geometry is manifest there. The ansatz (17) can be written as

$$ds^2 = L^2 \left[d\rho^2 + e^{2A(\rho)} \left(-H(\rho) \left(dx^+ + \frac{dx^-}{2H(\rho)} \right)^2 + \frac{(dx^-)^2}{4H(\rho)} + (dx^i)^2 \right) \right] , \tag{33}$$

where $H(\rho) \equiv B(\rho) e^{2(z-1)A(\rho)}$. Since $A(\rho) \to r/L$ and $B \to B_{(0)} e^{-2r/L}$ as $\rho \to \infty$ for both the IIB SUGRA and the M-theory cases ($z = 2$), we find $H(\rho) \to B_{(0)}$ in the UV.[7] Hence the metric in UV is AdS,

$$ds^2 = L^2 \left[d\rho^2 + e^{2\rho/L} \left(-d\tau^2 + d\sigma^2 + (dx^i)^2 \right) \right] , \tag{34}$$

where $\tau \equiv B_{(0)}^{1/2} x^+ + B_{(0)}^{-1/2} x^-/2$ and $\sigma \equiv B_{(0)}^{-1/2} x^-/2$. The coefficient $B_{(0)}$, which will be obtained from numerical solutions and should be of order-one, is related to this "twisting" of the AdS geometry to generate the Schrödinger geometry.

3.2 Numerical solutions

We solve the equations of motion numerically by using the shooting method to obtain domain-wall solutions. We begin with the case of the IB SUGRA, and will back to the M-theory case after that. We focus only on the leading behavior of the asymptotic behavior of the fields in the UV and the IR. The subleading terms could be computed with sufficient numerical accuracy.[8]

We have one parameter whose value can be chosen as the initial condition in the UV: $\eta_{(0)}$. There is a scaling symmetry that allows us to set $a_{(1)} = 1$ without loss of generality unless it is zero. Given these, we can obtain a domain-wall solution where the IR geometry is the Schrödinger geometry whose IR behavior is characterized by the value of a_0. We will show that the a_0 is simply determined as a function of $\eta_{(0)}$.

Fig. 2 shows the domain-wall solution in the five-dimensional SUGRA (8) with $\eta_{(0)} = 1$. In the IR, the a_0 and B grow along with η, and the domain-wall solution interpolate the AdS in the UV and the Schrödinger geometry in the IR. In the absence of the gauge field, the IR geometry is just another light-like AdS where the theory is in a non-supersymmetric vacuum. The choice of $\eta_{(0)}$ in Fig. 2 is just for demonstration; so it is interesting to see how a_0 behaves when $\eta_{(0)}$ varied. The result of is shown in Fig. 3, and we find that the plot points

[7] It is not suitable to write the metric as in the form (33) in the IR because $H(\rho) \to 0$ as $\rho \to -\infty$.

[8] This might be interesting from the point of view of the thermodynamic relation in Appendix A.

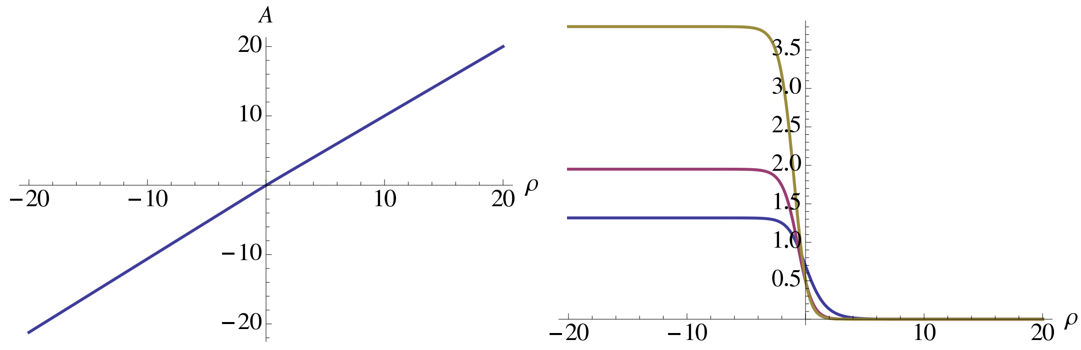

Figure 2: Domain-wall solution interpolating AdS and Schrödinger geometries in a five-dimensional SUGRA with $\eta_{(0)} = 1$. Left: The plot of $A(\rho)$. Right: The blue (bottom), the purple (middle) and the olive (top) lines correspond to $\eta(\rho)$, $a(\rho)$ and $B(\rho)$, respectively.

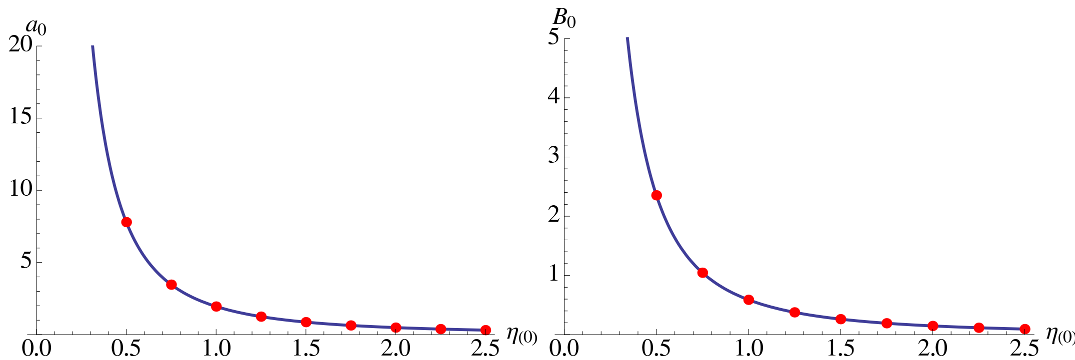

Figure 3: Left: The plot of a_0 as a function of $\eta_{(0)}$. The red dots correspond to numerical result, and they can be fitted by $1.95/\eta_{(0)}^2$ shown in the blue curve. Right: The plot of $B_{(0)}$ as a function of $\eta_{(0)}$. The plot points can be fitted by $0.588/\eta_{(0)}^2$.

can be fitted by $a_0 = 1.95/\eta_{(0)}^2$ and $B_{(0)} = 0.588/\eta_{(0)}^2$.

The domain-wall solution in the case of M-theory can be computed in similar way as in the case of type IIB SUGRA. The solution is shown in Fig. 4, where $\eta_{(0)} = 3/2$ is chosen as a demonstration. We find that a_0 is given as a function of $\eta_{(0)}$, $a_0 = 4.77/\eta_{(0)}^2$, and $B_{(0)} = 0.967/\eta_{(0)}^2$ is obtained.

The Ricci scalar and the Kretschmann scalar are regular and monotonic function of ρ along the RG flow. In the type IIB SUGRA case, $R = -20/L^2$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 40/L^4$ in the UV, and $R = -22.5 = -20/\ell^2$ and $RR = 50.625 = 40/\ell^4$ in the IR. This agrees with the case of AdS-AdS domain-walls.

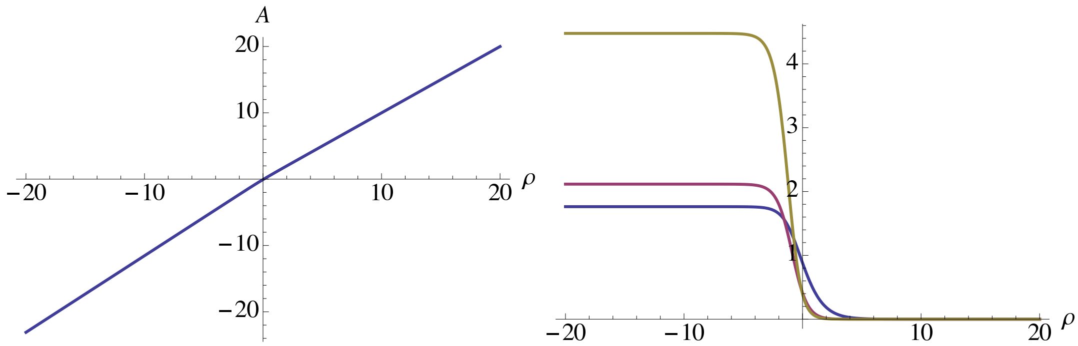

Figure 4: Domain-wall solution interpolating AdS and Schrödinger geometries in a four-dimensional SUGRA with $\eta_{(0)} = 3/2$. Left: Plot of $A(\rho)$. Right: The blue (bottom), the purple (middle) and the olive (top) lines correspond to $\eta(\rho)$, $a(\rho)$ and $B(\rho)$, respectively.

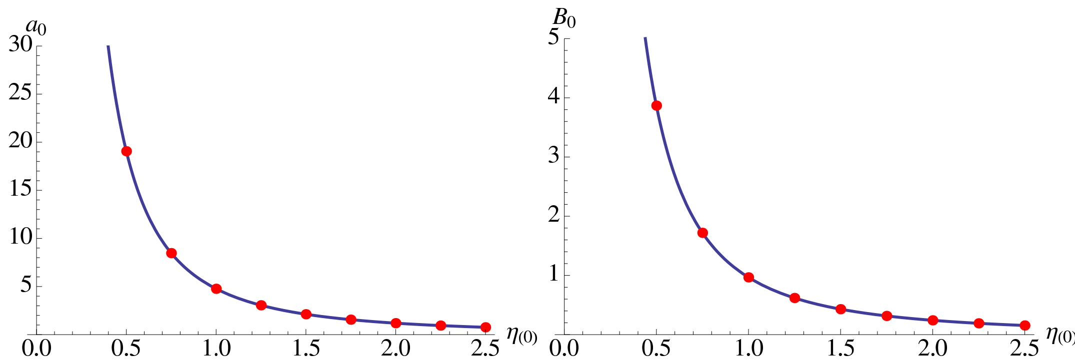

Figure 5: Left: Plot of a_0 as a function of $\eta_{(0)}$. The red dots (numerical results) can be fitted by $4.77/\eta_{(0)}^2$ shown in the blue curve. Right: Plot of $B_{(0)}$ as a function of $\eta_{(0)}$. The plot points can be fitted by $0.967/\eta_{(0)}^2$.

4 Discussions

We constructed RG flow solutions interpolating AdS and Schrödinger geometries by using Abelian Higgs models obtained in type IIB supergravity and in M-theory. We found that $z = 2$ Schrödinger geometries are realized at the minima of the scalar potential of these models, where the scalar field obtained nonzero vacuum expectation value, while it was not possible to embed the Lifshitz geometries into these models. The Lifshitz geometry has been constructed in four-dimensional $\mathcal{N} = 2$ SUGRA [16, 17] which can be lifted to string/M-theory (see also [13, 15]). It would be interesting to investigate consistently truncated Abelian Higgs models which allow the Lifshitz geometries as solutions.

The RG flow was of the type induced by a scalar field source $\eta_{(0)}$ in the UV corresponding to an operator deformation of the dual CFT. The solution is a one-parameter family of $\eta_{(0)}$, and the IR value of the massive gauge field was controlled by $\eta_{(0)}$. What we obtained can be phrased as a realization of emergent non-relativistic symmetry in the IR in zero-temperature. Our solutions, however, is singular when the deformation is turned off $\eta_{(0)} = 0$ as was shown in Fig. 3 and 5. This implies that our RG flow solutions cannot be obtained as a deformation of the AdS-AdS domain-walls of [28] despite the fact that both are the solutions of the same theory. It will be interesting to see how the domain-wall solution we obtained is related to the context of holographic superconductors [33, 34]. The domain-wall solution here is induced by the operator deformation in the UV. On the other hand, a non-zero chemical potential gives VEV to the operator dual to the scalar in the context of holographic superconductors. It will be challenging to construct the Schrödinger geometry by the RG flow induced by the VEV deformation in the UV. Our solutions would be a special limit of the anisotropic domain-walls recently constructed in [35, 36], which might provide us a clear understanding of the dual theories of our solutions.

Since the boundary is the AdS spacetime, we would be able to carry out holographic renormalization to understand the features of the dual field theory.[9] Our solutions are similar to the GPPZ flow [38] where the gravity dual of an operator deformation of $\mathcal{N} = 4$ SYM was studied. The holographic renormalization of the GPPZ flow was carefully carried out in [39], and it is straightforward to repeat the same analysis for our cases. Incongruously, the $U(1)$ current is conserved despite the operator deformation. This might be because the expectation value of the $U(1)$ charged operator dual to the source $\eta_{(0)}$ would be zero, *i.e.*, $\eta_{(1)} = 0$. The value of the $\eta_{(1)}$ is completely determined by the behavior of the superpotential around $\eta = 0$. Although the superpotential we used is numerically generated and it does not have a good asymptotic expansion around $\eta = 0$, the $\eta_{(1)}$ is likely to be zero within the numerical error. Another hint for the existence of the $U(1)$ symmetry is the conserved current we found in Appendix A that results from the scaling symmetry of our solutions. The conserved current could be interpreted as the thermodynamic relation of the dual field theory, and there might be a conserved charge of the form (47) that apparently comes from the bulk gauge field. Unfortunately, a usual procedure of the holographic renormalization does not give us such a contribution since the Maxwell action vanishes when the on-shell solution is substituted. To obtain a non-zero contribution to the on-shell Maxwell action, the A_--component on the light-like AdS spacetime will be necessary. It is not clear how to account for the conservation of the $U(1)$ current (coming from the bulk gauge symmetry) as well as the conserved charge (47) in the thermodynamic relation (this is not necessarily the same as the $U(1)$ symmetry mentioned above). Our solutions might be a singular limit of

[9] A holographic renormalization on the Schrödinger geometries was discussed in [37].

general solutions including the A_--component, so it would be necessary to relax our ansatz to clarify the property of our solutions.

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A Conserved charges and thermodynamic relation

We would like to evaluate the conserved charges from our solutions that are asymptotically AdS space. We follow the argument given in Appendix C.2 of [6]. Given the Fefferman-Graham coordinates of the AdS_{D+1} space

$$ds^2_{D+1} = L^2(d\rho^2 + e^{2\rho} g_{ab}(x,\rho) dx^a dx^b) \ , \tag{35}$$

where g_{ab} has the following expansion form

$$g(x,\rho) = g_{(0)} + g_{(2)} e^{-2\rho} + \cdots + g_{(D)} e^{-D\rho} - 2h_{(D)} \rho e^{-D\rho} + \cdots \ , \tag{36}$$

the coefficient $h_{(D)}$ is related with the Weyl anomaly of the boundary field theory, and the boundary stress-energy tensor reads

$$T_{ab} = \frac{DL}{16\pi G_N} (g_{(D)})_{ab} \ , \tag{37}$$

for the flat boundary $(g_{(0)})_{ab} = \eta_{ab}$ [40, 41]. For our cases, the metric is asymptotically light-like AdS space by shifting the light-like coordinates

$$\tilde{x}^+ = x^+ \ , \qquad \tilde{x}^- = x^- + B_{(0)} x^+ \ . \tag{38}$$

The stress-energy tensor vanishes except the ++ component

$$T_{++} = -\frac{(d+2)L}{16\pi G_N} B_{(1)} \ . \tag{39}$$

Then, the energy density and the particle number density, associated to the Killing vectors $\partial_{\tilde{x}^+}$ and $\partial_{\tilde{x}^-}$, respectively, are

$$\mathcal{E} = T_{++} = -\frac{(d+2)L}{16\pi G_N} B_{(1)} \ , \qquad \mathcal{N} = 0 \ . \tag{40}$$

One can find the conserved current in a similar manner to [42]

$$j = e^{dA}\left(e^{2A}(e^{2(z-1)A}B)' - \phi\phi'\right) \ , \qquad \phi \equiv e^{zA} a \ , \tag{41}$$

that satisfies $\partial_\rho j(\rho) = 0$ up to the equations of motion. It is associated to the scale invariance of the solution of the form (17)

$$A \to A - \log\lambda \ , \qquad x^i \to \lambda x^i \ , \qquad x^+ \to \lambda^z x^+ \ , \qquad x^- \to \lambda^{2-z} x^- \ . \tag{42}$$

The conservation of the current gives rise to the relation between the parameters in UV and IR regions

$$j(\rho = \infty) = j(\rho = -\infty) \ . \tag{43}$$

This gives us an additional constraint for the parameters as follows:

$$\frac{d+2}{d} B_{(1)} = a_{(1)} a_{(2)} \ . \tag{44}$$

In the non-relativistic scale-invariant theory, the energy density and pressure are related by

$$z\mathcal{E} = d\mathcal{P} \ . \tag{45}$$

Then, the thermodynamic relation gives rise to

$$\frac{d+z}{d}\mathcal{E} = T\mathcal{S} + \mu\mathcal{Q} \ , \tag{46}$$

where T and $\mathcal{S}$ are the temperature and entropy density, and μ and $\mathcal{Q}$ are the chemical potential and charge density associated with some symmetry, respectively. Since there is no temperature in our solutions and $B_{(1)}$ is proportional to the energy density $\mathcal{E}$, we obtain the following relation

$$\mu\mathcal{Q} = -\frac{(d+2)L}{16\pi G_N} a_{(1)} a_{(2)} \ . \tag{47}$$

This implies the existence of a conserved charge in our solutions, but the $U(1)$ symmetry is supposed to be explicitly broken by the operator deformation. We have the particle number symmetry in our solutions, and the density is zero as was given in (40). Then there seems to be no corresponding charges to (47).

This thermodynamic relation is unchanged in the (τ, σ)-coordinate. The components of the gauge field in this coordinate can be identified as $A_\tau = B_{(0)}^{-1/2} A_+$ and $A_\sigma = -B_{(0)}^{-1/2} A_+$ in $\rho \to \infty$, and hence the asymptotic behavior of A_τ is

$$A_\tau = L(a_{(1)} B_{(0)}^{-1/2} + a_{(2)} B_{(0)}^{-1/2} e^{-d\rho/L} + \cdots) \ . \tag{48}$$

On the other hand, the expansion of the metric gives $g_{(D)\tau\tau} = B_{(1)}/B_{(0)}$. Therefore, (44) is not disrupted by the coordinate transformation. In the (τ, σ)-coordinates, the situation is that the A_σ as well as the A_τ are turned on in UV.

References

[1] J. M. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge Theory Correlators from Non-Critical String Theory*, *Phys. Lett.* **B428** (1998) 105–114, [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter Space and Holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150].

[4] D. T. Son, *Toward an AdS/Cold Atoms Correspondence: a Geometric Realization of the Schrödinger Symmetry*, *Phys. Rev.* **D78** (2008) 046003, [arXiv:0804.3972].

[5] K. Balasubramanian and J. McGreevy, *Gravity Duals for Non-Relativistic CFTs*, *Phys. Rev. Lett.* **101** (2008) 061601, [arXiv:0804.4053].

[6] J. Maldacena, D. Martelli, and Y. Tachikawa, *Comments on String Theory Backgrounds with Non- Relativistic Conformal Symmetry*, *JHEP* **10** (2008) 072, [arXiv:0807.1100].

[7] C. P. Herzog, M. Rangamani, and S. F. Ross, *Heating Up Galilean Holography*, *JHEP* **11** (2008) 080, [arXiv:0807.1099].

[8] A. Adams, K. Balasubramanian, and J. McGreevy, *Hot Spacetimes for Cold Atoms*, *JHEP* **11** (2008) 059, [arXiv:0807.1111].

[9] S. Kachru, X. Liu, and M. Mulligan, *Gravity Duals of Lifshitz-Like Fixed Points*, *Phys. Rev.* **D78** (2008) 106005, [arXiv:0808.1725].

[10] M. Taylor, *Non-Relativistic Holography*, arXiv:0812.0530.

[11] S. S. Gubser and A. Nellore, *Ground States of Holographic Superconductors*, *Phys. Rev.* **D80** (2009) 105007, [arXiv:0908.1972].

[12] K. Balasubramanian and K. Narayan, *Lifshitz Spacetimes from AdS Null and Cosmological Solutions*, *JHEP* **08** (2010) 014, [arXiv:1005.3291].

[13] A. Donos and J. P. Gauntlett, *Lifshitz Solutions of D=10 and D=11 Supergravity*, arXiv:1008.2062.

[14] R. Gregory, S. L. Parameswaran, G. Tasinato, and I. Zavala, *Lifshitz Solutions in Supergravity and String Theory*, *JHEP* **12** (2010) 047, [arXiv:1009.3445].

[15] A. Donos, J. P. Gauntlett, N. Kim, and O. Varela, *Wrapped M5-Branes, Consistent Truncations and AdS/Cmt*, *JHEP* **12** (2010) 003, [arXiv:1009.3805].

[16] D. Cassani and A. F. Faedo, *Constructing Lifshitz Solutions from AdS*, arXiv:1102.5344.

[17] N. Halmagyi, M. Petrini, and A. Zaffaroni, *Non-Relativistic Solutions of $\mathcal{N}=2$ Gauged Supergravity*, arXiv:1102.5740.

[18] S. S. Gubser, C. P. Herzog, S. S. Pufu, and T. Tesileanu, *Superconductors from Superstrings*, *Phys. Rev. Lett.* **103** (2009) 141601, [arXiv:0907.3510].

[19] J. P. Gauntlett, J. Sonner, and T. Wiseman, *Holographic Superconductivity in M-Theory*, *Phys. Rev. Lett.* **103** (2009) 151601, [arXiv:0907.3796].

[20] J. P. Gauntlett and O. Varela, *Universal Kaluza-Klein Reductions of Type IIB to $\mathcal{N}=4$ Supergravity in Five Dimensions*, *JHEP* **06** (2010) 081, [arXiv:1003.5642].

[21] D. Cassani, G. Dall'Agata, and A. F. Faedo, *Type IIB Supergravity on Squashed Sasaki-Einstein Manifolds*, *JHEP* **05** (2010) 094, [arXiv:1003.4283].

[22] H.-C. Kim, S. Kim, K. Lee, and J. Park, *Emergent Schrödinger Geometries from Mass-Deformed CFT*, *JHEP* **08** (2011) 111, [arXiv:1106.4309].

[23] E. O Colgain, O. Varela, and H. Yavartanoo, *Non-Relativistic M-theory Solutions Based on Kähler- Einstein Spaces*, *JHEP* **07** (2009) 081, [arXiv:0906.0261].

[24] H. Ooguri and C.-S. Park, *Supersymmetric Non-Relativistic Geometries in M-theory*, *Nucl. Phys.* **B824** (2010) 136–153, [arXiv:0905.1954].

[25] J. Jeong, H.-C. Kim, S. Lee, E. O Colgain, and H. Yavartanoo, *Schrodinger invariant solutions of M-theory with Enhanced Supersymmetry*, *JHEP* **03** (2010) 034, [arXiv:0911.5281].

[26] A. Donos and J. P. Gauntlett, *Solutions of Type IIB and D=11 Supergravity with Schrödinger(Z) Symmetry*, *JHEP* **07** (2009) 042, [arXiv:0905.1098].

[27] L. J. Romans, *New Compactifications of Chiral $\mathcal{N} = 2$ $D = 10$ Supergravity*, *Phys. Lett.* **B153** (1985) 392.

[28] S. S. Gubser, S. S. Pufu, and F. D. Rocha, *Quantum Critical Superconductors in String Theory and M- Theory*, *Phys. Lett.* **B683** (2010) 201–204, [arXiv:0908.0011].

[29] C. N. Pope and N. P. Warner, *Two New Classes of Compactifications of $D = 11$ Supergravity*, *Class. Quant. Grav.* **2** (1985) L1.

[30] C. N. Pope and N. P. Warner, *An $SU(4)$ Invariant Compactification of $D = 11$ Supergravity on a Stretched Seven Sphere*, *Phys. Lett.* **B150** (1985) 352.

[31] J. P. Gauntlett, S. Kim, O. Varela, and D. Waldram, *Consistent Supersymmetric Kaluza–Klein Truncations with Massive Modes*, *JHEP* **04** (2009) 102, [arXiv:0901.0676].

[32] K. Skenderis and P. K. Townsend, *Gravitational Stability and Renormalization-Group Flow*, *Phys. Lett.* **B468** (1999) 46–51, [hep-th/9909070].

[33] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, *Building a Holographic Superconductor*, *Phys. Rev. Lett.* **101** (2008) 031601, [arXiv:0803.3295].

[34] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, *Holographic Superconductors*, *JHEP* **12** (2008) 015, [arXiv:0810.1563].

[35] D. Arean, M. Bertolini, C. Krishnan, and T. Prochazka, *Type IIB Holographic Superfluid Flows*, *JHEP* **1103** (2011) 008, [arXiv:1010.5777].

[36] D. Arean, M. Bertolini, C. Krishnan, and T. Prochazka, *Quantum Critical Superfluid Flows and Anisotropic Domain Walls*, arXiv:1106.1053.

[37] M. Guica, K. Skenderis, M. Taylor, and B. C. van Rees, *Holography for Schrödinger Backgrounds*, *JHEP* **02** (2011) 056, [arXiv:1008.1991].

[38] L. Girardello, M. Petrini, M. Porrati, and A. Zaffaroni, *The Supergravity Dual of $\mathcal{N} = 1$ Super Yang-Mills Theory*, *Nucl. Phys.* **B569** (2000) 451–469, [hep-th/9909047].

[39] M. Bianchi, D. Z. Freedman, and K. Skenderis, *Holographic Renormalization*, *Nucl. Phys.* **B631** (2002) 159–194, [hep-th/0112119].

[40] S. de Haro, S. N. Solodukhin, and K. Skenderis, *Holographic Reconstruction of Spacetime and Renormalization in the AdS/CFT Correspondence*, *Commun. Math. Phys.* **217** (2001) 595–622, [hep-th/0002230].

[41] K. Skenderis, *Asymptotically Anti-de Sitter Spacetimes and Their Stress Energy Tensor*, *Int. J. Mod. Phys.* **A16** (2001) 740–749, [hep-th/0010138].

[42] I. R. Klebanov, S. S. Pufu, and T. Tesileanu, *Membranes with Topological Charge and AdS4/CFT3 Correspondence*, *Phys. Rev.* **D81** (2010) 125011, [arXiv:1004.0413].