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Nonlocal metric formulations of MOND with sufficient lensing

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We demonstrate how to construct purely metric modifications of gravity which agree with general relativity in the weak field regime appropriate to the solar system, but which possess an ultra-weak field regime when the gravitational acceleration becomes comparable to $a_0 \sim 10^{-10}$ m/s^2. In this ultra-weak field regime, the models reproduce the MOND force without dark matter and also give enough gravitational lensing to be consistent with existing data. Our models are nonlocal and might conceivably derive from quantum corrections to the effective field equations.

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I. INTRODUCTION

Although Einstein's gravitational field equations are in remarkable agreement with all solar-system and binary-pulsar tests [1], they lead to a cosmological model which needs that the energy density of the Universe is strongly dominated by components, dark matter and dark energy, which have so far eluded direct detection. In particular, there is not enough baryonic matter to explain the observed properties of galactic dynamics using standard Einsteinian (or actually Newtonian) gravitational equations. The usual solution to this problem is to suppose that the vast majority of nonrelativistic matter in the universe consists of some weakly interacting particle we have not yet detected. Although there are several reasonable candidates for what this dark matter might be (see, e.g., [2]), it is worthwhile considering the alternative: It is possible that the field equations break down at galactic scales, i.e., that gravity is modified at distances relevant for dealing with galactic and inter-galactic dynamics.

Along this line, Milgrom proposed a simple phenomenological law [3] which leads to successful explanations of various observations, and to predictions [4] which turned out to be confirmed *a posteriori*. Milgrom's proposal, Modified Newtonian Dynamics (MOND), stipulates that a test particle at a distance r from a mass M will experience a gravitational acceleration given by the Newtonian expression $a_N = GM/r^2$ as long as a_N is (much) larger than a critical acceleration a_0, while the same particle will undergo the MOND acceleration $a_{\mathrm{MOND}} = \sqrt{a_N a_0} = \sqrt{GMa_0}/r$ when a_N is smaller than a_0. (Although Milgrom's proposal can be viewed as a change in Newton's 3rd law [5] it is more often imagined as a modification of gravity, which is the view we shall take.) It turns out that the value [6]

$$a_0 \approx 1.2 \times 10^{-10}\,\mathrm{m.s}^{-2} \tag{1}$$

allows an excellent fit of galaxy rotation curves using reasonable mass-to-luminosity ratios [7], without the need for non-baryonic dark matter [8]. MOND's ability to explain certain observed regularities of galactic structure contrasts favorably with dark matter, for which these regularities must either be accidental or else the result of some yet-to-be-discovered attractor solution in structure formation. Indeed, in the case of rotationally supported systems, MOND provides a simple explanation for (i) the Tully-Fisher relation [9], which states that the observed limiting rotation velocity of galaxies, v_∞, scales as the fourth root of the baryonic mass of the galaxy (see [10] for a recent dramatic confirmation of this relation); (ii) Milgrom's law, stating that the need for dark matter always seems to occur when the gravitational acceleration falls to about a_0 [11]; (iii) Freeman's law, namely that the surface density never exceeds a_0/G [12]; and (iv) Sancisi's law, i.e., that bumps in the rotation curves are correlated to the baryonic mass [12]. For pressure-supported systems, MOND also explains their typical size $R = \sqrt{GM/a_0}$ and predicts a stellar velocity dispersion $\sigma \propto (GMa_0)^{1/4}$ [13, 14], explaining thus the Faber-Jackson relation [15]. MOND also was able to predict properties of low surface brightness galaxies that were eventually confirmed by observation [16, 17]. Recently, Ref. [18] used a large catalog of widely separated binary star systems [19] as evidence for the breakdown of Kepler's third law at the MOND acceleration scale a_0.

On the other hand, the original MOND formulation does have some difficulties. In particular, a single acceleration scale a_0 fitted to galaxy rotation curves does not account for velocity dispersions in the cores of galactic clusters, which still need some amount of dark matter [20]. Similarly, X-ray and weak lensing data from the bullet cluster [21] indicate that dark matter exists at a different location from the gas. (Because weak lensing data is involved, one must make some assumption about how lensing occurs in MOND gravity, and this might affect the negative conclusion [22–24].) A reasonable fraction of dark baryons and/or massive neutrinos is thus still required at cluster scales, even if the MOND scheme happens to be an actual law of Nature at galaxy scales. This would not be in contradiction with observation, since most of the baryons in the Universe have not yet been detected. Dark baryons are actually also required to explain the observed peaks in the CMB spectrum.

The most serious problem of MOND is that it is not a complete theory, so that testing it often implies making guesses about its predictions for lensing or for cosmological evolution. This has not been for a lack of efforts, and a large number of theoretical constructions have been proposed over almost three decades to promote MOND to a consistent relativistic field theory. One major problem has always been simultaneously reproducing the Tully-Fisher relation and giving a sufficient amount of weak lensing. That problem was finally surmounted in 2004 by the tensor-vector-scalar (TeVeS) model constructed by Bekenstein (after years of work with Milgrom and Sanders) [25–30], in which the MOND force (implying the Tully-Fisher relation) is mediated by a scalar field, and where the presence of a unit timelike vector field helps in obtaining the right amount of light deflection from galaxies and clusters. The model has been shown to give better agreement with cosmological data than many believed any relativistic extension of MOND could do [31–37].

In its original formulation [28], TeVeS suffers from several theoretical and experimental difficulties [38–40], notably a serious instability [41, 42]. The latest version of TeVeS [43, 44], inspired by the Einstein-Aether framework [45–50], seems to avoid this instability and predicts post-Newtonian parameters consistent with solar-system tests. However, it still needs an unnaturally fine-tuned function of the scalar kinetic term in its action to be also consistent with binary-pulsar tests [38]. (The extended Vainshtein mechanism recently proposed in [51] is a way to avoid this difficulty.) Out of the many alternative models which have been proposed in the literature (for example, see [52]), the recent bi-metric theory [53] is a particularly promising and elegant one, although its detailed properties (notably its stability) remain to be fully understood.

As promising as we consider TeVeS to be, its dependence on other fields to carry part of the gravitational force is somewhat counter to the spirit of relativity. In the present paper, we re-examine pure-metric formulations of MOND along the general lines previously considered in [54].[1] Our aim is not to here produce the ultimate theory but rather just to show what form any pure metric generalization of MOND must take in order to combine two key features for a static, spherically symmetric and pressureless source which contains no dark matter:

- reproduce the MOND force law in the ultra-weak field regime of accelerations comparable to a_0; and
- produce enough weak lensing to be compatible with observations.

We first derive the form the MOND corrections to the Lagrangian must take in order to combine these properties when specialized to a static and spherically symmetric geometry. Then we demonstrate that no local curvature scalar has this form. However, nonlocal scalars do exist which take the correct form, and we exhibit some. As anticipated in [54], the Lagrangian inevitably becomes cubic in the weak fields, raising concerns about stability which we discuss briefly in the conclusion.

This paper is organized as follows. In Sec. II, we discuss some basic phenomenological properties used to deal with static, spherically symmetric systems, and we show how a pure metric action reproducing the MOND phenomenology can be devised for such systems. In Sec. III, we demonstrate that such an action cannot be local, i.e., it cannot be a function of only the metric and a finite number of its derivatives. Section IV introduces the main ingredients needed to construct a suitable nonlocal action for gravity. In Sec. V we exhibit a nonlocal model having the properties discovered in Sec. II, i.e., which reproduces the MOND dynamics at large distances, including enough weak lensing, while tending towards general relativity at small distances. Our conclusions are given in Sec. VI.

II. PHENOMENOLOGY

The point of this section is to derive the form that the MOND modification to the gravitational Lagrangian must take when specialized to the ultra-weak field regime of a static, spherically symmetric geometry,

$$ds^2 = -B(r)c^2dt^2 + A(r)dr^2 + r^2 d\Omega^2 \ . \tag{2}$$

We begin by reviewing how the equations of general relativity work for a source which would, of course, need to consist mostly of dark matter. In the ultra-weak field limit, these equations imply relations for the two linearized potentials, $a(r) \equiv A(r) - 1$ and $b(r) \equiv B(r) - 1$. One of these relations determines how the potentials depend upon the source and the other fixes how they depend upon each other. Our metric interpolation of MOND consists of changing how the potentials depend upon the source but not much how they depend upon one another. As the section closes we consider the form the MOND correction to the gravitational Lagrangian must take in order to substitute our MOND equations for those of general relativity.

We assume a perfect fluid source,

$$T_\mu^\nu = \mathrm{diag}\,(-\rho, P, P, P)\,, \tag{3}$$

where $\rho(r)$ and $P(r)$ are respectively the energy density and pressure. Only two of the ten field equations are independent in this geometry; the rest are either trivial or implied by conservation. Defining G as Newton's constant, the tt and rr Einstein equations equations are,

$$\frac{G_{tt}}{B} = \frac{A'}{rA^2} + \Big(\frac{A-1}{r^2 A}\Big) = \frac{8\pi G\rho}{c^4}\ , \tag{4}$$

$$\frac{G_{rr}}{A} = \frac{B'}{rAB} - \Big(\frac{A-1}{r^2 A}\Big) = \frac{8\pi G P}{c^4}\ . \tag{5}$$

[1] By "pure-metric", we mean that the full gravitational interaction is described by the dynamics of a single metric tensor $g_{\mu\nu}$, without introducing explicit extra fields like scalars or vectors, although such degrees of freedom may actually be hidden in some excitations of $g_{\mu\nu}$. For instance, we would call pure-metric the class of $f(R)$ models, although it is well known they are equivalent to specific scalar-tensor theories. Our phrase pure-metric should also be distinguished from what is called a "metric theory" in [1], meaning there that matter is minimally coupled to a single metric tensor $g_{\mu\nu}$. What we call pure-metric is a subclass of such metric theories, but we also impose that the kinetic term of gravity itself is a functional of only $g_{\mu\nu}$.

Equation (4) can be integrated to give us the rr component,

$$A(r) = \Big[1 - \frac{2GM(r)}{c^2 r}\Big]^{-1} \; , \tag{6}$$

where the enclosed mass is

$$M(r) \equiv \frac{4\pi}{c^2} \int_0^r dr' \, {r'}^2 \rho(r') . \tag{7}$$

The second equation (5) could also be integrated but we shall not need to do this.

Now consider the regime of zero pressure and very weak potentials, for which the linearized potentials take the form,

$$a(r) \approx \frac{2GM(r)}{c^2 r} \approx r b'(r) . \tag{8}$$

These relations can be expressed in many ways but a convenient form, for our purposes, is as one equation for how the potentials depend upon the source,

$$r b'(r) \approx \frac{2GM(r)}{c^2 r} \; , \tag{9}$$

and another equation for how the two potentials depend upon each other,

$$a(r) \approx r b'(r) . \tag{10}$$

The first equation (9) is what tells us that explaining cosmic motions requires dark matter, whereas the second equation (10) tells us that the amount of weak lensing is consistent with the data, assuming cosmic motions are explained.

For circular geodesic motion at fixed radius r with angular velocity $\dot{\phi}$, one can show

$$r B'(r) = r b'(r) = \frac{2 r^2 \dot{\phi}^2}{c^2} = \frac{2 v^2}{c^2} \; . \tag{11}$$

We emphasize that relation (11) depends only upon the geometry (2) and minimal coupling to matter (that we will always assume within the present paper), without any assumption about the gravitational field equations which produce it. The Tully-Fisher relation implies that the rotational speed $v(r) = r\dot{\phi}$ tends to a constant which goes as the fourth root of the source luminosity. MOND imposes the Tully-Fisher relation by changing equation (9) to [8],

$$r b'(r) \longrightarrow \frac{2\sqrt{a_0 G M(r)}}{c^2} \; , \tag{12}$$

where the right arrow indicates that the relation applies in the ultra-weak field regime of low accelerations.

Relation (12) contains the physics we want, but it is not yet in the form of a modification to just the left hand side of the gravitational field equations (4-5). To reach that form we need to isolate the local energy density $\rho(r)$ by first squaring, then differentiating and shifting some factors from right to left,

$$\frac{c^2}{2 a_0 r^2} \Big((r b')^2 \Big)' = \frac{8\pi G \rho}{c^4} \; . \tag{13}$$

The other MOND equation can be written in a variety of ways because the right hand side vanishes for the relevant case of zero pressure. The weak lensing data is also not good enough to justify insisting upon precisely (10), so we would be happy with $a(r) = k r b'(r)$ for any positive, order one constant k. This suggests the second MOND equation should take the form

$$\frac{c^2}{a_0 r^3} \Big(k r b' - a \Big)^2 = 0 . \tag{14}$$

Note that it would not change the MOND phenomenology were we to multiply (14) by a constant; we could also add a constant times it to (13).

Relations (13-14) are the modified gravity equations we wish to attain in the ultra-weak field regime for a static, spherically symmetric and pressureless source. We now seek an ultra-weak field expansion of a Lagrangian $\mathcal{L}_{\rm MOND}$

which cancels that of general relativity $\mathcal{L}_{\rm EH}$ and substitutes cubic terms whose variation gives (13-14). Although one generally loses field equations by specializing the metric before variation, we shall recover the correct g_{tt} and g_{rr} equations [55, 56], in the ultra-weak field regime of course. The equations lost by specializing first are those associated with conservation.

After some judicious partial integrations, the Einstein-Hilbert Lagrangian takes the form

$$\mathcal{L}_{\rm EH} = \frac{c^4}{16\pi G} R\sqrt{-g} \longrightarrow \Bigl(\text{Surface term}\Bigr) + \frac{c^4}{16\pi G}\Biggl\{ -rab' + \frac{a^2}{2} + O(h^3)\Biggr\}, \tag{15}$$

where h stands for a and b. In the ultra-weak field regime, the MOND Lagrangian we seek should have a quadratic term that cancels the quadratic part of the Einstein-Hilbert Lagrangian, plus a cubic term which enforces our interpolation (13-14) of the MOND physics. The most general Lagrangian of this form is

$$\mathcal{L}_{\rm MOND} \longrightarrow \frac{c^4}{16\pi G}\Biggl\{ \Bigl[rab' - \frac{a^2}{2} + O(h^3)\Bigr] + \frac{c^2}{a_0}\Bigl[\frac{\alpha a^3}{r} + \beta a^2 b' + \gamma r a b'^2 + \delta r^2 b'^3 + O(h^4)\Bigr]\Biggr\}, \tag{16}$$

where α, β, γ and δ are dimensionless constants whose properties we shall constrain using the phenomenology of the ultra-weak field regime. A minor point which deserves comment is that the $O(h^3)$ corrections to the first square-bracketed expression in (16) differ from the cubic MOND terms in the second square-bracketed expression by a factor of $a_0 r/c^2$, which would only become of order one on horizon scales and is utterly negligible on galaxy scales.

The gravity Lagrangian is $\mathcal{L}_{\rm grav} \equiv \mathcal{L}_{\rm EH} + \mathcal{L}_{\rm MOND}$, and we wish to compute the variation of the associated action when specialized to a static, spherically symmetric geometry. Recall that the full Einstein equations for arbitrary geometry are obtained by varying the Einstein-Hilbert action as

$$\frac{16\pi G}{c^4\sqrt{-g}}\frac{\delta S_{\rm EH}}{\delta g^{\mu\nu}(x)} = G_{\mu\nu}(x) = \frac{8\pi G}{c^4} T_{\mu\nu}(x). \tag{17}$$

For a static, spherically symmetric geometry $\sqrt{-g} = r^2\sqrt{(1+a)(1+b)}$. We want $g_{tt} = -(1+a)$ and $g_{rr} = 1+b$, and we neglect higher powers of the weak fields, so the relevant variations for us (assuming zero pressure) are

$$\frac{16\pi G}{c^4 r^2}\frac{\delta S_{\rm grav}}{\delta b(r)} = -\frac{c^2}{a_0 r^2}\Biggl\{\beta(a^2)' + 2\gamma(rab')' + 3\delta(r^2 b'^2)'\Biggr\} = \frac{8\pi G\rho}{c^4}\ , \tag{18}$$

$$-\frac{16\pi G}{c^4 r^2}\frac{\delta S_{\rm grav}}{\delta a(r)} = -\frac{c^2}{a_0 r^2}\Biggl\{\frac{3\alpha a^2}{r} + 2\beta a b' + \gamma r b'^2\Biggr\} = 0. \tag{19}$$

Demanding that equation (19) should have the unique solution $a = krb'$ implies

$$\alpha = -\frac{\beta}{3k} \qquad \text{and} \qquad \gamma = -\beta k. \tag{20}$$

Substituting (20) into equation (18) and demanding that it give (13) implies

$$\delta = -\frac{1}{6} + \frac{1}{3}\beta k^2. \tag{21}$$

Hence the MOND Lagrangian we seek has the following expansion in the ultra-weak field regime:

$$\mathcal{L}_{\rm MOND} \longrightarrow \frac{c^4 r^2}{16\pi G}\Biggl\{ \Bigl[\frac{ab'}{r} - \frac{a^2}{2r^2} + O(h^3)\Bigr] + \frac{c^2}{a_0}\Bigl[\frac{\beta}{3k}\Bigl(kb' - \frac{a}{r}\Bigr)^3 - \frac{b'^3}{6} + O(h^4)\Bigr]\Biggr\}, \tag{22}$$

where the constant β must be nonzero but is otherwise arbitrary.

III. LOCAL TOOLS FOR MODEL BUILDING

In the previous section we considered static, spherically symmetric geometries in the ultra-weak field limit for which MOND ought to apply. Our result is that the form (22) for the MOND addition to the Einstein-Hilbert Lagrangian, allows us to reproduce the MOND force and sufficient lensing without dark matter. The burden of this section is that no local, invariant Lagrangian can have that form. Of course minus the Einstein-Hilbert Lagrangian reproduces the quadratic parts of (22), so it is the cubic terms which comprise the fundamental obstacle. Our analysis is a straightforward proof by exhaustion: we examine all local curvature scalars for static, spherically symmetric geometries in the ultra-weak field regime.

It is useful to expand the various curvatures in powers of the graviton field $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$. We can greatly simplify the analysis by working in Cartesian coordinates, which makes the affine connection vanish in the absence of curvature. The $3+1$ decomposition of the graviton field is

$$h_{00} = -b(r) \qquad , \qquad h_{0i} = 0 \qquad \text{and} \qquad h_{ij} = a(r)\widehat{r}^i\widehat{r}^j , \tag{23}$$

where $\widehat{r}^i \equiv x^i/r$. It is also useful to introduce the projector

$$\pi^{ij} \equiv \delta^{ij} - \widehat{r}^i\widehat{r}^j . \tag{24}$$

The nonzero components of the affine connection are

$$\Gamma^0{}_{0i} = \frac{b'}{2}\widehat{r}^i + O(h^2) \;\; , \;\; \Gamma^i{}_{00} = \frac{b'}{2}\widehat{r}^i + O(h^2) \;\; , \;\; \Gamma^i{}_{jk} = \frac{a'}{2}\widehat{r}^i\widehat{r}^j\widehat{r}^k + \frac{a}{r}\widehat{r}^i\pi^{jk} + O(hh'). \tag{25}$$

In denoting higher order corrections, we make no distinction between derivatives and inverse powers of r. So the term $O(hh')$ includes terms of the form hh' and h^2/r.

Our convention for the Riemann tensor is

$$R^\rho{}_{\sigma\mu\nu} \equiv \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\alpha}\Gamma^\alpha{}_{\mu\sigma} . \tag{26}$$

Its nonzero components for a static, spherically symmetric geometry are

$$R_{0i0j} = \frac{b''}{2}\widehat{r}^i\widehat{r}^j + \frac{b'}{2r}\pi^{ij} + O(h'^2), \tag{27}$$

$$R_{ijk\ell} = \frac{b'}{2r}\Big[\widehat{r}^i\widehat{r}^k\pi^{j\ell} - \widehat{r}^k\widehat{r}^j\pi^{\ell i} + \widehat{r}^j\widehat{r}^\ell\pi^{ik} - \widehat{r}^\ell\widehat{r}^i\pi^{kj}\Big] + \frac{a}{r^2}\Big[\pi^{ik}\pi^{j\ell} - \pi^{i\ell}\pi^{jk}\Big] + O(h'^2). \tag{28}$$

We define the Ricci tensor as $R_{\mu\nu} \equiv R^\rho{}_{\mu\rho\nu}$, and its nonzero components in our geometry are

$$R_{00} = \frac{b''}{2} + \frac{b'}{r} + O(h'^2), \tag{29}$$

$$R_{ij} = \Big[-\frac{b''}{2} + \frac{a'}{r}\Big]\widehat{r}^i\widehat{r}^j + \Big[-\frac{b'}{2r} + \frac{a'}{2r} + \frac{a}{r^2}\Big]\pi^{ij} + O(h'^2). \tag{30}$$

The Ricci scalar is $R \equiv g^{\mu\nu}R_{\mu\nu}$, and it works out to be

$$R = -b'' - \frac{2b'}{r} + \frac{2a'}{r} + \frac{2a}{r^2} + O(h'^2). \tag{31}$$

From the preceding analysis we note that every nonzero component of the curvature involves two derivatives (or inverse powers of r) acting on one or more weak field,

$$\text{Curvature} \sim h'' + O(h'^2). \tag{32}$$

No matter how the indices are contracted, N factors of the curvature must therefore have the form

$$\Big(\text{Curvature}\Big)^N \sim (h'')^N + O\Big((h')^2(h'')^{N-1}\Big). \tag{33}$$

The MOND correction (22) we seek involves powers of just one derivative acting on a single weak field,

$$\mathcal{L}_{\rm MOND} \sim \frac{c^4 r^2}{16\pi G}\Bigg\{(h')^2 + \frac{c^2}{a_0}(h')^3 + O(h^4)\Bigg\}. \tag{34}$$

The Ricci scalar (31) gives the quadratic terms because its linear part is a total derivative. However, the cubic terms of (34) not only have too few derivatives per weak field, they also contribute an *odd* total number of derivatives. The latter problem is much worse than the former because the leading weak field term in a curvature scalar might drop out — as it does for R — but nothing can change the total number of the derivatives it contains. Including differentiated curvatures increases the number of derivatives per weak field, and can in any case only add an even number of derivatives once all the indices are contracted to form a scalar.

That completes the main argument of this section but it is worth giving the nonzero components of the Einstein and Weyl tensors for future reference:

$$G_{00} = \frac{a'}{r} + \frac{a}{r^2} + O(h'^2), \tag{35}$$

$$G_{ij} = \Big[\frac{b'}{r} - \frac{a}{r^2}\Big]\widehat{r}^i\widehat{r}^j + \Big[\frac{b''}{2} + \frac{b'}{2r} - \frac{a'}{2r}\Big]\pi^{ij} + O(h'^2), \tag{36}$$

$$C_{0i0j} = -\frac{1}{12}\Big[b'' - \frac{b'}{r} + \frac{a'}{r} - \frac{2a}{r^2}\Big]\Big(\delta^{ij} - 3\widehat{r}^i\widehat{r}^j\Big) + O(h'^2), \tag{37}$$

$$C_{ijk\ell} = -\frac{1}{6}\Big[b'' - \frac{b'}{r} + \frac{a'}{r} - \frac{2a}{r^2}\Big]\Bigg\{\Big(\delta^{ik} - \frac{3}{2}\widehat{r}^i\widehat{r}^k\Big)\Big(\delta^{j\ell} - \frac{3}{2}\widehat{r}^j\widehat{r}^\ell\Big) - \Big(\delta^{i\ell} - \frac{3}{2}\widehat{r}^i\widehat{r}^\ell\Big)\Big(\delta^{jk} - \frac{3}{2}\widehat{r}^j\widehat{r}^k\Big)\Bigg\} + O(h'^2). \tag{38}$$

Note that all components of the Weyl tensor are proportional to the same linear combination of the weak fields, so that any scalar formed from $C_{\rho\sigma\mu\nu}$ will access this combination

$$C^{\rho\sigma\mu\nu}C_{\rho\sigma\mu\nu} = \frac{1}{3}\Big[-b'' + \frac{b'}{r} - \frac{a'}{r} + \frac{2a}{r^2}\Big]^2 + O(h''h'^2). \tag{39}$$

It is also worth noting some of the other scalars we can get:

$$R^2 = \Big[-b'' - \frac{2b'}{r} + \frac{2a'}{r} + \frac{2a}{r^2}\Big]^2 + O(h''h'^2), \tag{40}$$

$$R^{\rho\sigma\mu\nu}R_{\rho\sigma\mu\nu} - 4R^{\mu\nu}R_{\mu\nu} + R^2 = -\frac{4}{r^2}\Big(ab'' + a'b'\Big) + O(h''h'^2). \tag{41}$$

IV. NONLOCAL TOOLS FOR MODEL BUILDING

The new features that nonlocality brings to model building are that inverse differential operators reduce the number of derivatives, and that the gradient of the invariant volume of the past light-cone allows us to define a timelike 4-vector with which we can select particular components of the curvature. The first feature is necessary because, as discussed in the previous section, curvature scalars involve powers of two derivatives (or factors of $1/r$) acting on a weak field, whereas the MOND correction (22) we seek to realize as a scalar involves powers of only a single derivative of a weak field. The second property is needed to get *the right* weak fields.

A philosophical digression is necessary at this point. We do not maintain that physics is nonlocal at the fundamental level; we believe rather that nonlocality enters through quantum corrections to the effective field equations from loops of massless gravitons. These induce no macroscopic nonlocality in flat space background because their interactions are suppressed by derivatives, however, the situation is quite different when a cosmological constant is present. It has been argued that self-interactions between elements of the vast ensemble of infrared gravitons produced during primordial inflation show secular growth which eventually becomes nonperturbatively strong [57]. Nonlocal effective field equations for cosmology have been studied [58] as a way of abstracting these effects to the nonperturbative regime. Our work here will apply the very same nonlocal tools to build a model of structure formation. Although we work on a purely phenomenological level, it might be possible to derive a successful model from first principles using the Schwinger-Keldysh formalism [59]. Hence our nonlocal constructions will always be viewed as proceeding from causal evolution, based on the notion that the universe was released in a prepared state at some finite time. This last point is the key to being able to define a timelike 4-vector field and it must be accepted, even if one chooses to disregard our motivations and treat the models we propose on a purely phenomenological level.

A. The inverse scalar d'Alembertian

The scalar d'Alembertian is familiar to students of general relativity,

$$\Box \equiv \frac{1}{\sqrt{-g}} \partial_\mu \Big(\sqrt{-g} g^{\mu\nu} \partial_\nu \Big) . \tag{42}$$

We define the function $F(x)$ obtained by acting $\Box^{-1}$ on any function $f(x)$ (that is, $F(x) = \Box^{-1} f$) by the solution of the differential equation

$$\Box F(x) = f(x) , \tag{43}$$

subject to retarded boundary conditions. Specializing to the case of a static, spherically symmetric geometry, and a source function $f(r)$ which falls off at infinity, we obtain an equation which can be solved by integration,

$$\frac{1}{r^2 \sqrt{A(r) B(r)}} \frac{d}{dr} \left[r^2 \sqrt{\frac{B(r)}{A(r)}} \frac{dF(r)}{dr} \right] = f(r) \tag{44}$$

$$\Longrightarrow r^2 \sqrt{\frac{B(r)}{A(r)}} \frac{dF(r)}{dr} = \int_0^r dr' \, r'^2 \sqrt{A(r') B(r')} \, f(r') \tag{45}$$

$$\Longrightarrow F(r) = - \int_r^\infty \frac{dr'}{r'^2} \sqrt{\frac{A(r')}{B(r')}} \int_0^{r'} dr'' \, r''^2 \sqrt{A(r'') B(r'')} \, f(r'') . \tag{46}$$

In the weak field limit for a source $f(r)$ which is already first order we can set $A = B = 1$. We can also change the order of integration and perform the r' integration to get

$$\frac{1}{\Box} f = - \int_r^\infty \frac{dr'}{r'^2} \int_0^{r'} dr'' \, r''^2 \, f(r'') + O(h^2) \tag{47}$$

$$= - \int_0^r dr' \, r'^2 f(r') \times \frac{1}{r} - \int_r^\infty dr' \, r'^2 f(r') \times \frac{1}{r'} + O(h^2) \tag{48}$$

$$= - \frac{1}{4\pi} \int d^3x' \, \frac{f(\|\vec{x}'\|)}{\|\vec{x} - \vec{x}'\|} + O(h^2) . \tag{49}$$

Of course this is the usual Coulomb Green's function. We can make similar contact with the Lienard-Wiechert potential if we regard the system as released at some early time labeled $t = 0$,

$$\frac{1}{\Box} f = - \frac{c}{4\pi} \int_0^\infty dt' \int d^3x' \, \frac{\delta\Big(c(t - t') - \|\vec{x} - \vec{x}'\| \Big)}{\|\vec{x} - \vec{x}'\|} \times f(\|\vec{x}'\|) + O(h^2) \tag{50}$$

$$= - \frac{1}{4\pi} \int d^3x' \, \frac{\theta\Big(ct - \|\vec{x} - \vec{x}'\| \Big)}{\|\vec{x} - \vec{x}'\|} \times f(\|\vec{x}'\|) + O(h^2) . \tag{51}$$

The theta function in expression (51) is usually irrelevant for functions $f(r)$ which fall off rapidly, and for late times t. However, it plays an important role when the function $f(r)$ happens to be constant,

$$\frac{1}{\Box} f_0 = - f_0 \int_0^{ct} dr' r' + O(h^2) = - \frac{1}{2} f_0 (ct)^2 + O(h^2) . \tag{52}$$

It is well to remember that even our static, spherically symmetric systems are embedded in a larger cosmological background which had a beginning and is even now slightly time dependent.

B. A timelike 4-vector field

The preceding considerations are especially important for our second nonlocal building block: the invariant volume of the past light-cone. Suppose $\mathcal{S}$ is the Cauchy surface on which the initial state was released and let $\mathcal{M}$ stand for the spacetime manifold comprising $\mathcal{S}$ and its future. For a general metric $g_{\mu\nu}$ we define the invariant volume of the past light-cone from the spacetime point x^μ as

$$\mathcal{V}[g](x) = \int_{\mathcal{M}} d^4x' \sqrt{-g(x')}\, \theta\Big(-\sigma[g](x,x')\Big) \theta\Big(\mathcal{F}[g](x,x')\Big). \tag{53}$$

Here $\sigma[g](x,x')$ is the geodesic length function introduced by DeWitt and Brehme [60]. In expression (53), we note $\theta\Big(\mathcal{F}[g](x,x')\Big)$, a functional which is the invariant generalization of $\theta(x^0 - x'^0)$ needed to restrict the integration over x'^μ to the past of x^μ. ($\mathcal{F}$ stands for "forward" in this notation.) This functional is defined as one when the extension of the geodesic between x^μ and x'^μ eventually intersects the initial value surface $\mathcal{S}$, and zero otherwise.

The volume of the past light-cone is of great interest to us because it is guaranteed to grow when the point x^μ evolves in whatever is the timelike direction of the metric $g_{\mu\nu}$. Hence its gradient must be timelike and can be used to define a timelike vector field [58],

$$u^\mu[g](x) \equiv -\frac{g^{\mu\nu}(x)\partial_\nu \mathcal{V}[g](x)}{\sqrt{-g^{\alpha\beta}(x)\partial_\alpha \mathcal{V}[g](x)\partial_\beta \mathcal{V}[g](x)}} \; . \tag{54}$$

For the static, spherically symmetric geometry we have been considering it reduces to

$$u^\mu[g](x) \longrightarrow \frac{\delta^\mu_0}{\sqrt{B(r)}} \; . \tag{55}$$

It can therefore be used to pick out the timelike components of a tensor, just like the fundamental vector field $\mathcal{U}_\mu$ of TeVeS [28].

For our purposes it is better to exploit the close relation which exists between the volume of the past light-cone and the functional inverse of the Paneitz operator,

$$D_P[g] \equiv \square^2 + 2D_\mu \Big[R^{\mu\nu} - \frac{1}{3} g^{\mu\nu} R\Big] D_\nu \; . \tag{56}$$

This 4th order differential operator appears in conformal anomalies [61]. The relation between it and the volume of the past light-cone is that $8\pi/D_P$ acting on one agrees with $\mathcal{V}$ for arbitrary homogeneous and isotropic spacetimes [62],

$$\frac{8\pi}{D_P[\mathrm{FRW}]} 1 = \mathcal{V}[\mathrm{FRW}]. \tag{57}$$

Perturbations away from this background do not quite agree [62] but that is probably irrelevant for any use we might make of $\mathcal{V}[g](x)$.

The great advantage to defining our timelike vector field using the inverse of D_P is that we can avail ourselves of a simple partial integration trick [63, 64] for deriving causal and conserved field equations. To understand the trick, consider varying the product of a local functional of the metric $F[g]$ times some inverse differential operator $\mathcal{D}^{-1}$ — either $\square^{-1}$ or D_P^{-1} — acting on another local functional $G[g]$,

$$\frac{\delta}{\delta g^{\mu\nu}(x)} \Bigg(F[g] \frac{1}{\mathcal{D}[g]} G[g] \Bigg) = \frac{\delta F}{\delta g^{\mu\nu}} \frac{1}{\mathcal{D}} G - F \frac{1}{\mathcal{D}} \frac{\delta \mathcal{D}}{\delta g^{\mu\nu}} \frac{1}{\mathcal{D}} G + F \frac{1}{\mathcal{D}} \frac{\delta G}{\delta g^{\mu\nu}} \; . \tag{58}$$

The second and third terms on the right of expression (58) would make acausal contributions to the field equations which involve fields to the future of x^μ.

To see the acausality of expression (58) more clearly, let us expand out the final term, with all the implied integrations and coordinate dependence made explicit. In order to fix notation we express the term being varied in (58) as

$$F \frac{1}{\mathcal{D}} G \equiv \int d^4x' \, F(x') \int d^4x'' \, \mathcal{G}_{\mathrm{ret}}(x';x'') G(x''). \tag{59}$$

Here $\mathcal{G}_{\mathrm{ret}}(x';x'')$ is the retarded Green's function associated with the differential operator $\mathcal{D}$, and "retarded" means that it vanishes for $x''^0 > x'^0$. In this same language the final term on the right of (58) would be

$$F \frac{1}{\mathcal{D}} \frac{\delta G}{\delta g^{\mu\nu}} = \int d^4x' \, F(x') \int d^4x'' \, \mathcal{G}_{\mathrm{ret}}(x';x'') \frac{\delta G(x'')}{\delta g^{\mu\nu}(x)} \; . \tag{60}$$

Saying $G[g](x'')$ is local means it depends only on the metric and some finite number of its derivatives at x''^{μ}. Hence its variation with respect to $g^{\mu\nu}(x)$ is proportional to at most a finite number of derivatives of $\delta^4(x''-x)$. Of course this means we can perform the integration over x''^{μ} to get at most some derivatives acting on $\mathcal{G}_{\rm ret}(x';x) = \mathcal{G}_{\rm adv}(x;x')$. The remaining integration over x'^{μ} involves fields to the future of x^{μ}.

This sort of acausality is inevitable for any nonlocal action based on a single field. The Schwinger-Keldysh effective field equations avoid it by the same physical field being represented in a complicated way with two dummy fields. One first varies with respect to one of the dummy fields and then sets the two dummy fields equal, after which cancellations between various contributions result in there being no dependence upon dynamical variables to the future of x^{μ}. We shall circumvent this complication by having recourse to the simple trick of "partially integrating" the acausal terms of (58) so that their nonlocality is restricted to the past of x^{μ} [63, 64],

$$-F\frac{1}{\mathcal{D}}\frac{\delta\mathcal{D}}{\delta g^{\mu\nu}}\frac{1}{\mathcal{D}}G \longrightarrow -\left(\frac{\delta\mathcal{D}}{\delta g^{\mu\nu}}\frac{1}{\mathcal{D}}G\right)\frac{1}{\mathcal{D}}F, \tag{61}$$

$$F\frac{1}{\mathcal{D}}\frac{\delta G}{\delta g^{\mu\nu}} \longrightarrow \frac{\delta G}{\delta g^{\mu\nu}}\frac{1}{\mathcal{D}}F. \tag{62}$$

The result is manifestly causal. It is also conserved (if we include the variation of the measure factor) because we have just substituted, in the field equations, the causal retarded Green's function everywhere an acausal advanced Green's function appeared. Conservation requires only the differential equation, which both the advanced and retarded solutions obey. Of course this is just a trick; a true derivation from fundamental theory would require use of the Schwinger-Keldysh formalism [59]. However, the object of our study is the effective field equations, and they are perfectly valid as long as we consider them on a purely phenomenological level.

V. AN EXPLICIT MODEL

There are many ways to define a suitable relativistic generalization of $\mathcal{L}_{\rm MOND}$. A particularly elegant construction is based on two nonlocal building blocks,

$$X[g](x) \equiv g^{\mu\nu}\left[\partial_\mu\frac{1}{\square}\left(R_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}R\right)\right]\left[\partial_\nu\frac{1}{\square}\left(R_{\rho\sigma}u^\rho u^\sigma - \frac{1}{2}R\right)\right] \tag{63}$$

$$Y[g](x) \equiv g^{\mu\nu}\left[\partial_\mu\frac{1}{\square}\left(2R_{\alpha\beta}u^\alpha u^\beta\right)\right]\left[\partial_\nu\frac{1}{\square}\left(2R_{\rho\sigma}u^\rho u^\sigma\right)\right] . \tag{64}$$

Although these scalars are deeply nonlocal, even when specialized to static and spherically symmetric geometries, they give local, and very simple results, to lowest order in the weak field expansion. To derive these limits recall first the weak field, static and spherically symmetric results for R_{00} and R from expressions (29) and (31), respectively,

$$R_{00} \longrightarrow \frac{1}{2r^2}\left(r^2 b'\right)' + O(h^2) , \tag{65}$$

$$R \longrightarrow \frac{1}{r^2}\left(-r^2 b' + 2ra\right)' + O(h^2) . \tag{66}$$

The arrow indicates specialization to static, spherically symmetric geometries in the weak field limit. Note also that our statement that the residues are "$O(h^2)$" refers only to their dependence upon the weak fields, without regard to derivatives or powers of r.

The specialization of the 4-vector $u^\mu[g](x)$ to a static, spherically symmetric geometry is given by expression (55). Hence it is just $u^\mu[g](x) \longrightarrow \delta^\mu_0 + O(h)$ to the order we require, and we can write,

$$R_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}R \longrightarrow \frac{1}{r^2}\left(r^2 b' - ra\right)' + O(h^2) , \tag{67}$$

$$2R_{\alpha\beta}u^\alpha u^\beta \longrightarrow \frac{1}{r^2}\left(r^2 b'\right)' + O(h^2) . \tag{68}$$

These terms are both first order in the weak fields so can use expression (48) to implement the action of $\square^{-1}$ on them,

$$\frac{1}{\square}\left(R_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}R\right) \longrightarrow -\int_r^\infty dr' \left(b'(r') - \frac{a(r')}{r'}\right) + O(h^2) , \tag{69}$$

$$\frac{1}{\square}\left(2R_{\alpha\beta}u^\alpha u^\beta\right) \longrightarrow -\int_r^\infty dr'\, b'(r') + O(h^2) . \tag{70}$$

Of course the only derivatives that matter are with respect to the radial coordinate r,

$$\partial_\mu \frac{1}{\Box}\Big(R_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}R\Big) \longrightarrow \delta^r_\mu\Big(b' - \tfrac{a}{r}\Big) + O(h^2) \; , \tag{71}$$

$$\partial_\mu \frac{1}{\Box}\Big(2R_{\alpha\beta}u^\alpha u^\beta\Big) \qquad \longrightarrow \delta^r_\mu b' + O(h^2) \; . \tag{72}$$

The residue terms in these expressions are still nonlocal. However, substituting (71) and (72) into expressions (63) and (64) gives a local result to leading order,

$$X[g](x) \longrightarrow \Big(b' - \frac{a}{r}\Big) + O(h^2) \; , \tag{73}$$

$$Y[g](x) \longrightarrow b' + O(h^2) \; . \tag{74}$$

Choosing $k = 1$ in (22), we find that acceptable MOND equations would result from

$$\mathcal{L}_{\rm MOND} = \frac{c^4}{16\pi G}\left\{\frac{1}{2}\Big(-X + Y\Big) + \frac{c^2}{6a_0}\Big(|X|^{\frac{3}{2}} - |Y|^{\frac{3}{2}}\Big) + \dots\right\}\sqrt{-g} \; . \tag{75}$$

Relation (75) gives just the first two terms in the ultra-weak field expansion of the theory. That is all we can infer from the deep MONDian regime. There are many ways of extending the expansion to all orders to define the full theory. The chief requirement on any such extension is that it be suitably suppressed in comparison with general relativity for Newtonian accelerations much larger than a_0. Newtonian gravity seems to be valid in the solar system out to at least 80 Astronomical Units (the furthest of the Pioneer Probes), at which point $g_N/a_0 \sim 10^4$ [65]. In this regime we can take the Newtonian acceleration to be $g_N \sim c^2 b'(r) \sim c^2 a(r)/r$, so we need

$$\Big|\mathcal{L}_{\rm MOND}\Big| \ll \Big|\mathcal{L}_{\rm GR}\Big| \sim \frac{a_0^2}{16\pi G}\Big(\frac{g_N}{a_0}\Big)^2 , \tag{76}$$

for $g_N/a_0 \gtrsim 10^4$.

It is best to study the weak-field regime using dimensionless variables

$$x[g] \equiv \frac{c^2}{3a_0}|X[g]|^{\frac{1}{2}} \longrightarrow \frac{c^2}{3a_0}\Big|b' - \frac{a}{r}\Big| + O(h^2), \tag{77}$$

$$y[g] \equiv \frac{c^2}{3a_0}|Y[g]|^{\frac{1}{2}} \longrightarrow \frac{c^2}{3a_0}|b'| + O(h^2). \tag{78}$$

Whereas the variable $y[g]$ is of order one or smaller in the ultra-weak regime, it is of order $g_N/a_0 \gtrsim 10^4$ in the solar system. However, the variable $x[g]$ vanishes, to lowest order, in both regimes, so whatever function interpolates between the two regimes must involve $y[g]$.

With the variables (77)–(78), the ultra-weak field expansion of the MOND Lagrangian (75) takes the form

$$\mathcal{L}_{\rm MOND} = \frac{9a_0^2}{32\pi G}\Big(-x^2 + y^2 + x^3 - y^3 + \dots\Big)\sqrt{-g} \; . \tag{79}$$

We are therefore seeking an extension of the bracketed term in (79) which is suppressed, relative to y^2, for large y and $x \sim 0$, and whose corrections to y^3 are numerically small for $y \lesssim 1$. Of course many functions of x and y have this property. However, we also need to pass tests of post-Newtonian gravity in the solar system and in binary pulsars, therefore the suppression of (79) should be very efficient at small distances. An extra constraint on any possible extension of (79) is that its variation with respect to x (i.e., to the radial component of the metric, a) should allow the looked-for solution $x = 0$ (i.e., $a = rb'$). We just quote here two examples of such extensions having the required properties, and their associated behaviors for large y and $x \sim 0$:

$$(y - x) \times \Big(x + xy + y\Big)e^{-(x+y)} \longrightarrow y^2 e^{-y}, \tag{80}$$

$$\big(y^2 e^{-y} - x^2 e^{-x}\big)\, e^{-y^2} \longrightarrow y^2 e^{-y^2}. \tag{81}$$

It is easy to check that the predicted deviations from general relativity are exponentially small with respect to the tightest solar-system constraints, but that the MOND behavior (79) is predicted at large distances.

The MOND Lagrangian (22) was constructed in Sec. II in order to cancel the general relativistic predictions at large distances while imposing the precise physics we wished to reproduce. In particular, we saw that it was possible to predict any amount of weak lensing by changing the numerical value of the coefficient k. In the present section, we chose $k = 1$ to recover the same weak lensing as predicted by general relativity in presence of a dark matter halo. In such a case, it is not necessary to cancel the x^2 term coming from the Einstein-Hilbert action and to add a cubic x^3 as in (79) above. Indeed, the original x^2 term is enough to force $x = 0$, and we may thus consider a Lagrangian depending only on y, for instance

$$\mathcal{L}_{\rm MOND} = \frac{9a_0^2}{32\pi G}\, y^2 e^{-y}\sqrt{-g}\ . \tag{82}$$

Added to the Einstein-Hilbert term, this suffices to reproduce the MOND dynamics and enough weak lensing at large distances, while predicting fully negligible deviations from general relativity at small distances.

None of these Lagrangians (80-82) is analytic in a_0, but they all possess the key property of vanishing when a_0 goes to zero from above. To see this, note that they vanish for $x = 0 = y$, irrespective of a_0. Note also that neither x nor y can be negative, so if $y = k/a_0$ for some positive constant k, then the limiting form, for small a_0 of (80) and (82) vanishes like e^{-k/a_0}, while the limiting form of (81) vanishes like e^{-k^2/a_0^2}.

Our final comment on explicit models concerns the "external field effect" in which MONDian behavior of one system can be severely affected by another [66]. This property is deeply embedded in the nonlocal constructions of our scalars $X[g](x)$ and $Y[g](x)$. As one can see from their definitions (63-64), these scalars involve the nonlocal operator $\square^{-1}$ acting on curvature scalars which are themselves contracted into the normalized gradient $u^\mu[g](x)$ of the invariant volume of the past light-cone. In the static, spherically symmetric limit we have studied, $X[g](x)$ and $Y[g](x)$ depend only on the central gravitating source. However, they can be quite different, even in the static limit, when other sources are present. It is highly significant that they also depend upon *past history.* This holds out the possibility for reconciling problems in describing recently disturbed systems such as the Bullet Cluster [22, 23]. Of course we cannot, at this stage, claim that our model incorporates the external field effect in a desirable way; what actually happens beyond the static, spherically symmetric limit is a matter for future study.

VI. DISCUSSION

We have considered the problem of devising a pure metric interpolation of MOND, with neither dark matter nor additional fields, for static, spherically symmetric systems. In the deep MOND regime of small accelerations, gravity is described by two weak fields, $b(r) \equiv -g_{tt} - 1$ and $a(r) \equiv g_{rr} - 1$. In this regime the MOND force law is given by equation (12), and the requirement that there be enough weak lensing is roughly $a(r) = krb'(r)$ for some positive constant k of order one. Our first result is that the ultra-weak field limiting forms of the g_{tt} and g_{rr} equations are (13-14), subject only to the ambiguity of multiplying (14) by a constant or adding such a term to (13). Our second result is that reaching this form requires the full gravitational Lagrangian $\mathcal{L}_{\rm grav}$ to possess a MOND correction to the Einstein-Hilbert term, $\mathcal{L}_{\rm grav} = \mathcal{L}_{\rm EH} + \mathcal{L}_{\rm MOND}$, where the ultra-weak field expansion of this correction takes the form (22).

We then turned to how the MOND Lagrangian $\mathcal{L}_{\rm MOND}$ depends upon a general metric. Our third result is that no local curvature scalar can reproduce the ultra-weak field form (22). The reason is that curvature scalars involve powers of two derivatives of a weak field whereas the MOND correction (22) involves powers of only a single derivative of the weak fields.

Nonlocal models have the great advantage that they allow one to effectively remove derivatives. Our fourth result is that it is possible to construct invariant nonlocal models which degenerate to (22), for static and spherically symmetric geometries in the ultra-weak field limit. In fact there seem to be many ways to do this, some of which are laid out in section V. So it would be fair, at this stage, to say we are developing a class of models rather than a unique model.

As explained in section IV, our constructions involve two nonlocal building blocks: the inverse scalar d'Alembertian (42) and the timelike vector field $u^\mu[g](x)$ formed from normalizing the gradient of either the volume of the past light-cone (53) or the closely related inverse of the Paneitz operator (57). Unlike TeVeS, the timelike vector field of our class of models is not an independent variable but rather a nonlocal functional of the metric itself. In our view this nonlocality is not fundamental but should be viewed rather as the result of quantum corrections (perhaps from the epoch of primordial inflation) to the effective field equations. So one should always bear in mind that our class of models involves the universe being released in some prepared initial state at a finite time. A derivation from fundamental theory would be in the context of the Schwinger-Keldysh formalism [59]. In the purely phenomenological context of our current work, we employ the partial integration trick (62) introduced in [63, 64] to derive causal and conserved field equations.

Although the timelike vector field $u^\mu[g](x)$ will certainly introduce preferred frame effects, we believe these should only be significant in the ultra-weak field limit for which the MOND corrections become important. Even in this regime they should be suppressed by the square of a peculiar velocity divided by the speed of light. Typical peculiar velocities are several hundreds of kilometers per second, so the suppression factor should be about 10^{-6}, which is not likely to be observable. However, it may be very significant that our model depends upon events in the past light-cone. One consequence of this dependence is that recently disturbed systems such as the Bullet Cluster may be far from the static MOND limit.

With any of the full metric interpolations described in section V, it would be possible to study the important issues of cosmological evolution and stability. As anticipated in [54], our gravitational equations (13-14) are quadratic in the ultra-weak field regime, which means the gravitational Lagrangian is cubic. That poses an obvious potential problem for stability, although our fears on this score might be avoided by the absolute values needed for the fractional powers of nonlocal scalars we employ such as (77-78). It should also be pointed out that the notion of energy for a nonlocal model is subtle, and more study of this issue is certainly required. If our class of models should prove to be unstable it might be that the time scale is $c/a_0 \approx 6/H_0$, which does not seem to pose a problem for galaxy and cluster dynamics. It might even be that the instability merely forces the weak fields back into the regime of general relativity which is stable.

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