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Inflationary Scalars Don't Affect Gravitons at One Loop

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ABSTRACT

Primordial inflation results in the production of a vast ensemble of highly infrared, massless, minimally coupled scalars. We use a recent fully renormalized computation of the one loop contribution to the graviton self-energy from these scalars to show that they have no effect on the propagation of dynamical gravitons. Our computation motivates a conjecture for the first correction to the vacuum state wave functional of gravitons. We comment as well on performing the same analysis for the more interesting contribution from inflationary gravitons, and on inferring one loop corrections to the force of gravity.

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1 Introduction

Inflation produces a vast ensemble of infrared gravitons and massless, minimally coupled (MMC) scalars [1]. In the theory of inflationary cosmology these particles are the source of primordial tensor and scalar perturbations [2], the scalar component of which has been detected [3]. It is natural to wonder how this ensemble of quanta changes the propagation of free particles during inflation.

The effect of inflationary gravitons or scalars on the propagation of a particular kind of particle is governed by that particle's one-particle-irreducible (1PI) 2-point function. For scalars this is the self-mass-squared, $-iM^2(x;x')$; it is the self-energy for a fermion, $-i[{}_i\Sigma_j](x;x')$; for a vector it is the vacuum polarization, $-i[{}^\mu\Pi^\nu](x;x')$; and it is the self-energy for a graviton, $-i[{}^{\mu\nu}\Sigma^{\rho\sigma}](x;x')$. One first computes the renormalized contribution of inflationary gravitons or MMC scalars to the appropriate 1PI function, then uses this to quantum-correct the linearized effective field equations. For example, the linearized effective field equations of a MMC scalar are,

$$\partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi(x)\Big) - \int d^4x'\, M^2(x;x')\varphi(x') = 0 \; . \tag{1}$$

Many studies of this type have been made over the past decade. The one loop effects of inflationary scalars have been worked out on photons, assuming the scalars are charged [4], on fermions, assuming a Yukawa coupling [5], and on other scalars, assuming either that the scalars have a quartic self-interaction [6], that they interact electromagnetically [7], or that they interact with fermions [8]. The effects of inflationary gravitons have been worked out for MMC scalars [9] and for massless fermions [10].

What happens in each case seems to depend upon whether or not the highly infrared gravitons and scalars created by inflation can maintain a significant interaction with the particle in question. Because neither electromagnetic nor Yukawa charge weakens with redshift, the effects of inflationary scalars on photons and fermions is profound: both particles acquire a growing mass [4, 5]. The same is true for MMC scalars with a quartic self-interaction [6], but the redshift of photons and fermions means that nothing significant happens to either charged scalars [7] or Yukawa-coupled scalars [8]. Because the spin of infrared gravitons does not redshift, they induce a growing field strength on fermions [10]. However, gravitons only interact with a MMC scalar through the scalar's rapidly redshifting kinetic energy, and this results in no significant effect[9].

The purpose of this paper is study how inflationary scalars affect the propagation of free gravitons. We have already computed the fully renormalized, one loop contribution to the graviton self-energy from MMC scalars [11]. That result is summarized in section 2. In section 3 we solve the linearized effective field equations at one loop order. Section 4 gives our conclusions.

2 The Effective Field Equations

The purpose of this section is to present the effective field equation which we solve in the next section. We begin by reviewing some useful facts about the background geometry. We then give our recently derived result for the one loop MMC scalar contribution to the graviton self-energy [11]. The section closes with a discussion of the Schwinger-Keldysh effective field equations and how one solves them perturbatively.

2.1 The Background Geometry

Our background geometry is the open conformal coordinate submanifold of 4-dimensional de Sitter space. A spacetime point $x^\mu = (\eta, x^i)$ takes values in the ranges

$$-\infty < \eta < 0 \qquad \text{and} \qquad -\infty < x^i < +\infty \; . \tag{2}$$

In these coordinates the invariant element is,

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = a^2 \eta_{\mu\nu} dx^\mu dx^\nu \; , \tag{3}$$

where $\eta_{\mu\nu}$ is the Lorentz metric, the scale factor is $a = -1/H\eta$ and H is the Hubble constant.

It is worth observing that our locally de Sitter geometry should be a good approximation for primordial inflation. This can be quantified in terms of the parameter ϵ which measures how nearly constant the Hubble parameter is. For a general scale factor, not necessarily de Sitter, we define ϵ as,

$$\epsilon \equiv -a^{-1} \frac{d}{d\eta} \Bigl(\frac{da^{-1}}{d\eta} \Bigr)^{-1} \; . \tag{4}$$

For de Sitter ($a = -1/H\eta$) the result is $\epsilon = 0$. If one assumes single scalar inflation then the current upper bound on the tensor-to-scalar ratio [3] implies $\epsilon < 0.014$ at the time, near the end of inflation, when the largest observable perturbations experienced horizon crossing [12]. Because ϵ is expected to have been even smaller at earlier times, the de Sitter approximation of $\epsilon = 0$ seems quite reasonable.

The MMC scalar contribution to the graviton self-energy is de Sitter invariant and can be expressed using the Sitter length function $y(x; x')$,

$$y(x; x') \equiv a a' H^2 \Bigl[\|\vec{x} - \vec{x}'\|^2 - (|\eta - \eta'| - i\epsilon)^2 \Bigr] \; . \tag{5}$$

Except for the factor of $i\epsilon$ (whose purpose is to enforce Feynman boundary conditions) the function $y(x; x')$ is closely related to the invariant length $\ell(x; x')$ from x^μ to x'^μ,

$$y(x; x') = 4 \sin^2\Bigl(\frac{1}{2} H \ell(x; x') \Bigr) \; . \tag{6}$$

With this de Sitter invariant quantity $y(x; x')$, we can form a convenient basis of de Sitter invariant bi-tensors. Note that because $y(x; x')$ is de Sitter invariant, so too are covariant derivatives of it. With the metrics $g_{\mu\nu}(x)$ and $g_{\mu\nu}(x')$, the first three derivatives of $y(x;x')$ furnish a convenient basis of de Sitter invariant bi-tensors [7],

$$\frac{\partial y(x;x')}{\partial x^\mu} = Ha\Bigl(y\delta^0_\mu + 2a'H\Delta x_\mu\Bigr) \; , \tag{7}$$

$$\frac{\partial y(x;x')}{\partial x'^\nu} = Ha'\Bigl(y\delta^0_\nu - 2aH\Delta x_\nu\Bigr) \; , \tag{8}$$

$$\frac{\partial^2 y(x;x')}{\partial x^\mu \partial x'^\nu} = H^2 aa'\Bigl(y\delta^0_\mu\delta^0_\nu + 2a'H\Delta x_\mu \delta^0_\nu - 2a\delta^0_\mu H\Delta x_\nu - 2\eta_{\mu\nu}\Bigr) \; . \tag{9}$$

Here and subsequently $\Delta x_\mu \equiv \eta_{\mu\nu}(x-x')^\nu$.

Acting covariant derivatives generates more basis tensors, for example [7],

$$\frac{D^2 y(x;x')}{Dx^\mu Dx^\nu} = H^2(2-y)g_{\mu\nu}(x) \; , \tag{10}$$

$$\frac{D^2 y(x;x')}{Dx'^\mu Dx'^\nu} = H^2(2-y)g_{\mu\nu}(x') \; . \tag{11}$$

The contraction of any pair of the basis tensors also produces more basis tensors [7],

$$g^{\mu\nu}(x)\frac{\partial y}{\partial x^\mu}\frac{\partial y}{\partial x^\nu} = H^2\Bigl(4y - y^2\Bigr) = g^{\mu\nu}(x')\frac{\partial y}{\partial x'^\mu}\frac{\partial y}{\partial x'^\nu} \; , \tag{12}$$

$$g^{\mu\nu}(x)\frac{\partial y}{\partial x^\nu}\frac{\partial^2 y}{\partial x^\mu \partial x'^\sigma} = H^2(2-y)\frac{\partial y}{\partial x'^\sigma} \; , \tag{13}$$

$$g^{\rho\sigma}(x')\frac{\partial y}{\partial x'^\sigma}\frac{\partial^2 y}{\partial x^\mu \partial x'^\rho} = H^2(2-y)\frac{\partial y}{\partial x^\mu} \; , \tag{14}$$

$$g^{\mu\nu}(x)\frac{\partial^2 y}{\partial x^\mu \partial x'^\rho}\frac{\partial^2 y}{\partial x^\nu \partial x'^\sigma} = 4H^4 g_{\rho\sigma}(x') - H^2\frac{\partial y}{\partial x'^\rho}\frac{\partial y}{\partial x'^\sigma} \; , \tag{15}$$

$$g^{\rho\sigma}(x')\frac{\partial^2 y}{\partial x^\mu \partial x'^\rho}\frac{\partial^2 y}{\partial x^\nu \partial x'^\sigma} = 4H^4 g_{\mu\nu}(x) - H^2\frac{\partial y}{\partial x^\mu}\frac{\partial y}{\partial x^\nu} \; . \tag{16}$$

Our basis tensors are naturally covariant, but their indices can of course be raised using the metric at the appropriate point. To save space in writing this out we define the basis tensors with raised indices as differentiation with respect to "covariant" coordinates,

$$\frac{\partial y}{\partial x_\mu} \equiv g^{\mu\nu}(x)\frac{\partial y}{\partial x^\nu} \; , \tag{17}$$

$$\frac{\partial y}{\partial x'_\rho} \equiv g^{\rho\sigma}(x')\frac{\partial y}{\partial x'^\sigma} \; , \tag{18}$$

$$\frac{\partial^2 y}{\partial x_\mu \partial x'_\rho} \equiv g^{\mu\nu}(x)g^{\rho\sigma}(x')\frac{\partial^2 y}{\partial x^\nu \partial x'^\sigma} \; . \tag{19}$$

2.2 The Graviton Self-Energy

It is simple to infer the unrenormalized one loop scalar contribution to the graviton self-energy from the correlator of two stress tensors at noncoincident points [13]. However, an enormous amount of labor is necessary to extract enough derivative operators to segregate the ultraviolet divergences onto local counterterms, leaving a result which is integrable in the $D=4$ effective field equations. This fully renormalized result takes the form [11],

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') = \sqrt{-g(x)}\,\mathcal{P}^{\mu\nu}(x)\sqrt{-g(x')}\,\mathcal{P}^{\rho\sigma}(x')\Big\{\mathcal{F}_0(y)\Big\} \\ +\sqrt{-g(x)}\,\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)\sqrt{-g(x')}\,\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x')\Big\{\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Big(\frac{D-2}{D-3}\Big)\mathcal{F}_2(y)\Big\}, \tag{20}$$

where the bi-tensor $\mathcal{T}^{\alpha\kappa}$ is,

$$\mathcal{T}^{\alpha\kappa}(x;x') \equiv -\frac{1}{2H^2}\,\frac{\partial^2 y(x;x')}{\partial x_\alpha \partial x'_\kappa}\ . \tag{21}$$

The other quantities in this expression are the spin zero and spin two projectors, $\mathcal{P}^{\mu\nu}$ and $\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}$, respectively, and their associated structure functions, $\mathcal{F}_0(y)$ and $\mathcal{F}_2(y)$. We shall devote a paragraph to each.

The two projectors come from expanding the scalar and Weyl curvatures around de Sitter background,

$$R - D(D-1)H^2 \;\equiv\; \mathcal{P}^{\mu\nu}\kappa h_{\mu\nu} + O(\kappa^2 h^2)\ , \tag{22}$$

$$C_{\alpha\beta\gamma\delta} \;\equiv\; \mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}\kappa h_{\mu\nu} + O(\kappa^2 h^2)\ . \tag{23}$$

From (22) we have,

$$\mathcal{P}^{\mu\nu} = D^\mu D^\nu - g^{\mu\nu}\Big[D^2 + (D-1)H^2\Big]\ , \tag{24}$$

where D^μ is the covariant derivative operator in de Sitter background. The more difficult expansion of the Weyl tensor gives,

$$\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta} = \mathcal{D}^{\mu\nu}_{\alpha\beta\gamma\delta} + \frac{1}{D-2}\Big[g_{\alpha\delta}\mathcal{D}^{\mu\nu}_{\beta\gamma} - g_{\beta\delta}\mathcal{D}^{\mu\nu}_{\alpha\gamma} - g_{\alpha\gamma}\mathcal{D}^{\mu\nu}_{\beta\delta} + g_{\beta\gamma}\mathcal{D}^{\mu\nu}_{\alpha\delta}\Big] \\ + \frac{1}{(D-1)(D-2)}\Big[g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\Big]\mathcal{D}^{\mu\nu}\ , \tag{25}$$

where we define,

$$\mathcal{D}^{\mu\nu}_{\alpha\beta\gamma\delta} \;\equiv\; \frac{1}{2}\Big[\delta^{(\mu}_\alpha\delta^{\nu)}_\delta D_\gamma D_\beta - \delta^{(\mu}_\beta\delta^{\nu)}_\delta D_\gamma D_\alpha - \delta^{(\mu}_\alpha\delta^{\nu)}_\gamma D_\delta D_\beta + \delta^{(\mu}_\beta\delta^{\nu)}_\gamma D_\delta D_\alpha\Big]\ , \tag{26}$$

$$\mathcal{D}^{\mu\nu}_{\beta\delta} \;\equiv\; g^{\alpha\gamma}\mathcal{D}^{\mu\nu}_{\alpha\beta\gamma\delta} = \frac{1}{2}\Big[\delta^{(\mu}_\delta D^{\nu)}D_\beta - \delta^{(\mu}_\beta\delta^{\nu)}_\delta D^2 - g^{\mu\nu}D_\delta D_\beta + \delta^{(\mu}_\beta D_\delta D^{\nu)}\Big]\ , \tag{27}$$

$$\mathcal{D}^{\mu\nu} \;\equiv\; g^{\alpha\gamma}g^{\beta\delta}\mathcal{D}^{\mu\nu}_{\alpha\beta\gamma\delta} = D^{(\mu}D^{\nu)} - g^{\mu\nu}D^2\ . \tag{28}$$

The spin zero structure function is,

$$\mathcal{F}_0 = \frac{\kappa^2 H^4}{(4\pi)^4}\Biggl\{\frac{\Box}{H^2}\Bigl[\frac{1}{72}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr)\Bigr]-\frac{1}{12}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr)+\frac{1}{72}\times\frac{4}{y}+\frac{1}{6}\ln^2\Bigl(\frac{y}{4}\Bigr)$$
$$+\frac{1}{45}\times\frac{4}{4-y}\ln(\frac{y}{4})-\frac{1}{45}\ln(\frac{y}{4})+\frac{43}{216}\times\frac{4}{4-y}-\frac{5}{6}\times\frac{y}{4}\ln(1-\frac{y}{4})$$
$$+\frac{7}{90}\times\frac{4}{y}\ln(1-\frac{y}{4})-\frac{1}{20}\ln(1-\frac{y}{4})-\frac{7(12\pi^2+265)}{540}\times\frac{y}{4}$$
$$+\frac{84\pi^2-131}{1080}-\frac{1}{3}\times\frac{y}{4}\ln^2\Bigl(\frac{y}{4}\Bigr)+\frac{4}{9}\times\frac{y}{4}\ln\Bigl(\frac{y}{4}\Bigr)$$
$$-\frac{1}{30}(2-y)\Bigl[7\mathrm{Li}_2(1-\frac{y}{4})-2\mathrm{Li}_2(\frac{y}{4})+5\ln(1-\frac{y}{4})\ln(\frac{y}{4})\Bigr]\Biggr\}\,. \tag{29}$$

Here $\mathrm{Li}_2(z)$ is the dilogarithm function,

$$\mathrm{Li}_2(z)\equiv-\int_0^z dt\,\frac{\ln(1-t)}{t}=\sum_{k=1}^{\infty}\frac{z^k}{k^2}\,. \tag{30}$$

The same function also appears in the spin two structure function,

$$\mathcal{F}_2 = \frac{\kappa^2 H^4}{(4\pi)^4}\Biggl\{\frac{\Box}{H^2}\Bigl[\frac{1}{240}\times\frac{4}{y}\ln((\frac{y}{4}\Bigr)\Bigr]+\frac{3}{40}\times\frac{4}{y}\ln\Bigl(\frac{y}{4}\Bigr)-\frac{11}{48}\times\frac{4}{y}+\frac{1}{4}\ln^2\Bigl(\frac{y}{4}\Bigr)$$
$$-\frac{119}{60}\ln\Bigl(\frac{y}{4}\Bigr)+\frac{4096}{(4y-y^2-8)^4}\Biggl[\Biggl[-\frac{47}{15}\Bigl(\frac{y}{4}\Bigr)^8+\frac{141}{10}\Bigl(\frac{y}{4}\Bigr)^7$$
$$-\frac{2471}{90}\Bigl(\frac{y}{4}\Bigr)^6+\frac{34523}{720}\Bigl(\frac{y}{4}\Bigr)^5-\frac{132749}{1440}\Bigl(\frac{y}{4}\Bigr)^4+\frac{38927}{320}\Bigl(\frac{y}{4}\Bigr)^3$$
$$-\frac{10607}{120}\Bigl(\frac{y}{4}\Bigr)^2+\frac{22399}{720}\Bigl(\frac{y}{4}\Bigr)-\frac{3779}{960}\Biggr]\frac{4}{4-y}+\Bigl[\frac{193}{30}\Bigl(\frac{y}{4}\Bigr)^4-\frac{131}{10}\Bigl(\frac{y}{4}\Bigr)^3$$
$$+\frac{7}{20}\Bigl(\frac{y}{4}\Bigr)^2+\frac{379}{60}\Bigl(\frac{y}{4}\Bigr)-\frac{193}{120}\Bigr]\ln(2-\frac{y}{2})+\Bigl[-\frac{14}{15}\Bigl(\frac{y}{4}\Bigr)^5-\frac{1}{5}\Bigl(\frac{y}{4}\Bigr)^4$$
$$+\frac{19}{2}\Bigl(\frac{y}{4}\Bigr)^3-\frac{889}{60}\Bigl(\frac{y}{4}\Bigr)^2+\frac{143}{20}\Bigl(\frac{y}{4}\Bigr)-\frac{13}{20}-\frac{7}{60}\Bigl(\frac{4}{y}\Bigr)\Bigr]\ln(1-\frac{y}{4})$$
$$+\Bigl[-\frac{476}{15}\Bigl(\frac{y}{4}\Bigr)^9+160\Bigl(\frac{y}{4}\Bigr)^8-\frac{5812}{15}\Bigl(\frac{y}{4}\Bigr)^7+\frac{8794}{15}\Bigl(\frac{y}{4}\Bigr)^6$$
$$-\frac{18271}{30}\Bigl(\frac{y}{4}\Bigr)^5+\frac{54499}{120}\Bigl(\frac{y}{4}\Bigr)^4-\frac{59219}{240}\Bigl(\frac{y}{4}\Bigr)^3+\frac{1917}{20}\Bigl(\frac{y}{4}\Bigr)^2$$
$$-\frac{1951}{80}\Bigl(\frac{y}{4}\Bigr)+\frac{367}{120}\Bigr]\frac{4}{4-y}\ln(\frac{y}{4})+\Bigl[4\Bigl(\frac{y}{4}\Bigr)^7-12\Bigl(\frac{y}{4}\Bigr)^6+20\Bigl(\frac{y}{4}\Bigr)^5$$
$$-20\Bigl(\frac{y}{4}\Bigr)^4+15\Bigl(\frac{y}{4}\Bigr)^3-7\Bigl(\frac{y}{4}\Bigr)^2+\Bigl(\frac{y}{4}\Bigr)\Bigr]\frac{4-y}{4}\ln^2(\frac{y}{4})$$
$$+\Bigl[\frac{367}{30}\Bigl(\frac{y}{4}\Bigr)^4-\frac{4121}{120}\Bigl(\frac{y}{4}\Bigr)^3+\frac{237}{16}\Bigl(\frac{y}{4}\Bigr)^2+\frac{1751}{240}\Bigl(\frac{y}{4}\Bigr)-\frac{367}{120}\Bigr]\ln(\frac{y}{2})$$

$$+\frac{1}{64}(y^2-8)\Big[4(2-y)-(4y-y^2)\Big]\Big[\frac{1}{5}\mathrm{Li}_2(1-\frac{y}{4})+\frac{7}{10}\mathrm{Li}_2(\frac{y}{4})\Big]\Big]\Big\} \; . \tag{31}$$

Note that these results were derived for Bunch-Davies vacuum, which corresponds to a state which is minimum energy in the distant past [11]. This is the standard choice for inflationary perturbations, and the choice we must make in order to compute quantum corrections to the usual tree order results.

2.3 The Schwinger-Keldysh Effective Field Equations

Because the graviton self-energy is the 1PI graviton 2-point function, it gives the quantum correction to the linearized Einstein equation,

$$\sqrt{-g}\,\mathcal{D}^{\mu\nu\rho\sigma}h_{\rho\sigma}(x)-\int\! d^4x'\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x')h_{\rho\sigma}(x')=\frac{1}{2}\kappa\sqrt{-g}\,T^{\mu\nu}_{\rm lin}(x) \; , \tag{32}$$

Here $\mathcal{D}^{\mu\nu\rho\sigma}$ is the Lichnerowicz operator, specialized to de Sitter background

$$\begin{aligned}\mathcal{D}^{\mu\nu\rho\sigma}\equiv D^{(\rho}g^{\sigma)(\mu}D^{\nu)}-\frac{1}{2}\Big[g^{\rho\sigma}D^{\mu}D^{\nu}+g^{\mu\nu}D^{\rho}D^{\sigma}\Big] \\ +\frac{1}{2}\Big[g^{\mu\nu}g^{\rho\sigma}-g^{\mu(\rho}g^{\sigma)\nu}\Big]D^2+(D-1)\Big[\frac{1}{2}g^{\mu\nu}g^{\rho\sigma}-g^{\mu(\rho}g^{\sigma)\nu}\Big]H^2 \; ,\end{aligned} \tag{33}$$

and D^μ is the covariant derivative operator in the background geometry.

Two embarrassments would confront us were we to solve equation (32) using the self-energy of the previous sub-section:

- *Causality violation* — the field equation at x^μ involves the field at points x'^μ outside the past light-cone of x^μ; and
- *Reality violation* — the quantum-induced graviton field would acquire an imaginary part due to the nonzero imaginary part of the in-out self-energy.

Both features are the result of taking the in-out matrix element of the operator field equations. This isn't wrong, in fact it is exactly the right thing to do in the study of asymptotic scattering problems. However, there is no S-matrix in de Sitter space [14], so the more natural problem is to release the universe in a prepared initial state and then watch it evolve.

The correct effective field equations for releasing the universe in a prepared initial state are derived by taking the expectation value of the operator field equations in that state. They are given by the Schwinger-Keldysh formalism [15] which, for our problem, amounts to replacing the in-out self-energy in (32) by the sum of two of the four Schwinger-Keldysh self-energies,

$$\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x')\longrightarrow\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{++}(x;x')+\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{+-}(x;x') \; . \tag{34}$$

At the one loop order we are working $\left[{}^{\mu\nu}\Sigma^{\rho\sigma}\right]_{++}(x;x')$ agrees exactly with the in-out result given in the previous sub-section. To get $\left[{}^{\mu\nu}\Sigma^{\rho\sigma}\right]_{+-}(x;x')$, at this order, one simply adds a minus sign and replaces the de Sitter length function $y(x;x')$ everywhere with,

$$y(x;x') \longrightarrow y_{+-}(x;x') \equiv H^2 a(\eta)a(\eta')\Big[\|\vec{x}-\vec{x}'\|^2 - (\eta-\eta'+i\epsilon)^2\Big] \; . \tag{35}$$

It will be seen that the $++$ and $+-$ self-energies cancel unless the point x'^μ is on or inside the past light-cone of x^μ. That makes the effective field equation (32) causal. When x'^μ is on or inside the past light-cone of x^μ the $+-$ self-energy is the complex conjugate of the $++$ one, which makes the effective field equation (32) real. This also effects a great simplification in the structure functions because only those terms with branch cuts in y can make nonzero contributions, for example,

$$\ln(y_{++}) - \ln(y_{+-}) = 2\pi i\theta\Big(\eta-\eta' - \|\vec{x}-\vec{x}'\|\Big) \; . \tag{36}$$

2.4 Perturbative Solution

Because we only know the self-energy at one loop order, all we can do is to solve (32) perturbatively by expanding the graviton field and the self-energy in powers of κ^2,

$$h_{\mu\nu}(x) = h^{(0)}_{\mu\nu}(x) + \kappa^2 h^{(1)}_{\mu\nu}(x) + O(\kappa^4) \; . \tag{37}$$

Of course $h^{(0)}_{\mu\nu}(x)$ obeys the classical, linearized Einstein equation. Given this solution, the corresponding one loop correction is defined by the equation,

$$\sqrt{-g(x)}\,\mathcal{D}^{\mu\nu\rho\sigma}\kappa^2 h^{(1)}_{\rho\sigma}(x) = \int\! d^4x' \left[{}^{\mu\nu}\Sigma^{\rho\sigma}\right](x;x') h^{(0)}_{\rho\sigma}(x') \; . \tag{38}$$

The classical solution for a dynamical graviton of wave vector $\vec{k}$ is [16],

$$h^{(0)}_{\rho\sigma}(x) = \epsilon_{\rho\sigma}(\vec{k}) u(\eta,k) e^{i\vec{k}\cdot\vec{x}} \; , \tag{39}$$

where the tree order mode function is,

$$u(\eta,k) = \frac{H}{\sqrt{2k^3}}\Big[1 - \frac{ik}{Ha}\Big]\exp\Big[\frac{ik}{Ha}\Big] \; , \tag{40}$$

and the polarization tensor obeys all the same relations as in flat space,

$$0 = \epsilon_{0\mu} = k_i\epsilon_{ij} = \epsilon_{jj} \quad \text{and} \quad \epsilon_{ij}\epsilon^*_{ij} = 1 \; . \tag{41}$$

3 Computing the One Loop Source

The point of this section is to evaluate the one loop source term on the right hand side of equation (38) for a dynamical graviton (39-41). We begin by drawing inspiration from what happens in the flat space limit. Our de Sitter analysis commences by partially integrating the projectors. This results in considerable simplification but the plethora of indices is still problematic. To effect further simplification we extract and partially integrate another d'Alembertian, whereupon the x^μ projector can be acted on the residual structure function to eliminate four contractions. At this point we digress to derive some important identities concerning covariant derivatives of the Weyl tensor. The final reduction reveals zero net result.

3.1 The Flat Space Limit

The one loop contribution to the graviton self-energy from MMC scalars in a flat background was first computed by 't Hooft and Veltman in 1974 [17]. When renormalized and expressed in position space using the Schwinger-Keldysh formalism the result takes the form [18],

$$\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm flat}\Big](x;x') = \Pi^{\mu\nu}\Pi^{\rho\sigma}F_0(\Delta x^2) + \Big[\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{1}{3}\Pi^{\mu\nu}\Pi^{\rho\sigma}\Big]F_2(\Delta x^2) \; . \tag{42}$$

Here $\Pi^{\mu\nu} \equiv \partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2$ and the two structure functions are,

$$F_0(\Delta x^2) = \frac{i\kappa^2}{(4\pi)^4}\frac{\partial^2}{9}\Bigg[\frac{\ln(\mu^2\Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2\Delta x^2_{+-})}{\Delta x^2_{+-}}\Bigg] \; , \tag{43}$$

$$F_2(\Delta x^2) = \frac{i\kappa^2}{(4\pi)^4}\frac{\partial^2}{60}\Bigg[\frac{\ln(\mu^2\Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2\Delta x^2_{+-})}{\Delta x^2_{+-}}\Bigg] \tag{44}$$

The two coordinate intervals are,

$$\Delta x^2_{++} \equiv \Big\Vert\vec{x}-\vec{x}'\Big\Vert^2 - \Big(|x^0-x'^0|-i\epsilon\Big)^2 \; , \tag{45}$$

$$\Delta x^2_{+-} \equiv \Big\Vert\vec{x}-\vec{x}'\Big\Vert^2 - \Big(x^0-x'^0+i\epsilon\Big)^2 \; . \tag{46}$$

Of course this same form follows from taking the flat space limit of the de Sitter result summarized in the previous section.

In flat space, the mode function for a plane wave graviton with wave vector $\vec{k}$ is,

$$h^{\rm flat}_{\mu\nu}(x) = \epsilon_{\rho\sigma}(\vec{k})\frac{1}{\sqrt{2k}}\,e^{-ikx^0+i\vec{k}\cdot\vec{x}} \; . \tag{47}$$

The one loop correction to this (from MMC scalars) is sourced by,

$$\Big({\rm Source}\Big)^{\mu\nu}(x) = \int dx^4x'\,\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm flat}\Big](x;x')h^{\rm flat}_{\rho\sigma}(x') \; . \tag{48}$$

It might seem natural to extract the various derivatives with respect to x^μ from the integration, for example,

$$\int d^4x' \, \Pi^{\mu\nu} \Pi^{\rho\sigma} F_0(\Delta x^2) \times h^{\rm flat}_{\rho\sigma}(x') = \frac{i\kappa^2}{(4\pi)^4} \Pi^{\mu\nu}\Pi^{\rho\sigma} \frac{\partial^2}{9} \int d^4x' \Big[\frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \Big] \times h^{\rm flat}_{\rho\sigma}(x') \; . \qquad (49)$$

That would reduce the source (48) to a tedious set of integrations, followed by some equally tedious differentiations.

The point of this sub-section is that a more efficient strategy is to first convert all the x^μ derivatives to x'^μ derivatives — which can be done because they act on functions of Δx^2. Then ignore surface terms and partially integrate the x'^μ derivatives to act upon $h^{\rm flat}_{\rho\sigma}(x')$. For example, doing this for the spin zero contribution (49) gives,

$$\int d^4x' \, \Pi^{\mu\nu} \Pi^{\rho\sigma} F_0(\Delta x^2) \times h^{\rm flat}_{\rho\sigma}(x') \longrightarrow \frac{i\kappa^2}{(4\pi)^4} \int d^4x' \Big[\frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \Big] \times \frac{\partial'^2}{9} \Pi'^{\mu\nu} \Pi'^{\rho\sigma} h^{\rm flat}_{\rho\sigma}(x') \; . \qquad (50)$$

Because the graviton mode function is both transverse and traceless, we have $\Pi'^{\rho\sigma} h^{\rm flat}_{\rho\sigma}(x') = 0$. The spin two contribution is only a little more complicated,

$$\int d^4x' \Big[\Pi^{\mu(\rho} \Pi^{\sigma)\nu} - \frac13 \Pi^{\mu\nu} \Pi^{\rho\sigma} \Big] F_2(\Delta x^2) \times h^{\rm flat}_{\rho\sigma}(x') \longrightarrow \frac{i\kappa^2}{(4\pi)^4} \int d^4x' \Big[\frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \Big] \times \frac{\partial'^6}{60} h^{\mu\nu}_{\rm flat}(x') \; . \qquad (51)$$

This also vanishes because $\partial'^2 h^{\rm flat}_{\rho\sigma}(x') = 0$.

In expressions (50) and (51) we have employed a rightarrow, rather than an equals sign, because the surface terms produce by partial integration were ignored. There are no surface terms at spatial infinity in the Schwinger-Keldysh formalism because the $++$ and $+-$ terms cancel for spacelike separation. The $++$ and $+-$ contributions also cancel when $x'^0 > x^0$, so there are no future surface terms. However, there are nonzero contributions from the initial value surface.[1] We assume that all such contributions are absorbed into perturbative corrections to the initial state, such as has recently been worked out for a MMC scalar with quartic self-interaction [20].

3.2 Partial Integration

We now start to evaluate the one loop source term (38) for a dynamical graviton,

$$\int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma} \Big](x;x') h^{(0)}_{\rho\sigma}(x')$$

[1]For a two loop example, see [19].

$$= i\int d^4x' \sqrt{-g(x)}\, \mathcal{P}^{\mu\nu}(x) \sqrt{-g(x')}\, \mathcal{P}^{\rho\sigma}(x') \Bigl\{\mathcal{F}_0\Bigr\} h^{(0)}_{\rho\sigma}(x')$$

$$+ 2i\int d^4x' \sqrt{-g(x)}\, \mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x) \sqrt{-g(x')}\, \mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x') \Biggl\{\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\mathcal{F}_2\Biggr\} h^{(0)}_{\rho\sigma}(x') \; . \qquad (52)$$

In this expression and henceforth we simply write "$\mathcal{F}_0$" and "$\mathcal{F}_2$" to stand for the full Schwinger-Keldysh expressions,

$$\mathcal{F}_0 \equiv \mathcal{F}_0(y_{++}) - \mathcal{F}_0(y_{+-}) \qquad , \qquad \mathcal{F}_2 \equiv \mathcal{F}_2(y_{++}) - \mathcal{F}_2(y_{+-}) \; . \qquad (53)$$

The integral (52) can be simplified in two steps. First, the projectors $\mathcal{P}^{\mu\nu}(x)$ and $\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)$, which act on a function of x^μ, can be pulled outside the integration over x'^μ. Second, the projectors $\mathcal{P}^{\rho\sigma}(x')$ and $\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x')$, which act on x'^μ, can be partially integrated to act on the graviton wave function $h^{(0)}_{\rho\sigma}(x')$. After these two steps, the integral (52) becomes,

$$\int d^4x' \Bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}\Bigr](x;x') h^{(0)}_{\rho\sigma}(x')$$

$$= i\sqrt{-g(x)}\, \mathcal{P}^{\mu\nu}(x) \int d^4x' \sqrt{-g(x')}\, \mathcal{F}_0 \Bigl\{\mathcal{P}^{\rho\sigma}(x') h^{(0)}_{\rho\sigma}(x')\Bigr\}$$

$$+ 2i\sqrt{-g(x)}\, \mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x) \int d^4x' \sqrt{-g(x')}\, \mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\mathcal{F}_2 \Biggl\{\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x') h^{(0)}_{\rho\sigma}(x')\Biggr\}. \qquad (54)$$

Note that the spin zero term drops out due to the tranversality and tracelessness of the dynamical graviton, $h^{(0)}_{\rho\sigma}$:

$$\mathcal{P}^{\rho\sigma} h^{(0)}_{\rho\sigma} = \Bigl\{D^\rho D^\sigma - \Bigl[D^2 + (D-1)H^2\Bigr] g^{\rho\sigma}\Bigr\} h^{(0)}_{\rho\sigma} = 0 \; . \qquad (55)$$

Thus we only have the spin two term, which gives the linearized Weyl tensor,

$$\mathcal{P}^{\rho\sigma}_{\kappa\lambda\theta\phi}(x') h^{(0)}_{\rho\sigma}(x') = \delta C_{\kappa\lambda\theta\phi}(x') \; . \qquad (56)$$

The one loop source term then reduces to the integral,

$$\int d^4x' \Bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}\Bigr](x;x') h^{(0)}_{\rho\sigma}(x')$$

$$= 2i\sqrt{-g(x)}\, \mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x) \int d^4x' \sqrt{-g(x')}\, \mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\mathcal{F}_2 \delta C_{\kappa\lambda\theta\phi}(x') \; . \qquad (57)$$

3.3 Extracting Another d'Alembertian

A challenge to evaluating expression (57) is the complicated tensor structure of the external projector $\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)$ acting on the internal factors of $\mathcal{T}^{\alpha\kappa} \cdots \mathcal{F}_2$. Recall from the flat space limit that all of this was converted to derivatives with respect to x'^μ and then partially integrated onto the graviton wave function to give zero. To follow this on de Sitter we must make the structure function more convergent by extracting a factor of $\square'$ and then partially integrating it onto the graviton wave function. After this the external projector can be acted, which eliminates four indices, and a final further partial integration can be performed.

The first step is extracting the extra d'Alembertian,

$$\mathcal{F}_2 = \frac{\square'}{H^2}\widehat{\mathcal{F}}_2 \; . \tag{58}$$

We next commute the $\square'$ through the factor of $\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}$:

$$\begin{aligned}\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\frac{\square'}{H^2}\widehat{\mathcal{F}}_2 &= \Bigl(\frac{\square'}{H^2}+4\Bigr)\Bigl[\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\widehat{\mathcal{F}}_2\Bigr] \\ & -\frac{1}{H^2}\widehat{\mathcal{F}}_2'\Biggl\{\frac{\partial y}{\partial x_\alpha}\frac{\partial y}{\partial x'_\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}+\cdots+\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\frac{\partial y}{\partial x_\delta}\frac{\partial y}{\partial x'_\phi}\Biggr\} \\ & -\frac{1}{2H^2}\widehat{\mathcal{F}}_2\Biggl\{g^{\alpha\beta}\frac{\partial y}{\partial x'_\kappa}\frac{\partial y}{\partial x'_\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}+g^{\alpha\gamma}\frac{\partial y}{\partial x'_\kappa}\frac{\partial y}{\partial x'_\theta}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\delta\phi} \\ & \qquad +g^{\alpha\delta}\frac{\partial y}{\partial x'_\kappa}\frac{\partial y}{\partial x'_\phi}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}+g^{\beta\gamma}\frac{\partial y}{\partial x'_\lambda}\frac{\partial y}{\partial x'_\theta}\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\delta\phi} \\ & \qquad\qquad +g^{\beta\delta}\frac{\partial y}{\partial x'_\lambda}\frac{\partial y}{\partial x'_\phi}\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\gamma\theta}+g^{\gamma\delta}\frac{\partial y}{\partial x'_\theta}\frac{\partial y}{\partial x'_\phi}\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\Biggr\} \; .\end{aligned} \tag{59}$$

Exploiting the tracelessness of the Weyl tensor on any two indices, and its antisymmetry on the first two and last two indices, gives,

$$\begin{aligned}P^{\mu\nu}_{\alpha\beta\gamma\delta}\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\frac{\square'}{H^2}\widehat{\mathcal{F}}_2\delta C_{\kappa\lambda\theta\phi} &= P^{\mu\nu}_{\alpha\beta\gamma\delta}\frac{\square'}{H^2}\Bigl[\widehat{\mathcal{F}}_2\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Bigr]\delta C_{\kappa\lambda\theta\phi} \\ = P^{\mu\nu}_{\alpha\beta\gamma\delta}\Biggl\{4\widehat{\mathcal{F}}_2\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi} &- \frac{4}{H^2}\widehat{\mathcal{F}}_2'\frac{\partial y}{\partial x_\alpha}\frac{\partial y}{\partial x'_\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Biggr\}\delta C_{\kappa\lambda\theta\phi} \; .\end{aligned} \tag{60}$$

For the first term of (60) we can partially integrate the $\square'$ onto the linearized Weyl tensor. Then the one loop source term becomes

$$\begin{aligned}&\int d^4x' \Bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}\Bigr](x;x')h^{(0)}_{\rho\sigma}(x') \\ &= 2i\sqrt{-g(x)}\,\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)\int d^4x'\sqrt{-g(x')}\,\Biggl\{\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\widehat{\mathcal{F}}_2\frac{\square'}{H^2}\delta C_{\kappa\lambda\theta\phi}(x') \\ &\quad +\Bigl[4\widehat{\mathcal{F}}_2\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}-\frac{4}{H^2}\widehat{\mathcal{F}}_2'\frac{\partial y}{\partial x_\alpha}\frac{\partial y}{\partial x'_\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Bigr]\delta C_{\kappa\lambda\theta\phi}(x')\Biggr\} \; .\end{aligned} \tag{61}$$

This sets the stage for acting the outer projector.

3.4 Derivatives of the Weyl Tensor

At this point it is useful to make a short digression on the covariant derivatives of the Weyl tensor. In this sub-section we use $g_{\mu\nu}$ for the full metric, not the de Sitter background. All curvatures are similarly for the full metric.

The Bianchi identity tells us,

$$D_\epsilon R_{\alpha\beta\gamma\delta} + D_\gamma R_{\alpha\beta\delta\epsilon} + D_\delta R_{\alpha\beta\epsilon\gamma} = 0 \; . \tag{62}$$

If the stress-energy vanishes, all solutions to the Einstein equation obey,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -3H^2 g_{\mu\nu} \qquad \Longrightarrow \qquad R_{\mu\nu} = 3H^2 g_{\mu\nu} \; . \tag{63}$$

In $D = 3+1$ the Weyl tensor can be expressed in terms of the other curvatures as,

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{2}\Bigl(g_{\alpha\gamma} R_{\beta\delta} - g_{\gamma\beta} R_{\delta\alpha} + g_{\beta\delta} R_{\alpha\gamma} - g_{\delta\alpha} R_{\gamma\beta}\Bigr) + \frac{1}{6}\Bigl(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}\Bigr) R \; . \tag{64}$$

Now note that the covariant derivative of the metric vanishes. Substituting (63) in (64) implies,

$$D_\epsilon C_{\alpha\beta\gamma\delta} = D_\epsilon R_{\alpha\beta\gamma\delta} \; . \tag{65}$$

Combining this relation into (62) gives,

$$D_\epsilon C_{\alpha\beta\gamma\delta} + D_\gamma C_{\alpha\beta\delta\epsilon} + D_\delta C_{\alpha\beta\epsilon\gamma} = 0 \; . \tag{66}$$

Our first key identity derives from contracting α into ϵ, and exploiting the tracelessness of the Weyl tensor,

$$D^\alpha C_{\alpha\beta\gamma\delta} = 0 \; . \tag{67}$$

Our second identity derives from contracting D^ϵ into relation (66), commuting derivatives and then using relation (67),

$$\begin{aligned} \square C_{\alpha\beta\gamma\delta} &= -D_\rho D_\gamma C_{\alpha\beta\delta}{}^\rho + D_\rho D_\delta C_{\alpha\beta\gamma}{}^\rho \; , & (68)\\ &= 6H^2 C_{\alpha\beta\gamma\delta} - R^\rho{}_{\alpha\gamma}{}^\sigma C_{\rho\beta\delta\sigma} + R^\rho{}_{\gamma\beta}{}^\sigma C_{\rho\delta\alpha\sigma} \\ &\quad - R^\rho{}_{\beta\delta}{}^\sigma C_{\rho\alpha\gamma\sigma} + R^\rho{}_{\delta\alpha}{}^\sigma C_{\rho\gamma\beta\sigma} - R^{\rho\sigma}{}_{\gamma\delta} C_{\alpha\beta\rho\sigma} \; . & (69) \end{aligned}$$

Relations (67) and (69) hold, to all orders in the graviton field, for any solution to the source-free Einstein equations. Taking the first order in the graviton field amounts to just replacing the full Weyl tensor by the linearized Weyl $\delta C_{\alpha\beta\gamma\delta}$ we have been using, replacing the full covariant derivative operators by the covariant derivatives in de Sitter background and replacing the full Riemann tensor by its de Sitter limit. When these things are done the two identities become,

$$\begin{aligned} D^\alpha \delta C_{\alpha\beta\gamma\delta} &= 0 + O(h^2) \; , & (70)\\ \square \delta C_{\alpha\beta\gamma\delta} &= 6H^2 \delta C_{\alpha\beta\gamma\delta} + O(h^2) \; . & (71) \end{aligned}$$

Note also that if the stress-energy had been nonzero the right hand sides of relations (70) and (71) would have contained simple combinations of derivatives of the stress tensor.

3.5 The Final Reduction

We are now ready to act the outer projector on the remaining terms,

$$\int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x')h^{(0)}_{\rho\sigma}(x') = 2i\sqrt{-g(x)} \int d^4x'\sqrt{-g(x')}\, \delta C_{\kappa\lambda\theta\phi}(x')$$
$$\Big\{\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)\Big[10\widehat{\mathcal{F}}_2\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi} - \frac{4}{H^2}\widehat{\mathcal{F}}'_2\frac{\partial y}{\partial x_\alpha}\frac{\partial y}{\partial x'_\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Big]\Big\} \; . \tag{72}$$

The second line of this expression is quite complicated by itself, but it is greatly simplified when contracted into the linearized Weyl tensor,

$$\delta C_{\kappa\lambda\theta\phi}(x')\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)\Big[10\widehat{\mathcal{F}}_2\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi} - \frac{4}{H^2}\widehat{\mathcal{F}}'_2\frac{\partial y}{\partial x_\alpha}\frac{\partial y}{\partial x'_\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Big]$$
$$= \delta C_{\kappa\lambda\theta\phi}(x')\Big\{\frac{\partial y}{\partial x'_\kappa}\frac{\partial y}{\partial x'_\theta}\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\phi}f_1(y) + \frac{\partial y}{\partial x'_\kappa}\frac{\partial y}{\partial x'_\phi}\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\theta}f_2(y)$$
$$+\frac{\partial y}{\partial x'_\lambda}\frac{\partial y}{\partial x'_\theta}\mathcal{T}^{\kappa(\mu}\mathcal{T}^{\nu)\phi}f_3(y) + \frac{\partial y}{\partial x'_\lambda}\frac{\partial y}{\partial x'_\phi}\mathcal{T}^{\kappa(\mu}\mathcal{T}^{\nu)\theta}f_4(y)\Big\} \; . \tag{73}$$

Here the functions $f_i(y)$ are,

$$f_1 = -125\widehat{\mathcal{F}}_2 + 115(2-y)\widehat{\mathcal{F}}'_2 - (68 - 116y + 29y^2)\widehat{\mathcal{F}}''_2 - 2(2-y)(4y-y^2)\widehat{\mathcal{F}}'''_2$$
$$f_2 = -\frac{75}{2}\widehat{\mathcal{F}}_2 + \frac{69}{2}(2-y)\widehat{\mathcal{F}}'_2 - (28 - 44y + 11y^2)\widehat{\mathcal{F}}''_2 - (2-y)(4y-y^2)\widehat{\mathcal{F}}'''_2$$
$$f_3 = -\frac{85}{2}\widehat{\mathcal{F}}_2 + \frac{15}{2}(2-y)\widehat{\mathcal{F}}'_2$$
$$f_4 = -5\widehat{\mathcal{F}}_2 - 13(2-y)\widehat{\mathcal{F}}'_2 - \frac{5}{2}(4y-y^2)\widehat{\mathcal{F}}''_2 \tag{74}$$

Changing the dummy indices in (73) gives,

$$\delta C_{\kappa\lambda\theta\phi}(x')\mathcal{P}^{\mu\nu}_{\alpha\beta\gamma\delta}(x)\Big[10\widehat{\mathcal{F}}_2\mathcal{T}^{\alpha\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi} - \frac{4}{H^2}\widehat{\mathcal{F}}'_2\frac{\partial y}{\partial x_\alpha}\frac{\partial y}{\partial x'_\kappa}\mathcal{T}^{\beta\lambda}\mathcal{T}^{\gamma\theta}\mathcal{T}^{\delta\phi}\Big]$$
$$= \frac{\partial y}{\partial x'_\kappa}\frac{\partial y}{\partial x'_\theta}\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\phi}f(y)\delta C_{\kappa\lambda\theta\phi}(x') \; . \tag{75}$$

Here the function $f(y)$ is,

$$f(y) = -50\widehat{\mathcal{F}}_2 + 60(2-y)\widehat{\mathcal{F}}'_2 - (40 - 62y + \frac{31}{2}y^2)\widehat{\mathcal{F}}''_2 - (2-y)(4y-y^2)\widehat{\mathcal{F}}'''_2 \; . \tag{76}$$

The final reduction is accomplished by one more partial integration. Let us define the integral $I[f]$ of a function $f(y)$ by the relations,

$$\frac{\partial y}{\partial x'_\kappa}f(y) \equiv \frac{\partial}{\partial x'_\kappa}I[f](y) \quad \text{such that} \quad \frac{\partial I[f]}{\partial y} = f(y) \; . \tag{77}$$

Then the one loop source becomes,

$$\begin{aligned}
&\int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x')h^{(0)}_{\rho\sigma}(x') \\
&= 2i\sqrt{-g(x)}\int d^4x'\sqrt{-g(x')}\frac{\partial y}{\partial x'_\kappa}f(y)\frac{\partial y}{\partial x'_\theta}\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\phi}\delta C_{\kappa\lambda\theta\phi}(x') \qquad (78)\\
&= -2i\sqrt{-g(x)}\int d^4x'\sqrt{-g(x')}I[f]\Big\{\frac{D^2y}{Dx'_\kappa Dx'_\theta}\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\phi}\delta C_{\kappa\lambda\theta\phi}(x') \\
&\qquad +\frac{D\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\phi}}{Dx'_\kappa}\frac{\partial y}{\partial x'_\theta}\delta C_{\kappa\lambda\theta\phi}(x')+\frac{\partial y}{\partial x'_\theta}\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\phi}D^\kappa\delta C_{\kappa\lambda\theta\phi}(x')\Big\}\ . \qquad (79)
\end{aligned}$$

The first and second terms include the metric,

$$\frac{D^2y}{Dx'_\kappa Dx'_\theta}=H^2(2-y)g^{\kappa\theta}(x'),\quad \frac{D\mathcal{T}^{\lambda(\mu}\mathcal{T}^{\nu)\phi}}{Dx'_\kappa}=\frac{1}{2}\frac{\partial y}{\partial x_{(\mu}}\mathcal{T}^{\nu)(\phi}g^{\lambda)\kappa}(x')\ , \qquad (80)$$

so they give zero when contracted into the linearized Weyl tensor. The third term vanishes by the transversality of the linearized Weyl tensor (for dynamical gravitons only) which we showed in (67). Hence the one loop source term for a dynamical graviton is zero:

$$\int d^4x' \Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x')h^{(0)}_{\rho\sigma}(x')=0\ . \qquad (81)$$

Before concluding we should comment on the validity of our result (81), in view of the enormous difference between de Sitter and the actual expansion history of the universe. Of course equation (32) is correct for any geometry, but we only know the graviton self-energy for de Sitter background. This does not make any difference for cosmologically observable tensor perturbations for two reasons:

- As explained section 2.1, de Sitter is an excellent approximation to primordial inflation up until cosmologically observable perturbations experience first horizon crossing. After this time the de Sitter approximation breaks down, but those perturbations are almost constant.

- Our result (57) is valid for any geometry, and the linearized Weyl tensor vanishes for constant perturbations. So there is no contribtuion from the portion of the integration which derives from times after the end of inflation.

To see the second point, note that general coordinate invariance requires matter contributions to the graviton self-energy to take the form (20), provided one uses expressions (22-23) to define the projectors for a general metric, and provided the general form of expression (21) is related to the geodetic length function through (6). That form is all we required to derive equation (57).

4 Conclusions

We have found that the inflationary production of MMC scalars has no effect on dynamical gravitons at one loop order. There is nothing very surprising about this result. It is exactly what happens in flat space [17]. Although the scalar contribution to the graviton self-energy is enormously more complex in de Sitter than in flat space, we showed in section 3 that all of this complexity can be absorbed into surface integrations over the initial time. It is plausible that these surface integrations can be regarded as perturbative redefinitions of the initial state which involve two scalars and one graviton. The null effect of flat space certainly has this interpretation, which implies the same for the highest derivative part of the de Sitter result. What has yet to be proved — and so must be labeled a conjecture — is that the lower derivative, intrinsically de Sitter parts have the same interpretation. Checking this requires a computation like that recently completed for the self-interacting scalar [12].

That is the math behind our result; the physics is that ultraviolet virtual scalars affect gravitons the same as in flat space, and infrared scalars carry too little stress-energy to have much effect. The effect of ultraviolet scalars is limited, as on flat space, to inducing higher derivative counterterms. Although primordial inflation produces many scalars, they are all highly infrared so they interact only weakly with gravtions. (This seems to be why inflationary gravitons have no significant effect on MMC scalars [9].) One might worry that a very infrared graviton would still suffer some effect from absorbing a comparably infrared scalar. To understand why this is not so, let us model the process by simply replacing the graviton's co-moving wave number k with a new one k',

$$0 = \ddot{u}(t,k) + 3H\dot{u}(t,k) + \frac{k^2}{a^2(t)} u(t,k) \longrightarrow \ddot{u}(t,k) + 3H\dot{u}(t,k) + \frac{k'^2}{a^2(t)} u(t,k) \; . \tag{82}$$

The effect on the mode function is negligible after both $1/a^2$ terms have redshifted into insignificance.

Both math and physics suggest that inflationary gravitons might do something interesting to other gravitons. The graviton contribution to the graviton self-energy has been derived at one loop order [21] so the computation can be made. Of course one can reduce the effect to a temporal surface term, as we did in section 3, but it seems likely that this surface term will depend upon the observation time η so that it cannot be absorbed into a perturbative correction to the initial state. The reason for this is that the graviton contribution contains de Sitter-breaking, infrared logarithms [21], unlike the scalar contribution. The physical principle involved would be that gravitons possess spin and even very infrared gravitons continue to interact via the spin-spin coupling which doesn't exist for scalars. This is presumably why inflationary gravitons induce a secular enhancement of the field strength of massless fermions [10].

It would also be interesting to investigate how inflationary scalars affect the force of gravity. That can be done by solving (38) to correct for the linearized response to a stationary point mass M [22],

$$h_{00}^{(0)}(x) = a^2 \times \frac{2GM}{a\|\vec{x}\|} \; , \; h_{0i}^{(0)}(x) = 0 \; , \; h_{ij}^{(0)}(x) = a^2 \times \frac{2GM}{a\|\vec{x}\|} \times \delta_{ij} \; . \tag{83}$$

The same reduction procedures we laid out in section 3 can be applied in this case except that:

- The spin zero projector $\mathcal{P}^{\rho\sigma}(x')$ does not annihilate (83); and
- The linearized stress tensor does not vanish.

Because the linearized stress tensor is proportional to $\delta^3(\vec{x'})$, we should be able to reduce the computation to a single integration over η'.

Note that the virtual scalars of flat space do induce a correction to the classical potential [23, 24] and we expect one as well on de Sitter background. On dimensional grounds the flat space result must (and does) take the form,

$$\Phi_{\rm flat} = -\frac{GM}{r}\Biggl\{1 + {\rm constant} \times \frac{G}{r^2} + O(G^2)\Biggr\} \; . \tag{84}$$

On de Sitter background there is a dimensionally consistent alternative provided by the Hubble constant H and by the secular growth driven by continuous particle production,

$$\Phi_{\rm dS} = -\frac{GM}{r}\Biggl\{1 + {\rm constant} \times GH^2 \ln(a) + O(G^2)\Biggr\} \; . \tag{85}$$

If such a correction were to occur its natural interpretation would be as a time dependent renormalization of the Newton constant. The physical origin of the effect (if it is present) would be that virtual infrared quanta which emerge near the source tend to collapse to it, leading to a progressive increase in the source.

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References

[1] L. P. Grishchuck, Sov. Phys. JETP **40** (1975) 409; L. H. Ford and L. Parker, Phys. Rev. **D16** (1977) 1601.

[2] A. A. Starobinsky, JET Lett. **30** (1979) 682; V. F. Mukhanov and G. V. Chibisov, JETP Lett. **33** (1981) 532.

[3] E. Komatsu et al., Astrophys. J. Suppl. **192** (2011) 18, arXiv:1001.4538.

[4] T. Prokopec, O. Tornkvist and R. P. Woodard, Phys. Rev. Lett. **89** (2002) 101301, astro-ph/0205331; Annals Phys. **303** (2003) 251, gr-qc/0205130; T. Prokopec and R. P. Woodard, Am. J. Phys. **72** (2004) 60, astro-ph/0303358; Annals Phys. **312** (2004) 1, gr-qc/0310056.

[5] T. Prokopec and R. P. Woodard, JHEP **0310** (2003) 059, astro-ph/0309593; B. Garbrecht and T. Prokopec, Phys. Rev. **D73** (2006) 064036, gr-qc/0602011; S. P. Miao and R. P. Woodard, Phys. Rev. **D74** (2006) 044019, gr-qc/0602110.

[6] T. Brunier, V. K. Onemli and R. P. Woodard, Class. Quant. Grav. **22** (2005) 59, gr-qc/0408080; E. O. Kahya and V. K. Onemli, Phys. Rev. **D76** (2007) 043512, gr-qc/0612026.

[7] E. O. Kahya and R. P. Woodard, Phys. Rev. **D72** (2005) 104001, gr-qc/0508015; Phys. Rev. **D74** (2006) 084012, gr-qc/0608049.

[8] L. D. Duffy and R. P. Woodard, Phys. Rev. **D72** (2005) 024023, hep-ph/0505156.

[9] E. O. Kahya and R. P. Woodard, Phys. Rev. **D76** (2007) 124005, arXiv:0709.0536; Phys. Rev. **D77** (2008) 084012, arXiv:0710.5282.

[10] S. P. Miao and R. P. Woodard, Class. Quant. Grav. **23** (2006) 1721, gr-qc/0511140; Phys. Rev. **D74** (2006) 044019, gr-qc/0602110; Class. Quant. Grav. **25** (2008) 145009, arXiv:0803.2377.

[11] S. Park and R. P. Woodard, Phys. Rev. **D83** (2011) 084049, arXiv:1101.5804.

[12] E. O. Kahya, V. K. Onemli and R. P. Woodard, Phys. Lett. **B694** (2010) 101, arXiv:1006.3999.

[13] G. Perez-Nadal, A. Roura and E. Verdaguer, JCAP **1005** (2010) 036, arXiv:0911.4870.

[14] E. Witten, "Quantum gravity in de Sitter space," in *New Fields and Strings in Subnuclear Physics: Proceedings of the International School of Subnuclear Physics (Subnuclear Series, 39)*, ed. A Zichichi (World Scientific, Singapore, 2002), hep-th/0106109; A. Strominger, JHEP **0110** (2001) 034, hep-th/0106113.

[15] J. Schwinger, J. Math. Phys. **2** (1961) 407; K. T. Mahanthappa, Phys. Rev. **126** (1962) 329; P. M. Bakshi and K. T. Mahanthappa, J. Math. Phys. **4** (1963) 1; J. Math. Phys. **4** (1963) 12; L. V. Keldysh, Sov. Phys. JETP **20** (1965) 1018; K. C. Chou, Z. B. Su, B. L. Hao and L. Yu, Phys. Rept. **118** (1985) 1; R. D. Jordan, Phys. Rev. **D33** (1986) 444; E. Calzetta and B. L. Hu, Phys. Rev. **D35** (1987) 495.

[16] N. C. Tsamis and R. P. Woodard, Phys. Lett. **B292** (1992) 269.

[17] G. 't Hooft and M. Veltman, Annales Poincare Phys. Theor. **A20** (1974) 69.

[18] L. H. Ford and R. P. Woodard, Class.and Quant. Grav. **22** (2005) 1637, gr-qc/0411003.

[19] N. C. Tsamis and R. P. Woodard, Ann. Phys. **238** (1995) 1.

[20] E. O Kahya, V. K. Onemli and R. P. Woodard, Phys. Rev. **D81** (2010) 023508, arxXiv:0904.4811.

[21] N. C. Tsamis and R. P. Woodard, Phys. Rev. **D54** (1996) 2621, hep-ph/9602317.

[22] N. C. Tsamis and R. P. Woodard, Commun. Math. Phys. **162** (1994) 217.

[23] A. F. Radkowski, Ann. Phys. **56** (1970) 319; D. M. Capper, M. J. Duff and L. Halperin, Phys. Rev. **D10** (1974) 461; D. M. Capper and M. J. Duff, Nucl. Phys. **B82** (1974) 147; D. M. Capper, Nuovo Cimento **A25** (1975) 29; M. J. Duff, Phys. Rev. **D9** (1974) 1837; H. Hamber and S. Liu, Phys. Lett. **B357** (1995) 51, hep-th/9505182; M. J. Duff and J. T. Liu, Phys. Rev. Lett. **85** (2000) 2052, hep-th/0003237; J. F. Donoghue, Phys. Rev. Lett. **72** (1994) 2996, gr-qc/9310024; Phys. Rev. **D50** (1994) 3874, gr-qc/9405057; I. J. Muzinich and S. Kokos, Phys. Rev. **D52** (1995) 3472, hep-th/9501083; A. Akhundov, S. Belucci and A. Shiekh, Phys. Lett. **B395** (1997) 16, gr-qc/9611018; I. B. Khriplovich and G. G. Kirilin, J. Exp. Theor. Phys. **95** (2002) 981, gr-qc/0207118; J. Exp. Theor. Phys. **98** (2004) 1063, gr-qc/0402018. N. E. J. Bjerrum-Bohr, J. F. Donoghue and B. R. Holstein, Phys. Rev. **D67** (2003) 084033, Erratum-ibid. **D71** (2005) 069903, hep-th/0211072; N. E. J. Bjerrum-Bohr, Phys. Rev. **D66** (2002) 084023, hep-th/0206236; B. R. Holstein and A. Ross, "Spin Effects in Long Range Gravitational Scattering," arXiv:0802.0716; A. Campos and E. Verdaguer, Phys. Rev. **D49** (1994) 1861, gr-qc/9307027; F. C. Lombardo and F. D. Mazzitelli, Phys. Rev. **D55** (1997) 3889, gr-qc/9609073; R. Martin and E. Verdaguer, Phys. Rev. **D61** (2000) 124024, gr-qc/0001098; A. Satz, F. D. Mazzitelli and E. Alvarez, Phys. Rev. **D71** (2005) 064001, gr-qc/0411046; D. A. R. Dalvit and F. D. Mazzitelli, Phys. Rev. **D50** (1994) 1001, gr-qc/9402003; E. D. Carlson, P. R. Anderson, A. Fabbri, S. Fagnocchi, W. H. Hirsch and S. Klyap, Phys. Rev. **D82** (2010) 124070, arXiv:1008.1433; A. Marunovic and T. Prokopec, Phys. Rev. **D83** (2011) 104039, arXiv:1101.5059.

[24] Sohyun Park and R. P. Woodard, Class. Quant. Grav. **27** (2010) 245008, ArXiv:1007.2662.